IN 538

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MEMORANDUM FOR AFRL/DEHP

ATTENTION: Dr. Carl E. Baum

FROM: AFRL/DEOB-PA (6-6246)

SUBJECT: Security Review Case Number(s): DE 98-368

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JUDY L. JOHNSTON
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1. DE 98-368
2. ___________

Interaction Notes

Note 538

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# Measures of Cross-Polarization for Discrimination of Symmetrical Targets

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Abstract

A radar signature of interest for certain symmetrical targets on or near (under) the earth surface is a null in the cross polarization in the usually $h$, v radar coordinates. This paper discusses measures of this cross polarization in both narrowband and broadband/transient contexts. These involve angles in the eigenvectors of the backscattering dyadic as well as norms of this dyadic and the cross-polarization part.

1. Introduction

In a recent paper [5] it was observed that the backscattering dyadic

$$\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i,s) = \begin{pmatrix} \tilde{\Lambda}_{b_{h,h}}(\vec{1}_i,s) & \tilde{\Lambda}_{b_{h,v}}(\vec{1}_i,s) \\ \tilde{\Lambda}_{b_{v,h}}(\vec{1}_i,s) & \tilde{\Lambda}_{b_{v,v}}(\vec{1}_i,s) \end{pmatrix}$$

$$= \tilde{\overleftrightarrow{\Lambda}}_b^T(\vec{1}_i,s) \quad \text{(reciprocity)}$$

$$\tilde{\Lambda}_{b_{h,v}}(\vec{1}_i,s) = \tilde{\Lambda}_{b_{v,h}}(\vec{1}_i,s) \tag{1.1}$$

$$\vec{1}_i = \text{direction of incidence (radar to target)}$$

$$= -\vec{1}_0$$

$$\vec{1}_0 = \text{direction of scattering (target to radar)}$$

$$\sim \equiv \text{Laplace transform (two-sided)}$$

$$s = \Omega + j\omega \equiv \text{Laplace-transform variable or complex frequency}$$

in the usual $h$, v radar coordinates, is diagonal (zero cross pol), i.e.

$$\tilde{\Lambda}_{b_{h,v}}(\vec{1}_i,s) = 0 \tag{1.2}$$

provided the target has $C_{\infty a}$ symmetry (body of revolution with axial symmetry planes) with $\vec{1}_z$ (perpendicular to the ground surface) as the symmetry axis, independent of $\vec{1}_i$ . This further assumes that the ground can also be approximated as being consistent with this body of revolution. Note that for horizontal polarization we have

$$\vec{1}_h \cdot \vec{1}_z = 0 \quad , \quad \vec{1}_h \ \text{real} \tag{1.3}$$

and for vertical polarization we have

$$\vec{1}_v \cdot \vec{1}_h = 0 \quad , \quad \vec{1}_v \ \text{real}$$

$$\vec{1}_v \cdot \vec{1}_z \neq 0 \quad \text{except in special cases} \tag{1.4}$$

For convenience the dependence of $\tilde{\overleftrightarrow{\Lambda}}_b$ on $\vec{1}_i$ and $s$ are now suppressed.

The backscattering dyadic is diagonalized as

$$\tilde{\overleftrightarrow{\Lambda}}_b = \sum_{\beta=1}^{2} \tilde{\lambda}_\beta \tilde{\overrightarrow{1}}_\beta \tilde{\overrightarrow{1}}_\beta$$

$$\tilde{\overleftrightarrow{\Lambda}}_b \cdot \tilde{\overrightarrow{1}}_\beta = \tilde{\lambda}_\beta \overrightarrow{1}_\beta = \overrightarrow{1}_\beta \cdot \tilde{\overleftrightarrow{\Lambda}}_b \tag{1.5}$$

$$\tilde{\overrightarrow{1}}_{\beta_1} \cdot \tilde{\overrightarrow{1}}_{\beta_2} = 1_{\beta_1,\beta_2} = \begin{cases} 1 \text{ for } \beta_1 = \beta_2 \\ 0 \text{ for } \beta_1 \neq \beta_2 \end{cases} \text{ (orthonormal)}$$

These eigenvectors are in general two-component complex. Note that in general

$$\left| \tilde{\overrightarrow{1}}_\beta \right| = \left\| \tilde{\overrightarrow{1}}_\beta \right\|_2 \neq 1 \tag{1.6}$$

except in the special case of real eigenvectors. As discussed in [6 (Appendix A)] the above is a valid representation except in the special case that

$$\tilde{\Lambda}_{b_{h,h}} - \Lambda_{b_{v,v}} = \pm j\, 2\, \tilde{\Lambda}_{b_{h,v}} \tag{1.7}$$

Considering $\Psi$ as an angle in the $h$, v plane, positive (counterclockwise) from the positive $h$ axis (or $\overrightarrow{1}_h$ direction) we have

$$\tilde{\overrightarrow{1}}_\beta = \cos(\tilde{\psi}_\beta) \overrightarrow{1}_h + \sin(\tilde{\psi}_\beta) \overrightarrow{1}_v \tag{1.8}$$

where each $\tilde{\psi}_\beta$ can be in general complex and we adopt the convention

$$\tilde{\psi}_2 = \tilde{\psi}_1 + \frac{\pi}{2} \tag{1.9}$$

Noting that $\overrightarrow{1}_\beta$ can equally well be replaced by $-\overrightarrow{1}_\beta$ we can adopt some convention that $\overrightarrow{1}_1$ is oriented as having a real part which is positive with respect to $\overrightarrow{1}_h$ $(-\pi/2$ to $\pi/2)$ or $\overrightarrow{1}_v$ $(0$ to $\pi)$. There is also the ambiguity of which eigenvalue/eigenvector to label 1 and which to label 2. One could choose the largest $|\tilde{\lambda}_\beta|$ as $|\tilde{\lambda}_1|$, but this is arbitrary.

Here the scattering dyadic is regarded as a function of frequency, but it can be regarded as a convolution operator (real valued and symmetric) in time domain. The aforementioned $C_{\infty a}$ symmetry property applies to both frequency and time domains, i.e., zero cross polarization passes right through the

Laplace transform and its inverse. So one can also consider time-domain measures and measures involving a broad band of frequencies. Noing that real objects do not have perfect symmetry one may wish to limit the band of frequencies used to those for which certain imperfections in the symetry are either minimized or enhanced, depending on the specific application. One may also wish to select the frequency band so that the scattering from the target is maximized (to maximize signal-to-clutter ratio), such as by exciting important natural frequencies [5].

Note that in time domain [6] we have

$$\overleftrightarrow{\Lambda}_b(t) = \begin{pmatrix} \Lambda_{b_{h,h}}(t) & \Lambda_{b_{h,v}}(t) \\ \Lambda_{b_{v,h}}(t) & \Lambda_{b_{v,v}}(t) \end{pmatrix}$$
$$= \overleftrightarrow{\Lambda}_b^T(t)$$
$$= \sum_{\beta=1}^{2} \lambda_\beta^{(t)}(t)\, \vec{1}_\beta^{(t)}(t)\, \vec{1}_\beta^{(t)}(t) \tag{1.10}$$
$$\lambda_\beta^{(t)}(t) \;,\; \vec{1}_\beta^{(t)}(t) \text{ real}$$
$$\vec{1}_{\beta_1}^{(t)}(t) \cdot \vec{1}_{\beta_2}^{(t)}(t) = 1_{\beta_1,\beta_2} \quad \text{(orthonormal)}$$

where the superscript "t" is used to distinguish these eigenvalues from the frequency-domain forms which are not in general Laplace/Fourier transforms of each other. Since the temporal dyadic is real valued and symmetric the eigenvalues and eigenvectors can be constructed as purely real and it is always diagonalizable. Note that this dyadic is part of a covolution operator as

$$\vec{E}_f(r,t) = \frac{1}{4\pi r}\, \overleftrightarrow{\Lambda}_b(t) \circ \vec{E}^{(inc)}\left(\vec{0}, t - \frac{r}{c}\right) \tag{1.11}$$
$$\circ \equiv \text{convolution with respect to time}$$

and it is in this form that we need to consider bounds.

With these preliminaries we are in a position to ask how close to the ideal case in (1.2) a particular measurement comes. We need some measure of this to be able to use this phenomenon as a target discriminant. Already some measurements indicate this to be an important discriminant [12].

## 2. *Measures of Narrowband Cross Polarization*

### 2.1 Angle-based measures

When there is zero cross polarization then $\tilde{\psi}_1$ and $\tilde{\psi}_2$ are real with values 0, π/2 (or possibly including π and 3π/2 depending on convention. So we can use

$$\begin{aligned}\cos(\tilde{\psi}) &= \cos(\mathrm{Re}(\tilde{\psi}))\cosh(\mathrm{Im}(\tilde{\psi})) - j\sin(\mathrm{Re}(\tilde{\psi}))\sinh(\mathrm{Im}(\tilde{\psi})) \\ \sin(\tilde{\psi}) &= \sin(\mathrm{Re}(\tilde{\psi}))\cosh(\mathrm{Im}(\tilde{\psi})) + j\cos(\mathrm{Re}(\tilde{\psi}))\sinh(\mathrm{Im}(\tilde{\psi}))\end{aligned} \tag{2.1}$$

and consider the real part of these functions noting the periodicity over $0 \le \mathrm{Re}(\psi) \le 2\pi$. Taking $\psi_1$ and $\psi_2$ we can subtract integer multiples of π/2 to shift the real part of either into our range of interest, say

$$-\frac{\pi}{4} < \mathrm{Re}\begin{pmatrix}\tilde{\psi}_1 \\ 2\end{pmatrix} - n\frac{\pi}{2} \le \frac{\pi}{4} \tag{2.2}$$

and use this to define a shifted angle $\tilde{\psi}_s$ with

$$\begin{aligned}\mathrm{Re}(\tilde{\psi}_s) &\equiv \mathrm{Re}\begin{pmatrix}\tilde{\psi}_1 \\ 2\end{pmatrix} - n\frac{\pi}{2} \\ \mathrm{Im}(\tilde{\psi}_s) &= \mathrm{Im}(\tilde{\psi}_1) = \mathrm{Im}(\tilde{\psi}_2)\end{aligned} \tag{2.3}$$

Our measure in this case is how close $\tilde{\psi}_s$ is to 0, and we can look at both $\mathrm{Re}(\tilde{\psi}_s)$ and $\mathrm{Im}(\tilde{\psi}_s)$ if desirable. For example

$$\mathrm{Re}(\tilde{\psi}) \ne 0 \quad \text{with} \quad \mathrm{Im}(\tilde{\psi}) = 0 \tag{2.4}$$

might be looked at as a coordinate rotation through a real angle. Such might be the case if $\tilde{\psi}_s$ were just a few degrees and had some measurement error in the alignment of the horizontal direction $\vec{1}_h$. The local earth surface may be slightly tilted anyway.

### 2.2 Norm-based measures

One might consider $|\tilde{\Lambda}_{b_{h,v}}|$, but to what should it be compared to obtain some dimensionless number? One measures the "magnitude" of a matrix by some kind of norm [1, 7]. These are many such norms, but the ones of most interest are those associated with some vector norm as

$$\|\overleftrightarrow{\Lambda}_b\| = \sup_{\vec{E}^{(inc)} \neq 0} \frac{\|\overleftrightarrow{\Lambda}_b \cdot \vec{E}^{(inc)}\|}{\|\vec{E}^{(inc)}\|} \tag{2.5}$$

In this form the incident electric field $\vec{E}^{(inc)}$ which is perpendicular to $\vec{1}_i$ is varied over all polarizations (including complex) to find the largest scattering as measured by the same norm.

Consider a "unit" vector $\vec{1}_\psi$ wth

$$\begin{aligned} \vec{1}_\psi \cdot \vec{1}_i &= 0 \quad (\vec{1}_i \ \ real) \\ \|\vec{1}_\psi\| &= 1 \end{aligned} \tag{2.6}$$

where the unit size depends on the particular norm. A common norm is the p-norm, which when referred to the $h$, v coordinates is

$$\left\|\vec{1}_\psi\right\|_p = \left[\left|1_{\psi_h}\right|^p + \left|1_{\psi_v}\right|^p\right]^{\frac{1}{2}} \tag{2.7}$$

Commonly used special cases include

$$\begin{aligned} \left\|\vec{1}_\psi\right\|_1 &= \left|1_{\psi_h}\right| + \left|1_{\psi_v}\right| \\ \left\|\vec{1}_\psi\right\|_2 &= \left|\vec{1}_\psi\right| = \left[1_{\psi_h} \cdot \vec{1}_\psi^{\,*}\right]^{\frac{1}{2}} \\ \left\|\vec{1}_\psi\right\|_\infty &= \max\left[\left|1_{\psi_h}\right| \ , \ \left|1_{\psi_v}\right|\right] \end{aligned} \tag{2.8}$$

The associated matrix norms include

$$\begin{aligned} \left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_1 &= \max\left[\left|\tilde{\Lambda}_{b_{h,h}}\right| + \left|\tilde{\Lambda}_{b_{v,h}}\right| \ , \ \left|\tilde{\Lambda}_{b_{h,v}}\right| + \left|\tilde{\Lambda}_{b_{v,v}}\right|\right] \\ &= \text{maximum column magnitude sum} \end{aligned}$$

$$\left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_2 = \left[\chi_{\max}\left(\overleftrightarrow{\tilde{\Lambda}}_b^{\dagger}\cdot\overleftrightarrow{\Lambda}\right)\right]^{\frac{1}{2}}$$

$$\dagger \equiv *T \quad \text{(conjugate transpose)} \tag{2.9}$$

$\chi_{\max} \equiv$ maximum eigenvalue (all eigenvalues real and non-negative)

$$\left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_\infty = \max\left[\left|\tilde{\Lambda}_{b_{h,h}}\right| + \left|\tilde{\Lambda}_{b_{h,v}}\right| \; , \; \left|\tilde{\Lambda}_{b_{v,h}}\right| + \left|\tilde{\Lambda}_{b_{v,v}}\right|\right]$$

$$= \left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_1 \quad \text{(due to reciprocity)}$$

Depending on the specific problem at hand one may wish to choose a particular one of these norms, or even one not on this list.

Of these various norms the 2-norm has the advantage of being roll invariant, i.e., indepenent of the $h$, v coordinates (but $\vec{1}_i$ still fixed). Note, however, that these norms are defined for complex $\vec{1}_\psi$, an important point for complex matrices. Since real vectors are a subset of complex vectors we have

$$\left\|\overleftrightarrow{\tilde{\Lambda}}_b \cdot \vec{1}_\psi\right\|_2 \le \left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_2 \quad \text{for } \vec{1}_\psi \text{ real} \tag{2.10}$$

so the associated matrix norm still provides an upper bound. This suggests that one might use the measure

$$\nu_2 \equiv \frac{\left|\tilde{\Lambda}_{b_{h,v}}\right|}{\left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_2} \tag{2.11}$$

Of course by changing subscripts any other norm may be used.

As discussed in Appendix A, the 2-norm is bounded both above and below by the root span. Noting that the matrix size is small (2 × 2) we have

$$2^{-\frac{1}{2}}\,\mathrm{rsp}(\overleftrightarrow{\tilde{\Lambda}}_b) \le \left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_2 \le \mathrm{rsp}(\overleftrightarrow{\tilde{\Lambda}}_b)$$

$$\mathrm{rsp}(\overleftrightarrow{\tilde{\Lambda}}_b) = \left[\left|\tilde{\Lambda}_{b_{h,h}}\right|^2 + 2\left|\tilde{\Lambda}_{b_{h,v}}\right|^2 + \left|\tilde{\Lambda}_{b_{v,v}}\right|^2\right]^{\frac{1}{2}} \tag{2.12}$$

Then we can define another measure as

$$v_{rs} = \frac{\left|\tilde{\Lambda}_{b_{h,v}}\right|}{\mathrm{rsp}(\tilde{\overleftrightarrow{\Lambda}}_b)} \tag{2.13}$$

As shown in Appendix A the root span is invariant to an orthogonal transformation such as a real coordinate rotation, i.e.,

$$\begin{aligned} &\mathrm{rsp}\left(\begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix} \cdot \tilde{\overleftrightarrow{\Lambda}}_b \cdot \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix}\right) \\ &= \mathrm{rsp}\,(\tilde{\overleftrightarrow{\Lambda}}_b) \end{aligned} \tag{2.14}$$

corresponding to a rotation of the coordinate axes by an angle $\psi$. Then root span is also roll invariant, like the 2-norm. Also root span is a matrix norm, but not an associated matrix norm (such as the 2-norm) which gives a tighter bound when dealing with products of matrices and vectors. However, root span is easier to compute.

## 3. Application of Narrowband Measures to Canonical Scattering Models

### 3.1 Rotational scatterer

As discussed in [6] such targets have

$$
\begin{aligned}
&\overset{\leftrightarrow}{\tilde{\Lambda}}_b = \tilde{\Lambda}_b \overset{\leftrightarrow}{1}_i = \tilde{\Lambda}_b \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ with longitudinal coordinate suppressed} \\
&\overset{\leftrightarrow}{1}_i = \overset{\leftrightarrow}{1}_0 = \overset{\leftrightarrow}{1} - \vec{1}_i \vec{1}_i = \overset{\leftrightarrow}{1} - \vec{1}_0 \vec{1}_0 \equiv \text{transverse identity} \\
&\overset{\leftrightarrow}{1} = \vec{1}_h \vec{1}_h + \vec{1}_v \vec{1}_v + \vec{1}_i \vec{1}_i \\
&\quad = \vec{1}_h \vec{1}_h + \vec{1}_v \vec{1}_v + \vec{1}_i \vec{1}_i \equiv \text{identity} \\
&\left(\vec{1}_h, \vec{1}_v, \vec{1}_0\right) = \left(\vec{1}_h, \vec{1}_v, \vec{1}_0\right) \equiv \textit{ right-handed coordinate directions} \\
&\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{\lambda} = \tilde{\Lambda}_b
\end{aligned}
\tag{3.1}
$$

Physically this means that scattered field is oriented parallel to the incident field. This model applies to any scatterer with $C_N$ symmetry ($N$-fold symmetry axis along $\vec{1}_i$ ) for $N \geq 3$ [13-14]. The scatterer may, in addition, have axial symmetry planes or even general $O_3$ symmetry (orthogonal group in three dimensions), but this is not required.

With the double degeneracy of the eigenvalues the eigenvectors are not uniquely defined. Hence $\psi_s$ is not defined and an angle-based measure is not useful.

For norm-based measures we have

$$
\left\| \overset{\leftrightarrow}{\tilde{\Lambda}}_b \right\| = \left| \tilde{\Lambda}_b \right| \tag{3.2}
$$

for all associated matrix norms. We also have

$$
\text{rsp}\left( \overset{\leftrightarrow}{\tilde{\Lambda}}_b \right) = 2^{\frac{1}{2}} \left| \tilde{\Lambda}_b \right| \tag{3.3}
$$

The norm-based measures previously defined have

$$
\nu_2 = 0 \quad , \quad \nu_{rs} = 0 \tag{3.4}
$$

since $\widetilde{\Lambda}_{b_{h,v}}$ is zero.

### 3.2 Line scatterer

This is defined by

$$\overset{\leftrightarrow}{\widetilde{\Lambda}}_b = \widetilde{\lambda}_1 \; \vec{\widetilde{1}}_1 \; \vec{\widetilde{1}}_1 = \widetilde{\lambda}_1 \begin{pmatrix} \cos^2(\widetilde{\psi}_1) & \cos(\widetilde{\psi}_1)\sin(\widetilde{\psi}_1) \\ \cos(\widetilde{\psi}_1)\sin(\psi_1) & \sin^2(\widetilde{\psi}_1) \end{pmatrix} \tag{3.5}$$

In this case the angle $\widetilde{\psi}_s$ is well defined in (2.3). While physically one may be thinking of a line scatterer for which $\widetilde{\psi}_1$ is a real angle (and frequency independent), this need not be the general case. Here an angle-based measure is easily interpretable with $\widetilde{\psi}_s$ near zero meaning small cross polarization.

For norm-based measures we have

$$\begin{aligned} \widetilde{\Lambda}_{b_{h,h}} &= \widetilde{\lambda}_1 \left| \vec{\widetilde{1}}_1 \cdot \vec{1}_h \right|^2 = \widetilde{\lambda}_1 \cos^2(\widetilde{\psi}_1) \\ \widetilde{\Lambda}_{b_{v,v}} &= \widetilde{\lambda}_1 \left| \vec{\widetilde{1}}_1 \cdot \vec{1}_v \right|^2 = \widetilde{\lambda}_1 \sin^2(\widetilde{\psi}_1) \\ \widetilde{\Lambda}_{b_{h,v}} &= \widetilde{\lambda}_1 \left[ \vec{\widetilde{1}}_1 \cdot \vec{1}_h \right] \left[ \vec{\widetilde{1}}_1 \cdot \vec{1}_v \right] = \frac{\widetilde{\lambda}_1}{2} \sin(2\,\widetilde{\psi}_1) \\ \widetilde{\lambda}_1 &= \widetilde{\Lambda}_{h,h} + \widetilde{\Lambda}_{v,v} \end{aligned} \tag{3.6}$$

For the 2-norm we have

$$\begin{aligned} \overset{\leftrightarrow}{\widetilde{\Lambda}}{}_b^{\dagger} \cdot \overset{\leftrightarrow}{\widetilde{\Lambda}}_b &= \widetilde{\lambda}_1^* \, \widetilde{\lambda}_1 \left[ \vec{\widetilde{1}}{}_1^* \cdot \vec{\widetilde{1}}_1 \right] \vec{\widetilde{1}}{}_1^* \, \vec{\widetilde{1}}_1 \\ &= \left|\widetilde{\lambda}_1\right|^2 \left| \vec{\widetilde{1}}_1 \right|^2 \vec{\widetilde{1}}{}_1^* \, \vec{\widetilde{1}}_1 \\ \left| \vec{\widetilde{1}}_1 \right|^{-1} \vec{\widetilde{1}}_1 &\equiv \text{right eigenvector} \\ \left| \vec{\widetilde{1}}_1 \right|^{-1} \vec{\widetilde{1}}{}_1^* &\equiv \text{left eigenvector} \end{aligned} \tag{3.7}$$

$$\left|\tilde{\lambda}_1\right|^2 \left|\vec{\tilde{1}}_1\right|^4 = \text{eigenvalue}$$

$$\left\|\tilde{\overleftrightarrow{\Lambda}}_b\right\|_2 = \left|\tilde{\lambda}_1\right| \left|\vec{\tilde{1}}_1\right|^2 = \left|\tilde{\lambda}_1\right| \vec{\tilde{1}}_1^{\,*} \cdot \vec{\tilde{1}}_1$$

which for $\vec{\tilde{1}}_1$ real is just

$$\left\|\tilde{\overleftrightarrow{\Lambda}}_b\right\|_2 = \left|\tilde{\lambda}_1\right| \tag{3.8}$$

For the root span we have

$$\operatorname{rsp}\left(\tilde{\overleftrightarrow{\Lambda}}_b\right) = \left[\operatorname{tr}\left(\tilde{\overleftrightarrow{\Lambda}}_b^{\,t} \cdot \tilde{\overleftrightarrow{\Lambda}}_b\right)\right]^{\frac{1}{2}} = \left|\tilde{\lambda}_1\right| \vec{1}_1^{\,*} \cdot \vec{1}_1 = \left\|\tilde{\overleftrightarrow{\Lambda}}_b\right\|_2 \tag{3.9}$$

In this case the root span and 2-norm are the same. We then have

$$\nu_2 = \nu_{rs} = \frac{\left|\vec{\tilde{1}}_1 \cdot \vec{1}_h\right| \left|\vec{\tilde{1}}_1 \cdot \vec{1}_v\right|}{\vec{\tilde{1}}_1^{\,*} \cdot \vec{\tilde{1}}_1} = \frac{1}{2} \left|\sin(2\tilde{\psi}_1)\right| \text{ for } \tilde{\psi}_1 \text{ real} \tag{3.10}$$

which goes to zero for $\tilde{\psi}_s \to 0$.

### 3.3 Dihedral scatterer

This is defined by

$$\tilde{\overleftrightarrow{\Lambda}}_b = \tilde{\lambda}\left[\vec{\tilde{1}}_1 \vec{\tilde{1}}_1 - \vec{\tilde{1}}_2 \vec{\tilde{1}}_2\right] \;,\quad \tilde{\lambda}_1 = -\tilde{\lambda}_2 = \tilde{\lambda} \tag{3.11}$$

For a physical dihedral under symmetrical illumination ( $\vec{1}_i$ parallel to a symmetry plane) the eigenvectors are real (and frequency independent). The model, of course, can be more general, applying to other kinds of structures as well. (For example, one could have two equal-length thin wires at right angles to each other, displaced by a quarter wavelength with two axial symmetry planes for $S_{4a}$ rotation-reflection symmetry.) The angle measure $\tilde{\psi}_s$ in (2.3) is not ambiguous in this case.

For norm-based measures we have

$$
\begin{aligned}
\tilde{\Lambda}_{b_{h,h}} &= \tilde{\lambda}\left[\cos^2(\tilde{\psi}_1) - \sin^2(\tilde{\psi}_1)\right] \\
\tilde{\Lambda}_{b_{v,v}} &= -\tilde{\lambda}\left[\cos^2(\tilde{\psi}_1) - \sin^2(\tilde{\psi}_1)\right] \\
\tilde{\Lambda}_{b_{h,v}} &= \tilde{\lambda}\; 2\cos(\tilde{\psi}_1)\sin(\tilde{\psi}_1) \\
tr\left(\overleftrightarrow{\tilde{\Lambda}}_b\right) &= \tilde{\lambda}_1 + \tilde{\lambda}_2 = 0 = \tilde{\Lambda}_{b_{h,h}} + \tilde{\Lambda}_{b_{v,v}} \\
\det\left(\overleftrightarrow{\tilde{\Lambda}}_b\right) &= -\tilde{\lambda}^2 = \tilde{\Lambda}_{b_{h,h}}\tilde{\Lambda}_{b_{v,v}} - \tilde{\Lambda}_{b_{h,v}}
\end{aligned}
\tag{3.12}
$$

For the 2-norm we have

$$
\overleftrightarrow{\tilde{\Lambda}}_b^{\dagger} \cdot \overleftrightarrow{\tilde{\Lambda}}_b = \tilde{\lambda}^*\tilde{\lambda}\left[\vec{\tilde{1}}_1^*\vec{\tilde{1}}_1^* - \vec{\tilde{1}}_2^*\vec{\tilde{1}}_2^*\right] \cdot \left[\vec{\tilde{1}}_1\vec{\tilde{1}}_1 - \vec{\tilde{1}}_2\vec{\tilde{1}}_2\right] \tag{3.13}
$$

which for real eigenvectors (real $\tilde{\psi}_1$) is

$$
\begin{aligned}
\overleftrightarrow{\tilde{\Lambda}}_b^{t} \cdot \overleftrightarrow{\tilde{\Lambda}}_b &= \left|\tilde{\lambda}\right|^2\left[\vec{\tilde{1}}_1\vec{\tilde{1}}_1 + \vec{\tilde{1}}_2\vec{\tilde{1}}_2\right] = \left|\tilde{\lambda}\right|^2 \overleftrightarrow{1}_t \\
\left\|\overleftrightarrow{\tilde{\Lambda}}_b\right\|_2 &= \left|\tilde{\lambda}\right|
\end{aligned}
\tag{3.14}
$$

For complex $\tilde{\psi}_1$, this is somewhat more complicated. For the root span we have

$$
\begin{aligned}
\text{rsp}\left(\overleftrightarrow{\tilde{\Lambda}}_b\right) &= \left[\left|\tilde{\Lambda}_{b_{h,h}}\right|^2 + 2\left|\tilde{\Lambda}_{b_{h,v}}\right|^2 + \left|\tilde{\Lambda}_{b_{v,v}}\right|^2\right]^{\frac{1}{2}} \\
&= \left[2\left|\tilde{\Lambda}_{b_{h,h}}\right|^2 + 2\left|\tilde{\Lambda}_{b_{v,v}}\right|^2\right]^{\frac{1}{2}} \\
&= \left[\text{tr}\left(\overleftrightarrow{\tilde{\Lambda}}_b^{t} \cdot \overleftrightarrow{\tilde{\Lambda}}_b\right)\right]^{\frac{1}{2}} \\
&= 2^{\frac{1}{2}}\left|\tilde{\lambda}\right| \text{ for } \tilde{\psi}_1 \text{ real}
\end{aligned}
\tag{3.15}
$$

We then have

$$\nu_2 = \frac{\left|\widetilde{\Lambda}_{b_{h,v}}\right|}{\left\|\overleftrightarrow{\widetilde{\Lambda}}_b\right\|_2}$$

$$\nu_2 = \frac{\left|\widetilde{\Lambda}_{b_{h,v}}\right|}{\widetilde{\lambda}} = \left|\sin(2\widetilde{\psi}_1)\right| \text{ for } \widetilde{\psi}_1 \text{ real}$$

$$\nu_{rs} = \frac{\left|\widetilde{\Lambda}_{b_{h,v}}\right|}{rsp\left(\overleftrightarrow{\widetilde{\Lambda}}_b\right)} = \frac{\left|\widetilde{\Lambda}_{b_{h,v}}\right|}{\left[2\left|\widetilde{\Lambda}_{b_{h,h}}\right|^2 + 2\left|\widetilde{\Lambda}_{b_{h,v}}\right|^2\right]^{\frac{1}{2}}} \tag{3.16}$$

$$\nu_{rs} = \frac{\left|\widetilde{\Lambda}_{b_{h,v}}\right|}{2^{\frac{1}{2}}\left|\widetilde{\lambda}\right|} = 2^{-\frac{1}{2}} \sin(2\widetilde{\psi}_1) \text{ for } \widetilde{\psi}_1 \text{ real}$$

3.4 Helical scatterer

Another commonly used canonical scatterer is the ideal helix characterized by [9, 10]

$$\overleftrightarrow{\widetilde{\Lambda}}_b = \widetilde{\Lambda}_b\begin{pmatrix} 1 & \pm j \\ \pm j & -1 \end{pmatrix} = \widetilde{\Lambda}_b\begin{pmatrix} 1 \\ \pm j \end{pmatrix}\begin{pmatrix} 1 \\ \pm j \end{pmatrix}$$
$$\pm \Rightarrow \begin{pmatrix} \text{right} \\ \text{left} \end{pmatrix} \text{ circular polarization} \tag{3.17}$$

the polarization being referred to the wave scattered from an incident linearly polarized wave. The incident and scattered waves have the convention

$$\vec{E}^{(inc)} \quad \text{proportional to } \begin{pmatrix} 1 \\ \mp j \end{pmatrix}$$
$$\vec{E}^{(sc)} \quad \text{proportional to } \begin{pmatrix} 1 \\ \pm j \end{pmatrix} \tag{3.18}$$
$$\text{for } \begin{pmatrix} \text{right} \\ \text{left} \end{pmatrix} \text{ circular polarization}$$

in our $h$, v coordinate system. Note the opposite signs for incident and scattered field. This is related to the definition of right/left circular polarization by the sense of the screw formed by the tip of the electric vector in space at the various positions along the direction (axis) of propagation at some snapshot (instant) in time.

As this screw passes the observer, the rotation of the electric vector has opposite sense depending on whether it is propagating in the $\vec{1}_i$ or $\vec{1}_0 (= -\vec{1}_i)$ direction. At the observer

$$\begin{aligned}\begin{pmatrix}1\\ \pm j\end{pmatrix} \Rightarrow \begin{pmatrix}\text{clockwise}\\ \text{counterclockwise}\end{pmatrix} \text{ rotation}\\ = \begin{pmatrix}\text{negative}\\ \text{positive}\end{pmatrix} \text{ rotation}\end{aligned} \tag{3.19}$$

for the observer looking in the $\vec{1}_i$ direction. Note that an incident right/left-handed polarization produces a purely right/left-handed scattered field. For other types of scatterers, this can be mixed. For a rotational scatterer with $\overleftrightarrow{\Lambda}_b$ proportional to $(1_{n,m})$ an incident right/left-handed polarization produces a left/right-handed scattered field, reversing the sense of the polarization.

Note that this canonical scatterer is precisely a case that cannot be diagonalized [6 (Appendix A)]. It has

$$\begin{aligned}\text{tr}\left(\tilde{\overleftrightarrow{\Lambda}}_b\right) = \det\left(\tilde{\overleftrightarrow{\Lambda}}_b\right) = 0\\ \tilde{\lambda}_1 = \tilde{\lambda}_2 = 0\end{aligned} \tag{3.20}$$

with only one independent eigenvector

$$(\tilde{x}_n)_1 = (\tilde{x}_n)_2 = \text{constant times}\begin{pmatrix}1\\ \pm \text{j}\end{pmatrix} \tag{3.21}$$

Such an eigenvector of a symmetric matrix cannot be normalized as in (1.5) since its dot product with itself is zero. Nevertheless, it has a dyadic representation as in (3.17).

For norm-based measures we have

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}_b^{\dagger} \cdot \tilde{\overleftrightarrow{\Lambda}}_b = 2\left|\tilde{\Lambda}_b\right|^2 \begin{pmatrix}1\\ \mp j\end{pmatrix}\begin{pmatrix}1\\ \pm j\end{pmatrix} = 2\left|\tilde{\Lambda}_b\right|^2 \begin{pmatrix}1 & \pm j\\ \mp j & 1\end{pmatrix}\\ \chi_\ell\left(\tilde{\overleftrightarrow{\Lambda}}_b^{\dagger} \cdot \tilde{\overleftrightarrow{\Lambda}}_b\right) = [1 \pm j]\; 2^{\frac{1}{2}} \left|\tilde{\Lambda}_b\right|^2\\ \left\|\tilde{\overleftrightarrow{\Lambda}}_b\right\|_2 = 2^{\frac{1}{2}} \left|\tilde{\Lambda}_b\right| \,,\; \text{rsp}\left(\tilde{\overleftrightarrow{\Lambda}}_b\right) = 2^{\frac{1}{2}} \left|\tilde{\Lambda}_b\right|\end{aligned} \tag{3.22}$$

We then have

$$\nu_2 = 2^{-\frac{1}{2}} \quad , \quad \nu_{rsp} = \frac{1}{2} \tag{3.23}$$

Such a "helical" scatterer has a large cross-pol component.

4. Measures of Broadband/Transient Cross Polarization

The target signature of zero cross polarization, as discussed in Section 1, for $C_{\infty a}$ symmetry is frequency independent, and hence applies to temporal waveforms as well. For real targets, however, this symmetry is not in general perfect, and one may wish to limit the band of frequencies over which one looks at this signature. Also our instrumentation (radar) will have limitations in frequency as well. These issues need to be addressed in forming measures of the cross polarization.

4.1 Angle-based measures

As in Section 2.1, one can use the angle $\tilde{\psi}_s$ as a measure, but as a function of frequency one will need to characterize it by some average, peak, etc. (some norm) of $|\tilde{\psi}_s(j\omega)|$ over some band of freqencies. As in (1.10) one can also define such an angle (real in this case) from the time-domain eigenvectors. Again one will need to define some kind of norm over the frequency band of interest. In time domain, this implies some sort of convolution weight (filter) before applying the p-norm or whatever. However, this seems considerably more complicated than norms based on the scattering-operator components. Furthermore one might wish to weight the answer by the strength (in some norm) of the scattering dyadic so that frequencies with small scattering (and more susceptible to noise errors) are weighted less than those frequencies with a large scattering.

4.2 Norm-based measures

For scalar time-domain convolution operators we have the associated operator norm [7]

$$\left\| \Lambda_{b_{h,v}}(t) \circ \right\| = \sup_{f(t) \neq 0} \frac{\left\| \Lambda_{b_{h,v}}(t) \circ f(t) \right\|}{\left\| f(t) \right\|} \tag{4.1}$$

with special cases

$$\begin{aligned} \left\| \Lambda_{b_{h,v}}(t) \circ \right\| &= \left\| \Lambda_{b_{h,v}}(t) \right\|_1 = \int_{-\infty}^{\infty} \left| \Lambda_{b_{h,v}}(t) \right| dt \\ &= \left\| \Lambda_{b_{h,v}}(t) \circ \right\|_{\infty} \\ \left\| \Lambda_{b_{h,v}}(t) \circ \right\|_2 &= \sup_{\omega} \left| \tilde{\Lambda}_{b_{h,v}}(j\omega) \right| \equiv \left| \tilde{\Lambda}_{b_{h,v}}(j\omega_{\max}) \right| \end{aligned} \tag{4.2}$$

The 2-norm is particularly interesting, being the largest magnitude for any frequency (ω real). For band-limited measurements this form can be useful and we can define

$$\left\| \Lambda_{b_{h,v}}(t) \circ \right\|_{2bl} = \sup_{\omega_1 \le \omega \le \omega_2} \left| \tilde{\Lambda}_{b_{h,v}}(j\omega) \right| \tag{4.3}$$

Instead of a sharp cut off at $\omega_1$ and $\omega_2$ one may wish to define

$$\left\| \Lambda_{b_{h,v}}(t) \circ \right\|_{2bl} \equiv \sup_{\omega} \left| \tilde{g}(j\omega)\, \tilde{\Lambda}_{b_{h,v}}(j\omega) \right| = \left\| g(t) \circ \Lambda_{h,v}(t) \circ \right\|_2 \tag{4.4}$$

as a weighted 2-norm [8] where the weight function $\tilde{g}(s)$ is a causal filter and $|\tilde{g}(j\omega)|$ is approximately 1 for $\omega_1 < \omega < \omega_2$ and rolls off smoothly below $\omega_1$ and above $\omega_2$.

For dyadic convolution operators we have the associated norm [2]

$$\left\| \overleftrightarrow{\Lambda}_b(t) \circ \right\| = \sup_{\vec{f}(t) \ne \vec{0}} \frac{\left\| \overleftrightarrow{\Lambda}_b(t) \circ \vec{f}(t) \right\|^{\frac{1}{2}}}{\left\| \vec{f}(t) \right\|} \tag{4.5}$$

with the 2-norm given by

$$\begin{aligned} \left\| \overleftrightarrow{\Lambda}_b(t) \circ \right\|_2 &= \sup_{\omega} \left[ \chi_{\max} \left( \tilde{\overleftrightarrow{\Lambda}}{}_b^{\dagger}(j\omega) \cdot \tilde{\overleftrightarrow{\Lambda}}_b(j\omega) \right) \right]^{\frac{1}{2}} \\ &= \left[ \chi_{\max} \left( \tilde{\overleftrightarrow{\Lambda}}{}_b^{\dagger}(j\omega_{\max}) \cdot \tilde{\overleftrightarrow{\Lambda}}_b(j\omega_{\max}) \right) \right]^{\frac{1}{2}} \end{aligned} \tag{4.6}$$

As discussed in [2] this is an example of a natural norm since it is time-translation invariant and roll invariant. (Another norm discussed there, the m-norm, also has these properties.) For band-limited measurements the supremum can be taken over $\omega_1 \le \omega \le \omega_2$ as in (4.3) or (4.4). For the 2-norm we then have the measure

$$\nu_{2bl} = \frac{\left\|\Lambda_{b_{h,v}}(t) \circ\right\|_{2bl}}{\left\|\overset{\leftrightarrow}{\tilde{\Lambda}}_b(t) \circ\right\|_{2bl}}$$

$$= \frac{\sup_{\omega}\left|\tilde{g}(j\omega)\right| \left|\tilde{\Lambda}_{b_{h,v}}(j\omega)\right|}{\sup_{\omega}\left|\tilde{g}(j\omega)\right| \left[\chi_{\max}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}{}_b^{\dagger}(j\omega) \cdot \overset{\leftrightarrow}{\tilde{\Lambda}}{}_b^{\dagger}(j\omega)\right)\right]^{\frac{1}{2}}} \tag{4.7}$$

Again, we can consider the root span. For present purposes let us first note that for all ω

$$2^{-\frac{1}{2}} \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \le \left\|\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right\|_2 \le \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \tag{4.8}$$

This generalizes to $\omega_{\max}$ in the 2-norm as

$$2^{-\frac{1}{2}} \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \le \left\|\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right\|_2 \le \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \tag{4.9}$$

Now we have

$$\operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega_{\max})\right) \le \sup_{\omega} \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \equiv \operatorname{mrsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \tag{4.10}$$

since $\omega_{\max}$ is defined by the 2-norm, not the root span. On the other hand, the root span is much easier to calculate than the 2-norm. One then needs to consider this over an appropriate band of frequencies, as done previously. Note that

$$\operatorname{rsp}\left(\tilde{g}(j\omega)\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) = \left|\tilde{g}(j\omega)\right| \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \tag{4.11}$$

this allows one to also have a band-limited form of the maximum of the root span as

$$\operatorname{brsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \equiv \sup_{\omega} \left|\tilde{g}(j\omega)\right| \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right) \tag{4.12}$$

This then gives another measure of the broadband cross polarization as

$$\begin{aligned} \nu_{brs} &= \frac{\left\|\Lambda_{b_{h,v}}(t)\circ\right\|_{2bl}}{\operatorname{brsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right)} \\ &= \frac{\sup_{\omega}\left|\tilde{g}(j\omega)\right| \left|\tilde{\Lambda}_{b_{h,v}}(j\omega)\right|}{\sup_{\omega}\left|\tilde{g}(j\omega)\right| \operatorname{rsp}\left(\overset{\leftrightarrow}{\tilde{\Lambda}}_b(j\omega)\right)} \\ &\le \nu_{2bl} \end{aligned} \tag{4.13}$$

## 4.3 Residue-based measures

As discussed in [5] the target class of interest here ($C_{\infty a}$ symmetry) has a set of eigenmodes and hence natural modes with $\cos(m\phi)$ and $\sin(m\phi)$ dependence on the azimuthal angle $\phi$ about the symmetry axis. This gives a double degeneracy (two independent natural modes for each natural frequency $s_\alpha$) for $m \ge 1$. For $m = 0$ there is in general (except for accidental dgeneracies [4]) only one natural mode for each $s_\alpha$. This property can be exploited.

The backscattering dyadic can be represented in the form of the singularity expansion method (SEM) as [3]

$$\begin{aligned} \overset{\leftrightarrow}{\tilde{\Lambda}}_b(\vec{1}_i,s) &= \sum_{\alpha} \frac{e^{-(s-s_\alpha)t_i}}{s-s_\alpha} \vec{c}_\alpha(\vec{1}_i)\,\vec{c}_\alpha(\vec{1}_i) \\ &\quad + \text{possible entire function} \\ t &\equiv \text{initial or turn-on time} \end{aligned} \tag{4.14}$$

where the entire function and turn-on time need not concern us here. The vector $\vec{c}_\alpha$ is a function of $\vec{1}_1$ (or $\vec{1}_i$ and $-\vec{1}_0$ in the bistatic case), but for our case of $C_{\infty a}$ target symmetry this vector is independent of $\vec{1}_i$ when referred to the $h$, v coordinates (which are functions of $\vec{1}_i$).

First observe that the lack of an $h$, v component in the scattering implies

$$\vec{c}_\alpha = \vec{1}_h \text{ or } \vec{1}_v \text{ only} \tag{4.15}$$

In particular we have [5]

$$\vec{c}_\alpha = \begin{cases} \vec{1}_h & \text{for antisymmetric (H) modes} \\ \vec{1}_v & \text{for symmetric (E) modes} \end{cases} \tag{4.16}$$

For $m = 0$, a particular $s_\alpha$ has generally only one of these orientations, but for $m \geq 1$ both types of $\vec{c}_\alpha$ are present for a particular $s_\alpha$ (double degeneracy). If one can extract the natural frequencies $s_\alpha$ and the pole residues $\vec{c}_\alpha\,\vec{c}_\alpha$ from the scattering data then the $\vec{c}_\alpha$ can be used as a target discriminant for this kind of symmetry.

Putting this into a norm context for a cross-polarization measure, we have the residue norm or r-norm [7]. For scalar functions, this takes the form

$$\begin{aligned} h(t) &= \sum_n R_n e^{s_n t} u(t) \\ \|h(t)\|_r &= \sum_n |R_n| \end{aligned} \tag{4.17}$$

which of course generalizes directly to dyadics if $|R_n|$ is replaced by an appropriate matrix norm. of course one ned not sum over all the residues, but merely take the largest, perhaps again limited by some frequency band, now interpreted as some domain in the s-plane. Since we are neglecting any entire function we are actually considering only the pole part of the response for present purposes.

So let us consider any particular pole of interest for our norm. For a non-degenerate pole ($m = 0$) we have

$$\begin{aligned} \overleftrightarrow{\Lambda}_\alpha &= \vec{c}_\alpha\,\vec{c}_\alpha \\ \Lambda_{h,v;\alpha} &= \left[\vec{1}_h \cdot \vec{c}_\alpha\right]\left[\vec{c}_\alpha \cdot \vec{1}_v\right] \end{aligned} \tag{4.18}$$

which for our symmetrical target is zero. Also we have

$$\left\| \overleftrightarrow{\Lambda}_\alpha \right\| = \left[ \chi_{\max}\left( \vec{c}_\alpha^{\,*} \vec{c}_\alpha^{\,*} \cdot \vec{c}_\alpha \vec{c}_\alpha \right) \right]^{\frac{1}{2}} = |\vec{c}_\alpha|^2 \tag{4.19}$$

giving a 2-norm residue measure as

$$\nu_{2r} = \frac{\left\| \left[ \vec{1}_h \cdot \vec{c}_\alpha \right] \left[ \vec{c}_\alpha \cdot \vec{1}_{\mathrm{v}} \right] \right\|}{|\vec{c}_\alpha|^2} = \frac{|c_{h_\alpha}|\,|c_{\mathrm{v}_\alpha}|}{|c_{h_\alpha}|^2 + |c_{\mathrm{v}_\alpha}|^2} \tag{4.20}$$

For a doubly degenerate pole ( $m \geq 1$ ) the situation is more complicated in that

$$\begin{aligned} \overleftrightarrow{\Lambda}_\alpha &= \vec{c}_{\alpha_1} \vec{c}_{\alpha_1} + \vec{c}_{\alpha_2} \vec{c}_{\alpha_2} \\ \vec{c}_{\alpha_1} \cdot \vec{c}_{\alpha_2} &= 0 \quad \text{(orthogonal)} \end{aligned} \tag{4.21}$$

where now one of the $\vec{c}_{\alpha_i}$ has only an $h$ component and the other only a v component for our ideal $C_{\infty a}$ target. In this case we can form

$$\begin{aligned} \Lambda_{h,\mathrm{v};\alpha} &= \left[ \vec{1}_h \cdot \vec{c}_{\alpha_1} \right]\left[ \vec{c}_{\alpha_1} \cdot \vec{1}_{\mathrm{v}} \right] + \left[ \vec{1}_h \cdot \vec{c}_{\alpha_2} \right]\left[ \vec{c}_{\alpha_2} \cdot \vec{1}_{\mathrm{v}} \right] \\ \nu_{2,\alpha} &= \frac{\left| c_{h_{\alpha_1}} c_{\mathrm{v}_{\alpha_1}} + c_{h_{\alpha_2}} c_{\mathrm{v}_{\alpha_2}} \right|}{\left\| \overleftrightarrow{\Lambda}_\alpha \right\|_2} \end{aligned} \tag{4.22}$$

For the special case that the $\vec{c}_{\alpha_i}$ are both scalar constants times real unit vectors as

$$\vec{c}_{\alpha_\ell} = c_{\alpha_\ell} \vec{1}_{\alpha_\ell} \quad , \quad \vec{1}_{\alpha_2} = -\vec{1}_i \times \vec{1}_{\alpha_1} \tag{4.23}$$

we have

$$\Lambda_{h,v;\alpha} = \left| \sum_{\ell=1}^{2} c_{\alpha_\ell}^2 \quad \vec{1}_h \cdot \vec{1}_{\alpha_\ell} \vec{1}_{\alpha_\ell} \cdot \vec{1}_v \right|$$

$$= \left| c_{\alpha_\ell}^2 - c_{\alpha_\ell}^2 \right| \left| \vec{1}_h \cdot \vec{1}_{\alpha_\ell} \vec{1}_{\alpha_\ell} \cdot \vec{1}_v \right|$$

$$\left\| \overleftrightarrow{\Lambda}_\alpha \right\|_2 = \left[ \chi_{\max} \left( \overleftrightarrow{\Lambda}_\alpha^* \cdot \overleftrightarrow{\Lambda}_\alpha \right) \right]^{\frac{1}{2}}$$

$$= \left[ \chi_{\max} \left( |c_{\alpha_1}|^4 \vec{1}_{\alpha_1} \vec{1}_{\alpha_1} + |c_{\alpha_2}|^4 \vec{1}_{\alpha 2} \vec{1}_{\alpha_2} \right)^{\frac{1}{2}} \right]$$

$$= \max \left( |c_{\alpha_1}|^2 \quad , \quad |c_{\alpha_2}|^2 \right) \tag{4.24}$$

$$\nu_{2,\alpha} = \frac{\left| c_{\alpha_\ell}^2 - c_{\alpha_\ell}^2 \right| \left| \vec{1}_h \cdot \vec{1}_{\alpha_\ell} \vec{1}_{\alpha_\ell} \cdot \vec{1}_v \right|}{\max\limits_{\ell} \left( |c_{\alpha_\ell}|^2 \right)}$$

The root span can be applied here as well, for which we have

$$\operatorname{rsp}\left( \overleftrightarrow{\Lambda}_\alpha \right) = \left[ \operatorname{tr} \left( |c_{\alpha_1}|^4 \vec{1}_{\alpha_1} \vec{1}_{\alpha_1} + |c_{\alpha_2}|^4 \vec{1}_{\alpha_2} \vec{1}_{\alpha_2} \right) \right]^{\frac{1}{2}}$$

$$= \left[ |c_{\alpha_1}|^4 + |c_{\alpha_2}|^4 \right]^{\frac{1}{2}}$$

$$= \left\| \overleftrightarrow{\Lambda}_\alpha \right\|_2 \tag{4.25}$$

$$\nu_{rs,\alpha} = \frac{\left| c_{\alpha_1}^2 - c_{\alpha_2}^2 \right| \left| \vec{1}_h \cdot \vec{1}_{\alpha_1} \vec{1}_{\alpha_1} \cdot \vec{1}_v \right|}{\left[ |c_{\alpha_1}|^4 + |c_{\alpha_2}|^4 \right]^{\frac{1}{2}}}$$

$$\leq \nu_{2,\alpha}$$

## 4.4 Application to canonical scattering models

Much of the discussion in Section 3 applies in the broadband/transient case as well. For the rotational scatterer (Section 3.1) the symmetry makes the zero $\tilde{\Lambda}_{h,v}$ apply for all frequencies. For the line scatterer (Section 3.2), if $\tilde{\psi}_1$ is a real, fequency-independent angle (e.g., a wire oriented at this angle) then the orientation of the unit vector $\vec{1}_1$ is frequency independent and the broadband response has the same property for all frequencies. If, however, $\tilde{\psi}_1$ is frequency dependent, then the broadband response is better characterized as a norm over the frequency band. Similarly for the dihedral scatterer (Section 3.3), a physical dihedral with frequency-independent characteristics (at least over a band of frequencies) is characterized in the same way as for the narrowband response.

For the helical scatterer (Section 3.4), however, we have a fundamental problem. In time-domain the scattering dyadic is a real-valued convolution operator. The frequency-domain scattering dyadic must then be conjugate symmetric, i.e.,

$$\begin{aligned} \tilde{\overleftrightarrow{\Lambda}}_b^*(s) &= \tilde{\overleftrightarrow{\Lambda}}_b(s^*) \\ \tilde{\overleftrightarrow{\Lambda}}_b(-j\omega) &= \tilde{\overleftrightarrow{\Lambda}}_b^*(j\omega) \end{aligned} \tag{4.26}$$

Applying this to (3.17) gives for some frequency $\omega_0$ where this model applies, the result

$$\tilde{\overleftrightarrow{\Lambda}}_b(-j\omega_0) = \tilde{\Lambda}_b(-j\omega_0)\begin{pmatrix} 1 & \mp j \\ \mp j & 1 \end{pmatrix} \tag{4.27}$$

It follows the $\tilde{\overleftrightarrow{\Lambda}}_b(s)$ *must* be frequency dependent. For real $s$ (= $\Omega$) the dyadic (including any scalar coefficient) is *real*. The circular-polarization characteristic must apply to only a limited set of frequencies, and this model cannot in general characterize broadband scattering.

## 5 Concluding Remarks

As we have seen, the measure of cross polarization in $h$, v coordinates is somewhat different for narrowband and broadband applications. In narrowband applications one can diagonalize $\overset{\leftrightarrow}{\tilde{\Lambda}}_b$ to find appropriate angles, except in the case of rotational scatterers. In broadband applications such angles are not appropriate except in cases where they do not rotate with frequency/time. The use of norms, such as 2-norm and root span, can be applied in both domains.

The norms of the scattering dyadic which we have used are roll invariant. In the numerator of the norm-based measures we have a norm of $\tilde{\Lambda}_{b_{h,v}}$. There can be errors in our estimation of the $\vec{1}_h$ direction. In particular the local earth near the target of interest may not be perfectly horizontal. In such a case it may be preferable to define horizontal by the earth near the target. This corresponds to a (small?) rotation of the $h$, v coordinates. In such a rotation one may observe a reduction of the cross polarization. Of course, this applies for real rotation angles. If there is a complex angle $(\tilde{\psi}_s)$ in the scattering, then this comes from some other source (including a non-symmetrical target). There is also the problem of noise in the data.

## Appendix A. Some matrix Inequalities

Considering an N × N matrix ($a_{n,m}$) we can form a positive semi-definite Hermitian matrix as

$$\begin{aligned} (a_{n,m})^{\dagger} \cdot (a_{n,m}) &= (c_{n,m}) \\ c_{n,m} &= \sum_{n'=1}^{N} a^{*}_{n',n} \; a_{n',m} \end{aligned} \qquad \text{(A.1)}$$

with trace [11]

$$\begin{aligned} \mathrm{tr}((c_{n,m})) &= \sum_{n=1}^{N} c_{n,n} = \sum_{n=1}^{N}\sum_{n'=1}^{N} a^{*}_{n',n} a_{n',n} \\ &= \sum_{n=1}^{N}\sum_{m=1}^{N} |a_{n,m}|^2 \\ &= \mathrm{sp}((a_{n,m})) \\ &= \sum_{\ell=1}^{N} \chi_\ell((b_{n,m})) = \sum_{\ell=1}^{N} \chi_\ell\left((a_{n,m})^{\dagger} \cdot (a_{n,m})\right) \\ \chi_\ell &\geq 0 \ \text{(eigenvalues of matrix argument)} \\ \mathrm{sp} &\equiv \text{span (like tr} \equiv \text{trace)} \end{aligned} \qquad \text{(A.2)}$$

For convenience we also define

$$\begin{aligned} \mathrm{rsp} &\equiv \text{root span} = [\mathrm{sp}]^{\frac{1}{2}} \\ \mathrm{rsp}((a_{n,m})) &= \left[\sum_{n=1}^{N}\sum_{m=1}^{N} |a_{n,m}|^2\right] \end{aligned} \qquad \text{(A.3)}$$

Noting that [1]

$$\begin{aligned} \|(a_{n,m})\|_2 &= \left[\chi_{\max}\left((a_{n,m})^{\dagger} \cdot (a_{n,m})\right)\right]^{\frac{1}{2}} \geq 0 \\ \chi_{\max} &\equiv \text{maximum eigenvalue} \end{aligned} \qquad \text{(A.4)}$$

with equality to zero for only the zero matrix, we have

$$\left\|(a_{n,m})\right\|_2 \le \left[\sum_{\ell=1}^{N} \chi_\ell\left((a_{n,m})^\dagger \cdot (a_{n,m})\right)\right]^{\frac{1}{2}} = \mathrm{rsp}\left((a_{n,m})\right)$$

$$\mathrm{rsp}\left((a_{n,m})\right) \le \left[N \ \ \chi_\ell\left((a_{n,m})^\dagger \cdot (a_{n,m})\right)\right]^{\frac{1}{2}} = N^{\frac{1}{2}} \left\|(a_{n,m})\right\|_2 \qquad \text{(A.5)}$$

So we have bracketed the root span by the 2-norm as

$$\left\|(a_{n,m})\right\|_2 \le \mathrm{rsp}\left((a_{n,m})\right) \le N^{\frac{1}{2}} \left\|(a_{n,m})\right\|_2 \qquad \text{(A.6)}$$

For small N then the root span is not too much larger than the 2-norm. Similarly the root span can bound the 2-norm, both above and below, as

$$N^{-\frac{1}{2}} \mathrm{rsp}\left((a_{n,m})\right) \le \left\|(a_{n,m})\right\|_2 \le \mathrm{rsp}\left((a_{n,m})\right) \qquad \text{(A.7)}$$

Consider an N × N unitary matrix

$$(u_{n,m})^\dagger = (u_{n,m})^{-1} \qquad \text{(A.8)}$$

Observe then that

$$\begin{aligned} &\mathrm{sp}\left((u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^\dagger\right) \\ &= \mathrm{tr}\left((u_{n,m})\cdot(a_{n,m})^\dagger\cdot(u_{n,m})^\dagger\cdot(u_{n,m})\cdot(a_{n,m})\cdot(u_{n,m})^\dagger\right) \\ &= \mathrm{tr}\left((u_{n,m})\cdot(a_{n,m})^\dagger\cdot(a_{n,m})\cdot(u_{n,m})^\dagger\right) \end{aligned} \qquad \text{(A.9)}$$

This is a similarity transformation of $(a_{n,m})^\dagger \cdot (a_{n,m})$ which conserves eigenvalues. Trace being the sum of the eigenvalues we then have

$$\begin{aligned}&\mathrm{sp}\left(\left(u_{n,m}\right)\cdot\left(a_{n,m}\right)\cdot\left(u_{n,m}\right)^{\dagger}\right)\\&=\mathrm{tr}\left(\left(a_{n,m}\right)^{\dagger}\cdot\left(a_{n,m}\right)\right)\\&=\mathrm{sp}\left(\left(a_{n,m}\right)\right)\end{aligned}\tag{A.10}$$

so span and also rot span are invariant to a unitary transformation. An orthogonal transformation, such as a real coordinate rotation, has $(u_{n,m})$ real and the above still applies.

One can interpret the root span as a *vector* norm where the vector consists of the $N^2$ components $a_{n,m}$ arranged in an appropriate order. In this form the root span is a vector 2-norm with the properties of such a norm. Specifically

$$\begin{aligned}&\mathrm{rsp}\left(\left(a_{n,m}\right)\right) = 0 \ \mathit{iff}\ \left(a_{n,m}\right) = \left(0_{n,m}\right)\\&\mathrm{rsp}\left(\alpha\ \left(a_{n,m}\right)\right) = |\alpha|\ \mathrm{rsp}\left(\left(a_{n,m}\right)\right)\\&\mathrm{rsp}\left(\left(a_{n,m}\right) + \left(b_{n,m}\right)\right) \le \mathrm{rsp}\left(\left(a_{n,m}\right)\right) + \mathrm{rsp}\left(\left(b_{n,m}\right)\right)\end{aligned}\tag{A.11}$$

For a matrix norm we also need the norm of a product. For this we have first

$$\begin{aligned}\left|\sum_{n'=1}^{N} a_{n,n'}\ b_{n',m}\right| &= \left|\begin{pmatrix}a_{n,1}\\a_{n,2}\\\vdots\\a_{n,N}\end{pmatrix}\cdot\begin{pmatrix}b_{n,1}\\b_{n,2}\\\vdots\\b_{n,N}\end{pmatrix}\right|\\&\le \left|\begin{pmatrix}a_{n,1}\\a_{n,2}\\\vdots\\a_{n,N}\end{pmatrix}\right|^{\frac{1}{2}}\left|\begin{pmatrix}b_{1,m}\\b_{2,m}\\\vdots\\b_{N,m}\end{pmatrix}\right|^{\frac{1}{2}}\\&= \left[\sum_{n'=1}^{N}\left|a_{n,n'}\right|^2\right]^{\frac{1}{2}}\left[\sum_{n'=1}^{N}\left|b_{n',m}\right|^2\right]^{\frac{1}{2}}\end{aligned}\tag{A.12}$$

where the matrix row and column have been reinterpreted as vectors (only variable index $n'$) for application of the Schwarz inequality, a special case of the Hölder inequality [7]. Now we find

$$\begin{aligned}
\operatorname{sp}\big((a_{n,m}) \cdot (b_{n,m})\big) &= \sum_{n=1}^{N}\sum_{m=1}^{N} \left|\sum_{n'=1}^{N} a_{n,n'}\, b_{n',m}\right|^2 \\
&\le \sum_{n=1}^{N}\sum_{m=1}^{N}\left[\left[\sum_{n'=1}^{N}|a_{n,n'}|^2\right]\left[\sum_{n'=1}^{N}|b_{n',m}|^2\right]\right] \\
&= \left[\sum_{n=1}^{N}\sum_{n'=1}^{N}|a_{n,n'}|^2\right]\left[\sum_{n'=1}^{N}\sum_{m=1}^{N}|b_{n',m}|^2\right] \\
&= \operatorname{sp}\big((a_{n,m})\big)\ \operatorname{sp}\big((b_{n,m})\big)
\end{aligned} \tag{A.13}$$

Noting the non-negative nature of span we can take square roots giving

$$\operatorname{rsp}\big((a_{n,m}) \cdot (b_{n,m})\big) \le \operatorname{rsp}\big((a_{n,m})\big)\ \operatorname{rsp}\big((a_{n,m})\big) \tag{A.14}$$

thereby showing that root span is a matrix norm (for square matrices). However, it is not an associated matrix norm which is defined by

$$\|(a_{n,m})\| \equiv \sup_{(x_n) \ne (0_n)} \frac{\|(a_{n,m}) \cdot (x_n)\|}{\|(x_n)\|} \tag{A.15}$$

This is seen by the fact that for all associated matrix norms we have

$$\|(1_{n,m})\| = 1 \tag{A.16}$$

while for root span we have

$$\operatorname{rsp}\big((1_{n,m})\big) = N^{\frac{1}{2}} \tag{A.17}$$

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