Interaction Notes

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Telegrapher Equations for Arbitrary Frequencies and Modes – Radiation of an Infinite, Lossless Transmission Line

Jürgen Nitsch and Sergey Tkachenko

Otto-von-Guericke University Magdeburg
Institute for Fundamental Electrical Engineering
and Electromagnetic Compatibility

Abstract

Maxwell's equations for an infinite, lossless transmission line consisting of a perfectly conducting wire above a perfectly conducting ground are transformed into telegrapher equations with new generalized per-unit-length parameters of the conductor. These new line parameters are complex-valued, frequency and source dependent, and contain the radiation resistance. Their explicit expressions depend on the chosen gauge, but there is also a gauge-independent representation for them. In the quasi-static approach of the Maxwell-Telegrapher equations the line parameters become real-valued, and radiation is absent. A Poynting vector analysis leads to a deeper physical understanding and interpretation of the new parameters.

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I. Introduction

The electromagnetic (EM) interaction of multiconductor transmission line structures with high frequency EM sources (up to several GHz) becomes an increasing topic of current research. This is due to a rapid development in the information and communication technology and the accompanying necessity to guarantee a smooth EM operation of all connected devices and systems. Since radiation phenomena occur more frequently and lead to EMC-relevant perturbation effects, they have to be included in the EM analysis of electrical and electronic systems. In particular, the effective simulation of new systems in the design phase becomes a cost-saving factor. There, the demand for numerical programs which can efficiently calculate the interaction of complex EM systems with high frequency fields is one resulting consequence. In this context the use of the telegrapher equations for nonuniform multiconductor transmission lines [1, 2] seems to be an adequate means. They have to be, however, extended to become valid for arbitrary frequencies and modes. This was done by Haase and Nitsch [3]. Different from their approach, in the present paper we deal with a simple line configuration, an infinite, uniform transmission line above perfectly conducting ground, and show that the Maxwell equations for this line can be cast into the form of the telegrapher equations, by keeping the source fixed but changing the classical line parameters to generalized, source-dependent, complex-valued ones.

In Section II we calculate the new line parameters in a gauge-independent way, using the Helmholtz decomposition [4] for the electric field. Their relation to the radiation resistance is established on the basis of a Poynting vector analysis for the radiating infinite line (Section IV). We also perform a quasi-static approach for the infinite line (Section III) and obtain solutions without radiation fields. In particular, the corresponding parameters are real.

Our generalized description of transmission lines can be extended to include multiconductor lines of finite length with losses. Then, when incorporated into an existing field-theoretical computer program for complex systems as a module for very efficient calculations of linear structures, simulations of electronic systems in the GHz-regime become essentially faster. The present paper, in a first step, gives insight into new physical phenomena which are connected and inherent in the new parameters.

II. The Telegrapher Equations in the Coulomb Gauge

In this section we rely on the Helmholtz-decomposition of vector fields $\vec{V}(\vec{r})$ into a longitudinal part, $\vec{V}_{\parallel}(\vec{r})$, and a transverse part, $\vec{V}_{\perp}(\vec{r})$, such that

$$\vec{V}(\vec{r}) = \vec{V}_{\parallel}(\vec{r}) + \vec{V}_{\perp}(\vec{r})$$

and (1 a,b,c)

$$\vec{\nabla}\times\vec{V}_{\parallel}(\vec{r}) = \vec{0}, \qquad \vec{\nabla}\cdot\vec{V}_{\perp}(\vec{r}) = 0$$

holds in real space. In Fourier space these equations read after a (3-dimensional) spatial Fourier transform

$$\vec{\mathcal{V}}(\vec{k}) = \int d^3\vec{r}\; \vec{V}(\vec{r})\, e^{-j\vec{k}\vec{r}} \qquad \text{(2 a,b)}$$

$$\vec{V}(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3\vec{k}\, \vec{\mathcal{V}}(\vec{k})\, e^{j\vec{k}\vec{r}}$$

as follows:

$$\vec{\mathcal{V}}(\vec{k}) = \vec{\mathcal{V}}_{\|}(\vec{k}) + \vec{\mathcal{V}}_{\perp}(\vec{k}) \tag{3 a,b}$$

and

$$\vec{k} \times \vec{\mathcal{V}}_{\|}(\vec{k}) = \vec{0}, \quad \vec{k} \cdot \vec{\mathcal{V}}_{\perp}(\vec{k}) = 0$$

At all points $\vec{k}$, in the reciprocal space, $\vec{\mathcal{V}}_{\|}(\vec{k})$ is obtained by projection of $\vec{\mathcal{V}}(\vec{k})$ onto the unit vector $\vec{e}_{\vec{k}} = \vec{k}/k$ in the direction $\vec{k}$. Thus one has:

$$\vec{\mathcal{V}}_{\|}(\vec{k}) = \vec{e}_k \left(\vec{e}_k \cdot \vec{\mathcal{V}}(\vec{k}) \right) \tag{4 a,b}$$

and

$$\vec{\mathcal{V}}_{\perp}(\vec{k}) = \vec{\mathcal{V}}(\vec{k}) - \vec{\mathcal{V}}_{\|}(\vec{k})$$

Observe, that in reciprocal space ($\vec{k}$ - space) the relationship between a vector field $\vec{\mathcal{V}}(\vec{k})$ and its longitudinal and transversal components is of *local* nature, whereas in the real space their relationship is *not local*: $\vec{V}_{\perp}(\vec{r})$, e.g., depends on values $\vec{V}(\vec{r}\,')$ of $\vec{V}$ at all other points $\vec{r}\,'$.

After these introductory remarks, we turn to the Maxwell equations in frequency space where we indicate the decomposition of the electric field into its longitudinal and transverse parts:

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega\varepsilon_0 \vec{E}_{\|}(\vec{r}) + j\omega\varepsilon_0 \vec{E}_{\perp}(\vec{r}) \tag{5}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = \vec{\nabla} \times \vec{E}_{\perp}(\vec{r}) = -j\omega\mu_0 \vec{H}(\vec{r}) \tag{6}$$

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \vec{\nabla} \cdot \vec{E}_{\|}(\vec{r}) = \frac{1}{\varepsilon_0} \rho(\vec{r}) \tag{7}$$

$$\vec{\nabla} \cdot \vec{H}(\vec{r}) = 0 \tag{8}$$

As usual we introduce the potentials, $\varphi(\vec{r})$ and $\vec{A}(\vec{r})$, and choose the Coulomb gauge (indicated by the index C)

$$\vec{\nabla} \cdot \vec{A}(\vec{r}) \equiv \vec{\nabla} \cdot \vec{A}_C(\vec{r}) = 0 = \vec{\nabla} \cdot \vec{A}_{C\perp}(\vec{r}) \tag{9}$$

$$\left(\vec{A}_{C\|}(\vec{r}) \cong \vec{0} \right)$$

to simplify the solution of Maxwell's equations.
Then the fields can be expressed in terms of the potentials:

$$\vec{E}_{\|}(\vec{r}) = -\vec{\nabla}\varphi_C, \qquad \vec{E}_{\perp}(\vec{r}) = -j\omega\vec{A}_C, \qquad \vec{H}(\vec{r}) = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}_C(\vec{r}) \tag{10}$$

Inserting these equations into the Maxwell equations delivers two non-trivial equations:

$$\left(\Delta \vec{A}_C(\vec{r}) + \frac{\omega^2}{c^2} \vec{A}_C(\vec{r}) \right) = -\mu_0 \left(\vec{J}_{\|}(\vec{r}) + \vec{J}_{\perp}(\vec{r}) \right) + j\frac{\omega}{c^2} \vec{\nabla}\varphi_C(\vec{r}) \tag{11}$$

with $$\vec{J}_{\|}(\vec{r}) = \frac{jk}{\mu_0 c}\vec{\nabla}\varphi_C(\vec{r}), \qquad (k := \omega/c) \tag{12}$$

and

$$-\vec{\nabla}\cdot\vec{\nabla}\varphi_C(\vec{r}) = -\Delta\varphi_C(\vec{r}) = \frac{1}{\varepsilon_0}\rho(\vec{r}) \tag{13}$$

Transformation of (11) and (13) into the reciprocal Fourier space yields

$$\varphi_C(\vec{k}') = \frac{\rho(\vec{k}')}{\varepsilon_0 k'^2} \tag{14}$$

$$\mathcal{A}_C(\vec{k}') = -\frac{\mu_0 \mathcal{J}(k')}{k^2 - k'^2} + \frac{k/c\,\vec{k}'\varphi_C(\vec{k}')}{k^2 - k'^2} =: \mathcal{A}_{CJ}(\vec{k}') + \mathcal{A}_{C\varphi}(\vec{k}') \tag{15}$$

For the current density $\vec{J}(\vec{r})$ we use a simple distribution only along the z-axis

$$\vec{J}(\vec{r}) = \vec{e}_z I_0 e^{-j\tilde{k}_{inc}z}\delta(x)\delta(y) \tag{16}$$

where $\tilde{k}_{inc}$ stems from the incident plane wave and is connected with the wave number k via the angle of incidence θ (we consider an excitation by a vertically polarized plane wave, see Fig. 1):

$$\tilde{k}_{inc} = k\cos\theta \tag{17}$$

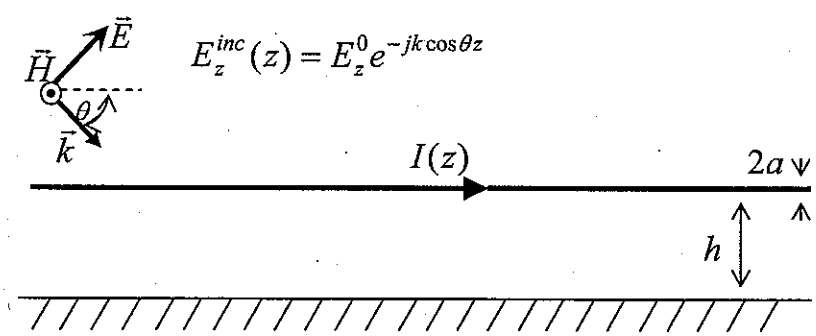


Fig. 1: Geometry of the problem.

Thus, we easily find the Fourier transform of this current density

$$\mathcal{J}(\vec{k}') = \vec{e}_z I_0 2\pi\,\delta\left(k_z' - \tilde{k}_{inc}\right) \tag{18}$$

From this we derive with the aid of the continuity equation the charge density $\rho(\vec{k}')$ and get for $\varphi_C(\vec{k}')$:

$$\varphi_C(\vec{k}') = \frac{2\pi k'_z I_0}{\omega \varepsilon_0 k'^2} \delta\left(k'_z - \widetilde{k}_{inc}\right) \tag{19}$$

Now we explicitly know all sources in equations (14) and (15) and can perform the back-transformation into the local space. These calculations become a little bit lengthy and cumbersome and also need some observations concerning the integration contour in the complex plane. At this place we only can present the results. However, the more detailed calculations will be published elsewhere [5]. For $\varphi_C(\vec{r})$ we obtain (see also [6])

$$\varphi(\vec{r}) = \frac{1}{2\pi}\left(\frac{\widetilde{k}_{inc}}{k}\right)\eta_0 I_0 e^{-j\widetilde{k}_{inc}z} K_0\left(\widetilde{k}_{inc}\rho\right) \tag{20}$$

with $\eta_0 := \sqrt{\mu_0/\varepsilon_0}$, $\rho = \left(x^2 + y^2\right)^{1/2}$ distance perpendicular to the line, and K_0 a modified Bessel function. For $\vec{A}_{CJ}(\vec{r})$ we find after a longer calculation .

$$\vec{A}_{CJ}(\vec{r}) = \vec{e}_z \frac{\mu_0 I_0}{2\pi} e^{-j\widetilde{k}_{inc}z} \begin{cases} -\dfrac{j\pi}{2} H_0^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\,\rho\right) & , \quad k > \widetilde{k}_{inc} \\ K_0\left(\sqrt{\widetilde{k}_{inc}^2 - k^2}\,\rho\right) & , \quad \widetilde{k}_{inc} > k \end{cases} \tag{21}$$

Here $H_0^{(2)}$ denotes a special Hankel function (see, e.g. [6]). In our approximation we always have $k^2 > \widetilde{k}_{inc}^2$. Therefore the first row in the eq. (21) applies.

Since for our further calculations we only need the z-component A_z of the vector potential, it is sufficient to evaluate

$$A_{C\varphi z}(\vec{r}) = -\frac{I_0 \mu_0}{2\pi} \frac{\widetilde{k}_{inc}^2}{k^2} e^{-j\widetilde{k}_{inc}z}\left[-\frac{j\pi}{2} H_0^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\,\rho\right) - K_0\left(\widetilde{k}_{inc}\rho\right)\right] \tag{22}$$

In order to get the total z-component of the vector potential we have to form the sum of the z-component in eq. (21) and of eq. (22). Also we must extend our above results to a wire above a perfectly conducting ground by using the mirror principle. Then we obtain, instead of eqs. (20)-(22) :

$$\varphi_C(\vec{r}, j\omega) = \frac{I_0 \eta_0}{4\pi}\left(\frac{\widetilde{k}_{inc}}{k}\right) e^{-j\widetilde{k}_{inc}z} g_0\left(\widetilde{k}_{inc}, \rho_1, \rho_2\right) \tag{23}$$

and

$$A_{Cz}(\vec{r}) = \left(\frac{I_0 \mu_0}{4\pi}\right) e^{-j\widetilde{k}_{inc}z}\left\{\frac{k^2 - k_{inc}^2}{k^2} g_k\left(\widetilde{k}_{inc}, \rho_1, \rho_2\right) + \left(\frac{k_{inc}^2}{k^2}\right) g_0\left(\widetilde{k}_{inc}, \rho_1, \rho_2\right)\right\} \tag{24}$$

Here we have used the abbreviations:

$$g_0\left(\widetilde{k}_{inc}, \rho_1, \rho_2\right) := 2\left(K_0\left(\widetilde{k}_{inc}\rho_1\right) - K_0\left(\widetilde{k}_{inc}\rho_2\right)\right) \tag{25}$$

and $g_k\left(\widetilde{k}_{inc},\rho_1,\rho_2\right):=\begin{cases}-j\pi\left[H_0^{(2)}\left(\sqrt{k^2-\widetilde{k}_{inc}^2}\,\rho_1\right)-H_0^{(2)}\left(\sqrt{k^2-\widetilde{k}_{inc}^2}\,\rho_2\right)\right], & k>\widetilde{k}_{inc}\\ \left[K_0\left(\sqrt{\widetilde{k}_{inc}^2-k^2}\,\rho_1\right)-K_0\left(\sqrt{\widetilde{k}_{inc}^2-k^2}\,\rho_2\right)\right], & \widetilde{k}_{inc}>k\end{cases}$ (26)

where $\rho_1=\left((x-h)^2+y^2\right)^{1/2}$ and $\rho_2=\left((x+h)^2+y^2\right)^{1/2}$ are distances to an observation point perpendicular to the wire and its image, respectively.

Now we are prepared to express explicitly the z-component of the electric field in terms of $E_{\|z}$ and $E_{\perp z}$. Choosing for he local coordinates ρ_1 and ρ_2 the surface of the wire, i.e. $\rho_1=a$ and $\rho_2=2h$ (note that the thin-wire approximation is used), then we obtain

$$E_{\|z}(a,h)=j\frac{I_0\eta_0}{4\pi}\frac{\widetilde{k}_{inc}^2}{k}e^{-j\widetilde{k}_{inc}z}G_0\left(\widetilde{k}_{inc}\right) \tag{27}$$

$$E_{\perp z}(a,h)=-jk\frac{I_0\eta_0}{4\pi}e^{-j\widetilde{k}_{inc}z}\left\{\frac{k^2-\widetilde{k}_{inc}^2}{k^2}G_k\left(\widetilde{k}_{inc}\right)+\frac{\widetilde{k}_{inc}^2}{k^2}G_0\left(\widetilde{k}_{inc}\right)\right\} \tag{28}$$

The capital-letter G-functions replace those from eqs.(23) and (24) where we have fixed the local values ρ_1 and ρ_2 on the surface of the wire.

Note that $\vec{E}_{\|}$ and $\vec{E}_{\perp}$ are *gauge independent* quantities. Therefore their representation in eqs.(27) and (28) does not depend on our choice of the Coulomb gauge. Also in the Lorenz gauge we would obtain the same result.

Next we want to correlate the longitudinal and transverse electric field to the line parameters per-unit-length relying on the representation of the differential-power density by the induced-EMF (IEMF) method [7]:

$$-E_zI^*=-E_{\|z}I^*-E_{\perp z}I^*=:Z'|I|^2=Z'_{\|}|I|^2+Z'_{\perp}|I|^2 \tag{29}$$

Insertion of the fields (eqs.(27) and (28)) and the current

$$I=I_0e^{-j\widetilde{k}_{inc}z} \tag{30}$$

into eq. (29) yields:

$$Z'_{\|}(j\omega)=\frac{\widetilde{k}_{inc}^2}{4\pi\varepsilon_0 j\omega}G_0\left(\widetilde{k}_{inc}\right)=:\frac{\widetilde{k}_{inc}^2}{j\omega C'_C(j\omega)} \tag{31}$$

with $$C'_C(j\omega)=4\pi\varepsilon_0G_0^{-1}(\widetilde{k}_{inc}) \tag{32}$$

and $$Z'_{\perp}(j\omega)=\frac{jk\eta_0}{4\pi}\left\{\frac{k^2-\widetilde{k}_{inc}^2}{k^2}G_k\left(\widetilde{k}_{inc}\right)+\frac{\widetilde{k}_{inc}^2}{k^2}G_0\left(\widetilde{k}_{inc}\right)\right\}=:j\omega L'_C(j\omega) \tag{33}$$

with $$L'_C(j\omega)=\frac{\mu_0}{4\pi}\left\{\frac{k^2-\widetilde{k}_{inc}^2}{k^2}G_k\left(\widetilde{k}_{inc}\right)+\frac{\widetilde{k}_{inc}^2}{k^2}G_0\left(\widetilde{k}_{inc}\right)\right\} \tag{34}$$

We may denote C'_C as a generalized capacitance per-unit-length and L'_C as generalized inductance per-unit-length of our lossless conductor above ground. The quantity C'_C is a pure real quantity which does not depend on k. This is not quite surprising since it was calculated from an instantaneous field. L'_C is complex-valued, and we will show that the imaginary part of it is correlated with the radiation resistance. An expansion of L'_C and C'_C for small arguments, i.e. $ka \ll 1$ and $2kh \ll 1$, leads to the well-known classical static transmission line parameters

$$C'_C(j\omega) \cong \frac{2\pi\varepsilon_0}{\ln(2h/a)} \qquad \text{and} \qquad L'_C(j\omega) \cong \frac{\mu_0}{2\pi}\ln(2h/a) \tag{35}$$

Also the case of grazing ($\theta = 0$) incidence is of interest. Then the parameters become inverse to each other and assume almost quasi-static values like for a TEM-mode excitation [8]:

$$C'^{\theta=0}_C(j\omega) = 4\pi\varepsilon_0 G_0^{-1}\left(\tilde{k}_{inc}\right), \qquad L'^{\theta=0}_C(j\omega) = \frac{\mu_0}{2\pi} G_0\left(\tilde{k}_{inc}\right) \tag{36}$$

In concluding this section we justify the notation of L'_C and C'_C as line parameters showing the close analogy to the classical transmission line equations. We refer to eqs. (23) and (24) where we have chosen the local values on the surface of the wire:

$$\varphi_C(z) = \frac{\eta_0}{4\pi}\left(\frac{\tilde{k}_{inc}}{k}\right) I(z) G_0(\tilde{k}_{inc}) \tag{37}$$

$$A_{Cz}(z) = \frac{\eta_0}{4\pi} I(z) \left\{ \frac{k^2 - \tilde{k}^2_{inc}}{k^2} G_k(\tilde{k}_{inc}) + \frac{\tilde{k}^2_{inc}}{k^2} G_0(\tilde{k}_{inc}) \right\} \tag{38}$$

The derivation of $\varphi_C(z)$ w.r.t. z yields (use eqs. (30) and (37))

$$\frac{\partial I(z)}{\partial z} + j\omega C'_C \varphi_C(z) = 0 \tag{39}$$

the first telegrapher equation. The second telegrapher equation is derived from the boundary condition:

$$\left(E_z + E_z^{inc}\right)\Big|_{\text{wire surface}} = 0 \tag{40}$$

or, equivalently,

$$\frac{\partial \varphi_C(z)}{\partial z} + j\omega A_{Cz}(z) = E_z^{inc} \tag{41}$$

Insertion of (38) into (41) gives

$$\frac{\partial \varphi_C(z)}{\partial z} + j\omega L'_C \, I(z) = E_z^{inc} \tag{42}$$

Thus we have proven that the Maxwell equations can be transformed into telegrapher equations with generalized line parameters. Therefore all known solution procedures for the telegrapher equations can also be applied to solve equations (39) and (42).

III. Maxwell's Equations without the Transverse Displacement Current - The Quasi-Static Approach

This section is devoted to the investigation of the influence of the transverse part of the displacement current on the solutions for the electromagnetic potentials and the line parameters. Frequently, in the literature, the cancellation of the transverse part of the displacement current density in Maxwell's equations is called "the quasi-static approximation" and the parameters $G_0(\widetilde{k}_{inc})$ are called "quasi-static parameters" [9, 10]. In the course of this section we will derive the quasi-static solutions and the corresponding line parameters and establish their relation to $G_0(\widetilde{k}_{inc})$.

In equations (5)-(8) we only modify Ampere's law by suppression of the transverse part of the displacement current density

$$\vec{\nabla} \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega\varepsilon_0 \vec{E}_{\|}(\vec{r}) + \underbrace{\cancel{j\omega\varepsilon_0 \vec{E}_{\perp}}(\vec{r})}_{\text{cancelled}} \tag{43}$$

All other equations remain unchanged, and instead of eq. (11) we now obtain

$$-\Delta \vec{A}_C(\vec{r}) = \mu_0 \, \vec{J}(\vec{r}) - j\omega\mu_0\varepsilon_0 \, \vec{\nabla}\varphi_C(\vec{r}) \tag{44}$$

The Fourier transform of this equation into the reciprocal space reads

$$\vec{\mathcal{A}}_C(\vec{k}') = \frac{\mu_0 \vec{\mathcal{J}}(k')}{k'^2} - \frac{k}{c}\frac{\vec{k}'\varphi_C(\vec{k}')}{k'^2} \tag{45}$$

Now we solve this equation in analogy to the previous case and get for the vector potential components

$$A_{CJz}(\vec{r}) = \frac{\mu_0 I_0}{2\pi} e^{-j\widetilde{k}_{inc}z} K_0\left(\widetilde{k}_{inc}\rho\right) \tag{46}$$

$$A_{C\varphi z}(\vec{r}) = -\frac{1}{4\pi}\mu_0 I_0 e^{-j\widetilde{k}_{inc}z}\left(\widetilde{k}_{inc}\rho\right)K_1\left(\widetilde{k}_{inc}\rho\right) \tag{47}$$

The solution for the scalar potential $\varphi_C(\vec{r})$ remains the same as in eq. (20). Therefore also the longitudinal field $E_{\|z}$ does not change (see eq. (27)). Only $E_{\perp z}$ is modified and reads

$$E_{\perp z}(\vec{r}) = -j\omega\left(A_{CJz} + A_{C\varphi_C z}\right)$$
$$= -j\omega\left[\frac{\mu_0}{2\pi} I_0\, e^{-j\tilde{k}_{inc}z} K_0\left(\tilde{k}_{inc}\rho\right) - \frac{1}{4\pi}\mu_0 I_0\left(\tilde{k}_{inc}\rho\right) e^{-j\tilde{k}_{inc}z} K_1\left(\tilde{k}_{inc}\rho\right)\right] \tag{48}$$

and for the wire above perfectly conducting ground becomes:

$$E_{\perp z}(r,h) = -jk\frac{\eta_0 I_0}{4\pi} e^{-j\tilde{k}_{inc}z}\left[g_0\left(\tilde{k}_{inc},\rho_1,\rho_2\right) - \tilde{g}_0\left(\tilde{k}_{inc},\rho_1,\rho_2\right)\right] \tag{49}$$

with

$$\tilde{g}_0\left(\tilde{k}_{inc},\rho_1,\rho_2\right) = \left[(\rho_1\tilde{k}_{inc})K_1(\rho_1\tilde{k}_{inc}) - (\rho_2\tilde{k}_{inc})K_1(\rho_2\tilde{k}_{inc})\right] \tag{50}$$

On the surface of the wire we have $\rho_1 = a$ and $\rho_2 = 2h$, and $\tilde{g}_0$ is replaced by $\tilde{G}_0\left(\tilde{k}_{inc},a,h\right)$. Analogous steps to eqs. (29) to (34) now lead to the *quasi-static* line parameters:

$$C'_C(j\omega) = 4\pi\varepsilon_0 G_0^{-1}(\tilde{k}_{inc},a,h) \tag{51}$$

(unchanged, like in (32))

and

$$L'_C(j\omega) = \frac{\mu_0}{4\pi}\left[G_0\left(\tilde{k}_{inc},a,h\right) - \tilde{G}_0\left(\tilde{k}_{inc},a,h\right)\right] \tag{52}$$

In Fig. 2 we display a simple example for the different line parameters.

Obviously, *the quasi-static* inductance per-unit-length is quite different from the result (34) and in particular from eq. (36). In the quasi-static approximation the line parameters are not inverse to each other. This, however, happens in the case of small arguments, i.e. $\tilde{k}_{inc}a \ll 1$ and $\tilde{k}_{inc}2h \ll 1$. Then $\tilde{G}_0\left(\tilde{k}_{inc},a,h\right) \cong 0$ and we have the same result as in eq.(36). Note that the quasi-static parameters are real functions, and therefore we do not have radiation losses.

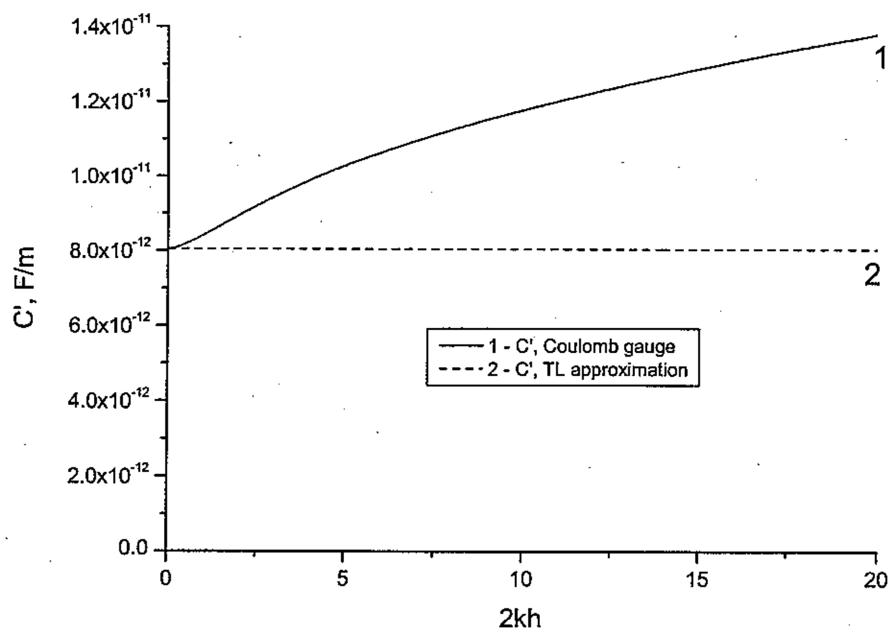
IV. The Poynting Vector of an Infinite Line

In the previous section we have derived new, generalized line parameters which occur in the generalized transmission line equations (41) and (42). Our interpretation of the line parameters as generalized per-unit-length capacitance and inductance was based on the differential-power density representation with the aid of an impedance function and the square of the current magnitude (compare eq.(29)). It was mentioned that the real part of this per-unit-length impedance equals the radiation resistance. The proof of this statement is the subject of this section.

From the electromagnetic potentials [11] in the Lorenz gauge

$$\vec{A}_L(\vec{r}) = \frac{\mu_0}{4\pi}\vec{e}_z I_0 e^{-j\tilde{k}_{inc}z} g_k\left(\tilde{k}_{inc},\rho_1,\rho_2\right) \tag{53}$$

and



a)

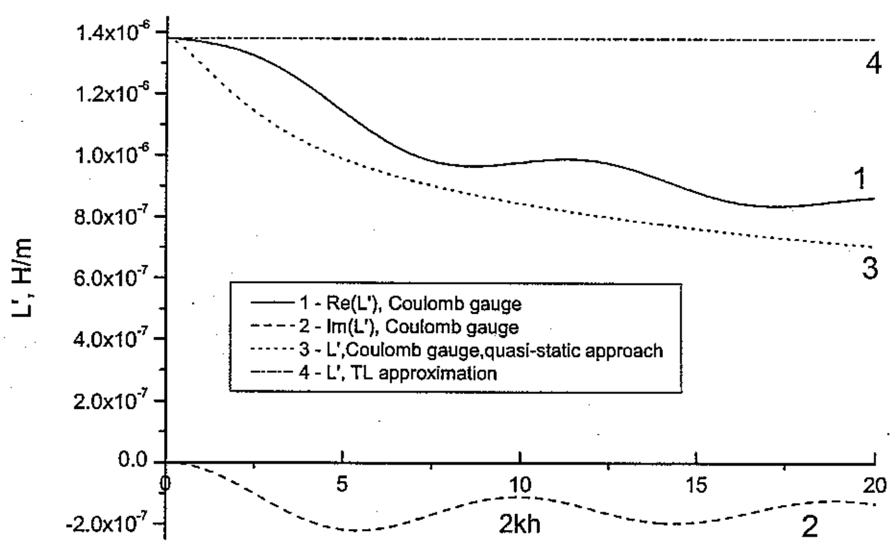


Fig. 2 Frequency dependency of generalized line parameters ($h = 0.5$ m, $a = 0.001$ m, $\theta = 45°$). a) Capacitance per-unit-length; b) Inductance per-unit-length.

$$\varphi_L(\vec{r}) = \frac{\eta_0}{4\pi} I_0\left(\frac{\widetilde{k}_{inc}}{k}\right) e^{-j\widetilde{k}_{inc}z} g_k\left(\widetilde{k}_{inc}, \rho_1, \rho_2\right) \tag{54}$$

with

$$\vec{\rho}_1 = (x-h)\vec{e}_x + y\vec{e}_y; \qquad \vec{\rho}_2 = (x+h)\vec{e}_x + y\vec{e}_y \tag{55}$$

we calculate the magnetic field $\vec{H}$ and the electric field $\vec{E}$, respectively:

$$\vec{H}(\vec{r}) = \frac{j I_0\sqrt{k^2 - \widetilde{k}_{inc}^2}}{4} e^{-j\widetilde{k}_{inc}z}\left\{H_1^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_1\right)\left[\vec{e}_{\rho_1}, \vec{e}_z\right] - H_1^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_2\right)\left[\vec{e}_{\rho_2}, \vec{e}_z\right]\right\} \tag{56}$$

The electric field is represented in two components:

$$E_z(\vec{r}) = \frac{-j\eta_0}{4\pi} I_0\left(\frac{k^2 - \widetilde{k}_{inc}^2}{k}\right) e^{-j\widetilde{k}_{inc}z} g_k\left(\widetilde{k}_{inc}, \rho_1, \rho_2\right) \tag{57}$$

and the transverse part to the conductor

$$\vec{E}_\rho(\vec{r}) = \frac{-j\eta_0 I_0}{4}\left(\frac{\widetilde{k}_{inc}}{k}\right)\sqrt{k^2 - \widetilde{k}_{inc}^2}\, e^{-j\widetilde{k}_{inc}z}\left\{H_1^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_1\right)\vec{e}_{\rho_1} - H_1^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_2\right)\vec{e}_{\rho_2}\right\} \tag{58}$$

Here we have introduced the unit vectors $\vec{e}_{\rho_1} := \left(\frac{\vec{\rho}_1}{\rho_1}\right)$ and $\vec{e}_{\rho_2} := \left(\frac{\vec{\rho}_2}{\rho_2}\right)$.

As usual, we now calculate the Poynting vector

$$\vec{S}(\vec{r}) = \frac{1}{2}\left[\vec{E}(\vec{r}), \vec{H}^*(\vec{r})\right] = \frac{1}{2}\left[E_z\vec{e}_z, \vec{H}^*\right] + \frac{1}{2}\left[\vec{E}_\rho, \vec{H}^*\right] =: \vec{S}_\perp + S_z\vec{e}_z \tag{59}$$

and decompose it into two summands. The first term, $\vec{S}_\perp$, represents the power density perpendicular to the conductor, the second term, $S_z\vec{e}_z$, is the power density which is conducted along the wire. For $\vec{S}_\perp$ we find

$$\begin{aligned}\vec{S}_\perp(\rho_1, \rho_2) = \frac{j\eta_0|I_0|^2}{32}\frac{\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\right)^3}{k}&\left\{H_0^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_1\right) - H_0^{(2)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_2\right)\right\}\\ &\left\{H_1^{(1)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_1\right)\vec{e}_{\rho_1} - H_1^{(1)}\left(\sqrt{k^2 - \widetilde{k}_{inc}^2}\rho_2\right)\vec{e}_{\rho_2}\right\}\end{aligned} \tag{60}$$

Integration over the area of the wire surface yields

$$\oint \vec{S}_\perp(\rho_1, \rho_2)\,d\vec{\sigma} = \frac{\eta_0|I_0|^2}{8}\frac{\left(k^2 - \widetilde{k}_{inc}^2\right)}{k}\left\{H_0^{(2)}\left(a\sqrt{k^2 - \widetilde{k}_{inc}^2}\right) - H_0^{(2)}\left(2h\sqrt{k^2 - \widetilde{k}_{inc}^2}\right)\right\} \tag{61}$$

Comparison of this expression with the expression for $Z'(j\omega)$ (see eqs. (29), (31) and (33))

$$Z'(j\omega)=\frac{\eta_0}{4\pi}\frac{\left(k^2-\tilde{k}_{inc}^2\right)}{jk}(\pi j)\left\{H_0^{(2)}\left(a\sqrt{k^2-\tilde{k}_{inc}^2}\right)-H_0^{(2)}\left(2h\sqrt{k^2-\tilde{k}_{inc}^2}\right)\right\} \tag{62}$$

leads us to equation

$$\oint \vec{S}_\perp(\rho_1,\rho_2)d\vec{\sigma}=\frac{Z'(j\omega)}{2}|I_0|^2 \tag{63}$$

where the factor two in the denominator stems from the time averaging procedure for the Poynting vector. Obviously, the real part of Z' represents the radiation resistance whereas its imaginary part is related to the stored energy in the near fields of the conductor.

The transported energy along wire direction can be evaluated using the power density

$$\begin{gathered}S_z(\rho_1,\rho_2)\vec{e}_z=\frac{\eta_0|I_0|^2}{32}\left(\frac{\tilde{k}_{inc}}{k}\right)\sqrt{k^2-\tilde{k}_{inc}^2}\,\vec{e}_z\\ \left|H_1^{(2)}\left(\sqrt{k^2-\tilde{k}_{inc}^2}\rho_1\right)\vec{e}_{\rho_1}-H_1^{(2)}\left(\sqrt{k^2-\tilde{k}_{inc}^2}\rho_2\right)\vec{e}_{\rho_2}\right|^2\end{gathered} \tag{64}$$

We recognize that this function is real and positive for $\tilde{k}_{inc}>0$, and also constant along the z-axis. Thus no energy is stored in this direction, almost all is led in the close neighbourhood along the conductor.

V. Conclusion

We have shown that the Maxwell equations of an ideal conductor above perfectly conducting ground can be cast into the form of telegrapher equations with generalized line parameters per-unit-length. The representation of these new line parameters turned out to be gauge-dependent. However, a definition of the generalized line parameters with respect to a Helmholtz decomposition of the electric field yields gauge-independent expressions for them. Since the Coulomb gauge is compatible with such a decomposition, the results obtained in this gauge for the parameters are also gauge-independent. The imaginary part of the new inductance per-unit-length, $\mathrm{Im}(\omega L_C')$, equals the radiation resistance of the infinite line. This proof was given by a Poynting vector analysis (comp. eq. (63)).

From classical transmission line theory it is known that for lossless lines (where only the TEM mode is assumed to propagate) the line parameters are inverse to each other. More precisely, one has [8]

$$\mathbf{L}'=\varepsilon_0\mu_0\mathbf{C}'^{-1} \tag{65}$$

with so-called "static" or "quasi-static" parameters. In our new approach we only obtain this result in the low-frequency approximation (see eq. (35)) or in case of a grazing incidence of the exciting plane wave (see eq.(36)).

Special emphasis was laid on the quasi-static approach for the fields and the new line parameters. As expected, due to the missing retardation of the fields, the line parameters turn out to be real, but do not fulfil eq. (65).

Note that the line parameters also depend on the source. It might be desirable to describe the properties of the line solely by its geometrical and inherent physical parameters, independent of its excitation. It is possible to meet these requirements. A corresponding theory for the multiconductor lines of arbitrary configuration which interact with very high frequency sources has been established by Haase and Nitsch [3,12]. This leads to an iterative solution procedure of the Telegrapher equations during the course of which the sources have to be redefined at any iteration step. In our present representation the sources are kept fixed and the parameters have to be adjusted to them.

The extension of the presented theory to multiconductor lines with losses and infinite length is straightforward. For finite multiconductor lines the theory is expected to become more involved and will to be the subject of our future investigations.

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