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# Perturbation Formula for the Internal Resonances of a Dielectric Object Embedded in a Low-Impedance Medium

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Abstract

A simple perturbation formula is presented for characterizing the natural frequencies of a dielectric object embedded in an isotropic, homogeneous medium of low wave impedance. For this situation the natural frequencies of the object can be obtained from the interior "cavity" resonances of the same object when immersed in a perfect conductor. Since the cavity modes are assumed to be known or easily measurable for a given body of interest, the presented formulation allows determination of an object's natural frequencies when it is embedded in any external medium of sufficiently low wave impedance. Considering that knowledge of an object's natural resonances can be used in a target identification scheme, the perturbation technique described here may be useful in the development of technologies to identify buried dielectric targets under appropriate soil conditions.

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I. Introduction

For the identification of a buried target using a singularity-based method, the natural resonances of the target must be known. The determination of target resonances for a perfectly conducting target in a homogeneous space has received considerable attention [1]. The natural modes (exterior resonances) of a target in any isotropic, homogeneous environment can be related to the resonances of the target when in free space via a scaling procedure [2]. This simplifies the identification problem, assuming that the medium external to the object is known, and that the target's free space resonances have been determined.

For the identification of a dielectric target, a natural frequency scaling procedure can also be employed. In this note, though, we will focus on the development of a perturbation formula for natural resonances of a target embedded in a medium having a wave impedance small compared to that of the target. This situation is of practical significance since many targets of interest have low relative permittivity (such as a dielectric mine with $\epsilon \approx 2.5\,\epsilon_0$), while the permittivity of common soils may often be in the range $\epsilon \approx 10\epsilon_0 - 30\epsilon_0$, with conductivities ranging from $10^{-3}$ to $10^{-2}$ typical. Although low permittivity soils may also be commonly found, attention is focused here on relatively high permittivity soils with conductivities in the useful range stated above. If the conductivity is too large, electromagnetic methods will not be applicable due to signal attenuation.

For soils of sufficiently high permittivity, the natural resonances of the target are simply related to the interior resonances of the same target when surrounded by an exterior medium of vanishing wave impedance. A previous note [3] considered this problem, where simple perturbation formulas were derived for the natural resonances of an infinite dielectric slab and sphere. Here we present a general perturbation formula for the internal resonances, applicable to any shape target embedded in an exterior region having low wave impedance. Additional material related to this problem can be found in [4].

II. Perturbation Formulas

Consider a lossless object characterized by $\epsilon_2, \mu_0$ embedded in a lossy medium having $\epsilon_1, \mu_0, \sigma_1$ with $\sigma_1$ the (real) conductivity. Working in the two-sided Laplace transform domain ($s \leftrightarrow t$) with $s = \Omega + j\omega$ the complex frequency, the following definitions are useful

$$\gamma_1 = \{s\mu_0[\sigma_1 + s\epsilon_1]\}^{\frac{1}{2}} = s\sqrt{\mu_0\epsilon_1}\left[1 + \frac{\sigma_1}{s\epsilon_1}\right]^{\frac{1}{2}}$$

= propagation constant in exterior medium

$$\gamma_2 = s\sqrt{\mu_0\epsilon_2}$$

= propagation constant in target

$$Z_1 = \left[\frac{s\mu_0}{\sigma_1 + s\epsilon_1}\right]^{\frac{1}{2}} = \sqrt{\frac{\mu_0}{\epsilon_1}}\left[1 + \frac{\sigma_1}{s\epsilon_1}\right]^{-\frac{1}{2}}$$

= wave impedance in exterior medium

$$Z_2 = \sqrt{\frac{\mu_0}{\epsilon_2}}$$

= wave impedance in target.

Let the ratio of internal to external wave impedances be given as

$$\xi(s) = \frac{Z_2(s)}{Z_1(s)} = \frac{\gamma_1(s)}{\gamma_2(s)} = \epsilon_r^{-\frac{1}{2}} \left[ 1 + \frac{\sigma_1}{s\epsilon_1} \right]^{\frac{1}{2}} \tag{1}$$

where $\epsilon_r \equiv \epsilon_2 / \epsilon_1$. The case of small $\epsilon_r$ (large $\xi$) can be thought of as a perturbation from the condition of a perfect electrical boundary around the target. For the perfectly conducting surround ($|\xi| \to \infty$), the object has a denumerably infinite set of interior or cavity resonances, located along the $j\omega$ axis in the complex s-plane. For large but finite $\xi$, these resonances are shifted into the LHP by some amount. Considering a typical dielectric target embedded in a lossy soil, it is clear that many situations may exist where the impedance ratio is large and finite. It is for that class of problem which was considered in [3], and will be further developed here.

The work in [3] considered the perturbation from the cavity resonance condition described above for an infinite slab of half-thickness "a" and a sphere having radius "a." A summary of the method developed in [3] will be given here, with details of the derivations available in that note. The main idea is to solve the problem of scattering from the geometry of interest exactly (normal incidence for the slab), for arbitrary $\epsilon_1, \epsilon_2$. From that solution, cavity modes can be obtained by letting $\xi \to \infty$, and the desired perturbation from the cavity value found by an appropriate expansion of the solution for large but finite $\xi$. The resulting formulas for the frequency perturbation of the $\alpha$*-th* mode are given below [3], where the superscript on E or H indicates the reference direction (surface normal or coordinate) as appropriate for the field in question.

*Slab*:

$$\Delta s_\alpha T_a = -\xi^{-1}(s_\alpha^0)$$

*Sphere*:

$$\Delta s_\alpha^{E^r} T_a = -\xi^{-1}(s_\alpha^{0,E^r}) \left[ \frac{\left(s_\alpha^{0,E^r} T_a\right)^2}{\left(s_\alpha^{0,E^r} T_a\right)^2 + n(n+1)} \right] \tag{2}$$

$$\Delta s_\alpha^{H^r} T_a = -\xi^{-1}(s_\alpha^{0,H^r})$$

where

$$\xi^{-1}(s_\alpha^{0,F}) = \epsilon_r^{1/2}\left[1 + \frac{\sigma_1 T_a}{s_\alpha^{0,F} T_a\, \epsilon_1}\right]^{-\frac{1}{2}} \approx \epsilon_r^{1/2}\left[1 - \frac{\sigma_1 T_a}{2 s_\alpha^{0,F} T_a\, \epsilon_1}\right] . \tag{3}$$

The approximation in (3) is valid when $\left|\left(s_\alpha^{0,F}\epsilon_1\right)/\sigma_1\right| \gg 1$ as discussed in [3]. In the above, $F$ signifies the type of mode and $T_a = \sqrt{\epsilon_2 \mu_0}\, a$ is a normalization factor. The complex natural frequency is obtained as $s_\alpha^F = s_\alpha^{0,F} + \Delta s_\alpha^F$, with $s_\alpha^{0,F}$ representing the cavity mode resonance. For the slab, the unperturbed resonances (cavity modes) are obtained from $\sinh\left(2 s_\alpha^0 T_a\right) = 0$. For the sphere, the cavity resonances for the $E^r$-modes ($TM^r$) are given by the solution of $\left[s_\alpha^{0,E^r} T_a\ i_n\left(s_\alpha^{0,E^r} T_a\right)\right]' = 0$ where $i_n$ are the modified spherical Bessel functions and the prime denotes differentiation. The cavity resonances for the $H^r$-modes ($TE^r$) are given by the solution of $i_n\left(s_\alpha^{0,H^r} T_a\right) = 0$. A similar procedure can be applied to normal incidence scattering from a infinite dielectric cylinder of radius "a", resulting in

*Infinite Cylinder:*

$$\begin{aligned} \Delta s_\alpha^{E^z} T_a &= -\xi^{-1}(s_\alpha^{0,E^z}) = \Delta s_\alpha^{H^\psi} T_a \\ \Delta s_\alpha^{H^z} T_a &= -\xi^{-1}(s_\alpha^{0,H^z}) \left[\frac{\left(s_\alpha^{0,H^z} T_a\right)^2}{\left(s_\alpha^{0,H^z} T_a\right)^2 + n^2}\right] = \Delta s_\alpha^{E^\psi} T_a \end{aligned} \tag{4}$$

where $\psi$ is the radial coordinate in the cylindrical coordinate system. For this case, the cavity resonances are obtained from $I_n\left(s_\alpha^{0,E^z} T_a\right) = 0$ for the $E^z$-modes ($TM^z$), and from $I_n'\left(s_\alpha^{0,H^z} T_a\right) = 0$ for the $H^z$-modes ($TE^z$), where $I_n$ are the modified Bessel functions. It can be seen that the factor $\xi^{-1}$, evaluated at the interior resonance frequency of the unperturbed cavity mode, is common to all three geometries. The $E^r$-modes for the sphere and the $H^z$-modes for the cylinder require multiplication by an additional, primarily geometrical factor. The reason that the $E^r$-modes of the sphere and the $H^z$-modes (not the $E^z$-modes) of the cylinder require an additional multiplicative factor is due to the reference coordinate chosen. For the sphere, the radial coordinate is naturally chosen, and mode type is defined with respect to presence or absence of a radial field component. The radial field component is, of course, everywhere normal to the spherical surface. For the infinite cylinder, the z-axis naturally provides the reference coordinate, which is everywhere tangential to the cylindrical surface. If the modes are defined with respect to presence or absence of a radial component (everywhere normal to the surface in question), then the E-type modes for both the sphere ($E^r$) and infinite cylinder ($E^\psi$) require an additional multiplicative factor, whereas the H-type modes do not. If the z-axis is chosen as the reference coordinate, then the roles of the E and H-type modes for the cylinder become interchanged, compared with the radial ($\psi$) axis reference.

The method described in [3] is only amenable to objects which can be solved exactly in closed form for arbitrary $\epsilon_1, \epsilon_2$. Since most realistic dielectric target shapes of interest do not correspond to the above geometries (with the possible exception of the sphere), a more general formulation is desirable. The derivation of a general perturbation formula which should be fairly accurate for large $\xi$ is presented in the next section, although the result will be stated here as

$$\Delta s_\alpha^F T_a = -\xi^{-1}(s_\alpha^{0,F}) \frac{\frac{a}{2}\oint_S |\vec{H}_0|^2 ds}{\int_V |\vec{H}_0|^2 dV} \tag{5}$$

where the fields are the cavity mode fields evaluated at the cavity resonance, $\vec{H}_0 = \vec{H}_0(s_\alpha^{0,F})$. It can be seen that the formula predicts that for a general body the perturbation is always of the form $\xi^{-1}$, multiplied by a factor which is primarily geometrical. When applied to the slab, sphere, and cylinder geometries, the perturbation formula (5) exactly reproduces the results from the analysis method presented in [3], i.e., (2),(4).

Considering that many dielectric targets of interest, such as mines, have the shape of a finite-height cylinder, the formula (5) was applied to that geometry, as shown in Fig.1. The resulting frequency shifts are

$$\begin{aligned} \Delta s_\alpha^{E^z} T_a &= -\xi^{-1}(s_\alpha^{0,E^z}) \left[1 + \frac{2}{1+\delta_{q,0}} \frac{a}{d}\right] \\ \Delta s_\alpha^{H^z} T_a &= -\xi^{-1}(s_\alpha^{0,H^z}) \frac{\left[\left(\frac{nq\pi a}{d}\right)^2 + x'^4_{np} + \frac{2a}{d}\left(\frac{q\pi a}{d}\right)^2 \left(x'^2_{np} - n^2\right)\right]}{\left(s_\alpha^{0,H^z} T_a\right)^2 \left(n^2 - x'^2_{np}\right)} \end{aligned} \tag{6}$$

where $\delta_{m,n}$ is the Kronecker delta function. For the $E^z$-modes, $s_\alpha^{0,E^z} T_a = j\sqrt{x_{np}^2 + \left(\frac{q\pi a}{d}\right)^2}$ where $J_n(x_{np}) = 0$ defines the p-th resonance of the n-th order Bessel function, and q describes the variation along the height of the cylinder. The subscript $\alpha$ then denotes a combination of (n,p,q) integers. For the $H^z$-modes, $s_\alpha^{0,H^z} T_a = j\sqrt{x'^2_{np} + \left(\frac{q\pi a}{d}\right)^2}$ where $J_n'(x'_{np}) = 0$ provides the p-th resonance of the derivative of the n-th order Bessel function. As a check, for the infinite cylinder ($d/a \to \infty$) Eq.(6) reduces to the form (4).

A special case of interest is the flat disk with no z-variation (q=0),

$$\begin{aligned} \Delta s_\alpha^{E^z} T_a &= -\xi^{-1}(s_\alpha^{0,E^z}) \left[1 + \frac{a}{d}\right] \\ \Delta s_\alpha^{H^z} T_a &= -\xi^{-1}(s_\alpha^{0,H^z}) \left[\frac{\left(s_\alpha^{0,H^z} T_a\right)^2}{\left(s_\alpha^{0,H^z} T_a\right)^2 + n^2}\right] \end{aligned} \tag{7}$$

where typically ($a \approx d$) might be of interest but only setting q=0 is necessary to obtain (7).

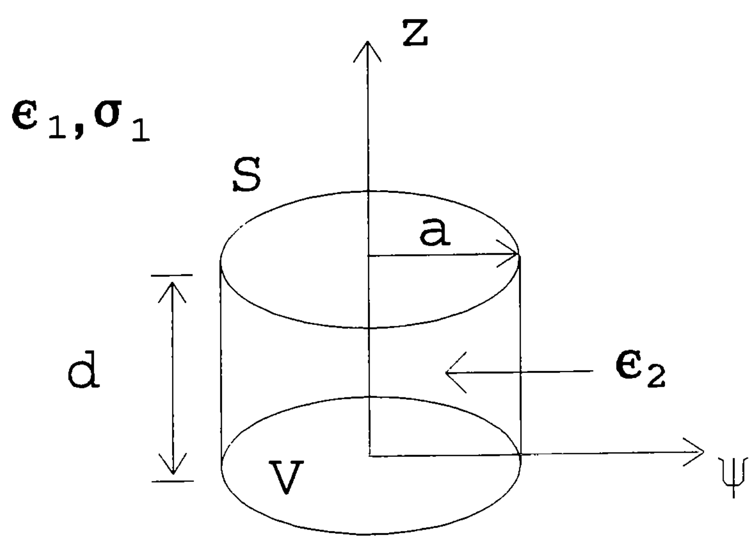


Fig. 1. Finite dielectric cylinder geometry.

It can be seen that for a thin disk (a ≫d) the frequency shifts for the $E^z$-modes are very large. This large damping is due to the surface-to-volume ratio becoming large, resulting in the inability of the cavity to store appreciable energy. It should also be noted that the $H^z$-modes are the same as those for the infinite cylinder, whereas the $E^z$-modes have the additional multiplicative factor $(1+a/d)$.

## III. Derivation of General Perturbation Formula

In this section the perturbation formula (5) will be derived. The method follows directly from the treatment in [5], where the usual application is to determine the effect of conductor wall loss in cavity problems. It will be shown that the formula is generally applicable to any low-impedance medium external to the object of interest.

Consider the geometry shown in Fig. 2. In Fig. 2(a), an object described by surface S enclosing volume V containing a medium electrically characterized by $(\epsilon_2, \mu_0)$ is shown immersed in a perfectly conducting background. The cavity formed by such an object will resonant at a pure imaginary frequency $s_0$, with corresponding modal fields $\vec{E}_0, \vec{H}_0$. In Fig. 2(b), the same object is embedded in a lossy medium characterized by $\epsilon_1, \mu_0$, with the permittivity being generally complex to account for loss. With no restrictions on the various material parameters, Maxwell's curl equations can be stated for each situation as

$$\begin{aligned} -\nabla\times\vec{E}_0 &= s_0\mu_0\vec{H}_0 \qquad & -\nabla\times\vec{E} &= s\mu_0\vec{H} \\ \nabla\times\vec{H}_0 &= s_0\epsilon_2\vec{E}_0 \qquad & \nabla\times\vec{H} &= s\epsilon_2\vec{E} \end{aligned} \quad . \tag{8}$$

Taking the dot product of $\vec{E}_0$ with $\nabla\times\vec{H}$, and of $\vec{H}$ with $\nabla\times\vec{E}_0$, and adding yields

$$\vec{E}_0\cdot\nabla\times\vec{H} - \vec{H}\cdot\nabla\times\vec{E}_0 = s\epsilon_2\vec{E}\cdot\vec{E}_0 + s_0\mu_0\vec{H}\cdot\vec{H}_0 \ . \tag{9}$$

Similarly, taking the dot product of $\vec{E}$ with $\nabla\times\vec{H}_0$, and of $\vec{H}_0$ with $\nabla\times\vec{E}$, and adding yields

$$\vec{E}\cdot\nabla\times\vec{H}_0 - \vec{H}_0\cdot\nabla\times\vec{E} = s_0\epsilon_2\vec{E}\cdot\vec{E}_0 + s\mu_0\vec{H}\cdot\vec{H}_0 \ . \tag{10}$$

Applying the vector identity $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\nabla\times\vec{A} - \vec{A}\cdot\nabla\times\vec{B}$ and subtracting (9) and (10) results in

$$\nabla\cdot(\vec{H}\times\vec{E}_0 - \vec{H}_0\times\vec{E}) = \left[\epsilon_2\vec{E}\cdot\vec{E}_0 - \mu_0\vec{H}\cdot\vec{H}_0\right](s - s_0) \ . \tag{11}$$

Upon volume integration of (11), and application of the divergence theorem,

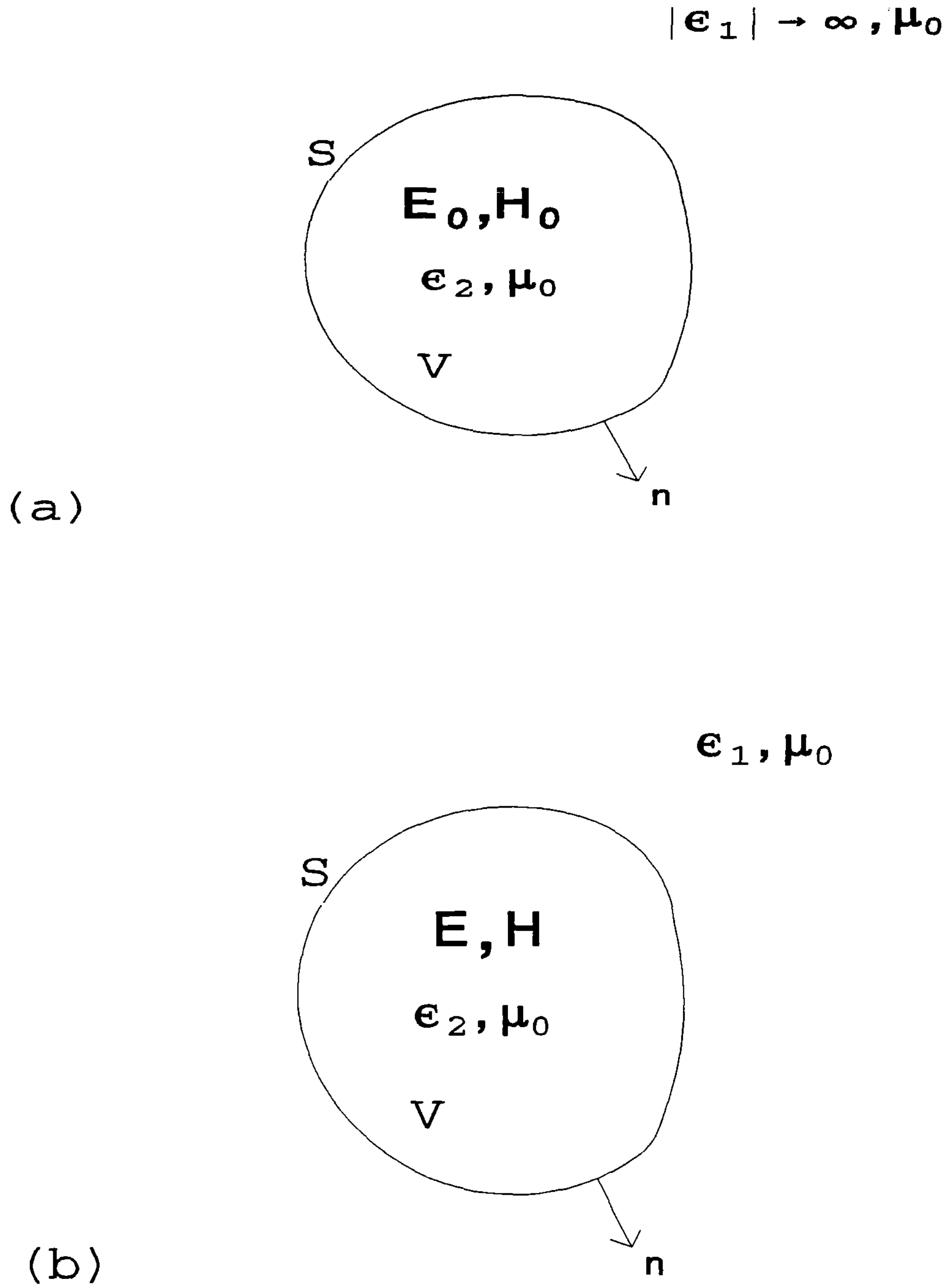


Fig. 2. 2(a) shows unperturbed (cavity) geometry, 2(b) depicts perturbed geometry.

$$\Delta s = s - s_0 = \frac{\oint_S \left(\vec{H}\times\vec{E}_0 - \vec{H}_0\times\vec{E}\right)\cdot d\vec{S}}{\int_V \left(\epsilon_2 \vec{E}\cdot\vec{E}_0 - \mu_0 \vec{H}\cdot\vec{H}_0\right) dV} \,. \tag{12}$$

Since $\hat{n}\cdot(\vec{H}\times\vec{E}_0) = -\vec{H}\cdot\hat{n}\times\vec{E}_0$, and $\hat{n}\times\vec{E}_0 = 0$ on S, we get

$$\Delta s = \frac{\oint_S \left(\vec{H}_0\cdot\hat{n}\times\vec{E}\right) dS}{\int_V \left(\epsilon_2 \vec{E}\cdot\vec{E}_0 - \mu_0 \vec{H}\cdot\vec{H}_0\right) dV} \tag{13}$$

which is an exact expression for the resonant frequency shift in terms of the unknown fields $\vec{E},\vec{H}$.

At this point two approximations are introduced to simplify the above expression. First, consider the typical situation that this type of formulation is applied to, that of a cavity with good but not perfectly conducting walls. For that situation, waves enter the medium surrounding the cavity approximately as plane waves propagating normal to the cavity surface [6]. In that case the field components at the cavity surface are related as $\hat{n}\times\vec{E} = Z_1\vec{H}_T$, with $\vec{H}_T$ being the tangential component of $\vec{H}$, and

$$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_1\left(1 + \frac{\sigma_1}{s\epsilon_1}\right)}} = \sqrt{\frac{\mu_0}{\epsilon_1}}\left[1 + \frac{\sigma_1}{s\epsilon_1}\right]^{-\frac{1}{2}} . \tag{14}$$

For good conductors ( $|\sigma_1 / s\epsilon_1| \gg 1$ ) (14) is often approximated as,

$$Z_1 = \sqrt{\frac{\mu_0}{\epsilon_1}}\left[1 + \frac{\sigma_1}{s\epsilon_1}\right]^{-\frac{1}{2}} = \sqrt{\frac{\mu_0 s}{\sigma_1}}\left[1 + \frac{s\epsilon_1}{\sigma_1}\right]^{-\frac{1}{2}} \approx \sqrt{\frac{\mu_0 s}{\sigma_1}}\left[1 - \frac{s\epsilon_1}{2\sigma_1}\right] \approx \sqrt{\frac{\mu_0 s}{\sigma_1}} . \tag{15}$$

Note also that $Z_1 = \xi^{-1} Z_2$ regardless of approximations, with $\xi$ large because of the conductivity term.

Now consider the case of interest here, where $\xi$ is large due to the real permittivity. For $\xi$ large enough, waves should enter the exterior medium approximately as plane waves propagating normal to the object's surface. In this case, the relation between the field components is still given by $\hat{n}\times\vec{E} = Z_1\vec{H}_T$. Since $\vec{H}$ has components normal and tangential to the object's surface, while $\vec{H}_0$ is purely tangential, (13) can be written as

$$\Delta s = \frac{Z_1 \oint_S \vec{H}\cdot\vec{H}_0 dS}{\int_V \left(\epsilon_2 \vec{E}\cdot\vec{E}_0 - \mu_0 \vec{H}\cdot\vec{H}_0\right) dV} . \tag{16}$$

The above expression still involves the unknown fields $\vec{E},\vec{H}$. As a first order approximation, assume that these fields can be replaced with their corresponding values for $Z_1 \to 0$, i.e., $\vec{E} \approx \vec{E}_0, \vec{H} \approx \vec{H}_0$ with $\vec{E}_0, \vec{H}_0$ purely real and imaginary, respectively. In that case, $\vec{E}_0\cdot\vec{E}_0 = |\vec{E}_0|^2$ and $\vec{H}_0\cdot\vec{H}_0 = -|\vec{H}_0|^2$, leading to

$$\Delta s = \frac{-Z_1 \oint_S |\vec{H}_0|^2 dS}{\int_V \left(\epsilon_2 |\vec{E}_0|^2 + \mu_0 |\vec{H}_0|^2\right) dV} . \tag{17}$$

Since at resonance the stored electric energy is equal to the stored magnetic energy,

$$\Delta s = \Delta s = \frac{-\frac{Z_2 \xi^{-1}}{2} \oint_S |\vec{H}_0|^2 dS}{\int_V \mu_0 |\vec{H}_0|^2 dV} . \tag{18}$$

Finally, applying the normalization factor $T_a$ to (18) results in

$$\Delta s_\alpha^F T_a = -\xi^{-1}(s_\alpha^{0,F}) \frac{\frac{a}{2}\oint_S |\vec{H}_0|^2 ds}{\int_V |\vec{H}_0|^2 dV} \tag{19}$$

where the explicit dependence of the $\alpha$-*th* cavity resonance frequency on mode type is included.

## IV. Conclusion

The natural resonances of a dielectric target embedded in a low-impedance medium are considered. A simple formula is presented which describes the perturbation of an object's natural frequencies from the interior cavity resonances of the same object when immersed in a perfect conductor.

The perturbation formula is valid for any object embedded in a low-impedance background, and is specifically applied to the dielectric disk shape, which resembles a large number of practical dielectric mines.

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