Interaction Notes

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An Observation Concerning the Entire Function in SEM Scattering

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Abstract

This paper first summarizes previous results concerning the entire function in the singularity-expansion-method (SEM) representation of the currents on a scatterer and of the scattered far fields. These gave lower bounds on the time width of such. The present paper extends this to find an upper bound for the temporal width of the scattering (far-field) entire function.

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1. Introduction

In the singularity expansion method (SEM) one of the hardest things to pin down has been the entire function (in the $s = \Omega + j\omega$ plane). In time domain, this is an early-time contribution not representable by complex exponentials (poles). This is discussed in some detail in [1] with extensive references. In the present paper we revisit this matter and gain additional insight into this entire function.

As usual we have

$$\sim \; \equiv \; \text{two-sided Laplace transform over time } t$$
$$s = \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency} \tag{1.1}$$

for going back and forth between time and frequency.

2. Class-1 Form of Interaction Representation

In [1] the currents on a body illuminated by a plane wave (an interaction problem) were considered. Summarizing we have an incident plane wave (electric field)

$$
\begin{aligned}
\vec{E}^{(inc)}(\vec{r},t) &= E_0 \, f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)\vec{1}_e \\
\tilde{\vec{E}}^{(inc)}(\vec{r},s) &= E_0 \, e^{-\gamma \vec{1}_i \cdot \vec{r}} \, \tilde{f}(s)\, \vec{1}_e
\end{aligned}
\tag{2.1}
$$

Referring to Fig. 2.1 we have

$$
\begin{aligned}
&\vec{1}_e \equiv \text{polarization (some weighted combination of radar coordinates } \vec{1}_h \text{ and } \vec{1}_v\text{)} \\
&\vec{1}_e \equiv \text{direction of incidence, } \vec{1}_e \cdot \vec{1}_i = 0 \\
&f(t) \equiv \text{incident waveform} \\
&\gamma = \frac{s}{c} \equiv \text{propagation constant} \\
&c = [\mu_0 \varepsilon_0]^{-\frac{1}{2}} \equiv \text{speed of light}
\end{aligned}
\tag{2.2}
$$

The interaction of this incident wave with the body is described by an integral equation of the form

$$
\begin{aligned}
\tilde{\vec{E}}_t^{(inc)}(\vec{r}_s,s) &= \left\langle \tilde{\overleftrightarrow{Z}}_t(\vec{r}_s, \vec{r}_s{}'; s); \tilde{\vec{J}}_s(\vec{r}_s{}', s) \right\rangle \\
&= \overleftrightarrow{1}_S(\vec{r}_s) \cdot \tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) \\
\overleftrightarrow{1}_S(\vec{r}_s) &= \overleftrightarrow{1} - \vec{1}_S(\vec{r}_s)\,\vec{1}_S(\vec{r}_s) \\
&\equiv \text{transverse dyadic to } \vec{1}_S(\vec{r}_s) \\
\vec{1}_S(\vec{r}_s) &\equiv \text{outward pointing normal to surface } S \text{ at surface coordinate } \vec{r}_s
\end{aligned}
\tag{2.3}
$$

In this form the scatterer is treated as perfectly conducting so that the integration is over S (coordinates $\vec{r}_s{}'$). However, this is not essential to the development. The above is also in the form of the E-field or impedance integral equation, but other forms can be used as well.

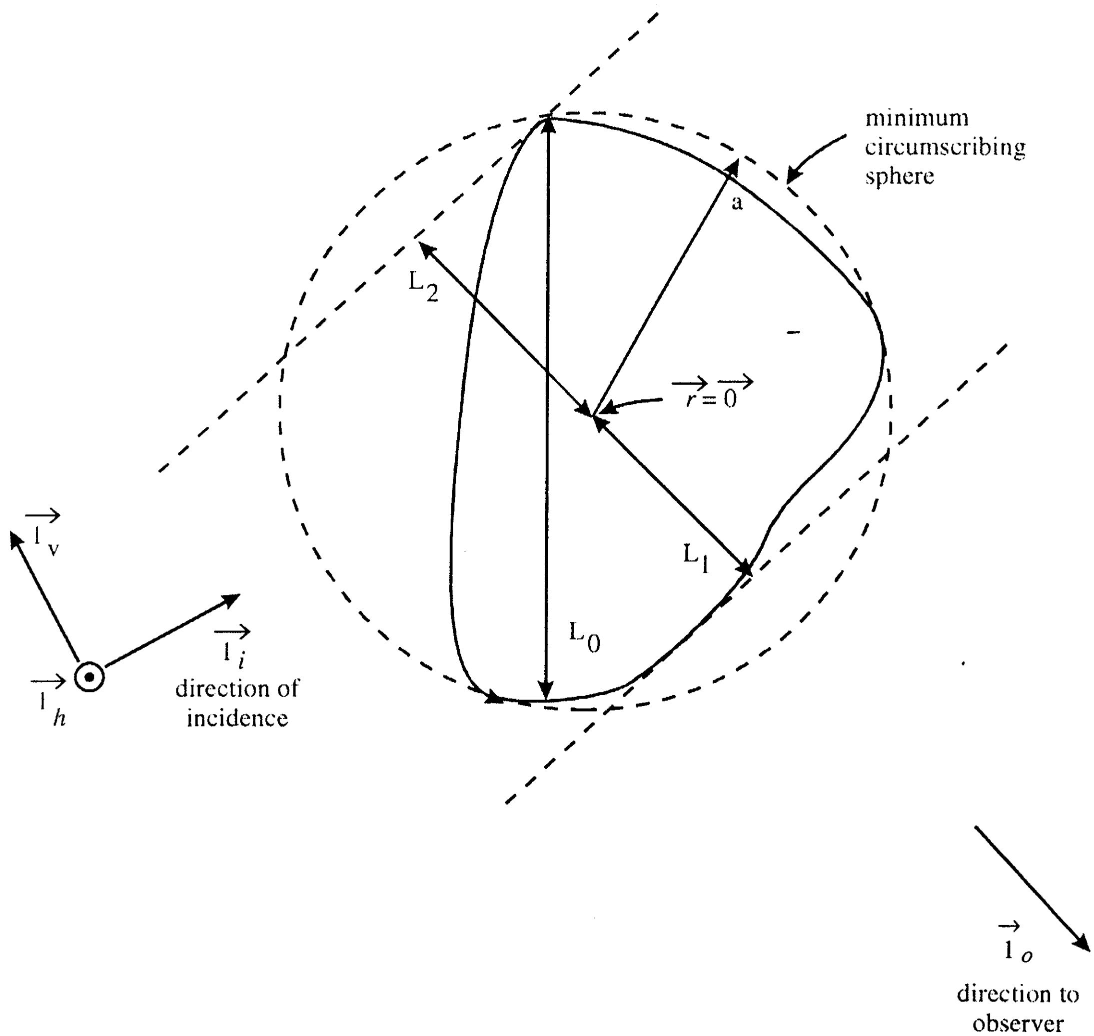

Fig. 2.1 Finite-Size Scatterer in Free Space Illuminated by Plane Wave

The formal solution to (2.3) is

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = \left\langle \tilde{\overleftrightarrow{Z}}_t^{-1}(\vec{r}_s,\vec{r}\,'_s;s); \tilde{\vec{E}}_t^{(inc)}(\vec{r}\,'_s,s)\right\rangle \tag{2.4}$$

In class-1 SEM form the solution is

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = E_0\tilde{f}(s)\left[\sum_\alpha \eta_\alpha(\vec{1}_i,\vec{1}_e)\,\vec{j}_{s_\alpha}(\vec{r}_s)\frac{e^{-[s-s_\alpha]t_i}}{s-s_\alpha} + \text{possible entire function}\right]$$

$$\tilde{\vec{J}}_s(\vec{r}_s,t) = E_0 f(t)\circ\left[\sum_\alpha \eta_\alpha(\vec{1}_i,\vec{1}_e)\,\vec{j}_{s_\alpha}(\vec{r}_s)\,e^{s_\alpha t}u(t-t_i) + \text{possible entire function (temporal form)}\right]$$

t_i = initial or turn-on time, typically taken as t_f the time the incident wave first touches the body

$\circ \equiv$ convolution with respect to time

$$\left\langle \tilde{\overleftrightarrow{Z}}_t(\vec{r}_s,\vec{r}\,'_s;s_\alpha); \vec{j}_{s_\alpha}(\vec{r}\,'_s)\right\rangle = \vec{0} \text{ (for natural frequencies and modes)}$$

$$\mathsf{U}_\alpha = \left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{d}{ds}\tilde{\overleftrightarrow{Z}}_t(\vec{r}_s,\vec{r}\,'_s;s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}\,'_s)\right\rangle^{-1} \tag{2.5}$$

$$\eta_\alpha(\vec{1}_i,\vec{1}_e) = \mathsf{U}_\alpha\,\vec{1}_e\cdot\vec{C}_\alpha(\vec{1}_i) \equiv \text{coupling coefficient}$$

$$\vec{C}_\alpha(\vec{1}_i) = \left\langle e^{-\gamma_\alpha \vec{1}\cdot\vec{r}\,'_s}, \vec{j}_{s_\alpha}(\vec{r}\,'_s)\right\rangle$$

$$\gamma_\alpha \equiv \frac{s_\alpha}{c}$$

Here first order poles have been used, appropriate to perfectly conducting bodies. Special cases of loaded bodies can have some higher-order poles, but this does not impact the argument.

As discussed in [1] a critical role is played by U_α. As one proceeds to sum over terms with more negative $\mathrm{Re}[s_\alpha]$ there is the asymptotic behavior

$$\mathsf{U}_\alpha = O_e(\gamma_\alpha L_0) = O_e(s_\alpha t_0) \text{ as } \mathrm{Re}(s_\alpha)\to -\infty$$

$$L_0 \equiv \text{maximum linear dimension of object} \tag{2.6}$$

$$t_0 \equiv \frac{L_0}{c}$$

Where the exponential order means a bound of the form $e^{s_\alpha t_0}$ times a function which grows more slowly than any exponential. We also have

$$\vec{C}_\alpha(\vec{1}_i) = O_e(-s_\alpha t_b)$$

$$t_b \equiv \text{time for wave to reach all of the body } (t \text{ back, positive}) \tag{2.7}$$

Ignoring $E_0 \tilde{f}(s)$, the residues then take the bound

$$O_e(s_\alpha t_0) O_e(-s_\alpha t_b) O(s_\alpha t_f) = O_e\left(s_\alpha \left[t_0 + t_f - t_b\right]\right)$$

$$t_f \equiv \text{time for wave to first reach any point on the body } (t \text{ front, negative})$$

Note that for all $\vec{1}_i$ we have

$$t_0 \geq t_b - t_f \tag{2.9}$$

Then with a bound on the number of poles smaller than an exponential as s_α sweeps to the left in the s plane the pole series converges. In time domain this implies convergence for all times $> t_f$. In order to avoid problems with convergence to a δ function (although this is often handled in the case of Fourier series) one can treat the step (or ramp) response by choice of $\tilde{f}(s)$ as s^{-1} etc., and including this in the pole residues.

While convergence of the class-1 series for all times $> t_f$ is a necessary condition for the series to be an accurate representation of the current, this does not preclude ipso facto the absence of an entire function to complete the representation. However, for this purpose we can appeal to the numerous calculations that have been made to see how many poles (damped sinusoids) are required to approximate the currents to some degree of accuracy. (See [3-5] and references therein.) For the various bodies (usually, but not always, perfectly conducting) it does not take many (say 10 or so) to reasonably agree with the numerically computed step response (often calculated by solution in the frequency domain followed by inverse Fourier transformation). Based on this let us define:

Simple scatterer (object) ≡ one for which the currents can be represented by class-1 poles (damped sinusoids) without an additional entire function.

Higher order poles can also be allowed. This recognizes that there may be exceptional cases of finite-size bodies in free space for which this does not apply.

3. Class-1 Form of Scattering Representation

In [1] the far fields scattered from a body were also considered. In that paper the convergence of the class-1 pole series for these fields was also considered, with different results from those for the currents.

The far scattering takes the form

$$
\begin{aligned}
\tilde{\vec{E}}_f(\vec{r},s) &= \frac{e^{-\gamma r}}{4\pi r}\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{0},s) \\
&= E_0\tilde{f}(s)\frac{e^{-\gamma r}}{4\pi r}\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s)\cdot\vec{1}_e \\
\vec{1}_o &\equiv \text{direction to observer (at } \vec{r}\,)
\end{aligned}
\tag{3.1}
$$

The scattering can also be described by

$$
\tilde{\vec{E}}_f(\vec{r},s) = -\frac{s\mu_0 e^{-\gamma r}}{4\pi r}\left\langle \overleftrightarrow{1}_o e^{\gamma\vec{1}_o\cdot\vec{r}_s};\tilde{\vec{J}}_s(\vec{r}_s,s)\right\rangle
$$

$$
\begin{aligned}
\overleftrightarrow{1}_o &= \overleftrightarrow{1} - \vec{1}_o\vec{1}_o \equiv \text{dyadic transverse to } \vec{1}_o \\
\overleftrightarrow{1}_i &\equiv \overleftrightarrow{1} - \vec{1}_i\vec{1}_i \equiv \text{dyadic transverse to } \vec{1}_i \\
\overleftrightarrow{1} &= \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z \equiv \text{three-dimensional identify}
\end{aligned}
\tag{3.2}
$$

From the integral-equation solution (2.4) we then have

$$
\begin{aligned}
\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s) &= -s\mu_0\left\langle \overleftrightarrow{1}_o e^{\gamma\vec{1}_o\cdot\vec{r}_s};\tilde{\overleftrightarrow{Z}}_t^{-1}(\vec{r}_s,\vec{r}\,'_s;s);\overleftrightarrow{1}_i e^{-\gamma\vec{1}_i\cdot\vec{r}\,'_s}\right\rangle \\
&= \tilde{\overleftrightarrow{\Lambda}}^T(-\vec{1}_i-\vec{1}_o;s) \text{ (reciprocity)}
\end{aligned}
\tag{3.3}
$$

which for backscattering (monostatic) reduces to

$$
\vec{1}_o = -\vec{1}_i \;, \quad \overleftrightarrow{1}_o = \vec{1}_i
$$

$$
\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i,s) \equiv \tilde{\overleftrightarrow{\Lambda}}(-\vec{1}_i,\vec{1}_i;s)
$$

$$= -s\mu_0 \left\langle \overleftrightarrow{1}_i e^{-\gamma \vec{1}_i \cdot \vec{r}_s} ; \tilde{\overleftrightarrow{Z}}_t(\vec{r}_s, \vec{r}\,'_s; s); \overleftrightarrow{1}_i e^{-\gamma \vec{1}_i \cdot \vec{r}_s} \right\rangle \tag{3.4}$$

$$= \tilde{\overleftrightarrow{\Lambda}}_b^T(\vec{1}_i, s) \quad \text{(symmetric)}$$

Writing the scattering dyadic in class-1 form as

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s) = \sum_\alpha \frac{e^{-[s-s_\alpha]t_p}}{s-s_\alpha} \vec{c}_\alpha(-\vec{1}_o)\, \vec{c}_\alpha(\vec{1}_i) + \text{entire function}$$

$$\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i, s) = \sum_\alpha \frac{e^{-[s-s_\alpha]t_p}}{s-s_\alpha} \vec{c}_\alpha(\vec{1}_i)\, \vec{c}_\alpha(\vec{1}_i) + \text{entire function} \tag{3.5}$$

$$\vec{c}_\alpha(\vec{1}_i) = w_\alpha \vec{C}_\alpha(\vec{1}_i)$$

$$W_\alpha = w_\alpha^2 = -s_\alpha \mu_0 U_\alpha$$

In [1] it is shown that the turn-on time t_p for the poles is, in typical cases, after the first signal to reach the observer at a retarded time $2t_f$ (negative). Otherwise the pole series diverges as one sums over poles with more negative $\mathrm{Re}[s_\alpha]$.

Specialize to the worst case for this divergence problem, namely backscattering. Referring to Fig. 2.1 we have

$$t_1 = -\frac{L_1}{c} = \text{retarded time corresponding to the closest position on the scatterer to the observer (negative)}$$

$$t_2 = \frac{L2}{c} = \text{retarded time corresponding to the farthest position on the scatterer from the observer (positive)} \tag{3.6}$$

$$t_3 \equiv t_2 - t_1$$

$$0 \le t_3 \le t_0 = \frac{L_0}{c}$$

Comparing to (2.7) the case of backscattering has

$$t_1 = t_f \quad , \quad t_2 = t_b \tag{3.7}$$

From (2.6), (2.7), and (3.5) we have a bound for the residues as

$$\mathrm{O}_e\left(s_\alpha t_p\right)\mathrm{O}_e\left(s_\alpha t_0\right)\mathrm{O}_e\left(-s_\alpha t_2\right)\mathrm{O}_e\left(-s_\alpha t_2\right) = \mathrm{O}_e\left(s_\alpha\left[t_p + t_0 - 2t_2\right]\right) \tag{3.8}$$

This gives convergence with

$$t_p > 2t_2 - t_0 \tag{3.9}$$

Now the first signal from the target to reach the observer is at $2t_1$ (negative) in retarded time. So we have a time

$$t_e = t_p - 2t_1 > 2\left[t_2 - t_1\right] - t_0 \tag{3.10}$$

as a time window when the pole series does not converge and gives a bound ~~on~~ how large t_p should be. Note for thin bodies (compared to L_0) that t_e can be negative. However, in a worst case sense $\vec{1}_i$ is aligned with L_0 giving $t_e = t_0$. So we have

$$0 \le t_e \le t_0 \tag{3.11}$$

including a limiting case in the case of worst-case alignment. The point is:

For a time window of width t_e tan entire function (temporal form) is required to adequately represent the far scattering.

Now let us approach the entire-function question from a different (new) direction. In Section 2 we defined a simple scatterer as one for which the currents can be represented by class-1 poles without an entire function, and noted that this has found to be the case by many examples. From (2.5) we then write in time domain.

$$\vec{J}_s(\vec{r},t) = E_0 f(t) \circ \left[\sum_\alpha \eta_\alpha\left(\vec{1}_i, \vec{1}_e\right)\vec{j}_{s_\alpha}(\vec{r}_s)e^{s_\alpha t}u\left(t-t_f\right)\right] \tag{3.12}$$

An alternate form is the step response from $\tilde{f}(s) = 1/s$ as

$$\begin{aligned} \tilde{\vec{J}}_s(\vec{r}_s,s) &= E_0 \sum_\alpha \eta_\alpha\left(\vec{1}_i, \vec{1}_e\right)\vec{j}_{s_\alpha}(\vec{r}_s)\frac{e^{-[s-s_\alpha]t_f}}{s_\alpha\left[s-s_\alpha\right]} + \text{pole at } s=0 \\ \vec{J}_s(\vec{r}_s,t) &= E_0 \sum_\alpha \eta_\alpha\left(\vec{1}_i, \vec{1}_e\right)\vec{j}_{s_\alpha}(\vec{r}_s)\frac{1}{s_\alpha}\, e^{s_\alpha t}\mathrm{u}\left(t-t_f\right) + \text{step function} \end{aligned} \tag{3.13}$$

It is this form that has been used for many of the numerical-verification examples. Note that the low-frequency far scattering is proportional to s^2 [2], making the step response in far scattering proportional to s. This insures that the late-time step-induced far scattering goes to zero at late time, even with a pole at $s = 0$.

From the formula for the far scattered fields (3.2) then we have

$$\tilde{\vec{E}}_f\left(\vec{r}, t+\frac{r}{c}\right) = \frac{\mu_0}{4\pi r}\overleftrightarrow{1}_0 \cdot \left[\frac{\partial}{\partial t}\int_S \vec{J}_s\left(\vec{r}_s, t+\frac{\vec{1}_o \cdot \vec{r}_s}{c}\right) dS\right] \tag{3.14}$$

Applying this termwise to (3.12) or (3.13) gives terms of the form

$$\begin{aligned} &\int_S e^{s_\alpha \theta\left[t + \frac{\vec{1}_o \cdot \vec{r}_s}{c}\right]} u\left(t - t_f + \frac{\vec{1}_o \cdot \vec{r}_s}{c}\right) dS \\ &= e^{s_\alpha t}\int_S e^{s_\alpha \frac{\vec{1}_o \cdot \vec{r}_s}{c}} u\left(t - t_f + \frac{\vec{1}_o \cdot \vec{r}_s}{c}\right) dS \\ &= e^{s_\alpha t}\int_S e^{s_\alpha \frac{\vec{1}_o \cdot \vec{r}_s}{c}} dS \quad \text{for } t \geq t_f + t_2 \\ t_2 &= \frac{L_2}{c} = \sup_S\left[-\frac{\vec{1}_o \cdot \vec{r}_s}{c}\right] \end{aligned} \tag{3.15}$$

So for times (retarded at the observer) greater than $t_f + t_2$ the scattered field (for $f(t)$ a delta- or step-function) can be represented by *damped sines only*. The temporal form of the entire function exists only for times less than this and after the first scattered signal reaches the observer.

Applying this to backscattering we have

$$\vec{1}_o = -\vec{1}_i \;, \quad t_f = t_1 \tag{3.16}$$

with pure damped sines for

$$t \geq t_1 + t_2 \tag{3.17}$$

and first signal reaching the observer (retarded)

$$t = 2t_1 \tag{3.18}$$

This gives an upper bound on the temporal width of the entire function as

$$t_1 + t_2 - 2t_1 = t_2 - t_1 = t_3 \tag{3.19}$$

which is one transit through the body in the direction of the observer. Noting that for thin bodies

$$0 \leq t_3 \leq t_0 \tag{3.20}$$

we have a result comparable to (3.11). Now we have an upper bound on the entire-function width, whereas previously we had a lower bound.

Summarizing we have for backscattering:

> The entire-function (temporal form) of $\overleftrightarrow{\Lambda}_b(\overrightarrow{1}_i,t)$ for *simple scatterers* has a temporal width of no greater than t_3, one transit through the body. This window begins at the time $2t_1$, the time of the first scattered signal.

4. Concluding Remarks

In [1] a lower bound was found for the temporal width of the entire function in scattering. Noting the lack of an entire function in interaction (currents on the body) for simple scatterers, we now have an upper bound of t_3 (one transit of the body) for the temporal width of the entire function. These results are complementary and more tightly pin down the scattering entire function as an early-time phenomenon.

There are related issues discussed in [1]. In particular there is the *identification time* t_d, defined as the retarded time for the observer to receive a scattered signal from all points on S (or more generally V for a volume type of scatterer). This occurs at a time of $2t_3$ in backscattering after first scattered signal in the simplest cases. Accounting for t_3 (at most) for the entire function this gives a time of t_3 during which a *unique* pole series does not apply, since more than one scatterer may give exactly the same scattering during this time, the differences between the two scatterers having not yet been "seen". Afterwards the pole series can correspond to only one target, unless we include various parts (e.g., inside) which experience no fields. For more general targets t_d can be longer, e.g., for perfectly conducting targets for which waves have to travel around the scatterer by non-straight geodesic paths (fat targets).

References

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