Interaction Notes

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Quadrupole Terms in Magnetic Singularity Identification, Part 2

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Abstract

This paper follows up on previous quadrupole considerations for magnetic singularity identification (MSI). Using the 2-norm over the unit sphere one can find an optimal center of a natural mode for minimum quadrupole contribution. Symmetry considerations are also extended to discrete two-dimensional rotation symmetry. A set of Appendices is included for many of the mathematical details.

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1. Introduction

This paper continues the treatment of the quadrupole terms in magnetic singularity identification (MSI) in [4] which we can conveniently reference as Part 1. In that paper the magnetic-dipole formulae were extended to magnetic-quadrupole terms as certain integrals over the magnetization vector, particularly in the form of natural modes. Then questions were addressed concerning optimal choice of coordinate origin to "minimize" the quadrupole term associated with modes having a magnetic-dipole moment. The present paper (Part 2) further extends this development, including the use of norms and the point symmetry groups.

Summarizing from Part 1, we have

$\vec{\chi}_\alpha(\vec{r}) \equiv$ magnetization (or magnetic-moment density) nautral mode

$\vec{m}_\alpha \equiv \int_V \vec{\chi}_\alpha(\vec{r})\,dV \equiv$ magnetic (dipole) moment

$\vec{H}_\alpha^{(sc,d)} = \vec{h}^{(d)}(\vec{r}) \cdot \vec{m}_\alpha \equiv$ magnetic-dipole part of magnetic-field natural mode

$$\vec{h}^{(d)}(\vec{r}) = \frac{1}{4\pi r^3}\left[3\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right]$$

$$r \equiv |\vec{r}| \;,\quad \vec{1}_r = \frac{\vec{r}}{r} \;\;(\text{for } r \neq 0)$$

$\overleftrightarrow{1} = \sum_{\ell=1}^{3} \vec{1}_\ell\,\vec{1}_\ell \equiv$ three-dimensional identity

$\vec{1}_1, \vec{1}_2, \vec{1}_3 \equiv$ any set of real orthogonal unit vectors

$\vec{H}_\alpha^{(sc,q)}(\vec{r}) = \left\langle \overleftrightarrow{h}^{(q)}(\vec{r},\vec{r}\,'); \vec{\chi}_\alpha(\vec{r}\,') \right\rangle \equiv$ magnetic-quadrupole part of magnetic-field natural mode

$$\overleftrightarrow{h}^{(q)}(\vec{r},\vec{r}\,') = \frac{3}{4\pi r^4}\left[\left[\vec{1}_r \cdot \vec{r}\,'\right]\left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right] - \vec{1}_r\,\vec{r}\,' - \vec{r}\,'\,\vec{1}_r\right] = \overleftrightarrow{h}^{(q)T}(\vec{r},\vec{r}\,')$$

Refer to Part 1 for details, including assumptions.

2. 2-Norm Over Solid Angles (Unit Sphere)

Part of the problem in choosing the coordinate origin (location of $\vec{r} = \vec{0}$) for minimizing the quadrupole term concerns its variability as one considers different angles around the object. This is contained in the dependence on $\vec{1}_r$. Which $\vec{1}_r$ should one choose? One approach to this problem is to take a norm over all values of $\vec{1}_r$ since a value of zero implies zero for "all" $\vec{1}_r$. Various norms are possible, such as the ∞-norm (peak). For present purposes and convenient analytic properties we will use the 2-norm.

So let us form

$$\left\| \vec{H}_\alpha^{(sc,q)}(\vec{r}) \right\|_2 = \left[\int_{S_1} \left| \vec{H}_\alpha^{(sc,q)}(r, \vec{1}_r) \right|^2 dS_1 \right]^{\frac{1}{2}} \tag{2.1}$$

S_1 = unit sphere

where r is held constant and $\vec{1}_r$ varies over the unit sphere. This gives

$$\begin{aligned} U_\alpha &\equiv \frac{16\pi^2 r^4}{9} \left\| \vec{H}_\alpha^{(sc,q)}(\vec{r}) \right\|_2^2 \\ &= \int_{S_1} \left| \left\langle \overleftrightarrow{V}(\vec{1}_r, \vec{r}\,'); \; \vec{\chi}_\alpha(\vec{r}\,') \right\rangle \right|^2 dS_1 \\ \overleftrightarrow{V}(\vec{1}_r, \vec{r}\,') &= \left[\vec{1}_r \cdot \vec{r}\,' \right] \left[5\, \vec{1}_r \vec{1}_r - \overleftrightarrow{1} \right] - \vec{1}_r \vec{r}\,' - \vec{r}\,' \vec{1}_r = \overleftrightarrow{V}^T(\vec{1}_r, \vec{r}\,') \end{aligned} \tag{2.2}$$

as the thing to minimize.

Rearrange (2.2) as

$$\begin{aligned} U_\alpha &= \int_{S_1} \left\langle \overleftrightarrow{V}(\vec{1}_r, \vec{r}\,'); \; \vec{\chi}_\alpha(\vec{r}\,') \right\rangle \cdot \left\langle \overleftrightarrow{V}(\vec{1}_r, \vec{r}\,''); \; \vec{\chi}_\alpha(\vec{r}\,'') \right\rangle dS_1 \\ &= \int_{S_1} \left\langle \vec{\chi}_\alpha(\vec{r}\,'); \; \overleftrightarrow{V}(\vec{1}_r, \vec{r}\,') \cdot \overleftrightarrow{V}(\vec{1}_r, \vec{r}\,''); \; \vec{\chi}_\alpha(\vec{r}\,'') \right\rangle dS_1 \end{aligned} \tag{2.3}$$

Interchange the order of integration over S_1 with the integrals over $\vec{r}'$ and $\vec{r}''$ to give

$$
\begin{aligned}
U_\alpha &= \left\langle \vec{\chi}_\alpha(\vec{r}'); \overleftrightarrow{W}_\alpha(\vec{r}',\vec{r}''); \vec{\chi}_\alpha(\vec{r}'') \right\rangle \\
\overleftrightarrow{W}(\vec{r}',\vec{r}'') &\equiv \int_{S_1} \overleftrightarrow{\chi}(\vec{1}_r,\vec{r}',\vec{r}'')\,dS_1 \\
\overleftrightarrow{X}(\vec{1}_r,\vec{r}',\vec{r}'') &= \overleftrightarrow{V}(\vec{1}_r,\vec{r}') \cdot \overleftrightarrow{V}(\vec{1}_r,\vec{r}'') \\
&= \left[\left[\vec{1}_r \cdot \vec{r}'\right]\left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right] - \vec{1}_r\,\vec{r}' - \vec{r}'\,\vec{1}_r\right] \\
&\quad \cdot \left[\left[\vec{1}_r \cdot \vec{r}''\right]\left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right] - \vec{1}_r\,\vec{r}'' - \vec{r}''\,\vec{1}_r\right] \qquad (2.4) \\
\overleftrightarrow{X}^T(\vec{1}_r,\vec{r}',\vec{r}'') &= \overleftrightarrow{X}(\vec{1}_r,\vec{r}'',\vec{r}') \\
\overleftrightarrow{W}^T(\vec{r}',\vec{r}'') &= \overleftrightarrow{W}(\vec{r}'',\vec{r}')
\end{aligned}
$$

Let us divide this into four parts for analysis as

$$
\begin{aligned}
\overleftrightarrow{X}(\vec{1}_r,\vec{r}',\vec{r}'') &= \sum_{m=1}^{4} \overleftrightarrow{X}^{(m)}(\vec{1}_r,\vec{r}',\vec{r}'') \\
\overleftrightarrow{W}(\vec{r}',\vec{r}'') &= \sum_{m=1}^{4} \overleftrightarrow{W}^{(m)}(\vec{r}',\vec{r}'') \\
\overleftrightarrow{W}^{(m)}(\vec{r}',\vec{r}'') &= \int_{S_1} \overleftrightarrow{X}^{(m)}(\vec{1}_r,\vec{r}',\vec{r}'')\,dS_1
\end{aligned}
$$

$$
\begin{aligned}
\overleftrightarrow{X}^{(1)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= \left[\vec{1}_r \cdot \vec{r}\,'\right]\left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right] \cdot \left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right] \cdot \left[\vec{1}_r \cdot \vec{r}\,''\right] \\
\overleftrightarrow{X}^{(2)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= \left[\vec{1}_r\,\vec{r}\,' + \vec{r}\,'\,\vec{1}_r\right] \cdot \left[\vec{1}_r\,\vec{r}\,'' + \vec{r}\,''\,\vec{1}_r\right] \\
\overleftrightarrow{X}^{(3)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= -\left[\vec{1}_r \cdot \vec{r}\,'\right]\left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right] \cdot \left[\vec{1}_r\,\vec{r}\,'' + \vec{r}\,''\,\vec{1}\right] \\
\overleftrightarrow{X}^{(4)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= -\left[\vec{1}_r\,\vec{r}\,' + \vec{r}\,'\,\vec{1}_r\right] \cdot \left[5\,\vec{1}_r\,\vec{1}_r - \overleftrightarrow{1}\right]\left[\vec{1}_r \cdot \vec{r}\,''\right]
\end{aligned}
\tag{2.5}
$$

The first term gives

$$
\begin{aligned}
\overleftrightarrow{X}^{(1)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= \left[\vec{r}\,' \cdot \vec{1}_r\,\vec{1}_r \cdot \vec{r}\,''\right]\left[15\,\vec{1}_r\,\vec{1}_r + \overleftrightarrow{1}\right] \\
&= \left[\vec{r}\,' \cdot \vec{1}_r\,\vec{1}_r \cdot \vec{r}\,''\right]\overleftrightarrow{1} + 15\,\vec{r}\,' \cdot \vec{1}_r\,\vec{1}_r\,\vec{1}_r\,\vec{1}_r \cdot \vec{r}\,'' \\
\overleftrightarrow{W}^{(1)}(\vec{r}\,',\vec{r}\,'') &= \frac{4\pi}{3}\,\vec{r}\,' \cdot \vec{r}\,''\,\overleftrightarrow{1} + 4\pi\left[\vec{r}\,'\,\vec{r}\,'' + \vec{r}\,''\,\vec{r}\,' + \vec{r}\,' \cdot \vec{r}\,''\,\overleftrightarrow{1}\right]
\end{aligned}
\tag{2.6}
$$

using the results in Apendix B, specifically (B.1) and (B.16). The second term gives

$$
\begin{aligned}
\overleftrightarrow{X}^{(2)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= \vec{r}\,' \cdot \vec{1}_r\,\vec{1}_r \cdot \vec{r}\,'' + \vec{r}\,'\,\vec{1}_r\,\vec{1}_r \cdot \vec{r}\,'' + \vec{r}\,' \cdot \vec{r}\,''\,\vec{1}_r\,\vec{1}_r + \vec{r}\,'\,\vec{r}\,'' \\
\overleftrightarrow{W}(\vec{r}\,',\vec{r}\,'') &= \frac{4\pi}{3}\,\vec{r}\,'\,\vec{r}\,'' + \frac{4\pi}{3}\,\vec{r}\,'\,\vec{r}\,'' + \frac{4\pi}{3}\,\vec{r}\,' \cdot \vec{r}\,''\,\overleftrightarrow{1} + 4\pi\,\vec{r}\,'\,\vec{r}\,'' \\
&= \frac{4\pi}{3}\left[5\,\vec{r}\,'\,\vec{r}\,'' + \vec{r}\,' \cdot \vec{r}\,''\,\overleftrightarrow{1}\right]
\end{aligned}
\tag{2.7}
$$

The third term gives

$$
\begin{aligned}
\overleftrightarrow{X}^{(3)}(\vec{1}_r,\vec{r}\,',\vec{r}\,'') &= -\left[\vec{1}_r \cdot \vec{r}\,'\right]\left[5\,\vec{1}_r\,\vec{r}\,'' + 5\left[\vec{1}_r \cdot \vec{r}\,''\right]\vec{1}_r\,\vec{1}_r - \vec{1}_r\,\vec{r}\,'',\vec{r}\,''\,\vec{1}_r\right] \\
&= -\left[\vec{r}\,' \cdot \vec{1}_r\right]\left[4\,\vec{1}_r,\vec{r}\,'' - \vec{r}\,''\,\vec{1}_r + 5\,\vec{1}_r\,\vec{1}_r\left[\vec{1}_r \cdot \vec{r}\,''\right]\right]
\end{aligned}
$$

$$= -4\vec{r}' \cdot \vec{1}_r\vec{1}_r\vec{r}'' + \vec{r}''\vec{1}_r\vec{1}_r \cdot \vec{r}' - 5\vec{r}' \cdot \vec{1}_r\vec{1}_r\vec{1}_r$$

$$\overleftrightarrow{W}^{(3)}(\vec{r}', \vec{r}'') = -\frac{16\pi}{3}\vec{r}'\vec{r}'' + \frac{4\pi}{3}\vec{r}''\vec{r}' - \frac{4\pi}{3}\left[\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' + \vec{r}' \cdot \vec{r}''\overleftrightarrow{1}\right] \tag{2.8}$$

$$= \frac{4\pi}{3}\left[-4\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' - \left[\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' + \vec{r}' \cdot \vec{r}''\overleftrightarrow{1}\right]\right]$$

The fourth term gives

$$\overleftrightarrow{\chi}^{(4)}(\vec{1}_r, \vec{r}', \vec{r}'') = -\left[5\vec{r}' \cdot \vec{1}_r\vec{1}_r\vec{1}_r - \vec{1}_r\vec{r}' + 5\vec{r}'\vec{1}_r - \vec{r}'\vec{1}_r\right]\left[\vec{1}_r \cdot \vec{r}''\right]$$

$$= -\left[4\vec{r}'\vec{1}_r - \vec{1}_r\vec{r}' + 5\vec{r}' \cdot \vec{1}_r\vec{1}_r\vec{1}_r\right]\left[\vec{1}_r \cdot \vec{r}''\right]$$

$$= -4\vec{r}'\vec{1}_r\vec{1}_r \cdot \vec{r}'' + \vec{r}'' \cdot \vec{1}_r\vec{1}_r\vec{r}' - 5\vec{r}' \cdot \vec{1}_r\vec{1}_r\vec{1}_r\vec{1}_r \cdot \vec{r}'' \tag{2.9}$$

$$\overleftrightarrow{W}^{(4)}(\vec{r}', \vec{r}'') = -\frac{16\pi}{3}\vec{r}'\vec{r}'' + \frac{4\pi}{3}\vec{r}''\vec{r}' - \frac{4\pi}{3}\left[\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' + \vec{r}' \cdot \vec{r}''\overleftrightarrow{1}\right]$$

$$= \frac{4\pi}{3}\left[-4\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' - \left[\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' + \vec{r}' \cdot \vec{r}''\overleftrightarrow{1}\right]\right]$$

Summing these gives

$$\overleftrightarrow{W}(\vec{r}', \vec{r}'') = \frac{4\pi}{3}\left[-3\vec{r}'\vec{r}'' + 2\vec{r}''\vec{r}' + \vec{r}' \cdot \vec{r}''\overleftrightarrow{1} + \left[\vec{r}'\vec{r}'' + \vec{r}''\vec{r}' + \vec{r}' \cdot \vec{r}''\overleftrightarrow{1}\right]\right]$$
$$= \frac{4\pi}{3}\left[-2\vec{r}'\vec{r}'' + 3\vec{r}''\vec{r}' + +3\vec{r}' \cdot \vec{r}''\overleftrightarrow{1}\right] \tag{2.10}$$

Going back a step to U_α we have

$$\frac{3}{4\pi}U_\alpha = -2\left\langle \vec{\chi}_\alpha(\vec{r}');\ \vec{\chi}_\alpha(\vec{r}'')\right\rangle$$
$$+3\left\langle \vec{\chi}_\alpha(\vec{r}');\ \vec{r}''\vec{r}';\ \vec{\chi}_\alpha(\vec{r}'')\right\rangle \tag{2.11}$$
$$+3\int_V\int_V \vec{r}' \cdot \vec{r}''\vec{\chi}_\alpha(\vec{r}'') \cdot \vec{\chi}_\alpha(\vec{r}')\,dV''\,dV'$$

Using

$$\vec{m}_\alpha = \int_V \vec{\chi}_\alpha(\vec{r})\,dV$$
$$q_\alpha \equiv \int_V \vec{r} \cdot \vec{\chi}_\alpha(\vec{r})\,dV \qquad (2.12)$$
$$\overleftrightarrow{Q}_\alpha \equiv \int_V \vec{r}\,\vec{\chi}_\alpha(\vec{r})\,dV$$

with all parameters real valued (scalars, vectors, dyadics), we have

$$\left\langle \vec{\chi}_\alpha(\vec{r}\,');\ \vec{r}\,'\vec{r}\,'';\ \vec{\chi}_\alpha(\vec{r}\,'') \right\rangle = \left\langle \vec{\chi}_\alpha(\vec{r}\,'); \vec{r}\,' \right\rangle \left\langle \vec{r}\,''; \vec{\chi}_\alpha(\vec{r}\,'') \right\rangle$$
$$\left\langle \vec{\chi}_\alpha(\vec{r}\,'');\ \vec{r}\,''\vec{r}\,';\ \vec{\chi}_\alpha(\vec{r}\,'') \right\rangle = \int_V\int_V \vec{\chi}_\alpha(\vec{r}\,') \cdot \vec{r}\,''\vec{\chi}_\alpha(\vec{r}\,'') \cdot \vec{r}\,'\,dV''\,dV'$$
$$= \int_V \vec{\chi}_\alpha(\vec{r}\,') \cdot \overleftrightarrow{Q}_\alpha \cdot \vec{r}\,'\,dV' = \int_V \vec{r}\,' \cdot \overleftrightarrow{Q}_\alpha^{T} \cdot \vec{\chi}_\alpha(\vec{r}\,')\,dV' \qquad (2.13)$$
$$\int_V\int_V \vec{r}\,' \cdot \vec{r}\,''\vec{\chi}_\alpha(\vec{r}\,'') \cdot \vec{\chi}_\alpha(\vec{r}\,')\,dV''\,dV'$$
$$= \int_V \vec{r}\,' \cdot \overleftrightarrow{Q}_\alpha \cdot \vec{\chi}_\alpha(\vec{r}\,')\,dV'$$

Combining these gives

$$U_\alpha = \frac{4\pi}{3}\left[-2q_\alpha^2 + 3\,Y_\alpha\right]$$
$$Y_\alpha \equiv \int_V \vec{r}\,' \cdot \left[\overleftrightarrow{Q}_\alpha + \overleftrightarrow{Q}_\alpha^{T}\right] \cdot \vec{\chi}_\alpha(\vec{r}\,')\,dV' \qquad (2.14)$$

as the thing to be minimized by optimal choice of coordinate origin.

3. Optimal Shift of Coordinates

Replacing $\vec{r}$ by $\vec{r}-\vec{r}_0$ so that $\vec{r}_0$ is the new coordinate center, we have

$$
\begin{aligned}
q_\alpha &= \int_V \left[\vec{r}-\vec{r}_0\right] \cdot \vec{\chi}_\alpha(\vec{r})dV = \int_V \vec{r} \cdot \vec{\chi}_\alpha(\vec{r})dV - \vec{r}_0 \cdot \vec{m}_\alpha \\
&\equiv q_\alpha^{(0)} - \vec{r}_0 \cdot \vec{m}_\alpha \\
\vec{Q}_\alpha &= \int_V \left[\vec{r}-\vec{r}_0\right] \vec{\chi}_\alpha(\vec{r})dV = \int_V \vec{r}\,\vec{\chi}_\alpha(\vec{r})dV - \vec{r}_0 \vec{m}_\alpha \\
&\equiv \vec{Q}_\alpha^{(0)} - \vec{r}_0 \vec{m}_\alpha \\
\vec{m}_\alpha &= \int_V \vec{\chi}_\alpha(\vec{r})dV \quad \text{(unchanged in the coordinate shift)} \\
Y_\alpha &= \int_V \left[\vec{r}-\vec{r}_0\right] \cdot \left[\vec{Q}_\alpha^{(0)} + \vec{Q}_\alpha^{(0)T} - \vec{r}_0 \vec{m}_\alpha - \vec{m}_\alpha \vec{r}_0\right] \cdot \vec{\chi}_\alpha(\vec{r})dV \\
&= \int_V \vec{r} \cdot \left[\vec{Q}_\alpha^{(0)} + \vec{Q}_\alpha^{(0)T}\right] \cdot \vec{\chi}_\alpha(\vec{r})dV - \vec{r}_0 \cdot \left[\vec{Q}_\alpha^{(0)} + \vec{Q}_\alpha^{(0)T}\right] \cdot \vec{m}_\alpha \\
&\quad - \int_V \vec{r} \cdot \left[\vec{r}_0 \vec{m}_\alpha + \vec{m}_\alpha \vec{r}_0\right] \cdot \vec{\chi}_\alpha(\vec{r})dV + \vec{r}_0^{\,2} \vec{m}_0^{\,2} + \left[\vec{r}_0 \cdot \vec{m}_\alpha\right]^2 \\
Y_\alpha^{(0)} &\equiv \int_V \vec{r} \cdot \left[\vec{Q}_\alpha^{(0)} + \vec{Q}_\alpha^{(0)T}\right] \cdot \vec{\chi}_\alpha(\vec{r})dV \\
r_0 &\equiv |\vec{r}_0| \;, \quad \vec{1}_0 = \frac{\vec{r}_0}{r_0} \\
m_\alpha &= |\vec{m}_\alpha| \;, \quad \vec{1}_\alpha = \frac{\vec{m}_\alpha}{m_\alpha}
\end{aligned}
\qquad (3.1)
$$

The zero superscripts indicate parameters evaluated before coordinate shift. Now form

$$
\begin{aligned}
U_\alpha &= \frac{4\pi}{3}\left[-2q_a^2 + 3Y_\alpha\right] \\
&= \frac{4\pi}{3}\left[-2\left[q_\alpha^{(0)} - \vec{r}_0 \cdot \vec{m}_\alpha\right]^2\right. \\
&\quad + 3\left[Y_\alpha^{(0)} - \vec{r}_0 \cdot \left[\overleftrightarrow{Q}_\alpha^{(0)} + \overleftrightarrow{Q}_\alpha^{(0)T}\right] \cdot \vec{m}_\alpha\right. \\
&\quad - \int_V \vec{r} \cdot \left[\vec{r}_0\,\vec{m}_\alpha + \vec{m}_\alpha\,\vec{r}_0\right] \cdot \vec{\chi}_\alpha(\vec{r})\,dV \\
&\quad \left.\left. + r_0^2\, m_\alpha^2 + \left[\vec{r}_0 \cdot \vec{m}_\alpha\right]^2\right]\right] \\
&= \frac{4\pi}{3}\left[-2\left[q_\alpha^{(0)} - \vec{r}_0 \cdot \vec{m}_\alpha\right]^2\right. \\
&\quad + 3\left[Y_\alpha^{(0)} - 2\,\vec{r}_0 \cdot \left[\overleftrightarrow{Q}_\alpha^{(0)} + \overleftrightarrow{Q}_\alpha^{(0)T}\right] \cdot \vec{m}_\alpha\right. \\
&\quad \left.\left. + r_0^2\, m_\alpha^2 + \left[\vec{r}_0 \cdot \vec{m}_\alpha\right]^2\right]\right] \\
&= \frac{4\pi}{3}\, m_\alpha^2\left[-2\left[m_\alpha^{-1} q_\alpha^{(0)} + \vec{r}_0 \cdot \vec{1}_\alpha\right]^2\right. \\
&\quad + 3\left[m_\alpha^{-2} Y_\alpha^{(0)} - 2\,\vec{r}_0 \cdot \left[m_\alpha^{-1}\left[\overleftrightarrow{Q}_\alpha^{(0)} + \overleftrightarrow{Q}_\alpha^{(0)T}\right]\right] \cdot \vec{1}_\alpha\right. \\
&\quad \left.\left. + r_0^2 + \left[\vec{r}_0 \cdot \vec{1}_\alpha\right]^2\right]\right]
\end{aligned}
\tag{3.2}
$$

So as to find a minimum of U_α, let us take a gradient with respect to $\vec{r}_0$ using various vector/dyadic identities (similar to those in Appendix B) as

$$\nabla_0 U_\alpha = \frac{4\pi}{3}\left[4\left[q_\alpha^{(0)} - \vec{r}_0 \cdot \vec{m}_0\right]\vec{m}_\alpha\right.$$

$$+ 3\left[-2\left[\overset{\leftrightarrow}{Q}{}_\alpha^{(0)} + \overset{\leftrightarrow}{Q}{}_\alpha^{(0)T}\right] \cdot \vec{m}_\alpha + 2\, m_\alpha^2\, \vec{r}_0\left[\vec{r}_0 \cdot \vec{m}_0\right]\vec{m}_0\right]\Bigg]$$

$$= \frac{4\pi}{3}\left[4 q_\alpha^{(0)}\vec{m}_\alpha + 2\,\vec{m}_\alpha\,\vec{m}_\alpha \cdot \vec{r}_0\right.$$

$$\left. - 6\left[\overset{\leftrightarrow}{Q}{}_\alpha^{(0)} + \overset{\leftrightarrow}{Q}{}_\alpha^{(0)T}\right] \cdot \vec{m}_\alpha + 6\, m_\alpha^2\, \vec{r}_0\right] \tag{3.3}$$

Setting this to the zero vector we have

$$\vec{0} = 2\, m_\alpha^{-1}\, q_\alpha^{(0)}\,\vec{1}_\alpha + \vec{1}_\alpha\,\vec{1}_\alpha \cdot \vec{r}_0$$

$$- 3\, m_\alpha^{-1}\left[\overset{\leftrightarrow}{Q}{}_\alpha^{(0)} + \overset{\leftrightarrow}{Q}{}_\alpha^{(0)T}\right] \cdot \vec{1}_\alpha + 3\,\vec{r}_0$$

$$\left[3\,\overset{\leftrightarrow}{1} + \vec{1}_\alpha\,\vec{1}_\alpha\right] \cdot \vec{r}_0 = 3\, m_\alpha^{-1}\left[\overset{\leftrightarrow}{Q}{}_\alpha^{(0)} + \overset{\leftrightarrow}{Q}{}_\alpha^{(0)T}\right] \cdot \vec{1}_\alpha \tag{3.4}$$

$$- 2\, m_\alpha^{-1}\, q_\alpha^{(0)}\,\vec{1}_\alpha$$

The reader can verify that

$$\left[\overset{\leftrightarrow}{1} + \frac{1}{3}\vec{1}_\alpha\,\vec{1}_\alpha\right]^{-1} = \overset{\leftrightarrow}{1} - \frac{1}{4}\vec{1}_\alpha\,\vec{1}_\alpha$$

$$\left[3\,\overset{\leftrightarrow}{1} + \vec{1}_\alpha\,\vec{1}_\alpha\right]^{-1} = \frac{1}{3}\left[\overset{\leftrightarrow}{1} - \frac{1}{4}\vec{1}_\alpha\,\vec{1}_\alpha\right] = \frac{1}{3}\,\overset{\leftrightarrow}{1} - \frac{1}{12}\vec{1}_\alpha\,\vec{1}_\alpha \tag{3.5}$$

This gives our solution (only one) for the optimum choice of $\vec{r}_0$ as

$$\vec{r}_0 = \left[\overset{\leftrightarrow}{1} - \frac{1}{4}\vec{1}_\alpha\,\vec{1}_\alpha\right] \cdot \left[m_\alpha^{-1}\left[\overset{\leftrightarrow}{Q}{}_\alpha^{(0)} + \overset{\leftrightarrow}{Q}{}_\alpha^{(0)T}\right] \cdot \vec{1}_\alpha\right.$$

$$\left. - \frac{1}{2}\, m_\alpha^{-1}\, q_\alpha^{(0)}\,\vec{1}_\alpha\right] \tag{3.6}$$

This minimizes U_α, but does not in general make it zero.

4. Conditions for $U_\alpha = 0$

From Part 1 (Section 4) we have an expression for the magnetic quadrupole term in (1.1). Setting this to zero for a natural mode gives

$$\vec{0} = \left\langle \left[\vec{1}_r \cdot \vec{r}'\right]\left[5\,\vec{1}_r\vec{1}_r - \vec{1}_r\right] - \vec{1}_r\vec{r}' - \vec{r}'\vec{1}_r;\ \vec{\chi}_\alpha(\vec{r}')\right\rangle \tag{4.1}$$

which must hold for all $\vec{1}_r$. Rearranging the integral gives

$$\begin{aligned}\vec{0} &= \iiint_V \left[\left[5\,\vec{1}_r\vec{1}_r - \vec{1}_r\right] \cdot \vec{\chi}_\alpha(\vec{r}')\vec{r}' \cdot \vec{1}_r - \vec{1}_r\vec{r}' \cdot \vec{\chi}_\alpha(\vec{r}') - \vec{r}'\vec{\chi}_\alpha(\vec{r}') \cdot \vec{1}_r\right] dV' \\ &= \left[5\,\vec{1}_r\vec{1}_r - \vec{1}_r\right] \cdot \overleftrightarrow{Q}_\alpha^T \cdot \vec{1}_r - \vec{1}_r q_\alpha - \overleftrightarrow{Q}_\alpha \cdot \vec{1}_r\end{aligned} \tag{4.2}$$

Split (4.2) into two parts. First operate by $\vec{1}_r \cdot$ on the left giving

$$\begin{aligned}0 &= 4\,\vec{1}_r \cdot \overleftrightarrow{Q}_\alpha^T \cdot \vec{1}_r - q_\alpha - \vec{1}_r \cdot \overleftrightarrow{Q}_\alpha \cdot \overleftrightarrow{1}_r \\ &= 3\,\vec{1}_r \cdot \overleftrightarrow{Q}_\alpha \cdot \vec{1}_r - q_\alpha\end{aligned} \tag{4.3}$$

Second operate on the left with $\overleftrightarrow{1}_r \cdot$ where

$$\overleftrightarrow{1}_r \equiv \overleftrightarrow{1} - \vec{1}_r\vec{1}_r \tag{4.4}$$

giving

$$\begin{aligned}\vec{0} &= -\vec{1}_r \cdot \overleftrightarrow{Q}_\alpha^T \cdot \vec{1}_r - \vec{1}_r \cdot \overleftrightarrow{Q}_\alpha \cdot \vec{1}_r \\ \vec{0} &= \vec{1}_r \cdot \left[\overleftrightarrow{Q}_\alpha + \overleftrightarrow{Q}_\alpha^T\right] \cdot \vec{1}_r\end{aligned} \tag{4.5}$$

We see various combinations of terms that have appeared in previous sections. Now we require that both (4.3) and (4.5) hold independent of $\overrightarrow{1}_r$.

From Appendix C we have the general form of $\overleftrightarrow{Q}_\alpha$ in (4.3) as

$$\overleftrightarrow{Q}_\alpha = a \overleftrightarrow{1} + \overleftrightarrow{C} \quad , \quad \overleftrightarrow{C}^T = - \overleftrightarrow{C} \tag{4.6}$$

Substituting this form in (4.5) gives

$$\overleftrightarrow{1} \bullet \left[\overleftrightarrow{Q}_\alpha + \overleftrightarrow{Q}_\alpha^T \right] \bullet \overrightarrow{1}_r = \overleftrightarrow{1} \bullet \left[2a \overleftrightarrow{1} \right] \bullet \overrightarrow{1}_r = \overrightarrow{0} \tag{4.7}$$

showing that this form also satisfies (4.5). Returning to (4.3) we have from Appendix C

$$\overleftrightarrow{1} \bullet \overleftrightarrow{Q}_\alpha \bullet \overrightarrow{1}_r = a \tag{4.8}$$

which implies

$$q_\alpha = 3a \tag{4.9}$$

For comparison we have the optimal choice of coordinate origin in (3.6) giving

$$\begin{aligned} \overrightarrow{r}_0 &= \left[\overleftrightarrow{1} - \frac{1}{4} \overrightarrow{1}_\alpha \overrightarrow{1}_\alpha \right] \bullet \left[m_\alpha^{-1} 2a \overleftrightarrow{1} \right] \bullet \overrightarrow{1}_\alpha \\ &\quad - \frac{1}{2} \overrightarrow{m}_\alpha^{-1} 3a \overrightarrow{1}_\alpha \\ &= \overrightarrow{0} \end{aligned} \tag{4.10}$$

as we might expect. Furthermore, the 2-norm considerations result in (2.14) to minimize U_α giving

$$U_\alpha = \frac{4\pi}{3}\left[-2[3a]^2 + 3\int_V \vec{r}' \cdot \left[2a \overleftrightarrow{1}\right] \cdot \vec{\chi}_\alpha(\vec{r}')dV'\right]$$

$$= \frac{4\pi}{3}\left[-18a^2 + 6a\, q_\alpha\right] \tag{4.11}$$

$$= 0$$

So the general form for $\overleftrightarrow{Q}_\alpha$ in (4.6), together with the associated form for q_α in (4.9) give us constraints on the form of $\vec{\chi}_\alpha(\vec{r})$ to have zero resulting quadrupole term ($U_\alpha = 0$). Remember that all parameters here are real valued scalars, vectors, and dyadics.

5. Special Cases

5.1 Displaced magnetic dipole

Consider a magnetic dipole at $\vec{r} = \vec{r}_1$, i.e., $\vec{m}_\alpha$ at $\vec{r}_1$, with

$$\vec{\chi}_\alpha(\vec{r}) = \vec{m}_\alpha \, \delta(\vec{r} - \vec{r}_1) \quad , \quad \vec{m}_\alpha = \int_V \vec{\chi}_\alpha(\vec{r}) dV \neq 0 \tag{5.1}$$

Then we have

$$\begin{aligned}
q_\alpha^{(0)} &= \int_V \vec{r} \cdot \vec{m}_\alpha \delta(\vec{r} - \vec{r}_1) dV = \vec{r}_1 \cdot \vec{m}_\alpha \\
\overleftrightarrow{Q}_\alpha^{(0)} &= \int_V \vec{r}\, \vec{m}_\alpha \delta(\vec{r} - \vec{r}_1) dV = \vec{r}_1 \vec{m}_\alpha \\
Y_\alpha^{(0)} &= \int_V \vec{r} \cdot \left[\overleftrightarrow{Q}_\alpha^{(0)} + \overleftrightarrow{Q}_\alpha^{(0)T} \right] \cdot \vec{m}_\alpha \delta(\vec{r} - \vec{r}_1) dV \\
&= \vec{r}_1 \cdot \left[\vec{r}_1 \vec{m}_\alpha + \vec{m}_\alpha \vec{r}_1 \right] \cdot \vec{m}_\alpha \\
&= \vec{r}_1^{\,2} \cdot \vec{m}_\alpha^{\,2} + \left[\vec{r}_1 \cdot \vec{m}_\alpha \right]^2 \\
r_1 &\equiv |\vec{r}_1| \quad , \quad m_\alpha^2 \equiv \vec{m}_\alpha \cdot \vec{m}_\alpha
\end{aligned} \tag{5.2}$$

Then we have

$$\begin{aligned}
U_\alpha &= \frac{4\pi}{3} \left[-2 \left[\vec{r}_1 \cdot \vec{m}_\alpha \right]^2 + 3 \left[r_1^2 m_\alpha^2 + \left[\vec{r}_1 \cdot \vec{m}_\alpha \right]^2 \right] \right] \\
&= \frac{4\pi}{3} \left[3\, r_1^2\, m_\alpha^2 + \left[\vec{r}_1 \cdot \vec{m}_\alpha \right]^2 \right]
\end{aligned} \tag{5.3}$$

Both terms being non-negative (squares of real numbers), this is made zero by the unique solution

$$\vec{r}_1 = \vec{0} \tag{5.4}$$

This result can also be approached via the coordinate shift discussed in Section 3. This gives an optimal value of $\vec{r}_0$ from (3.6) as

$$\begin{aligned}\vec{r}_0 &= \left[\overleftrightarrow{1} - \frac{1}{4}\vec{1}_\alpha \vec{1}_\alpha\right] \cdot \left[\vec{r}_1 \vec{1}_\alpha + \vec{1}_\alpha \vec{r}_1\right] \cdot \vec{1}_\alpha \\ &\quad - \frac{1}{2}\vec{1}_\alpha \vec{1}_\alpha \cdot \vec{r}_1 \\ &= \vec{r}_1 + \vec{1}_\alpha \vec{1}_\alpha \cdot \vec{r}_1 - \frac{1}{4}\vec{1}_\alpha \vec{1}_\alpha \cdot \vec{r}_1 - \frac{1}{4}\vec{1}_\alpha \vec{1}_\alpha \cdot \vec{r}_1 \\ &= \vec{r}_1\end{aligned} \tag{5.5}$$

which checks.

Comparing the result for a single magnetic dipole to the general allowable forms for $\overleftrightarrow{Q}_\alpha$ and q_α in Section 4 we have

$$\begin{aligned}q_\alpha &= 3a_1 = \vec{r}_1 \cdot \vec{m}_1 = 0 \\ \overleftrightarrow{Q}_\alpha &= a_1 \overleftrightarrow{1} + \overleftrightarrow{C}_1 = \vec{r}_1 \vec{m}_1 \quad , \quad \overleftrightarrow{C}_1^T = -\overleftrightarrow{C}\end{aligned} \tag{5.6}$$

which implies

$$\begin{aligned}a_1 &= 0 \\ \overleftrightarrow{C}_1^T &= \vec{m}_1 \vec{r}_1 \neq \vec{r}_1 \vec{m}_1 - \overleftrightarrow{C}_1 \\ \overleftrightarrow{C}_1 &= \overleftrightarrow{0}\end{aligned} \tag{5.7}$$

since even $\vec{m}_1$ and $\vec{r}_1$ parallel (required for equality) the sign is reversed unless $\vec{r}_1 = \vec{0}$. It is the requirement of a single dyad which makes both a_1 and $\overleftrightarrow{C}_1$ zero. (See also Appendix D.2.)

5.2 Two displaced magnetic dipoles

Let

$$\begin{aligned}\vec{\chi}_\alpha(\vec{r}) &= \vec{m}_\alpha^{(1)}\,\delta(\vec{r}-\vec{r}_1) + \vec{m}_1^{(2)}\,\delta(\vec{r}-\vec{r}_2) \\ \vec{m}_\alpha &= \vec{m}_\alpha^{(1)} + \vec{m}_\alpha^{(2)} \neq \vec{0}\ ,\quad \vec{m}_\alpha^{(1)} \neq \vec{0}\ ,\quad \vec{m}_\alpha^{(2)} \neq \vec{0}\end{aligned} \tag{5.8}$$

Constraining $\vec{r}_0 = \vec{0}$ we have

$$\begin{aligned}q_\alpha &= \vec{r}_1 \cdot \vec{m}_\alpha^{(1)} + \vec{r}_2 \cdot \vec{m}_\alpha^{(2)} \\ \overleftrightarrow{Q}_\alpha &= \vec{r}_1 \cdot \vec{m}_\alpha^{(1)} + \vec{r}_2 \cdot \vec{m}_\alpha^{(2)} \\ Y_\alpha &= \int_V \vec{r} \cdot \left[\overleftrightarrow{Q}_\alpha + \overleftrightarrow{Q}_\alpha^T\right] \cdot \vec{\chi}_\alpha(\vec{r})\,dV \\ &= \vec{r}_1 \cdot \left[\overleftrightarrow{Q}_\alpha + \overleftrightarrow{Q}_\alpha^T\right] \cdot \vec{m}_\alpha^{(1)} + \vec{r}_2 \cdot \left[\overleftrightarrow{Q}_\alpha^T + \overleftrightarrow{Q}_\alpha^T\right] \cdot \vec{m}_\alpha^{(2)}\end{aligned} \tag{5.9}$$

From Section 4, for zero quadrupole we have the general allowable form

$$\begin{aligned}\overleftrightarrow{Q}_\alpha &= a_2\overleftrightarrow{1} + \overleftrightarrow{C}_2\ ,\quad \overleftrightarrow{C}^T = -C \\ q_\alpha &= 3a_2\end{aligned} \tag{5.10}$$

giving

$$
\begin{aligned}
q_\alpha &= 3a_2 = \vec{r}_1 \cdot \vec{m}_1 + \vec{r}_2 \cdot \vec{m}_\alpha^{(2)} \\
\overleftrightarrow{Q}_\alpha &= a_2 \overleftrightarrow{1} + \overleftrightarrow{C}_2 = \vec{r}_1 \vec{m}_\alpha^{(1)} + \vec{r}_2 \cdot \vec{m}_\alpha^{(2)} \\
Y_\alpha &= a_2 \left[\vec{r}_1 \cdot \left[2a_2 \overleftrightarrow{1} \right] \cdot \vec{m}_1^{(1)} + \vec{r}_2 \cdot \left[2a_2 \overleftrightarrow{1} \right] \cdot \vec{m}_1 \right] \\
&= 2a_2^2 q_\alpha \\
U_\alpha &= \frac{3\pi}{4} \left[-2q_\alpha^2 + 3Y_\alpha \right] = 0 \\
\vec{r}_0 &= \vec{0}
\end{aligned}
\tag{5.11}
$$

From Appendix D we have the result (D.31) for the most general form that the sum of two dyads can take consistent with (5.10) is

$$
\begin{aligned}
&\overleftrightarrow{Q}_\alpha = \overleftrightarrow{C}_2 = \vec{c}_2 \times \overleftrightarrow{1} = \vec{r}_1 \vec{m}_\alpha^{(1)} + \vec{r}_2 \vec{m}_\alpha^{(2)} = \overleftrightarrow{C}_2^T \\
&\vec{r}_2 = b' \vec{m}_\alpha^{(1)} \;, \quad \vec{m}_\alpha^{(1)} = b \vec{r}_1 \\
&bb' = -1 \;, \quad a_2 = 0
\end{aligned}
\tag{5.12}
$$

This makes all four vectors comprising the two dyads that make up $\overleftrightarrow{Q}_\alpha$ coplanar, this plane being determined by $\vec{r}_1$ and $\vec{m}_\alpha^{(1)}$. This generalizes the result in Part 1 (Section 5) which has the four vectors all collinear ($\overleftrightarrow{C}_2 = \overleftrightarrow{0}$) as a special case of (5.12).

Our solution is illustrated in Fig. 5.1 where the plane of the page is taken as this plane. It may seem strange that $\vec{r}_1$ and $\vec{r}_2$ are not necessarily collinear through the coordinate origin $\vec{r} = \vec{0}$ to null the quadrupole term. One way to view this is to consider components of the magnetic dipoles. If z is perpendicular to this plane, then the x components do give a quadrupole, while the y components give the negative of this quadrupole for a net cancellation.

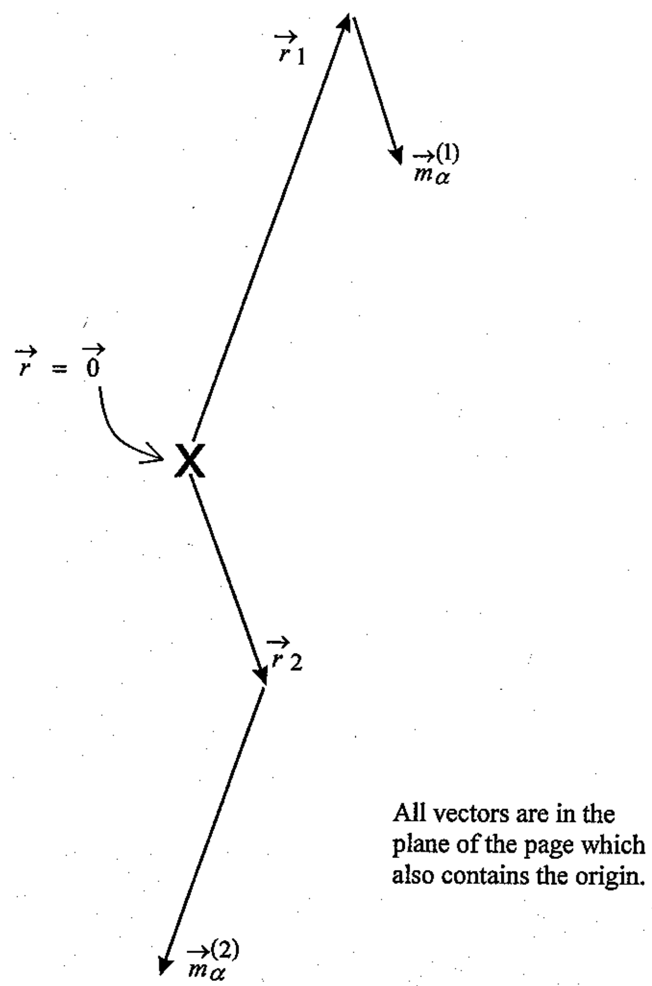


Fig. 5.1 Case of Two Displaced Magnetic Dipoles with Zero Quadrupole.

6. Symmetry Considerations

Part 1, Section 6, considers the effects of symmetry planes and two-dimensional continuous rotation symmetry. Here we delve deeper into the various point symmetries (reflections and rotations).

6.1 Symmetry planes

A single symmetry plane, say $z = 0$ is characterized by the two-element reflection group with dyadic representation

$$R_z = \left\{ \overleftrightarrow{1}, \overleftrightarrow{R}_z \right\} \ , \ \overleftrightarrow{R}_z^{\,2} = \overleftrightarrow{1}$$
$$\overleftrightarrow{R}_z = \overleftrightarrow{1} - 2\,\vec{1}_z\,\vec{1}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \overleftrightarrow{1}_z - \vec{1}_z\,\vec{1}_z \tag{6.1}$$

As with any symmetry group $\overleftrightarrow{\sigma}(\vec{r})$ and $\overleftrightarrow{\mu}(\vec{r})$ must be invariant to this transformation for it to apply.

A symmetry plane divides the electromagnetic response into two noncoupling parts, designated symmetric and antisymmetric. For magnetization natural modes, these being magnetic parameters, we have [8]

$$\vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r}^{(2)}) = \mp\, \overleftrightarrow{R}_z \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r}^{(1)}) \ , \ (\vec{r}^{(2)}) = \overleftrightarrow{R}_z \cdot \vec{r}^{(1)}) \tag{6.2}$$

For the magnetic moment Part 1 shows that

$$\vec{m}_{\alpha,sy} = \vec{1}_z\,\vec{1}_z \cdot \int_V \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV$$
$$= \vec{m}_{\alpha,sy} = \vec{1}_z \quad \text{(longitudinal)} \tag{6.3}$$
$$\vec{m}_{\alpha,as} = \vec{1}_z \cdot \int_V \vec{\chi}_{\alpha,as}(\vec{r})dV \quad \text{(transverse)}$$

This analysis also applies to multiple symmetry planes, which also give a symmetric/antisymmetric decomposition. As discussed in Part 1, $\vec{m}_\alpha$ cannot be symmetric with respect to two (and, by extension, three) perpendicular symmetry planes. Furthermore, $\vec{m}_\alpha$ can be antisymmetric with respect to two symmetry planes (say $x = 0$ and $y = 0$) and necessarily symmetric (if nonzero) with respect to a third ($z = 0$) symmetry plane.

Similar considerations apply to the quadrupole term. As in Part 1 divide V as

$$
\begin{aligned}
V &= V_+ \cup V_- \\
V_+ &\equiv \text{portion of } V \text{ for } z > 0 \\
V_- &\equiv \text{portion of } V \text{ for } z < 0
\end{aligned}
\tag{6.4}
$$

for the case of a $z = 0$ symmetry plane. Then for the quadrupole terms we have

$$
\begin{aligned}
q_{\alpha,\substack{sy\\as}} &= \int_{V} \vec{r} \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV + \int_{V} \vec{r} \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV \\
&= \int_{V_+} \vec{r} \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV + \int_{V_-} \vec{r} \cdot \vec{R}_z \cdot \left[\mp \overleftrightarrow{R}_z\right] \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV \\
&= [1 \mp 1] \int_{V_+} \vec{r} \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV \\
q_{a,sy} &= 0 \\
\overleftrightarrow{Q}_{\alpha,\substack{sy\\as}} &= \int_{V_+} \vec{r}\,\vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV + \int_{V_-} \vec{r} \cdot \vec{R}_z \left[\mp \overleftrightarrow{R}_z\right] \cdot \vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV \\
&= \overleftrightarrow{Q}^{(+)}_{\alpha,\substack{sy\\as}} \mp \overleftrightarrow{R}_z \cdot \overleftrightarrow{Q}^{(+)}_{\alpha,\substack{sy\\as}} \cdot \overleftrightarrow{R}_z \\
\overleftrightarrow{Q}^{(+)}_{\alpha,\substack{sy\\as}} &\equiv \int_{V_+} \vec{r}\,\vec{\chi}_{\alpha,\substack{sy\\as}}(\vec{r})dV
\end{aligned}
\tag{6.5}
$$

where the $\overleftrightarrow{Q}_\alpha$ result is found by writing out the reflection dyadic and collecting terms.

6.2 Inversion symmetry

This is described by the group

$$I = \left\{ \overleftrightarrow{1}, -\overleftrightarrow{1} \right\} \tag{6.6}$$

Again, $\vec{\chi}_\alpha$ can be split into two parts (labeled as ±)

$$\vec{\chi}_{\alpha,\pm}(-\vec{r}) = \pm \vec{\chi}_{\alpha,\pm}(\vec{r}) \tag{6.7}$$

noting that the plus subscript (like symmetric) is associated with the magnetic dipole (unless higher symmetries force it to zero). Again dividing V as in (6.4), but with the division by any plane passing through the origin, we have

$$\begin{aligned} \vec{m}_{\alpha,\pm} &= [1 \pm 1] \int_{V_+} \vec{\chi}_{\alpha,\pm}(\vec{r})dV \\ \vec{m}_{\alpha,-} &= \vec{0} \end{aligned} \tag{6.8}$$

So only the plus part gives a nonzero magnetic-dipole moment.

For the quadrupole term we have

$$\begin{aligned} q_{\alpha,\pm} &= \int_{V_+} \vec{r} \cdot \vec{\chi}_{\alpha,\pm}(\vec{r})dV + \int_{V_+} \vec{r} \cdot \vec{\chi}_{\alpha,\pm}(-\vec{r})dV \\ &= [1 \mp 1] \int_{V_+} \vec{r} \cdot \vec{\chi}_{\alpha,\pm}(\vec{r})dV \\ q_{a,+} &= 0 \\ \overleftrightarrow{Q}_{\alpha,\pm} &= \int_{V_+} \vec{r}\,\vec{\chi}_{\underset{as}{\alpha,\pm}}(\vec{r})dV - \int_{V_+} \vec{r}\,\vec{\chi}_{\alpha,\pm}(-\vec{r})dV \\ &= [1 \mp 1] \int_{V_+} \vec{r}\,\vec{\chi}_{\alpha,\pm}(\vec{r})dV \\ \overleftrightarrow{Q}_{\alpha,+} &= \overleftrightarrow{0} \end{aligned} \tag{6.9}$$

Thus we have:

+ subscript $\Rightarrow$ nonzero dipole, zero quadrupole

- subscript $\Rightarrow$ zero dipole, nonzero quadrupole

Again, "nonzero" terms can also be zero in special cases.

As an example an object with three (mutually perpendicular) planes (intersecting at the origin) also has inversion symmetry, and the above applies. These additional symmetries can have various modes $\vec{\chi}_\alpha$ with symmetric and antisymmetric parts with respect to each of the symmetry planes, as discussed previously. An example of such an object is a metal brick. One might think of some $\vec{m}_\alpha$ as antisymmetric with respect to some plane (say $z = 0$) and symmetric with respect to the other two as some kind of principal or lowest-order dipole mode with zero associated quadrupole. There are two other such dipole modes, antisymmetric with respect to the other two symmetry planes ($x = 0$ and $y = 0$).

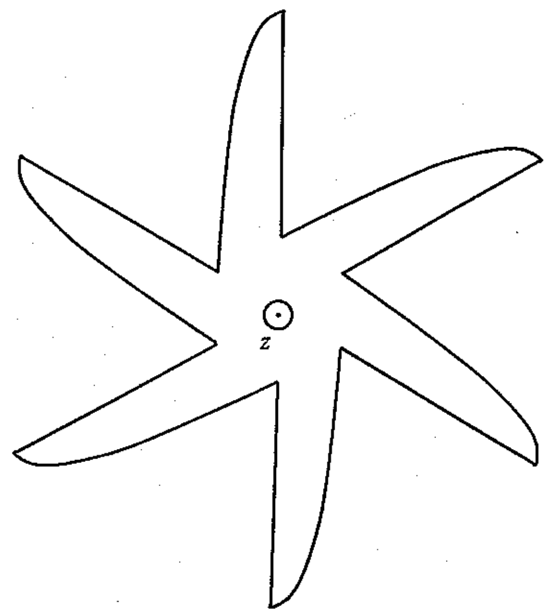
6.3 C_{Nt} symmetry with N even

Before considering more general rotations let N = even (N-fold rotation axis) with a transverse ($z = 0$) symmetry plane. Examples are in Fig. 6.1.

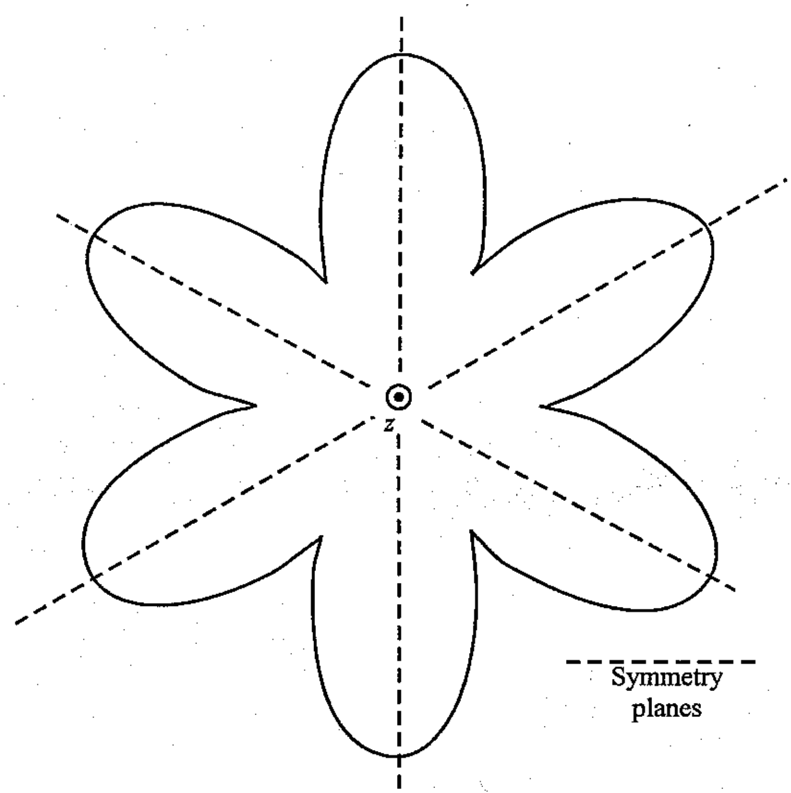
The existence of a transverse symmetry plane means that there is an associated $\vec{\chi}_{a,sy}$ with the additional property that

$$\vec{\chi}_{a,sy}(-\vec{r}) = -\overleftrightarrow{R}_z \cdot \vec{\chi}_{sy,a}(\vec{r}) = \vec{\chi}_{a,sy}(\vec{r}) \tag{6.10}$$

for N even. This is because every point on the body for $z > 0$ corresponds to a point for $\vec{r} \to -\vec{r}$, thereby giving the object inversion symmetry. The mode in (6.10) might be termed a principal longitudinal mode with no change on rotation by $2\pi/N$. Note that this applies only for $N \geq 2$. The magnetic dipole has only a z component and the associated quadrupole is zero. Note that $\vec{r} = \vec{0}$ is defined at the intersection of the rotation axis with the transverse symmetry plane.



A. C_{6t} symmetry (transverse symmetry plane parallel to page).



B. C_{6at} symmetry (addition of N axial symmetry planes).

Fig. 6.1 C_{Nt} Symmetry: N-Fold Rotation Axis with Transverse Symmetry Planes

These simple considerations also apply to transverse modes for the case of axial symmetry planes for N even as illustrated in Fig. 6.1B. Again, a symmetric $\vec{\chi}_{\alpha,sy}$ with respect to any of these axial symmetry planes implies inversion symmetry with plus sign, giving a magnetic dipole perpendicular to any chosen symmetry plane with zero associated quadrupole. Recall from Part 1, Table 6.1, that for C_N with $N \geq 3$ the transverse modes are doubly degenerate, i.e., two independent modes with the same natural frequency s_α.

6.4 C_{Nt} symmetry for general N

With cylindrical (Ψ, ϕ, z) coordinates with

$$x = \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \tag{6.11}$$

let us consider the properties of the natural modes. Figure 6.1 gives examples of such rotational targets. The reflection symmetry pairs points (Ψ,ϕ,z) with ($\Psi,\phi,-z$), which is not so simple as inversion symmetry. Let us then consider summing the integrands over the $2N$ points at $\pm z$ and $\phi + \frac{2\pi\ell}{N}$ for $\ell = 1$ to N (or zero to N-1). For cases that such sums are zero the corresponding integrals are zero.

C_N symmetry is given by

$$C_N = \left\{ \overleftrightarrow{C}(\phi_\ell) \middle| \ell = 0, 1, \cdots, N-1 \right\}$$

$$\overleftrightarrow{C}(\phi_\ell) = \left(C_{n,m}(\phi_\ell) \right) = \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) & 0 \\ \sin(\phi_\ell) & \cos(\phi_\ell) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) \\ \sin(\phi_\ell) & \cos(\phi_\ell) \end{pmatrix} \oplus (1_{1,1})$$

$$\phi_\ell = \frac{2\pi\ell}{N}$$

$$(1_{1,1}) \equiv 1 \times 1 \text{ unit matrix}$$

$$\overleftrightarrow{C}(0) = \overleftrightarrow{C}(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftrightarrow{1}$$

$$\overleftrightarrow{C}^{-1}(\phi) = \overleftrightarrow{C}^{T}(\phi) = \overleftrightarrow{C}(-\phi)$$

$$\det\left(\overleftrightarrow{C}(0)\right) = 1$$

$$\overleftrightarrow{C}(\phi') \cdot \overleftrightarrow{C}(\phi'') = \overleftrightarrow{C}(\phi' + \phi'') \tag{6.12}$$

This also has a convenient exponential form as

$$\overleftrightarrow{C}(\phi) = e^{\phi\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}} = e^{\phi\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}} \oplus (1_{1,1}) \tag{6.13}$$

6.4.1 Longitudinal natural mode

This has

$$\vec{\chi}_{a,sy}\left(\overleftrightarrow{R}_z \cdot \vec{r}\right) = -\overleftrightarrow{R}_z \cdot \vec{\chi}_{a,sy}(\vec{r})$$

$$\vec{m}_{\alpha,sy} = m_{\alpha,sy}\,\vec{1}_z = \int_V \vec{\chi}_{\alpha,sy}(\vec{r})dV \tag{6.14}$$

$$m_{\alpha,sy} = \vec{1}_z \cdot \int_V \vec{\chi}_{\alpha,sy}(\vec{r})dV$$

The longitudinal natural mode of interest is that with the rotational property (for all ℓ)

$$\vec{\chi}_{\alpha,sy}(\Psi,\phi+\phi_\ell,z) = \vec{\chi}_{\alpha,sy}(\overleftrightarrow{C}(\phi_\ell) \cdot \vec{r}) = \overleftrightarrow{C}(\phi_\ell) \cdot \vec{\chi}_{\alpha,sy}(\vec{r}) \tag{6.15}$$

Noting that

$$\vec{r} \cdot \vec{\chi}_{\alpha,sy}(\vec{r}) + \left[\overleftrightarrow{R}_z \cdot \vec{r}\right] \cdot \left[-\overleftrightarrow{R}_z \cdot \vec{\chi}_{\alpha,sy}(\vec{r})\right] = 0 \tag{6.16}$$

the sum over ℓ to form $q'_{\alpha,sy}$ is zero giving for the integral

$$q_{\alpha,sy} = 0 \tag{6.17}$$

The dyadic term $\overleftrightarrow{Q}_{\alpha,sy}$ is more complicated. Define longitudinal (z) and transverse (t) parts by

$$\begin{aligned} &\vec{r} = \vec{r}^{(t)} + z\,\vec{1}_z \;,\quad \vec{r}\cdot\vec{1}_z = 0 \\ &\vec{\chi}_{\alpha,sy}(\vec{r}) = \vec{\chi}^{(t)}_{\alpha,sy}(\vec{r}) + \chi^{(2)}_{\alpha,sy}\,\vec{1}_z\,,\quad \vec{\chi}^{(t)}_{\alpha,sy}(\vec{r})\cdot\vec{1}_z = 0 \end{aligned} \tag{6.18}$$

Then form

$$\begin{aligned} \vec{r}\,\vec{\chi}_{\alpha,sy}(\vec{r}) &= \vec{r}^{(t)}\vec{\chi}^{(t)}_{\alpha}(\vec{r}) + z\,\chi^{(z)}_{\alpha,sy}(\vec{r})\,\vec{1}_z\vec{1}_z \\ &\quad + \chi^{(z)}_{\alpha,sy}(\vec{r})\,\vec{r}\,\vec{1}_z + z\,\vec{1}_z\vec{\chi}^{(t)}_{\alpha}(\vec{r}) \end{aligned} \tag{6.19}$$

Noting that

$$\begin{aligned} &\overleftrightarrow{R}_z\cdot\vec{r} = \vec{r}^{(t)} - \vec{1}_z \\ &\overleftrightarrow{R}_z\cdot\vec{\chi}_{\alpha,sy}(\vec{r}) = \vec{\chi}^{(t)}_{\alpha,sy}(\vec{r}) - \chi^{(z)}_{\alpha,sy}\,\vec{1}_z \end{aligned} \tag{6.20}$$

then sum over points at z and $-z$ for the same (Ψ,ϕ) giving

$$\begin{aligned} &\vec{r}\,\vec{\chi}_{\alpha,sy}(\vec{r}) + \left[\overleftrightarrow{R}_z\cdot\vec{r}\right]\left[-\overleftrightarrow{R}_z\cdot\vec{\chi}_{\alpha,sy}(\vec{r})\right] \\ &= 2\,\chi^{(z)}_{\alpha,sy}(\vec{r})\,\vec{r}^{(t)}\vec{1}_z + 2z\,\vec{1}_z\,\vec{\chi}^{(t)}_{\alpha,sy}(\vec{r}) \end{aligned} \tag{6.21}$$

Summing over ℓ we have

$$\begin{aligned} \overleftrightarrow{Q}'_{\alpha,sy} &= 2\,\chi^{(z)}_{\alpha,sy}(\vec{r})\sum_{\ell=0}^{N-1}\left[\overleftrightarrow{C}(\phi_\ell)\cdot\vec{r}^{(t)}\right]\left[\overleftrightarrow{C}(\phi_\ell)\cdot\vec{1}_z\right] \\ &\quad + 2z\sum_{\ell=0}^{N-1}\left[\overleftrightarrow{C}(\phi_\ell)\cdot\vec{1}_z\right]\left[\overleftrightarrow{C}(\phi_\ell)\cdot\vec{\chi}^{(t)}_{\alpha,sy}(\vec{r})\right] \end{aligned}$$

$$= 2\, \vec{\chi}^{(t)}_{\alpha,sy}(\vec{r}) \left[\left[\sum_{\ell=0}^{N-1} \overleftrightarrow{C}(\phi_\ell) \right] \cdot \vec{r}^{(t)} \right] \vec{1}_z$$

$$+ 2z\, \vec{1}_z\, \vec{\chi}^{(t)}_{\alpha,sy}(\vec{r}) \cdot \left[\sum_{\ell=0}^{N-1} \overleftrightarrow{C}^{T}(\phi_\ell) \right] \tag{6.22}$$

In Appendix E, it is shown that these sums of rotation matrices are the zero dyadic for N ≥ 2. Thus we have

$$\overleftrightarrow{Q}'_{\alpha,sy} = \overleftrightarrow{0} \quad \text{for } N \geq 2 \tag{6.23}$$

with these sums zero then the integral over V is

$$\overleftrightarrow{Q}_{\alpha,sy} = \overleftrightarrow{0} \quad \text{for } N \geq 2 \tag{6.24}$$

and the quadrupole term is zero for all N ≥ 2, both even and odd.

6.4.2 Transverse natural modes

These have

$$\begin{aligned} &\vec{\chi}_{\alpha,as}(\overleftrightarrow{R}_z \cdot \vec{r}) = \overleftrightarrow{R}_z \cdot \vec{\chi}_{\alpha,as}(\vec{r}) \\ &\vec{m}_{\alpha,as} \cdot \vec{1}_z = 0 = \vec{1}_z \cdot \int_V \vec{\chi}_{\alpha,as}(\vec{r})\, dV \\ &\vec{m}_{\alpha,as} = \overleftrightarrow{1}_r \cdot \int_V \vec{\chi}_{\alpha,as}(\vec{r})\, dV \end{aligned} \tag{6.25}$$

The lowest order ones (having a nonzero magnetic-dipole moment) are doubly degenerate.

Dividing such modes into transverse and longitudinal parts as in (6.18) we have summing over points at $+z$ and $-z$

$$\vec{\chi}_{\alpha,as}(\vec{r}) + \overleftrightarrow{R}_z \cdot \vec{\chi}_{\alpha,as}(\vec{r}) = 2\,\overleftrightarrow{1}_z \cdot \vec{\chi}_{\alpha,as}(\vec{r})$$

$$\vec{r} \cdot \vec{\chi}_{\alpha,as}(\vec{r}) + \left[\overleftrightarrow{R}_z \cdot \vec{r}\right] \cdot \left[\overleftrightarrow{R}_z \cdot \vec{\chi}_{\alpha,as}(\vec{r})\right] = 2\,\vec{r} \cdot \vec{\chi}_{\alpha,as}$$

$$\vec{r}\,\vec{\chi}_{\alpha,as}(\vec{r}) + \left[\overleftrightarrow{R}_z \cdot \vec{r}\right]\left[\overleftrightarrow{R}_z \cdot \vec{\chi}_{\alpha,as}(\vec{r})\right] \tag{6.26}$$

$$= 2\,\vec{r}^{(t)}\,\vec{\chi}^{(t)}_{\alpha,as}(\vec{r}) + 2\,z\,\chi^{(z)}_{\alpha,as}(\vec{r})\,\vec{1}_z\,\vec{1}_z$$

These need to the summed over the ϕ_ℓ to find $q'_{\alpha,as}$ and $\overleftrightarrow{Q}_{\alpha,as}$.

Details concerning these rotational symmetry modes are given in [3]. Since these modes are also eigenmodes of the integral equation for the object response (with complex frequency s evaluated at the natural frequency s_α) then rotation of the mode by ϕ_ℓ gives another natural mode. For N ≥ 3 there are two independent modes, in terms of which the remainder can be constructed by linear combination. As a consequence of the nondepolarization theorem for axial incidence [8] we have that $\vec{m}_{\alpha,as}$ is collinear (parallel or antiparallel) with the transverse part of the incident magnetic field for N ≥ 3. In this case we interpret $\vec{m}_{\alpha,as}$ as the resultant magnetic dipole moment from some combination of the degenerate modes.

In [3] the Volume V occupied by the object is divided into N volumes V_ℓ. Starting from some angle, say $\phi = 0$, define V_ℓ as lying between $\phi_{\ell-1}$ and ϕ_ℓ. The coordinates $\vec{r}$ can be written as $\vec{r}^{(\ell)}$ for each V_ℓ where

$$\vec{r}^{(\ell)} = \overleftrightarrow{C}(\phi_\ell) \cdot \vec{r}^{(0)} = \vec{r}^{(N)} \tag{6.27}$$

Then eigenmodes, and thereby natural modes, are constructed in the form

$$\vec{\chi}^{(\ell)}_\alpha(\vec{r}^{(\ell)}) = \overleftrightarrow{C}(\phi_\ell) \cdot \vec{\chi}^{(0)}_\alpha(\vec{r}^{(0)})\,e^{-ju\phi_\ell}$$

$$u = 1, 2, \ldots, N \quad (\text{or } 0, 1, \ldots, N-1) \tag{6.28}$$

$$= \text{modal index (part of } \alpha)$$

Knowing that for MSI the natural modes can be taken as purely real, we can also take real and imaginary parts as eigenmodes, showing the double degeneracy of such modes, except in special cases, including $u = 0$ (or N), and $u = N/2$ for N even. This is associated with the fact that u and $-u$ (or $N - u$) give the same results.

At this point we can note that the case of $u = 0$ corresponds to the longitudinal natural mode for $\vec{m}_{sy}$ discussed in Section 6.4.1.

Now form the magnetic-dipole moment summing over terms from (6.26) and (6.28) for all ℓ as

$$\vec{m}'_{\alpha,as} = 2\left[\sum_{\ell=0}^{N-1} \overleftrightarrow{1}_z \cdot \overleftrightarrow{C}(\phi_\ell)\, e^{-ju\phi_\ell}\right] \cdot \vec{\chi}^{(0)}_{\alpha,as}(\vec{r}^{(0)}) \tag{6.29}$$

In Appendix E it is shown that this sum is nonzero only for $u = 1, N-1$. This corresponds to an aligning of the $\vec{\chi}_\alpha(\vec{r})$ from the various V_ℓ in a "parallel" fashion as one goes through all the ℓ values. Here, for simplification, we have summed over the complex form for $\vec{\chi}_\alpha(\vec{r})$ in (6.28). This sum can also be split into real and imaginary parts with the same result.

Consider then the quadrupole term for the same choices of u that give a nonzero dipole term. Sum over ℓ to include corresponding points in each V_ℓ gives for the scalar term

$$\begin{aligned}
q'_{\alpha,sy} &= \sum_{\ell=0}^{N-1} 2\,\vec{r}^{(\ell)} \cdot \vec{\chi}_{\alpha,as}(\vec{r}^{(\ell)}) \\
&= 2\sum_{\ell=0}^{N-1}\left[\overleftrightarrow{C}(\phi_\ell) \cdot \vec{r}^{(\ell)}\right] \cdot \left[\overleftrightarrow{C}(\phi_\ell) \cdot \vec{\chi}^{(0)}_\alpha(\vec{r}_0)\, e^{-ju\phi_\ell}\right] \\
&= 2\left[\sum_{\ell=0}^{N-1} e^{-ju\phi_\ell}\right] \vec{r}^{(0)} \cdot \vec{\chi}^{(0)}_\alpha(\vec{r}^{(0)}) \\
&= 0 \text{ for } N \geq 2 \text{ and } u = 1, N-1
\end{aligned} \tag{6.30}$$

using the result of (E.1). This gives for the integral

$$q_{\alpha,sy} = 0 \text{ for } N \geq 2 \text{ and non zero dipole} \tag{6.31}$$

For the dyadic term we have

$$\overset{\leftrightarrow}{Q}'_{\alpha,sy} = 2\sum_{\ell=0}^{N-1}\left[\overset{\leftrightarrow}{C}(\phi_\ell)\cdot\vec{r}^{(t,0)}\right]\left[\overset{\leftrightarrow}{C}(\phi_\ell)\cdot\vec{\chi}^{(t,0)}_{\alpha,sy}(\vec{r}^{(0)})\right]e^{-ju\phi_\ell}$$
$$+ 2z\,\vec{1}_z\,\vec{1}_z\,\chi^{(z,0)}_{\alpha,as}(\vec{r}^{(0)})\sum_{\ell=0}^{N-1}e^{-ju\phi_\ell} \tag{6.32}$$

This has two parts. The longitudinal (*z*, *z*) part has

$$\sum_{\ell=0}^{N-1}e^{-ju\phi_\ell} = 0 \text{ for } N \geq 2 \text{ and } u = 1, N-1 \tag{6.33}$$

which corresponds to the nonzero dipole case. The transverse part involves only the *x* and *y* indices of the rotation dyadics. This corresponds to the case in Appendix E (E.26) giving

$$\sum_{\ell=0}^{N-1}\left[\overset{\leftrightarrow}{C}(\phi_\ell)\cdot\vec{r}^{(t,0)}\right]\left[\overset{\leftrightarrow}{C}(\phi_\ell)\cdot\vec{\chi}^{(t,0)}_{\alpha,sy}(\vec{r}^{(0)})\right]e^{-ju\phi_\ell}$$
$$= (\vec{r}^{(t,0)})\cdot\left[\sum_{\ell=0}^{N-1}\overset{\leftrightarrow}{C}^T(\phi_\ell)\,\overset{\leftrightarrow}{C}(\phi_\ell)e^{-ju\phi_\ell}\right]\cdot\vec{\chi}^{(t,0)}_{\alpha,sy}(\vec{r}^{(0)}) \tag{6.34}$$
$$= \overset{\leftrightarrow}{0} \text{ for } u \neq 0 \text{ and } N = 2 \text{ and for } u \neq 2, N-2 \text{ for } N \geq 3$$

and the quadrupole moment (associated with nonzero transverse magnetic dipole moment) is zero provided $N = 2$ or $N \geq 4$.

7. Concluding Remarks

This paper (Part 2) has extended the basic properties of magnetic quadrupoles in MSI. For minimizing such terms associated with natural modes with nonzero magnetic dipoles, the 2-norm over the unit sphere is formed. This has led to the definition of the optimal coordinate center (or center of the natural mode) for minimizing the quadrupole term. The case of two displaced magnetic dipoles has been generalized for zero quadrupole. The symmetry considerations have been extended to discrete two-dimensional rotation with a transverse symmetry plane.

There may be other cases to consider, but this should help in optimal employment of MSI.

Appendix A. Some Properties of Polyadics

We are familiar with vectors $\vec{a}$ (or monads) and dyadics $\overleftrightarrow{A}$ (sums of dyads of form $\vec{a}\,\vec{b}$) [7 (App. 3)]. This is generalized to polyadics or n-adics of degree n by sums of the form

$$\overset{n}{\vec{P}} = \sum_{n'=1}^{m} \vec{a}_1^{\,(n')} \vec{a}_2^{\,(n')} \cdots \vec{a}_n^{\,(n')} \tag{A.1}$$

This is equivalent to a tensor of rank n [6 (ch. 16)], but we prefer the use of the term degree n to distinguish from the rank of a matrix (or dyadic). While the use of the vector symbol $\rightarrow$ is used for vectors in 3-space, the above can apply to any number of dimensions. The dyadic symbol

$$\overleftrightarrow{} \equiv \overrightarrow{}\overrightarrow{} \equiv \overset{2}{\rightarrow} \tag{A.2}$$

is generalized to $\overset{n}{\rightarrow}$.

Consider the partly symmetric form $\vec{f}\,\overset{n}{\vec{P}}\,\vec{f}$. If we dot product on both sides by arbitrary vectors we have

$$\begin{aligned} \vec{a} \cdot \vec{f}\,\overset{n}{\vec{P}}\,\vec{f} \cdot \vec{b} &= \left[\vec{a} \cdot \vec{f}\right] \overset{n}{\vec{P}} \left[\vec{f} \cdot \vec{b}\right] = \left[\vec{b} \cdot \vec{f}\right] \overset{n}{\vec{P}} \left[\vec{f} \cdot \vec{a}\right] \\ &= \vec{b} \cdot \vec{f}\,\overset{n}{\vec{P}}\,\vec{f} \cdot \vec{a} \end{aligned} \tag{A.3}$$

In particular we have for tetradics involving $\vec{1}_r$ (the unit vector in the $\vec{r}$ direction)

$$\vec{a} \cdot \vec{1}_r \vec{1}_r \vec{1}_r \vec{1}_r \cdot \vec{b} = \vec{b} \cdot \vec{1}_r \vec{1}_r \vec{1}_r \vec{1}_r \cdot \vec{a} \tag{A.4}$$

Including the dyadic identity

$$\overleftrightarrow{1} = \sum_{\ell=1}^{3} \vec{1}_\ell \, \vec{1}_\ell$$

$$\vec{1}_1,\ \vec{1}_2,\ \vec{1}_3 \equiv \text{mutually orthogonal real unit vectors} \tag{A.5}$$

we also have

$$\begin{aligned} \vec{a} \cdot \vec{1}_r \overleftrightarrow{1} \vec{1}_r \cdot \vec{b} &= \vec{b} \cdot \vec{1}_r \overleftrightarrow{1} \vec{1}_r \cdot \vec{a} \\ \vec{a} \cdot \overleftrightarrow{1} \overleftrightarrow{1} \cdot \vec{b} &= \vec{a} \vec{b} \end{aligned} \tag{A.6}$$

More generally we have for polyadics

$$\vec{a} \cdot \vec{f} \overset{n}{\vec{P}} \vec{g} \cdot \vec{b} = \vec{b} \cdot \vec{g} \overset{n}{\vec{P}} \vec{f} \cdot \vec{a} \tag{A.7}$$

This is transpose-like, except that $\overset{n}{\vec{P}}$ is *not* transposed, keeping the order of its vector constituents.

Appendix B: Some Polyadic Identities

With S_1 as the surface of the unit sphere we first summarize some identities from [2 (App. B)], including

$$\vec{1}_r \equiv \frac{\vec{r}}{r} \quad , \quad r = |\vec{r}|$$
$$\int_{S_1} dS_1 = 4\pi \quad , \quad \int_{S_1} \vec{1}_r \, dS_1 = \vec{0} \tag{B.1}$$
$$\int_{S_1} \vec{1}_r \, \vec{1}_r \, dS_1 = \frac{4\pi}{3} \overleftrightarrow{1}$$

For polyadics involving $\vec{1}_r$

$$\overset{n}{\vec{1}}_r \equiv \vec{1}_r \, \vec{1}_r \cdots \vec{1}_r \quad (n \text{ terms})$$
$$\vec{1}_r(-\vec{r}) = -\vec{1}_r(\vec{r}) \tag{B.2}$$
$$\overset{n}{\vec{1}}_r(-\vec{r}) = (-1)^n \, \overset{n}{\vec{1}}_r(\vec{r})$$

Dividing S_1 into

$$S_1 = S_{1+} \cup S_{1-}$$
$$\vec{r} = (x, y, z) = x\,\vec{1}_x + y\,\vec{1}_y + z\,\vec{1}_z \quad \text{(Cartesian coordinates)} \tag{B.3}$$
$$S_{1+} = S \text{ for } z \geq 0$$
$$S_{1-} = S \text{ for } z \leq 0$$

and pairing points at $\vec{r}$ and $-\vec{r}$ on S we have

$$\int_{S_1} \overset{n}{\vec{1}}_r \, dS_1 = \overset{n}{\vec{0}} \equiv \vec{0}\,\vec{0} \cdots \vec{0} \quad (n \text{ terms}) \text{ for } n \text{ odd} \tag{B.4}$$

Consider now the integral over the tetradic $\vec{1}_r \overleftrightarrow{1} \vec{1}_r$. Compute

$$\vec{a} \cdot \left[\int_{S_1} \vec{1}_r \overleftrightarrow{1} \vec{1}_r \, dS_1 \right] \cdot \vec{b} = \int_{S_1} \left[\vec{a} \cdot \vec{1}_r \right] \overleftrightarrow{1} \left[\vec{1}_r \cdot \vec{b} \right] dS_1$$

$$= \overleftrightarrow{1} \int_{S_1} \vec{a} \cdot \vec{1}_r \vec{1}_r \cdot \vec{b} \, dS_1 = \overleftrightarrow{1} \vec{a} \cdot \left[\int_{S_1} \vec{1}_r \vec{1}_r \, dS_1 \right] \cdot \vec{b} \tag{B.5}$$

$$= \overleftrightarrow{1} \vec{a} \cdot \left[\frac{4\pi}{3} \overleftrightarrow{1} \right] \cdot \vec{b} = \frac{4\pi}{3} \vec{a} \cdot \vec{b} \overleftrightarrow{1}$$

Outer (dyadic) multiply by $\vec{a}$ on the left, let $\vec{a} = \vec{1}_\ell$ successively with $\ell = 1, 2, 3$ (as in (A.5)), and sum over ℓ to obtain the dyadic identity giving

$$\left[\int_{S_1} \vec{1}_r \overleftrightarrow{1} \vec{1}_r \, dS_1 \right] \cdot \vec{b} = \frac{4\pi}{3} \vec{a} \cdot \vec{b} \overleftrightarrow{1} \tag{B.6}$$

A similar construction gives ($\vec{a}$ general)

$$\vec{a} \cdot \left[\int_{S_1} \vec{1}_r \overleftrightarrow{1} \vec{1}_r \, dS_1 \right] = \frac{4\pi}{3} \overleftrightarrow{1} \vec{a} \tag{B.7}$$

Going a step further, in (B.7) again take the outer product with $\vec{a}$ on the left, let $\vec{a}$ assume the role of the unit vectors and sum, giving

$$\int_{S_1} \vec{1}_r \overleftrightarrow{1} \vec{1}_r \, dS_1 = \frac{4\pi}{3} \sum_{\ell=1}^{3} \vec{1}_\ell \overleftrightarrow{1} \vec{1}_\ell \tag{B.8}$$

If desired $\overleftrightarrow{1}$ can be expanded in terms of the same unit vectors.

Now consider the tetrad $\overset{4}{\overrightarrow{1}}_r$. Begin with the dyadic integral formula [7]

$$\int_V \nabla \vec{v}(\vec{r})dV = \int_S \vec{1}_S \, \vec{v}(\vec{r})dS$$

$$\vec{1}_S \equiv \text{outward pointing unit normal on } S \text{ (which encloses } V\text{)} \tag{B.9}$$

Let

$$\vec{v}(\vec{r}) = \left[\vec{a} \cdot \vec{r}\right]\left[\vec{b} \cdot \vec{r}\right] \vec{r} \;, \quad \vec{a}, \vec{b} = \text{constant vectors} \tag{B.10}$$

Noting that

$$\nabla \vec{r} = \overleftrightarrow{1} \;, \quad \nabla \times \vec{r} = \vec{0} \tag{B.11}$$

this gives

$$\nabla \vec{v}(\vec{r}) = \nabla\left[\left[\vec{a} \cdot \vec{r}\right]\left[\vec{b} \cdot \vec{r}\right]\vec{r}\right]$$

$$= \left[\nabla\left[\left[\vec{a} \cdot \vec{r}\right]\left[\vec{b} \cdot \vec{r}\right]\right]\right]\vec{r} + \left[\vec{a} \cdot \vec{r}\right]\left[\vec{b} \cdot \vec{r}\right]\nabla \vec{r}$$

$$\nabla\left[\left[\vec{a} \cdot \vec{r}\right]\left[\vec{b} \cdot \vec{r}\right]\right] = \left[\vec{a} \cdot \vec{r}\right]\nabla\left[\vec{b} \cdot \vec{r}\right] + \left[\vec{b} \cdot \vec{r}\right]\nabla\left[\vec{a} \cdot \vec{r}\right]$$

$$\nabla\left[\vec{a} \cdot \vec{r}\right] = \vec{a} \times \left[\nabla \times \vec{r}\right] + \vec{r} \times \left[\nabla \times \vec{a}\right] + \left[\vec{a} \cdot \nabla\right]\vec{r} + \vec{r}\left[\nabla \cdot \vec{a}\right]$$

$$= \left[\vec{a} \cdot \nabla\right]\vec{r} = \vec{a} \cdot \overleftrightarrow{1} = \vec{a}$$

$$\nabla\left[\vec{b} \cdot \vec{r}\right] = \vec{b}$$

$$\nabla\left[\left[\vec{a} \cdot \vec{r}\right]\left[\vec{b} \cdot \vec{r}\right]\right] = \left[\vec{a} \cdot \vec{r}\right]\vec{b} + \left[\vec{b} \cdot \vec{r}\right]\vec{a} = \left[\vec{a}\,\vec{b} + \vec{b}\,\vec{a}\right] \cdot \vec{r}$$

$$\nabla \vec{v}(\vec{r}) = \left[\vec{a}\,\vec{b} + \vec{b}\,\vec{a}\right] \cdot \vec{r}\,\vec{r} + \left[\vec{a} \cdot \vec{r}\right]\left[\vec{r} \cdot \vec{b}\right]\overleftrightarrow{1} \tag{B.12}$$

Specializing to S_1 and V_1 and noting that

$$\vec{r} = r\,\vec{1}_r\,,\quad \vec{1}_S = \vec{1}_r\quad,\quad dV_1 = r^2\sin(\theta)d\phi d\phi\,\text{(spherical coordinates)} \tag{B.13}$$

gives

$$\begin{aligned}
\int_{V_1}\vec{r}\,\vec{r}\,dV_1 &= \int_{V_1} r^2\,\vec{1}_r\,\vec{1}_r\,dV_1 \\
&= \int_{V_1} r^2\,\vec{1}_r\,\vec{1}_r\,r^2\sin(\theta)d\phi d\theta dr \\
&= \left[\int_0^1 r^4 dr\right]\left[\int_{S_1}\vec{1}_r\,\vec{1}_r\,dS_1\right] = \frac{1}{5}\int_S \vec{1}_r\,\vec{1}_r\,dS_1 \\
&= \frac{4\pi}{15}\overleftrightarrow{1}
\end{aligned}$$

$$\begin{aligned}
\int_{V_1}\nabla\,\vec{v}(\vec{r})dV_1 &= \left[\vec{a}\,\vec{b} + \vec{b}\,\vec{a}\right]\cdot\int_{V_1}\vec{r}\,\vec{r}\,dV_1 \\
&\quad + \overleftrightarrow{1}\,\vec{a}\cdot\left[\int_{V_1}\vec{r}\,\vec{r}\,dV_1\right]\cdot\vec{b} \\
&= \frac{4\pi}{15}\left[\vec{a}\,\vec{b} + \vec{b}\,\vec{a} + \vec{a}\cdot\vec{b}\,\overleftrightarrow{1}\right]
\end{aligned} \tag{B.14}$$

We also have

$$\begin{aligned}
\int_{S_1}\vec{1}_r\,\vec{v}(\vec{r})dS_1 &= \int_{S_2}\left[\vec{a}\cdot\vec{1}_r\right]\vec{1}_r\,\vec{1}_r\left[\vec{1}_r\cdot\vec{b}\right]dS_1 \\
&= \vec{a}\cdot\left[\int_{S_1}\vec{1}_r^{\,4}\,dS_1\right]\cdot\vec{b} \\
&= \vec{b}\cdot\left[\int_{S_1}\vec{1}_r^{\,4}\,dS_1\right]\cdot\vec{a}
\end{aligned} \tag{B.15}$$

Combining, we have

$$\vec{a} \bullet \left[\int_{S_1}^{4} \vec{1}_r \, dS_1 \right] \bullet \vec{b} = \vec{b} \bullet \left[\int_{S_1}^{4} \vec{1}_r \, dS_1 \right] \bullet \vec{a}$$
$$= \frac{4\pi}{15} \left[\vec{a}\,\vec{b} + \vec{b}\,\vec{a} + \vec{a} \bullet \vec{b}\,\overleftrightarrow{1} \right] \tag{B.16}$$

Note the consistency of this symmetric result with (A.4)

As before, let $\vec{a}$ be outer multiplied on the left and assume the role of the orthogonal unit vectors which are then summed, giving

$$\left[\int_{S_1}^{4} \vec{1}_r \, dS_1 \right] \bullet \vec{b} = \frac{4\pi}{15} \left[\overleftrightarrow{1}\,\vec{b} + \vec{b}\,\overleftrightarrow{1} + \sum_{\ell=1}^{3} \vec{1}_\ell\,\vec{b}\,\vec{1}_\ell \right] \tag{B.17}$$

Outer multiply on the right by $\vec{b}$ and repeat the procedure to give

$$\int_{S_1}^{4} \vec{1}_r \, dS_1 = \frac{4\pi}{15} \left[\overleftrightarrow{1}\,\overleftrightarrow{1} + \sum_{\ell=1}^{3} \vec{1}_\ell\,\overleftrightarrow{1}\,\vec{1}_\ell + \sum_{\ell=1}^{3} \sum_{\ell'=1}^{3} \vec{1}_\ell\,\vec{1}_{\ell'}\,\vec{1}_\ell\,\vec{1}_{\ell'} \right] \tag{B.18}$$

Comparing this to (B.16), we see that the former is a more compact result. This last result can be verified by taking the dot products by $\vec{a}$ and $\vec{b}$ on left and right to recover (B.16).

Appendix C. Implications of $\vec{1}_r \cdot \overleftrightarrow{A} \cdot \vec{1}_r$ = Constant

Begin with

$$a = \vec{1} \cdot \overleftrightarrow{A} \cdot \vec{1}_r \tag{C.1}$$

independent of $\vec{1}_r$ and where a and $\overleftrightarrow{A}$ are a constant scalar and a constant dyadic, respectively. First note that by choice of $\vec{1}_r$ as the unit vectors $\vec{1}_x, \vec{1}_y, \vec{1}_z$ of Cartesian (*x*, *y*, *z*) coordinates we have the diagonal components of $\overleftrightarrow{A}$ as

$$a = A_{x,x} = A_{y,y} = A_{z,z} \tag{C.2}$$

So we can write

$$\overleftrightarrow{A} = a \overleftrightarrow{1} + \overleftrightarrow{C} \tag{C.3}$$

where $\overleftrightarrow{C}$ has only off-diagonal elements. Substituting in (C.1) gives

$$\begin{aligned} a &= \vec{1}_r \cdot \left[a \overleftrightarrow{1} + \overleftrightarrow{C} \right] \cdot \vec{1}_r = a + \vec{1}_r \cdot \overleftrightarrow{C} \cdot \vec{1}_r \\ 0 &= \vec{1}_r \cdot \overleftrightarrow{C} \cdot \vec{1}_r \end{aligned} \tag{C.4}$$

Expanding $\vec{1}_r$ in terms of the Cartesian unit vectors, using (θ,ϕ) of the usual spherical (r,θ,ϕ) coordinates, we have

$$\vec{1}_r = \vec{1}_x \sin(\theta)\cos(\phi) + \vec{1}_y \sin(\theta)\sin(\phi) + \vec{1}_z \cos(\theta) \tag{C.5}$$

Applying this to (C.4) we have

$$
\begin{aligned}
0 &= \left[C_{x,y} + C_{y,x}\right]\sin^2(\theta)\cos(\phi)\sin(\phi) \\
&+ \left[C_{x,z} + C_{z,x}\right]\sin(\theta)\cos(\theta)\sin(\phi) \\
&+ \left[C_{y,z} + C_{z,y}\right]\sin(\theta)\cos(\theta)\sin(\phi)
\end{aligned} \tag{C.6}
$$

Requiring that this apply for all (θ,ϕ) (or simply by choosing three appropriate pairs of θ and ϕ) we find

$$
\overset{\leftrightarrow}{C}^{T} = -\overset{\leftrightarrow}{C} \tag{C.7}
$$

which is an arbitrary (constant) antisymmetric dyadic. This completes the form in (C.3).

As an antisymmetric dyadic it can also be written in the form [7]

$$
\begin{aligned}
\overset{\leftrightarrow}{C} &= \overset{\leftrightarrow}{c} \times \overset{\leftrightarrow}{1} = \overset{\leftrightarrow}{1} \times \vec{c} = \begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_y \\ -c_y & c_x & 0 \end{pmatrix} \\
\overset{\leftrightarrow}{C}^{T} &= -\overset{\leftrightarrow}{C} = -\overset{\leftrightarrow}{c} \times \overset{\leftrightarrow}{1} = -\overset{\leftrightarrow}{1} \times \vec{c}
\end{aligned} \tag{C.8}
$$

This can be seen by writing out the components.

Note that the orientation of the Cartesian coordinates is arbitrary. Rotating one set of coordinates into another leaves the general form of $a\,\overset{\leftrightarrow}{1} + \overset{\leftrightarrow}{C}$ with $\overset{\leftrightarrow}{C}^{T} = -\overset{\leftrightarrow}{C}$ unchanged. Note that for real $\overset{\leftrightarrow}{A}$, both a and $\overset{\leftrightarrow}{C}$ are real.

Appendix D. Implications of Real $a\overleftrightarrow{1} + \overleftrightarrow{C}$ Constructed from Two Real Dyads

Let us take the form of $\overleftrightarrow{A}$ from Appendix C and try to construct it in the form

$$\begin{aligned} \overleftrightarrow{A} &= a\overleftrightarrow{1} + \overleftrightarrow{C} = \vec{p}^{(1)}\vec{q}^{(1)} + \vec{p}^{(2)}\vec{q}^{(2)} \\ \overleftrightarrow{C} &= \vec{c} \times \overleftrightarrow{1} = \overleftrightarrow{1} \times \vec{c} = -\overleftrightarrow{C}^{T} \end{aligned} \tag{D.1}$$

following (C.8). All above scalars, vectors, and dyadics are assumed real valued.

D.1 Removal of identity dyadic

As one should expect, two dyads are not adequate to span a three-dimensional space. To see this construct a $\vec{p}^{(3)}$ such that

$$\vec{q}^{(1)} \cdot \vec{p}^{(3)} = 0 \;,\; \vec{q}^{(2)} \cdot \vec{p}^{(3)} = 0 \tag{D.2}$$

this always being possible since $\vec{q}^{(1)}$ and $\vec{p}^{(2)}$ are at most constrained to a plane. Merely choose a $\vec{p}_3 \neq \vec{0}$ perpendicular to this plane (or line if $\vec{q}_1$ and $\vec{q}_2$ are not independent). Then we have

$$\overleftrightarrow{A} \cdot \vec{p}^{(3)} = \vec{0} \tag{D.3}$$

and $\overleftrightarrow{A}$ is singular, implying

$$\begin{aligned} \det(\overleftrightarrow{A}) = 0 &= \begin{vmatrix} a & -c_z & c_y \\ c_z & a & -c_x \\ -c_y & c_x & a \end{vmatrix} \\ &= a\left[a^2 + c_x^2\right] + c_2\left[c_z\, a - c_x\, c_y\right] \\ &\quad + c_y\left[c_z\, c_x + a\, c_y\right] \\ &= a\left[a^2 + c_x^2 + c_y^2 + c_z^2\right] \\ &= a\left[a^2 + \vec{c} \cdot \vec{c}\right] = a\left[a^2 + |\vec{c}|\right] \end{aligned} \tag{D.4}$$

With all these parameters real-valued we have three solutions:

$$a = 0, \quad \pm j |\vec{c}| \tag{D.5}$$

Since a must be real-valued we have

$$a = 0 \tag{D.6}$$

Consider now the zero diagonal elements in

$$\overleftrightarrow{A} = \overleftrightarrow{C} = \vec{p}^{(1)} \vec{q}^{(1)} + \vec{p}^{(2)} \vec{q}^{(2)} \tag{D.7}$$

giving

$$\begin{aligned} 0 &= p_x^{(1)} q_x^{(1)} + p_x^{(2)} q_x^{(2)} \\ &= p_y^{(1)} q_y^{(1)} + p_y^{(2)} q_y^{(2)} \\ &= p_z^{(1)} q_z^{(1)} + p_z^{(2)} q_z^{(2)} \end{aligned} \tag{D.8}$$

Summing over these implies

$$0 = \vec{p}^{(1)} \cdot \vec{q}^{(1)} + \vec{p}^{(2)} \cdot \vec{q}^{(2)} \tag{D.9}$$

but (D.8) is even more restrictive.

In addition we have the trace

$$\begin{aligned} tr(\overleftrightarrow{A}) = tr(\overleftrightarrow{C}) &= C_{x,x} + C_{y,y} = C_{z,z} = 0 \\ &= \sum_{\beta=1}^{3} \lambda_\beta \end{aligned} \tag{D.10}$$

$\lambda_\beta \equiv$ eigenvalues

The eigenvalues are found from

$$\det(\overleftrightarrow{C} - \lambda_\beta \overleftrightarrow{1}) = 0 \tag{D.11}$$

Noting that this has the same form as (D.4), we have

$$\lambda_\beta = 0\,,\ \pm j\,|\,\vec{c}\,| \tag{D.12}$$

D.2 Case of single dyad

If any one of the four vectors comprising $\overleftrightarrow{A}$ is zero (say $\vec{p}_z$ or $\vec{q}_z$), then we have

$$\overleftrightarrow{A} = \overleftrightarrow{C} = \vec{p}^{(1)}\vec{q}^{(1)} = -\overleftrightarrow{C}^T = -\vec{q}^{(1)}\vec{p}^{(1)} \tag{D.13}$$

implying that $\vec{p}^{(1)}$ and $\vec{q}^{(1)}$ are collinear (one being a scalar times the other). In turn this implies that

$$\overleftrightarrow{A} = \overleftrightarrow{C} = \overleftrightarrow{0}\,,\ \vec{c} = \vec{0} \tag{D.14}$$

and one of $\vec{p}^{(1)}$ and $\vec{q}^{(1)}$ must also be $\vec{0}$.

So a single dyad implies (D.14), and this is a simple special case.

D.3 Case of two nonzero dyads

Let us now assume that all four of the vectors comprising $\overleftrightarrow{A}$ are nonzero. Then we have two cases to consider. Either $\vec{c} = \vec{0}$ or $\vec{c} \neq \vec{0}$. Let us consider these two cases in turn

D.4 Case of $\vec{c} = \vec{0}$

In this case we have

$$\vec{C} = \overleftrightarrow{0} = \vec{p}^{(1)}\vec{q}^{(1)} + \vec{p}^{(2)}\vec{q}^{(2)} \tag{D.15}$$

Taking some nonzero vector $\vec{d}$ and dotting on the right side gives

$$\overleftrightarrow{0} = \vec{p}^{(1)}\left[\vec{q}^{(1)} \cdot \vec{d}\right] + \vec{p}^{(2)}\left[\vec{q}^{(2)} \cdot \vec{d}\right] \tag{D.16}$$

Orient $\vec{d}$ such that neither dot product is zero (always possible). Then we have

$$\vec{p}^{(1)} = f\,\vec{p}^{(2)}, \quad f \text{ real} \tag{D.17}$$

From (D.12) we then have

$$\begin{aligned} \overleftrightarrow{0} &= \vec{p}^{(2)}\left[f\,\vec{q}^{(1)} + \vec{q}^{(2)}\right] \\ \vec{0} &= f\,\vec{q}^{(1)} + \vec{q}^{(2)} \\ q^{(2)} &= -f\,q^{(1)} \end{aligned} \tag{D.18}$$

This case then reduces to a single dyadic with $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$ parallel or antiparallel, and similarly for $\vec{q}^{(1)}$ and $\vec{q}^{(2)}$.

D.5 Case of $\vec{c} \neq \vec{0}$

Since the case of $\vec{c} = \vec{0}$ corresponded to $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$, collinear and similarly for $\vec{q}^{(1)}$ and $\vec{q}^{(2)}$, let us make the opposite assumption for the present case. So let (D.17) and (D.18) be replaced by *inequalities*. Then (D.15) can be replaced by

$$\vec{c} \times \overleftrightarrow{1} = \overleftrightarrow{1} \times \vec{c} = \vec{p}^{(1)}\vec{q}^{(1)} + \vec{p}^{(2)}\vec{q}^{(2)} \neq \overleftrightarrow{0} \tag{D.19}$$

Dot product with $\vec{c}$ on the left giving

$$\vec{c} \cdot \overleftrightarrow{1} \times \vec{c} = \vec{c} \times \vec{c} = \vec{0} = \left[\vec{c} \cdot \vec{p}^{(1)} \right] \vec{q}^{(1)} + \left[\vec{c} \cdot \vec{p}^{(2)} \right] \vec{q}^{(2)} \tag{D.20}$$

Since $\vec{q}^{(1)}$ and $\vec{q}^{(2)}$ are not collinear and are nonzero we must have

$$\vec{c} \cdot \vec{p}^{(1)} = 0 \ , \quad \vec{c} \cdot \vec{p}^{(2)} = 0 \tag{D.21}$$

Since $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$ are not collinear they determine a plane perpendicular to $\vec{p}^{(1)} \times \vec{p}^{(2)}$. We then also have that $\vec{c}$ is perpendicular to this plane and

$$\begin{aligned} &\vec{c} = g\,\vec{p}^{(1)}\vec{p}^{(2)} \neq 0 \\ &g \neq 0 \quad (g \text{ real}) \end{aligned} \tag{D.22}$$

Similarly dot product with $\vec{c}$ on the right giving

$$\vec{c} \times \overleftrightarrow{1} \cdot \vec{c} = \vec{c} \times \vec{c} = \vec{0} = \vec{p}^{(1)} \left[\vec{q}^{(1)} \cdot \vec{c} \right] + \vec{p}^{(2)} \left[\vec{q}^{(2)} \cdot \vec{c} \right] \tag{D.23}$$

leading to the same conclusion for $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$ namely

$$\begin{aligned} &\vec{c} = h\,\vec{q}^{(1)} \times \vec{q}^{(2)} \\ &h \neq 0 \quad (h \text{ real}) \end{aligned} \tag{D.24}$$

Then the plane determined by $\vec{g}^{(1)}$ and $\vec{g}^{(2)}$ is the same as that determined by $\vec{p}^{(1)}$ and $\vec{p}^{(2)}$. Thus *all four vectors are coplanar.*

Without loss of generality let us rotate the coordinates ∋ $\vec{c}$ is aligned with the *z* axis as

$$\vec{c} = c_z \vec{1}_z \tag{D.25}$$

Then the four vectors comprising $\overleftrightarrow{A}$ all have only *x* and *y* components, i.e., they can be considered as lying in the (*x*, *y*) plane where they can be considered as two-component vectors.

Now by rotating the coordinates ∋

$$\vec{p}^{(1)} = p_x^{(1)} \vec{1}_x \tag{D.26}$$

we have

$$\begin{aligned} \overleftrightarrow{1} \times \vec{c} &= p_x^{(1)} \vec{1}_x \vec{q}^{(1)} + \vec{p}^{(2)} \vec{q}^{(2)} \\ \vec{1}_y \times \left[\overleftrightarrow{1} \times \vec{c} \right] &= \vec{1}_y \times \vec{c} = \vec{1}_y \times \vec{1}_z c_z = -\vec{1}_x c_z \\ &= \left[\vec{1}_x \cdot \vec{p}^{(2)} \right] \vec{q}^{(2)} \end{aligned} \tag{D.27}$$

Thus $\vec{q}^{(2)}$ has only an x component and we can set

$$\vec{q}^{(2)} = b \vec{p}^{(2)} \quad , \quad b \neq 0 \quad (b \text{ real}) \tag{D.28}$$

Similarly, by rotating the coordinates ∋ $q^{(1)}$ is parallel to the x axis we find

$$\vec{p}^{(2)} = b' \vec{q}^{(1)} \quad , \quad b \neq 0 \quad (b \text{ real}) \tag{D.29}$$

Then we have

$$
\begin{aligned}
\overleftrightarrow{C} &= \vec{p}^{(1)}\vec{q}^{(1)} + b\,b'\vec{q}^{(1)}\vec{p}^{(1)} \\
&= -\overleftrightarrow{C}^{T} = -\vec{q}^{(1)}\vec{p}^{(1)} - b\,b'\vec{p}^{(1)}\vec{q}^{(1)}
\end{aligned}
\tag{D.30}
$$

This in turn gives

$$
\begin{aligned}
b\,b' &= -1 \\
\overleftrightarrow{A} = \overleftrightarrow{C} &= \vec{p}^{(1)}\vec{q}^{(1)} + \left[b'\,q^{(1)}\right]\left[b\,\vec{p}^{(1)}\right] \\
&= \vec{p}^{(1)}\vec{q}^{(1)} - \vec{q}^{(1)}\vec{p}^{(1)}
\end{aligned}
\tag{D.31}
$$

as the most general solution for $\overleftrightarrow{A}$ constructed from the sum of two dyads. By a coordinate rotation this applies for arbitrary orientation of the axes. Note that if $\vec{p}^{(1)} = \mathrm{v}\,\vec{q}^{(1)}$ so that the vectors are collinear, then we obtain $\overleftrightarrow{A} = \overleftrightarrow{0}$, $\vec{c} = \vec{0}$, consistent with the previous results in Appendix D.4.

Again aligning the z axis with $\vec{c}$ we find

$$
\begin{aligned}
\overleftrightarrow{C}\cdot\overleftrightarrow{C} &= c_z^2\,\vec{1}_z \times \overleftrightarrow{1} \times \vec{1}_z = \begin{pmatrix} 0 & c_z & 0 \\ -c_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & c_z & 0 \\ -c_z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} c_z^2 & 0 & 0 \\ 0 & c_z^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \overleftrightarrow{C}\cdot\left[\vec{p}^{(1)}\vec{q}^{(1)} - \vec{q}^{(1)}\vec{p}^{(1)}\right] = c_z\left[\left[\vec{1}_z \times \vec{p}^{(1)}\right]\vec{q}^{(1)} - \left[\vec{1}_z \times \vec{q}^{(1)}\right]\vec{p}^{(1)}\right] \\
&= c_z\left[\begin{pmatrix} -\vec{p}^{(1)} \\ p_x^{(1)} \\ 0 \end{pmatrix}\begin{pmatrix} q_x^{(1)} \\ q_y^{(1)} \\ 0 \end{pmatrix} - \begin{pmatrix} -q_y^{(1)} \\ q_x^{(1)} \\ 0 \end{pmatrix}\begin{pmatrix} p_x^{(1)} \\ p_y^{(1)} \\ 0 \end{pmatrix}\right] \\
&= c_z\begin{pmatrix} p_x^{(1)}q_y^{(1)} - p_y^{(1)}q_x^{(1)} & 0 & 0 \\ 0 & p_x^{(1)}q_y^{(1)} - p_y^{(1)}q_x^{(1)} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
c_z = p_y^{(1)}q_x^{(1)} - p_x^{(1)}q_y^{(1)}
\tag{D.32}
$$

as one way to compute $\vec{c}$ from $\vec{p}^{(1)}$ and $q^{(1)}$. Of course one can write out the components of $\overleftrightarrow{C}$ in (D.31) for arbitrary coordinate orientations.

Appendix E. Certain Sums of Rotation Dyadics

Similar to [1] we have the sum of a geometric series of scalars ($N \geq 1$) as

$$\begin{aligned}
\sum_{\ell=0}^{N-1} e^{-j\frac{2\pi u\ell}{N}} &= \left[1-e^{-j2\pi u}\right]\left[1-e^{-j\frac{2\pi u}{N}}\right]^{-1} \\
&= \begin{cases} 0 \ \text{for} \ u \neq 0 \ \text{and} \ N \geq 2 \\ N \ \text{otherwise} \end{cases} \\
u &= 0, 1, 2, \cdots, N-1 \ (\text{or } 1 \text{ to } N)
\end{aligned} \tag{E.1}$$

Consider the sum of dyadics

$$\begin{aligned}
\sum_{\ell=0}^{N-1} \overleftrightarrow{C}(\phi_\ell) &= \sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \\
&= \sum_{\ell=0}^{N-1} \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) & 0 \\ \sin(\phi_\ell) & \cos(\phi_\ell) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\phi_\ell &= \frac{2\pi\ell}{N}
\end{aligned} \tag{E2}$$

This can be reduced to

$$\begin{aligned}
\sum_{\ell=0}^{N-1} \overleftrightarrow{C}(\phi_\ell) &= \sum_{\ell=0}^{N-1} \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) \\ \sin(\phi_\ell) & \cos(\phi_\ell) \end{pmatrix} \oplus (1_{1,1}) \\
&= \left[\sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \oplus (1_{1,1})\right] \\
&= \left[\sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}\right] \oplus \sum_{\ell=0}^{N-1} (1_{1,1})
\end{aligned} \tag{E.3}$$

So we need the two parts of the direct sum. The second is simply

$$\sum_{\ell=0}^{N-1}\left(1_{1,1}\right) = N\left(1_{1,1}\right) = \left(N\ 1_{1,1}\right) \tag{E.4}$$

The sum of 2 x 2 matrices can be solved by noting that it is a geometric series of matrices. In general we have for N x N matrices

$$\begin{gathered}\sum_{\ell=0}^{k-1}\left(a_{n,m}\right)^{\ell} = \left[\left(1_{n,m}\right) - \left(a_{n,m}\right)^{k}\right] \bullet \left[\left(1_{n,m}\right) - \left(a_{n,m}\right)\right]^{-1} \\ k \geq 1 \\ \left(1_{n,m}\right) - \left(a_{n,m}\right) = \text{nonsingular matrix}\end{gathered} \tag{E.5}$$

This can be verified by moving the inverse matrix to the left and rearranging terms. Note that all matrices here commute.

Evaluating (E.5) for $k = N \geq 1$ we have

$$\sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{2\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}\right] \bullet \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}\right]^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } N \geq 2 \tag{E.6}$$

To see this note that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{2\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos(2\pi) & -\sin(2\pi) \\ \sin(2\pi) & \cos(2\pi) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{E.7}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{\phi_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1-\cos(\phi_1) & \sin(\phi_1) \\ -\sin(\phi_1) & 1-\cos(\phi_1) \end{pmatrix} \tag{E.8}$$

which is singular (no inverse) provided the determinant is nonzero giving

$$\det\left(\begin{pmatrix} 1-\cos(\phi_1) & \sin(\phi_1) \\ -\sin(\phi_1) & 1-\cos(\phi_1) \end{pmatrix}\right) = \left[1-\cos(\phi_1)\right]^2 + \sin^2(\phi_1)$$

$$= 2\left[1-\cos(\phi_1)\right] = 2\left[1-\cos\left(\frac{2\pi}{N}\right)\right] \tag{E.9}$$

$$= 0 \text{ only if } N = 1$$

For $N = 1$ we have

$$\sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } N = 1 \tag{E.10}$$

So we have

$$\sum_{\ell=0}^{N-1} \overset{\leftrightarrow}{C}(\phi_\ell) \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N \end{pmatrix} = N\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \left(1_{1,1}\right) & \text{for } N \geq 2 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overset{\leftrightarrow}{1} & \text{for } N = 2 \end{cases} \tag{E.11}$$

As symmetric matrices for the sums, the same applies to the sums of the transposes.

Another sum combines the scalar and dyadic terms as

$$\sum_{\ell=0}^{N-1} \overset{\leftrightarrow}{C}(\phi_0) e^{-ju\phi_\ell} \sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}} \bullet e^{-ju\phi_\ell \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$$= \sum_{\ell=0}^{N-L} e^{\phi_\ell \begin{pmatrix} -ju & -1 & 0 \\ 1 & -ju & 0 \\ 0 & 0 & -ju \end{pmatrix}} \tag{E.12}$$

$$= \sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} -ju & -1 \\ 1 & -ju \end{pmatrix}} \oplus \left[e^{-ju\phi_\ell}\left(1_{1,1}\right)\right]$$

$$u = 1, 2, \cdots, N-1$$

(Note the commutation of the 3 x 3 matrices.) This leaves two sums to find.

The scalar sum ((3,3) element in (E.9) is given in (E.1). This leaves the sum of 2 x 2 matrices as

$$\sum_{\ell=0}^{N-1} e^{\phi_\ell \begin{pmatrix} -ju & -1 \\ 1 & -ju \end{pmatrix}} = \sum_{\ell=0}^{N-1} e^{-ju\phi_\ell} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{-j2\pi u} e^{2\pi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \right] \cdot \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{-ju\phi_1} e^{\phi_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \right]^{-1} \tag{E.13}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{-ju\phi_1} e^{\phi_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} \text{ non singular}$$

The above result then does *not* apply if [5]

$$\begin{aligned}
0 &= \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - e^{-ju\phi_1} \begin{pmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{pmatrix} \right) \\
&= \left[1 - e^{-ju\phi_1} \cos(\phi_1) \right]^2 + e^{-j2u\phi_1} \sin^2(\phi_1) \\
&= 1 + e^{-ju\phi_1} - 2e^{-ju\phi_1} \cos(\phi_1) \\
0 &= \cos(u\phi_1) - \cos(\phi_1) \\
&= -2 \sin\left(\frac{[u+1]\phi_1}{2} \right) \sin\left(\frac{[u-1]\phi_1}{2} \right)
\end{aligned} \tag{E.14}$$

This requires either

$$\begin{aligned}
\frac{[u+1]\phi_1}{2} &= [u+1]\frac{\pi}{N} = \pi m \quad , \quad m \text{ an integer} \\
u &= mN - 1 = N - 1 \quad (\text{from } u = 0, \cdots, N\text{-}1)
\end{aligned} \tag{E.15}$$

or

$$\begin{aligned}
\frac{[u+1]\phi_1}{2} &= [u-1]\frac{\pi}{N} = \pi m' \quad , \quad m' \text{ an integer} \\
u &= m'N + 1 = 1 \text{ or } 0 \text{ if } N = 1 \quad (\text{from } u = 0, \cdots, N\text{-}1)
\end{aligned} \tag{E.16}$$

Note that the case of $u = 0$ for $N = 1$ corresponds to (E.10).

For these cases that (E.13) does not give the zero dyadic we need for $u = 1$

$$\sum_{\ell=0}^{N-1} e^{-j2\phi_\ell} = \left[1-e^{-j2\phi_1 N}\right]\left[1-e^{-j2\phi_1}\right]^{-1}$$

$$= \left[1-e^{-j4\pi}\right]\left[1-e^{-j\frac{4\pi}{N}}\right]^{-1} \tag{E.17}$$

$$= \begin{cases} 0 & \text{for } N \geq 3 \\ 2 & \text{for } N \geq 2 \\ 1 & \text{for } N \geq 1 \end{cases}$$

Similarly for $u = N - 1$

$$e^{-j2[N-1]\phi_\ell} = e^{-j2N\phi_\ell}\, e^{j2\phi_\ell} = e^{j2\phi_\ell}$$

$$\sum_{\ell=0}^{N-1} e^{j2\phi_\ell} = \sum_{\ell=0}^{N-1} e^{-j2\phi_\ell} \tag{E.18}$$

Then we have for $u = 1$

$$\sum_{\ell=0}^{N-1} e^{-j\phi_\ell} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \sum_{\ell=0}^{N-1} \begin{pmatrix} e^{-j\phi_\ell}\cos(\phi_\ell) & e^{-j\phi_\ell}\sin(\phi_\ell) \\ e^{-j\phi_\ell}\sin(\phi_\ell) & e^{-j\phi_\ell}\cos(\phi_\ell) \end{pmatrix}$$

$$= \frac{1}{2}\sum_{\ell=0}^{N-1} \begin{pmatrix} 1+e^{-j2\phi_\ell} & j\left[1-e^{-j2\phi_\ell}\right] \\ -j\left[1-e^{-j2\phi_\ell}\right] & 1+e^{-j2\phi_\ell} \end{pmatrix}$$

$$= \frac{1}{2}\sum_{\ell=0}^{N-1} \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} + \frac{1}{2}\sum_{\ell=0}^{N-1} e^{-j2\phi_\ell} \begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix}$$

$$= \frac{N}{2}\begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} + \frac{1}{2}\sum_{\ell=0}^{N-1} e^{-j2\phi_\ell}$$

$$= \begin{cases} \frac{N}{2}\begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} & \text{for } N \geq 3 \\ 2\begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} & \text{for } N = 2 \\ \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix} & \text{for } N = 1 \end{cases} \tag{E.19}$$

For $u = N-1$ we have

$$\sum_{\ell=0}^{N-1} e^{-j[N-1]\phi_\ell} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \sum_{\ell=0}^{N-1} e^{j\phi_\ell} e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$= \begin{cases} \frac{N}{2}\begin{pmatrix} 1 & -j \\ j & 1 \end{pmatrix} & \text{for } N \geq 3 \\ 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } N = 2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } N = 1 \end{cases} \tag{E.20}$$

which is just the complex conjugate of (E.19). Dividing the sums into real and imaginary parts we have

$$\sum_{\ell=0}^{N-1} \cos(\phi_\ell) e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = \sum_{\ell=0}^{N-1} \cos([N-1]\phi_\ell) e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$= \begin{cases} \frac{N}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } N \geq 3 \\ 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } N = 2 \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } N = 1 \end{cases}$$

$$\sum_{\ell=0}^{N-1} \sin(\phi_\ell) e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}} = -\sum_{\ell=0}^{N-1} \sin([N-1]\phi_\ell) e^{\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

$$= \begin{cases} \frac{N}{2}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } N \geq 3 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } N = 1,2 \end{cases} \tag{E.21}$$

Substituting from (E.1) for the (3,3) term, and (E.13), (E.19), and (E.20) for the 2 x 2 block gives the final answer. Instead of complex sums, real sums in (E.20) can also be used.

Another kind of sum we can encounter involves the form

$$\sum_{\ell=0}^{N-1}\left[\overleftrightarrow{C}(\phi_\ell)\cdot\vec{a}\right]\left[\overleftrightarrow{C}(\phi_\ell)\cdot\vec{b}\right]e^{-ju\phi_\ell}=\sum_{\ell=0}^{N-1}\vec{a}\cdot\overleftrightarrow{C}^T(\phi_\ell)\;\overleftrightarrow{C}(\phi_\ell)\cdot\vec{b}\,e^{-ju\phi_\ell}$$

$$=\vec{a}\cdot\left[\sum_{\ell=0}^{N-1}\overleftrightarrow{C}^T(\phi_\ell)\;\overleftrightarrow{C}(\phi_\ell)\,e^{-ju\phi_\ell}\right]\cdot\vec{b} \tag{E.22}$$

so that we have a sum of tetradics. If we limit our consideration to the case that $\vec{a}$ and $\vec{b}$ are both transverse, i.e.,

$$\vec{1}_z\cdot\vec{a}=0=\vec{1}_z\cdot\vec{b} \tag{E.23}$$

then it is only the transverse (x, y) parts of the rotation dyadics that we need consider.

So let us consider the two-dimensional case

$$\overleftrightarrow{D}_{n,m}=\vec{1}_n\cdot\left[\sum_{\ell=0}^{N-1}\begin{pmatrix}\cos(\phi_\ell) & \sin(\phi_\ell)\\ -\sin(\phi_\ell) & \cos(\phi_\ell)\end{pmatrix}\begin{pmatrix}\cos(\phi_\ell) & -\sin(\phi_\ell)\\ \sin(\phi_\ell) & \cos(\phi_\ell)\end{pmatrix}e^{-ju\phi_\ell}\right]\cdot\vec{1}_m$$

$$=\sum_{\ell=0}^{N-1}\left[\begin{pmatrix}\cos(\phi_\ell) & -\sin(\phi_\ell)\\ \sin(\phi_\ell) & \cos(\phi_\ell)\end{pmatrix}\begin{pmatrix}\cos(\phi_\ell) & -\sin(\phi_\ell)\\ \sin(\phi_\ell) & \cos(\phi_\ell)\end{pmatrix}\cdot\vec{1}_m\right]e^{-ju\phi_\ell} \tag{E.24}$$

where n and m take on the roles of x and y giving four pairs to consider. The tradic $\overset{4}{\vec{D}}$ is comprised of four 2 x 2 dyadics.

The x, x or 1, 1 term is

$$\overleftrightarrow{D}_{1,1}=\sum_{\ell=0}^{N-1}\begin{pmatrix}\cos(\phi_\ell)\\ \sin(\phi_\ell)\end{pmatrix}\begin{pmatrix}\cos(\phi_\ell)\\ \sin(\phi_\ell)\end{pmatrix}e^{-ju\phi_\ell}$$

$$=\sum_{\ell=0}^{N-1}\begin{pmatrix}\cos^2(\phi_\ell) & \cos(\phi_\ell)\sin(\phi_\ell)\\ \cos(\phi_\ell)\sin(\phi_\ell) & \sin^2(\phi_\ell)\end{pmatrix}e^{-ju\phi_\ell}$$

$$=\frac{1}{2}\sum_{\ell=0}^{N-1}\begin{pmatrix}1+\cos(2\phi_\ell) & \sin(2\phi_\ell)\\ \sin(2\phi_\ell) & 1-\cos(2\phi_\ell)\end{pmatrix}e^{-ju\phi_\ell} \tag{E.24}$$

$$=\frac{1}{4}\sum_{\ell=0}^{N-1}\begin{pmatrix}2+e^{j2\phi_\ell}+e^{-j2\phi_\ell} & -je^{j2\phi_\ell}+je^{-j2\phi_\ell}\\ -je^{j2\phi_\ell}+je^{-j2\phi_\ell} & 2-e^{j2\phi_\ell}+e^{-j2\phi_\ell}\end{pmatrix}e^{-ju\phi_\ell}$$

So we need to consider sums of the form

$$\begin{aligned}
\sum_{\ell=0}^{N-1} e^{-ju\phi_\ell} &= \sum_{\ell=0}^{N-1} e^{\frac{-j2\pi u\ell}{N}} = \begin{cases} 0 \text{ for } u \neq 0 \text{ and } N \geq 2 \\ N \text{ otherwise} \end{cases} \\
\sum_{\ell=0}^{N-1} e^{-ju\phi_\ell} e^{j2\phi_\ell} &= \sum_{\ell=0}^{N-1} e^{-j[u-2]\phi_\ell} = \begin{cases} 0 \begin{cases} \text{for } u \neq 0 \text{ and } N = 2 \\ \text{for } u \neq 2 \text{ and } N \geq 3 \end{cases} \\ N \text{ otherwise} \end{cases} \\
\sum_{\ell=0}^{N-1} e^{-ju\phi_\ell} e^{-j2\phi_\ell} &= \sum_{\ell=0}^{N-1} e^{-j[u+2]\phi_\ell} = \begin{cases} 0 \begin{cases} \text{for } u \neq 0 \text{ and } N = 2 \\ \text{for } u \neq N = 2 \text{ and } N \geq 3 \end{cases} \\ N \text{ otherwise} \end{cases}
\end{aligned} \tag{E.25}$$

This gives a set of conditions for the various elements in (E.24) to be zero. Writing out the other three terms gives the same sums as in (E.25) and thus the same conditions to give zero. We can then conclude

$$\begin{aligned}
\overset{4}{\vec{D}} &= \left(\vec{D}_{n,m} \right) = \overset{4}{\vec{0}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } u \neq 0 \text{ and } N = 2 \text{ and for } u \neq 2, N-2 \text{ for } N \geq 3
\end{aligned} \tag{E.26}$$

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