Interaction Notes

Note 573

25 September 2002

Complex-Valued Transmission-Line Parameters and their Relation to the Radiation Resistance

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Abstract

In this paper the telegrapher equations are extended to general modes and very high frequencies to include radiation effects. It is shown that the new line parameters are gauge dependent. However, there is also a gauge-independent representation of these parameters. In this representation the per-unit-length capacitance is not correlated with the radiation resistance, only the per-unit-length inductance (strictly speaking, the imaginary part of it) constitutes it. The generalization to multiconductor transmission lines is straightforward.

This work was sponsored by the Deutsche Forschungsgemeinschaft DFG under contract number FOR 417.

I. Introduction

Apart from high-and highest-frequency technology, nowadays, also in the information technology solutions of the Maxwell-equations become of interest at frequencies far beyond 1 GHz. This, for example, is the case in design processes of electronic devices and systems or also in the field of EMC. There, more and more frequently, radiated perturbation effects do occur.

In electrical engineering one essential tool to describe linear structures (e.g. conductors) at not too high frequencies are the so-called telegrapher equations. These equations do not contain radiation phenomena and they are not applicable at high and very high frequencies. Nonetheless, one may ask the question whether the telegrapher equations can be extended and generalized to include radiation effects in the high-frequency regime, while retaining their formal structure. This question is not quite new. It was raised in a similar kind already in the thirties [1] and again taken up recently [2,3]. In these papers it is shown that the lumped elements in RLC-circuits experience an extension when the complete displacement current is included in the Maxwell-equations. They become complex-valued, and the (integral) radiation resistance is calculated from the imaginary parts of the inductance and capacitance. On the basis of this consideration, Haase and Nitsch [4,5] represented the telegrapher equations in a new form as a "full wave transmission line theory (FWTLT)" where the new line parameters also become complex-valued. Another ansatz to derive circuit elements from complete solutions of Maxwell's equations is given by the PEEC-method [6].

In the present paper we demonstrate with the aid of an example the generalization of the telegrapher equations to arbitrary modes and frequencies and investigate the physical meaning of the new line parameters. Since we succeed to preserve the structure of the telegrapher equations, all known solution procedures of these equations can be applied immediately. We consider an infinite, lossless uniform line above a perfectly conducting ground in the thin-wire approximation. This wire is excited by an incoming plane electromagnetic wave.

In the second Section we derive the new telegrapher equations with generalized line parameters in the Lorenz gauge. These first order differential equations are equivalent to the second order Pocklington equation. The third Section mainly is devoted to a gauge independent representation of the new per - unit - length capacitance and inductance, whereas in the fourth Section their relation to the radiation resistance is established. Finally, we conclude our paper with Section five.

II. Lossless, Infinite Line above Perfectly Conducting Ground - Exact Solution

We consider a lossless, infinite transmission line above a perfectly conducting ground which is excited by an incoming plane wave (see Fig. 1). This is a well-known, simple configuration, the solution of which can be analytically expressed. Our emphasis does not lie on the solution procedure rather than on the question of the possibility to generalize the use of transmission line parameters at very high frequencies, while keeping the transmission-line structure of the corresponding equations fixed. Therefore, some steps of our derivations are already known, but we shall briefly repeat them in order to make the results complete, stringent, and compact.

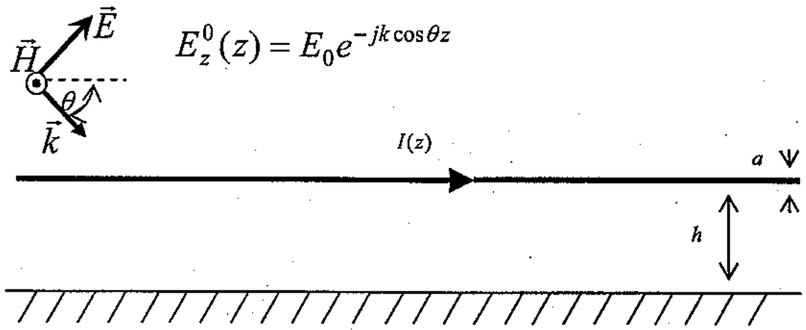


Fig. 1: Geometry of the problem.

For the excitation of the conductor in the thin-wire approach we only need the z-component $E_z^0(z)$ of the incoming field, which, for the case of a vertically – polarized plane wave, is given by

$$E_z^0(z) = E^i e^{-jk_1 z}\left(e^{jkh\sin\theta} - e^{-jkh\sin\theta}\right)\sin\theta = E_0 e^{-jk\cos\theta z} \qquad (1)$$

where θ is the angle of incidence ($0 < \theta < \pi$).

The scattered field $E_z(z)$ is calculated via the potentials $\varphi(z)$ and $A_z(z)$ in the Lorenz gauge

$$E_z = -j\omega A_z - \partial\varphi(z)/\partial z \qquad (2)$$

with

$$A_z(j\omega, z) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} g(z-z')I(z')\,dz' \qquad (3)$$

and

$$\varphi(j\omega, z) = \frac{j}{4\pi\varepsilon_0\omega}\frac{\partial}{\partial z}\int_{-\infty}^{\infty} g(z-z')I(z')\,dz' \qquad (4)$$

The Green's function $g(z)$ along the wire is given by

$$g(z) = \frac{e^{-jk\sqrt{z^2+a^2}}}{\sqrt{z^2+a^2}} - \frac{e^{-jk\sqrt{z^2+4h^2}}}{\sqrt{z^2+4h^2}} \qquad (5)$$

The letter h denotes the height of the wire above ground, and a is the radius of the wire. Insertion of the potentials into eq. (2) and the observation of the boundary condition $E_z + E_z^0 = 0$ (on the surface of the conductor) yield the two equations:

$$\frac{\partial}{\partial z}\varphi(z) + j\omega\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} g(z-z')I(z')\,dz' = E_z^0(z)$$

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty} g(z-z')I(z')\,dz' + j\omega\,4\pi\varepsilon_0\,\varphi(z) = 0 \qquad (6\text{ a,b})$$

These equations serve as the basis for the derivation of the familiar, classic telegrapher equations as well as for an exact solution. To obtain the telegrapher equations we make a low-frequency approximation for eq. (5) (i.e. $k \to 0$) and make use of the strong weighting property of the Green's function. Then we easily get the usual result

$$\frac{\partial}{\partial z}\varphi(z) + j\omega L' I(z) = E_z^0(z)$$
$$\frac{\partial}{\partial z} I(z) + j\omega C' \varphi(z) = 0 \qquad \text{(7 a,b)}$$

with the per-unit-length inductance L' and per-unit-length capacitance C' which read

$$L' = \frac{\mu_0}{2\pi}\ln(2h/a) \quad , \qquad C' = \frac{2\pi\varepsilon_0}{\ln(2h/a)} \qquad \text{(8 a,b)}$$

The solution of eqs. (7 a,b) (for the current $I(z)$ and potential $\varphi(z)$) is not the subject of our current consideration. We rather are interested in the *exact* solution of eqs. (6 a,b) and in the representation of the generalized line parameters (in the Lorenz gauge). To tackle this we first transform eqs. (6 a,b) into the one-dimensional k-space via a one dimensional Fourier-transformation and obtain

$$-jk_2\,\varphi(k_2) + j\omega\frac{\mu_0}{4\pi}G_k(k_2)I(k_2) = E_z^0(k_2)$$
$$-jk_2 I(k_2) + j\omega\, 4\pi\varepsilon_0\, G_k^{-1}(k_2)\varphi(k_2) = 0 \qquad \text{(9 a,b)}$$

with the Green's function in the k-space

$$G_k(k_2) = \begin{cases} -j\pi\left\{H_0^{(2)}\left(a\sqrt{k^2-k_2^2}\right) - H_0^{(2)}\left(2h\sqrt{k^2-k_2^2}\right)\right\} \;, & k > k_2 \\ 2\left\{K_0\left(a\sqrt{k_2^2-k^2}\right) - K_0\left(2h\sqrt{k_2^2-k^2}\right)\right\} \;, & k < k_2 \end{cases} \qquad \text{(10 a,b)}$$

Here $H_0^{(2)}(x)$ denotes the Hankel function of zeroth order and second kind, and $K_0(x)$ is the modified Bessel function of zeroth order.

Due to our simple excitation function, the Fourier transform of $E_z^0(z)$ becomes

$$E_z^0(k_2) = E_z^0\delta(k_2 - k_1), \qquad (k_1 \equiv \frac{\omega}{c}\cos(\theta)) \qquad \text{(11)}$$

and therefore G, in the z-representation, is not an integrodifferential operator (or a differential operator of infinite order: $G_k(k_2) \leftrightarrow G_k(\partial/\partial z)$) rather than a pure *number* $G_k(k_1)$. Therefore eqs. (9 a,b) simplify correspondingly (i.e. $G_k(k_2) = G_k(k_1)$ in (9 a,b)) and the back-transformation to the local space results in:

$$\frac{\partial}{\partial z}\varphi(z) + j\omega\frac{\mu_0}{4\pi}G_k(k_1)\,I(z) = E_z^0(z)$$
$$\frac{\partial}{\partial z}I(z) + j\omega\,4\pi\varepsilon_0\,G_k^{-1}(k_1)\,\varphi(z) = 0 \qquad (12\ \text{a,b})$$

Recognize that eqs. (7 a,b) and (12 a,b) are of the same structure, and they even become formally equal, if we define generalized line parameters

$$L' := \frac{\mu_0}{4\pi}G_k(k_1) \qquad \text{and} \qquad C' := 4\pi\varepsilon_0\,G_k^{-1}(k_1) \qquad (13,\ \text{a,b})$$

Remember, this result was obtained in the Lorenz gauge and it is valid for all frequencies. Different from the classic approach (8 a,b), the parameters are now complex-valued, and the meaning of the imaginary parts has to be analyzed. This becomes the subject of the following two sections. In Fig. 2 we observe that already below 100 MHz (100 MHz corresponds to $2kh = 2$ for the parameter values in the Figures) the new values significantly deviate from those of the transmission-line approximation. This holds for the real and imaginary parts of L' and C'. Also the gauge-dependency becomes quite obvious. Especially the appearance of the imaginary parts and their connection to the radiation resistance is a new remarkable property. For realistic systems at high frequencies ($f \geq 1$ GHz) one can expect differences between the classical and new parameters of about 10% to 20%. The line parameters in the Coulomb gauge will be presented in the next section.

III. Physical Interpretation of the New Line Parameters

Usually, and in particular in the low-frequency approximation, the per-unit-length capacitance is correlated with the scalar potential φ and the per-unit-length inductance with the vector potential $\vec{A}$. However, we have to keep in mind that the potentials are gauge-dependent, and therefore we obtain also gauge-dependent expressions for the line parameters. This is not very satisfactory and one may ask for a gauge–independent access to the line parameters. We can achieve such an approach as follows. Consider the, for our purposes, essential component of the electric field, $E_z(z)$, which we write in two different ways:

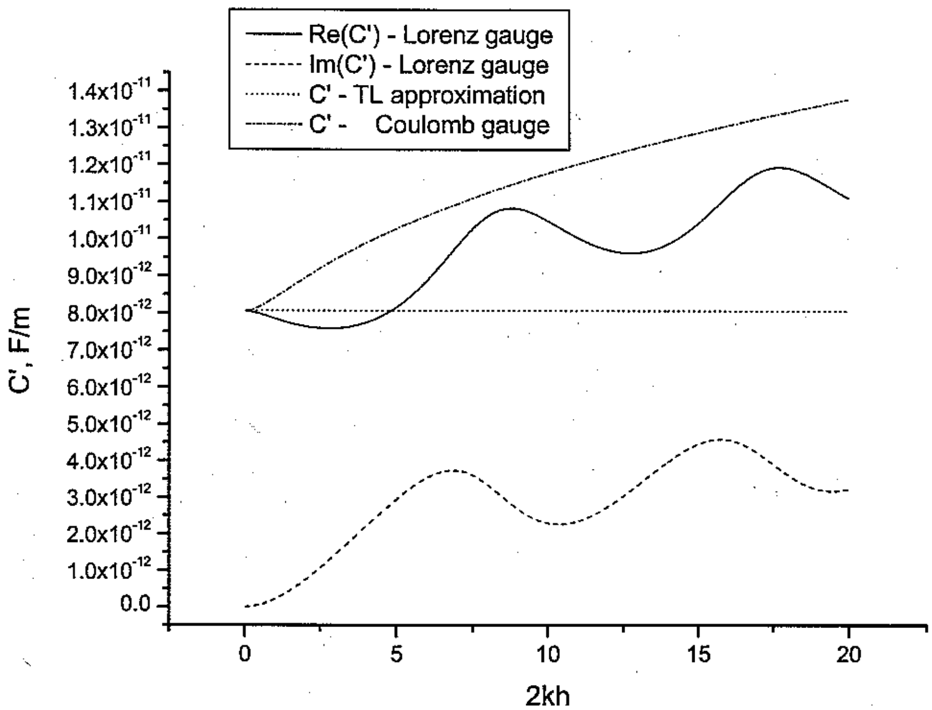
$$E_z(z) = -j\omega A_z - \frac{\partial}{\partial z}\varphi(z) =: \begin{cases} E_z^A(z) + E_z^\varphi(z) & \text{- gauge dependent decomposition} \\ \left(\vec{E}_{trans}\right)_z + \left(\vec{E}_l\right)_z & \text{- gauge independent decomposition} \end{cases}$$

(14 a,b)

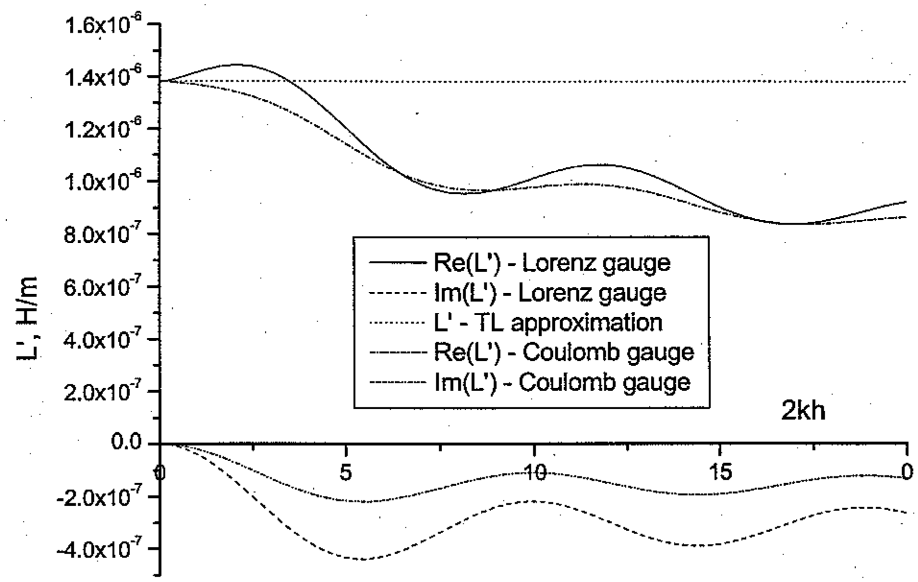
The second definition uses the Helmholtz – decomposition of the electric field

$$\vec{E} = \vec{E}_l + \vec{E}_{trans} \qquad (15,\ \text{a,b})$$

with $\qquad div\,\vec{E}_{trans} = 0, \qquad rot\,\vec{E}_l = \vec{0}$



a)



b)

Fig. 2 Frequency dependency of generalized line parameters ($h = 0.5$ m, $a = 0.001$ m, $\theta = 45°$). a) Capacitance per-unit-length; b) Inductance per-unit-length.

This decomposition results into two *gauge-independent* contributions: the longitudinal, instantaneous field $\vec{E}_l$ [7] and the transverse (to the $\vec{k}$ - vector in the $\vec{k}$ - space) electric field $\vec{E}_{trans}$ [7] which contains retarded terms as well as an $\vec{E}_l$ - compensating term. We only need to consider the z – components of these fields

$$\begin{aligned}(\vec{E}_l)_z &= (-\nabla\varphi)_z - j\omega(\vec{A}_l)_z \\ (\vec{E}_{trans})_z &= -j\omega(\vec{A}_{trans})_z\end{aligned} \tag{16 a,b}$$

and constitute $E_z(z)$ as

$$E_z(z) = \left\{-\frac{\partial}{\partial z}\varphi(z) - j\omega(\vec{A}_l)_z\right\} - j\omega(\vec{A}_{trans})_z \tag{17}$$

With this field and the current *I(z)* we can present the differential-power density by the induced-EMF (IEMF) method [8]:

$$-E_z(z)I^*(z) =: Z'(z)I(z)I^*(z) \tag{18}$$

where we have defined the per-unit-length impedance function $Z'(z)$. As usual, we now write Z' in terms of L' and C'

$$Z'(z) = j\omega L'(z) + \frac{k_1^2}{j\omega C'(z)} \tag{19}$$

Here, k_1 denotes a suitable, so far not fixed, factor (wave number) which has to be introduced to get the correct dimension. Insertion of eqs. (17) and (19) into (18) yields the gauge-independent expressions for L' and C':

$$L'(z) = \left(\frac{A_{trans\,z}(z)}{I(z)}\right), \qquad C'(z) = \left(\frac{k_1^2}{j\omega}\right)\frac{I(z)}{\left[\frac{\partial}{\partial z}\varphi(z) + j\omega A_{lz}(z)\right]} \tag{20 a,b}$$

Only the transverse part of the vector potential contributes to L', whereas the longitudinal part A_{lz} and the scalar potential occur in the expression for C'. Note, that our results, obtained in the Lorenz gauge, are not compatible with (20 a,b). They correspond to the gauge-dependent (gd) calculations

$$L'_{gd}(z) = \left(\frac{A_z(z)}{I(z)}\right), \qquad C'_{gd}(z) = \left(\frac{k_1^2}{j\omega}\right)\left(\frac{I(z)}{\frac{\partial}{\partial z}\varphi(z)}\right) \tag{21 a,b}$$

It is interesting to note that in the Coulomb-gauge the equations (20 a,b) and (21 a,b) coincide and lead to the same, gauge independent results. In the Coulomb gauge we derive

$$L'_C = \frac{\mu_0}{4\pi}\left\{\frac{k^2 - k_1^2}{k^2} G_k(k_1) + \frac{k_1^2}{k^2} G_0(k_1)\right\}, \qquad C'_C = 4\pi\,\varepsilon_0 G_0^{-1}(k_1) \qquad (22\text{ a,b})$$

with

$$G_k(k_1) = -j\pi\left\{H_0^{(2)}(ka\sin\theta) - H_0^{(2)}(2kh\sin\theta)\right\}$$
$$G_0(k_1) = 2\left\{K_0\left(ka|\cos\theta|\right) - K_0\left(2kh|\cos\theta|\right)\right\} \qquad (23\text{ a,b})$$

Note that the capacitance per-unit-length C'_C is a real function of k_1, whereas L'_C is complex-valued. Thus, only the near fields (reactive energy) contribute to C'_C, the radiation resistance (see next section) is solely connected with L'_C.

It is worth to be mentioned, and will be more explicitely elaborated elsewhere [9], that with eqs. (22 a,b) an equivalent set of equations to (12 a,b) can be derived:

$$\frac{\partial \varphi_C}{\partial z} + j\omega L'_C I(z) = E_z^0(z)$$
$$\frac{\partial I}{\partial z} + j\omega C'_C \varphi_C(z) = 0 \qquad (24\text{ a,b})$$

We have found two different sets of equations ((12 a,b) and (24 a,b)) of the same structure which describe the same physical phenomenon. Since all physical, observable quantities which are derived from these equations are equal (like e.g. $I(z), Z'(z)$, $E_z(z)$, etc.) it remains to decide whether one would prefer a gauge-independent representation for L' and C'. In this case the Coulomb-gauge would be preferable to the Lorenz-gauge. In view of a generalization of electrodynamics to quantum electrodynamics [7] also the Coulomb-gauge is used.

IV. The New Parameters and their Relation to the Radiation Resistance

This section is devoted to the investigation of the relation between the new parameters and the radiation resistance of the lossless, infinite line. To tackle this problem we first estimate the vector potential of the far field for coordinates perpendicular to the propagation direction of the current

$$I(z) = I_0 \cdot \exp\left(-jk_1 z\right) \qquad (25)$$

The vector potential at far distances orthogonal to the wire is obtained by an expansion of the appropriate Green's function for large arguments and results in [10] :

$$\vec{A} \underset{k\rho>>1}{=} \vec{e}_z \mu_0 I_0 \exp\left(-jk(z\cos\theta + \rho\sin\theta)\right)\left(\frac{j}{2\pi\rho k\sin\theta}\right)^{1/2} \sin(kh\sin\theta\cos\varphi) \qquad (26)$$

Here $\rho = \sqrt{x^2 + y^2}$ is the magnitude of a local vector which is directed to a point (x, y) lying in a plane perpendicular to the z-direction. With the aid of $\vec{A}(\rho, z)$ we calculate the magnetic and electric far-field components (which decrease like $\rho^{-1/2}$), $\vec{H}_{far}(\rho,z)$ and $\vec{E}_{far}(\rho,z)$, respectively and the corresponding time-averaged Poynting vector

$$\vec{S}=\frac{1}{2}\mathrm{Re}\left[\vec{E}_{far},\vec{H}^{*}_{far}\right]=\frac{\eta_0 k\sin(\theta)|I_0|^2}{4\pi\rho}\left(\cos(\theta)\vec{e}_\rho+\sin(\theta)\vec{e}_z\right)\cdot\sin^2(kh\sin\theta\sin\varphi) \qquad (27)$$

where $\eta_0=\sqrt{\mu_0/\varepsilon_0}$ denotes the wave impedance of free space. Observe, that the power flux density has a radial component ($\vec{e}_\rho$ - direction, into upper half-space) which corresponds to radiation and a z-component ($\vec{e}_z$ - direction) which corresponds to energy propagation along the wire. The power radiated per-unit-length by the wire is obtained through integration of the radial component of the Poynting vector in the upper half space ($-\pi/2\le\varphi\le\pi/2$):

$$P_S=\int_{-\pi/2}^{\pi/2}S_\rho\rho d\varphi=\frac{\eta_0 k\sin^2(\theta)|I_0|^2}{8}\left(1-J_0(2kh\sin\theta)\right) \qquad (28)$$

The uniformly distributed radiation resistance along the wire now is defined via the equation

$$P_S=:\frac{1}{2}R'_{rad}|I_0|^2 \qquad (29)$$

and gives

$$R'_{Rad}=\frac{\eta_0 k\sin^2(\theta)}{4}\left(1-J_0(2kh\sin\theta)\right) \qquad (30)$$

On the other hand this resistance is connected with the above introduced quantity $Z'(z)$ by

$$R'_{rad}=\mathrm{Re}\{Z'(z)\}=\mathrm{Re}\left\{j\omega L'+\frac{k_1^2}{j\omega C'}\right\} \qquad (31)$$

Of course, R'_{rad} is gauge-independent and related with the imaginary part of the Green's function in k-space :

$$R'_{rad}=-\omega\,\mathrm{Im}(L'_C)=-\frac{\mu_0}{4\pi}\omega\frac{\left(k^2-k_1^2\right)}{k^2}\mathrm{Im}(G_k(k_1))=$$

$$\frac{\eta_0 k}{4}\sin^2\theta\left[J_0(ka\sin\theta)-J_0(2kh\sin\theta)\right]\underset{ka\ll 1}{\approx}\frac{\eta_0 k\sin^2(\theta)}{4}\left[1-J_0(2kh\sin\theta)\right] \qquad (32)$$

In eq. (32) we have used the fact that in the thin-wire approximation $ka \ll 1$.

Our above investigation resembles somewhat a similar problem which already has been treated in the thirties [1] and has been again reinvented later on [2, 3]. In these papers it is shown that in RLC-circuits the lumped elements have to be generalized if the complete displacement current is included in the calculations. They thereby become complex-valued and the (integral) radiation resistance can be evaluated via the imaginary parts of the inductance and the capacitance. However, gauge dependencies were not discussed.

V. Conclusion

With our example of a lossless, infinite line above perfectly conducting ground we have shown that the telegrapher equations can be generalized to include higher modes and radiation effects. The new line parameters which occur turn out to be gauge dependent and complex-valued. The gauge dependency can be removed by using the Helmholtz decomposition of the electric field and in consequence also of the vector potential with a subsequent corresponding calculation of the line parameters. The imaginary parts of these parameters constitute the radiation resistance. They only appear if the complete displacement current (i.e. in particular also the transverse part of it) is taken into consideration in Maxwell's equations. Neglecting of the transverse displacement current ($\varepsilon_0 \, \partial \vec{E}_{trans} / \partial t$) leads to a quasi-static approach of the solution of Maxwell's equations and therefore does not contain retardation effects.

Also the excitation influences the radiation. The far-field Poynting vector has two components: one perpendicular to the conductor axis, the other parallel to it. In case that the wave vector of the incoming wave is parallel to the wire axis (we practically have a TEM current wave, induced in the wire), then there is no radiation[1]; all energy is guided (in the near zone) along the line. In the other extreme case ($\theta = \pi/2$), all energy is emitted in ρ-direction.
Eventually, we mention that the generalization to multiconductor lines is straight-forward. Also for finite lines our procedure can be applied in a kind of a perturbation approximation and will become the subject of another paper.

Acknowledgment
The authors are grateful to Prof. G. Wollenberg and to Dr. C. Baum for helpful discussions.

References

[1] W.Wessel, Über den Einfluss des Verschiebungsstromes auf den Wechselstromwiderstand, Zeitschr. f. techn. Physik, 472-475, 1936.

[2] W.Mathis, Theorie nichtlinearer Netzwerke, Springer-Verlag, Berlin-Heidelberg, 1987.

[3] W.Mathis, Über nicht-quasistationäre Effekte in elektrischen Schaltungen, Kleinheubacher Berichte 2000, Band 44, S. 125 -129, 2001.

[4] H.Haase, J.Nitsch, Full-wave transmission line theory (FWTLT) for the analysis of three – dimensional wire-like structures, Zurich Symposium on EMC, Feb.2001, pp.235-240.

[1] Strictly speaking, in this case the wave does not notice the wire, and just propagates like in free space.

[5] H.Haase, J.Nitsch, Investigation of simple nonuniform transmission line structures by a generalized transmission line theory, Zurich Symposium on EMC, Feb. 2003. (Accepted for publication.)

[6] S.V.Kochetov, G.Wollenberg, Considering the skin effect in wire-like 3D interconnection structures with arbitrary cross sections by a new modification of the PEEC method, Zurich Symposium on EMC, Feb. 2003. (Accepted for publication).

[7] C.Cohen-Tannoudji, J.Dupont-Roc, G.Grynberg, Photons and Atoms, Introduction to Quantum Electrodynamics, J. Wiley & Sons, Inc., New-York, 1997.

[8] E.K.Miller, PCs for AP and Other EM Reflections, IEEE Antennas and Propagation Magazine, Vol. 41, No. 2, April 1999, pp. 82-86.

[9] J. Nitsch, S.Tkachenko, Radiation of an infinite, lossless transmission line, The 2003 IEEE International Symposium on EMC, Istanbul, Turkey, May 2003.

[10] S.Tkachenko, J.Nitsch, Investigation of high-frequency coupling with uniform and non-uniform lines: Comparison of exact analytical results with those of different approximations, International Union of Radio Science XXVII General Assembly, Maastricht, the Netherlands, 7-24 Aug., 2002.