Interaction Notes

Note 517

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A Volumetric Eigenmode Expansion Method for Dielectric Bodies

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Abstract

An eigenmode expansion method (EEM) is developed for dielectric bodies residing completely within a homogeneous region of a generally inhomogeneous medium. The representation follows naturally from the EEM method previously developed for perfectly conducting bodies, including those with impedance loading. For the latter class of objects, the presence of loading shifts the eigenvalues from those of the unloaded case, but leaves the eigenmodes unchanged. It is observed from the governing integral equations that the dielectric body can be considered as a loading of the background space. As such, eigenmodes of homogeneous isotropic bodies are found to be independent of the material comprising the body, with eigenvalues dependent upon the material's characteristics in a simple fashion. Formulation of the eigenvalue problem is described for general dielectric bodies, and the EEM is applied to an infinite slab problem to demonstrate the method.

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Contents

I. Introduction

In this note, the eigenmode expansion method (EEM) previously described [1]-[3] is developed explicitly for dielectric bodies residing within a single homogeneous region of a generally inhomogeneous medium. The motivation for the problem considered here is the study of dielectric targets buried in the ground, such as a dielectric mine.

The EEM developed previously, and principally applied to perfectly conducting objects, has been found to be useful for a variety of problems related to object characterization, and has a strong connection with the singularity expansion method (SEM) [3],[4]. Synthesis of desirable object response characteristics through control of eigenimpedance values is also possible through proper impedance loading, and is simply described via the EEM.

It should be noted that the EEM is similar to the method of characteristic modes, which has been described for conducting and dielectric objects [5],[6]. The characteristic modes are determined through a weighted eigenvalue equation, and for a unity weight become the same as the eigenmodes described here. As opposed to the treatment in [6], this note focuses attention on the fact that eigenmodes can be found which are independent of the medium interior to the target. The separation of the relevant integral operator into target medium dependent and independent parts was noted in [6]. In [6], though, separation of the operator in the eigenvalue problem, which leads to target medium independent modes, was not carried out. Explicit separation of the operator into target medium dependent and independent parts allows classification of target modes which are only dependent upon the shape of the target and on the medium surrounding the target, and constitutes the principle contribution of this note.

II. Formulation of Volume Integral Equation and Corresponding Eigenvalue Problem

In the following development, all electromagnetic quantities will be assumed to obey Maxwell's equations in the two-sided Laplace transform domain, governed by the transform pair

$$F(s) = \int_{-\infty}^{\infty} F(t)\, e^{-st} dt$$
$$F(t) = \frac{1}{2\pi j} \int_{\Omega_0 - j\infty}^{\Omega_0 + j\infty} F(s)\, e^{st} ds \qquad (1)$$

where $s = \Omega + j\omega$, and Ω, ω are real quantities.

Consider the geometry shown in Fig.1, which depicts a closed surface S bounding volume V containing a generally inhomogeneous and anisotropic medium characterized by $\overleftrightarrow{\epsilon}(\vec{r}), \mu_0$. The medium external to S is assumed for simplicity to be isotropic but generally inhomogeneous. In Fig. 1 multiple planar homogeneous layers are shown, but the formulation presented here applies to a generally inhomogeneous medium. The layered exterior medium case corresponds to the buried dielectric body problem as long as the ground can be approximated as consisting of planar layers. In each layer, as well as internal to S, material loss can be accommodated by defining the permittivity to be complex as $\epsilon \rightarrow \epsilon + \sigma/s$ for real conductivity σ.

A wave impedance and propagation constant can be defined for the j-th exterior region, and formally extended to the interior region

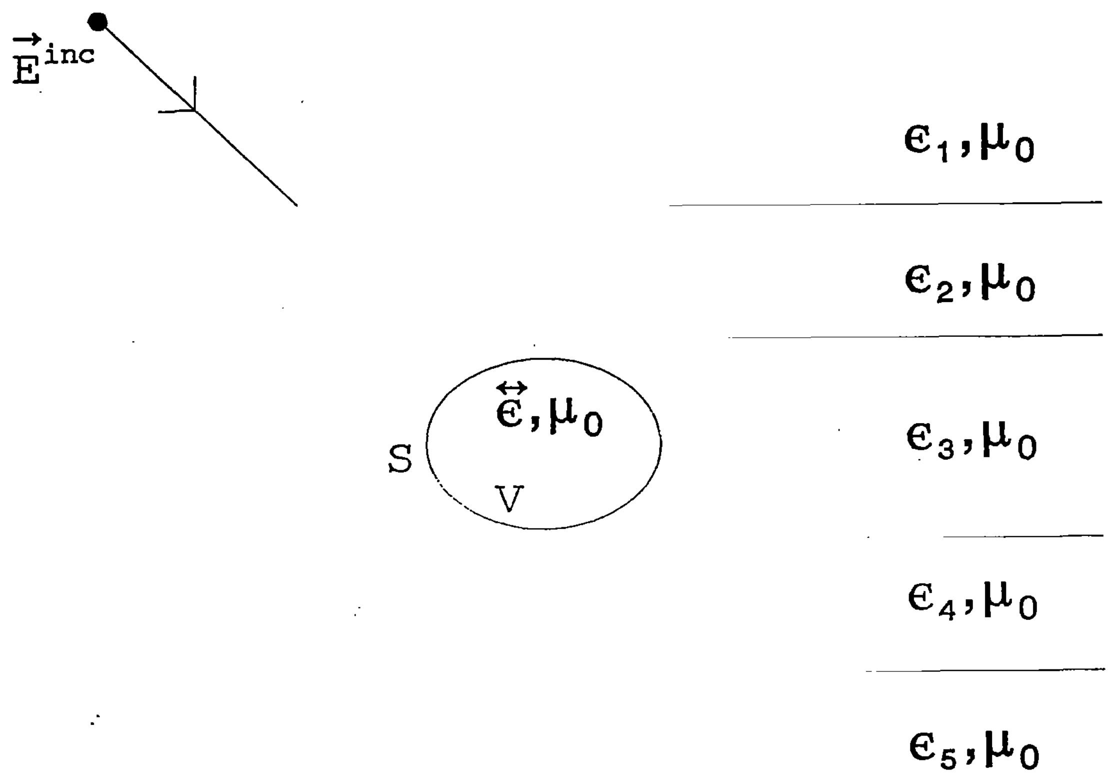

Fig. 1. Dielectric body in layered media.

$$\gamma_j = s[\mu_0 \epsilon_j]^{1/2}$$
$$Z_j = [\mu_0 \epsilon_j^{-1}]^{1/2} \qquad (2)$$

$$\vec{\gamma}(\vec{r}) = s[\mu_0 \vec{\epsilon}(\vec{r})]^{1/2}$$
$$\vec{Z}(\vec{r}) = [\mu_0 \vec{\epsilon}^{-1}(\vec{r})]^{1/2} \qquad (3)$$

noting the following combinations

$$\gamma_j Z_j = s\mu_0 \, , \; \vec{\gamma}\cdot\vec{Z} = \vec{I} s\mu_0 \, ,$$
$$\gamma_i Z_j^{-1} = s\epsilon_j , \; \vec{\gamma}\cdot\vec{Z}^{-1} = s\vec{\epsilon} \qquad (4)$$

The relationship between the electric field and an electric current source can be written as

$$\vec{E}(\vec{r},\gamma_o) = -\left\langle \vec{Z}^e(\vec{r}|\vec{r}',\gamma_o) \; ; \; \vec{J}(\vec{r}) \right\rangle \qquad (5)$$

where < , > denotes integration of the two terms separated by the comma over common spatial coordinates, and the symbol above the comma indicates the real scalar product. The propagation constant γ_o is mealy symbolic, and represents dependance on all of the exterior region layers, i.e., $\gamma_o \equiv \{\gamma_1, \gamma_2, \gamma_3, \ldots\}$. The electric dyadic Green's function, $-Z^e_{\alpha\beta}(\vec{r}|\vec{r}',\gamma_o)$, provides the α*-th* component of field at $\vec{r}$ due to the β*-th* component of a Hertzian current element at $\vec{r}'$. In general, the Green's dyadic can be written as

$$\vec{Z}^e(\vec{r}|\vec{r},\gamma_o) = s\mu_0 \left\{ \vec{g}(\vec{r}|\vec{r}',\gamma_o) + \vec{L}_\delta \, \gamma^{-2} \, \delta(\vec{r}-\vec{r}') \right\} \qquad (6)$$

where the first term is evaluated in a principle value sense for a specified exclusion volume [7]. The second term is the depolarizing dyad contribution, which depends on the shape of the exclusion volume as well as the material properties of the region which includes the source current (with the appropriate value of propagation constant assigned to γ). Details concerning (6), as well as general forms of the principal value term for homogeneous and layered media surround regions are provided in Appendix B.

Next, a volume integral equation for scattering from a dielectric body completely contained within one layer of the surround region can be formulated [8]. Let the region containing the body be region i characterized by ϵ_i, and the region interior to S be characterized by unsubscripted symbols $(\vec{\epsilon}, \vec{\gamma})$. One can replace the medium internal to S with a homogeneous isotropic medium characterized by ϵ_i and containing unknown volume polarization currents

$$\vec{J}(\vec{r},\gamma_o,\vec{\gamma}) = \{\vec{\gamma}\cdot\vec{Z}^{-1} - \gamma_i Z_i^{-1}\vec{I}\} \cdot \vec{E}(\vec{r},\gamma_o,\vec{\gamma})$$
$$= \vec{Z}_d^{-1}(\gamma_i,\vec{\gamma}) \cdot \vec{E}(\vec{r},\gamma_o,\vec{\gamma}) \qquad \ldots \; \forall \vec{r} \in V \qquad (7)$$

where

$$\vec{E}(\vec{r},\gamma_o,\vec{\gamma}) = \vec{E}^{inc}(\vec{r},\gamma_o) + \vec{E}^{s}(\vec{r},\gamma_o,\vec{\gamma}) \tag{8}$$

with $\vec{E}^{inc}$ being the incident field (with the dielectric object removed) and $\vec{E}^{s}$ is the scattered field due to currents excited by the material contrast.

Since

$$\vec{\vec{Z}}_d(\gamma_i,\vec{\gamma})\cdot\vec{J}(\vec{r},\gamma_o,\vec{\gamma}) = \vec{E}^{inc}(\vec{r},\gamma_o) + \vec{E}^{s}(\vec{r},\gamma_o,\vec{\gamma}) \tag{9}$$

with $\vec{E}^{s}$ given by (5), an IE is obtained as

$$\left\langle \vec{\vec{Z}}^{e}(\vec{r}|\vec{r}',\gamma_o) + \vec{\vec{Z}}_d(\gamma_i,\vec{\gamma})\,\delta(\vec{r}-\vec{r}')\ ;\ \vec{J}(\vec{r}',\gamma_o,\vec{\gamma})\right\rangle = \vec{E}^{inc}(\vec{r},\gamma_o)\ . \tag{10}$$

By suitable manipulation of (7), another form can be found as

$$\left\langle \vec{\vec{Z}}^{e}(\vec{r}|\vec{r}',\gamma_o)\cdot\vec{\vec{Z}}_d^{-1}(\gamma_i,\vec{\gamma}) + \vec{\vec{I}}\,\delta(\vec{r}-\vec{r}')\ ;\ \vec{E}(\vec{r}',\gamma_o,\vec{\gamma})\right\rangle = \vec{E}^{inc}(\vec{r},\gamma_o)\ . \tag{11}$$

In (10) and (11), as well as in the development to follow, the IE will be enforced over a range equal to its domain, e.g., $\vec{r}\in V$ for the general three-dimensional case. The forms (10) and (11) are equivalent, and both forms have been used for computational work in scattering theory. The volume IE was originally proposed by Richmond [9], with subsequent work performed by many investigators.

It should be noted that in the absence of material contrast, i.e., $\vec{\vec{\epsilon}} = \epsilon_i\vec{\vec{I}}$, the IE reduces to $\vec{E} = \vec{E}^{inc}$. It will be implicitly assumed in the following that material contrast exists to maintain the polarization current (7).

If the exterior region is comprised of reciprocal media as assumed here, the Green's function involved in the above IE obeys the symmetry property

$$\vec{\vec{Z}}^{e}(\vec{r}|\vec{r}',\gamma_o) = \vec{\vec{Z}}^{e\,T}(\vec{r}'|\vec{r},\gamma_o) \tag{12}$$

where the superscript T denotes the transpose operation. Thus, the Green's function is complex symmetric, but not Hermitian, for general complex frequencies. For the most general form of the impedance dyadic $\vec{\vec{Z}}_d$, a set of eigenvalue equations for (10) can be defined as

$$\begin{aligned} &\left\langle \vec{\vec{Z}}^{e}(\vec{r}|\vec{r}',\gamma_o) + \vec{\vec{Z}}_d(\gamma_i,\vec{\gamma})\,\delta(\vec{r}-\vec{r}')\ ;\ \vec{j}^{\,r}_{\beta}(\vec{r}',\gamma_o,\vec{\gamma})\right\rangle = \lambda^{j,r}_{\beta}(\gamma_o,\vec{\gamma})\vec{j}^{\,r}_{\beta}(\vec{r},\gamma_o,\vec{\gamma}) \\ &\left\langle \vec{j}^{\,l}_{\beta}(\vec{r}',\gamma_o,\vec{\gamma})\ ;\ \vec{\vec{Z}}^{e}(\vec{r}'|\vec{r},\gamma_o) + \vec{\vec{Z}}_d(\gamma_i,\vec{\gamma})\,\delta(\vec{r}-\vec{r}')\right\rangle = \lambda^{j,l}_{\beta}(\gamma_o,\vec{\gamma})\vec{j}^{\,l}_{\beta}(\vec{r},\gamma_o,\vec{\gamma}) \end{aligned} \tag{13}$$

and for (11) as

$$\left\langle \tilde{\tilde{Z}}^e(\bar{r}|\bar{r}',\gamma_o)\cdot\tilde{\tilde{Z}}_d^{-1}(\gamma_i,\tilde{\gamma})+\tilde{I}\,\delta(\bar{r}-\bar{r}')\ ;\ \bar{e}_\beta^r(\bar{r}',\gamma_o,\tilde{\gamma})\right\rangle=\lambda_\beta^{e,r}(\gamma_o,\tilde{\gamma})\bar{e}_\beta^r(\bar{r},\gamma_o,\tilde{\gamma})$$
$$\left\langle \bar{e}_\beta^l(\bar{r}',\gamma_o,\tilde{\gamma})\ ;\ \tilde{\tilde{Z}}^e(\bar{r}'|\bar{r},\gamma_o)+\tilde{\tilde{Z}}_d(\gamma_i,\tilde{\gamma})\,\delta(\bar{r}-\bar{r}')\right\rangle=\lambda_\beta^{e,l}(\gamma_o,\tilde{\gamma})\bar{e}_\beta^l(\bar{r},\gamma_o,\tilde{\gamma}) \tag{14}$$

The left and right eigenmodes in (13) share the same eigenvalues, i.e., $\lambda_\beta^{j,l}=\lambda_\beta^{j,r}=\lambda_\beta^j$, and form a biorthogonal set,

$$\left\langle \bar{j}_\beta^r\ ;\ \bar{j}_{\beta'}^l\right\rangle=0 \qquad \lambda_\beta^j\neq\lambda_{\beta'}^j$$
$$\left\langle \bar{j}_\beta^r\ ;\ \bar{j}_\beta^l\right\rangle\neq 0 \tag{15}$$

and similarly for the modes in (14). Since the above implies a real inner product, and the eigenmodes may in general be complex, the second inner product in (15) is not guaranteed to be nonvanishing. This is discussed in [1], where a convincing argument is made for the second of (15).

Note that generally the eigenvalues and eigenmodes are functions of $\gamma_o,\tilde{\gamma}$. The eigenvalue problem for an inhomogeneous, anisotropic body is considered in more detail in Appendix A. If we now consider the special case of a homogeneous, isotropic body, IE's (10) and (11) are modified by replacing $\tilde{\tilde{Z}}_d,\tilde{\tilde{Z}}_d^{-1}$ with $Z_d\tilde{I},Z_d^{-1}\tilde{I}$, respectively. The resulting kernels are then complex symmetric, in which case the resulting eigenvalue problems become

$$\left\langle \tilde{\tilde{Z}}^e(\bar{r}|\bar{r}',\gamma_o)+Z_d(\gamma_i,\gamma)\tilde{I}\,\delta(\bar{r}-\bar{r}')\ ;\ \bar{j}_\beta(\bar{r}',\gamma_o)\right\rangle=\lambda_\beta^j(\gamma_o,\gamma)\bar{j}_\beta(\bar{r},\gamma_o)$$
$$\left\langle \tilde{\tilde{Z}}^e(\bar{r}|\bar{r}',\gamma_o)Z_d^{-1}(\gamma_i,\gamma)+\tilde{I}\,\delta(\bar{r}-\bar{r}')\ ;\ \bar{e}_\beta(\bar{r}',\gamma_o)\right\rangle=\lambda_\beta^e(\gamma_o,\gamma)\bar{e}_\beta(\bar{r},\gamma_o) \tag{16}$$

since $\bar{j}_\beta^l=\bar{j}_\beta^r=\bar{j}_\beta$ for the symmetric case, and similarly for the other mode set. The eigenvalues $\lambda_\beta^{j,e}$ depend upon γ_o,γ, but the eigenmodes $\bar{j}_\beta,\bar{e}_\beta$ only depend upon the exterior region through $\gamma_o\equiv\{\gamma_1,\gamma_2,\gamma_3,\ldots\}$, and the shape of the object. The last observation follows from the spectral mapping theorem.

It is easy to show that the mode sets are related by

$$\bar{j}_\beta=\bar{e}_\beta$$
$$\lambda_\beta^j=Z_d\lambda_\beta^e \tag{17}$$

in that with those substitutions the first of (16) becomes equal to the second, and vice versa. This is also consistent with the fact that λ_β^j should have dimensions of ohms [1], since (10) maps currents to electric fields, whereas λ_β^e should be dimensionless since (11) maps electric fields to electric fields. For purposes here, attention will be primary directed at the first of (16), with results obtained being generally applicable to the second of (16) as well.

To be consistent with earlier notation [1], let $\lambda^j_\beta \equiv Z_\beta$ and note that rather than (16), we can solve

$$\left\langle \vec{\vec{Z}}^e(\vec{r}|\vec{r}',\gamma_o)\ ;\ \vec{j}_\beta(\vec{r}',\gamma_o)\right\rangle = Z^u_\beta(\gamma_o)\vec{j}_\beta(\vec{r},\gamma_o) \tag{18}$$

and obtain the eigenvalue as

$$Z_\beta(\gamma_o,\gamma) = Z^u_\beta(\gamma_o) + Z_d(\gamma_i,\gamma) \tag{19}$$

where $Z_d(\gamma_i,\gamma) = (\gamma Z^{-1} - \gamma_i Z_i^{-1}) = s(\epsilon - \epsilon_i)$ provides a simple shift factor for the eigenvalues.

The significance of (18) is that one can solve an eigenvalue problem which is only dependent on the exterior region medium and the shape of the body, independent of the type of medium internal to the body. Explicit dependance on the interior region medium only comes from the shift in the eigenvalue through the term $Z_d(\gamma_i,\gamma)$, which only depends on the object's permittivity and that of the layer containing the object.

A further observation can be made for specific types of depolarizing dyad terms. If $\vec{\vec{L}}_\delta = b\vec{\vec{I}}$, such as for spherical or cubical exclusion volumes where b=1/3, the eigenvalue problem becomes

$$\left\langle \vec{\vec{g}}(\vec{r}|\vec{r}',\gamma_o)\ ;\ \vec{j}_\beta(\vec{r}',\gamma_o)\right\rangle = \hat{Z}^u_\beta(\gamma_o)\vec{j}_\beta(\vec{r},\gamma_o) \tag{20}$$

which involves only the principle value part of the electric Green's dyadic, and the resulting eigenvalues are obtained as

$$Z_\beta(\gamma_o,\gamma) = s\mu_0 \hat{Z}^u_\beta(\gamma_o) + b\gamma_i^{-2} + Z_d(\gamma_i,\gamma)\ . \tag{21}$$

III. Eigenmode Expansion Properties

The following properties of (18), the volume eigenvalue equation, follow directly from the previous EEM development [1]. Consider the general eigenvalue problem

$$\left\langle \vec{\vec{Z}}(\vec{r}|\vec{r}',\gamma_o)\ ;\ \vec{j}_\beta(\vec{r}',\gamma_o)\right\rangle = Z_\beta(\gamma_o)\vec{j}_\beta(\vec{r},\gamma_o) \tag{22}$$

where $\vec{\vec{Z}}$ represents either $\vec{\vec{Z}}^e$ or $\vec{\vec{g}}$, and Z_β is either Z^u_β or $\hat{Z}^u_\beta$. It is easy to show that

$$\left\langle \vec{j}_{\beta'}(\vec{r},\gamma_o)\ ;\ \vec{j}_\beta(\vec{r},\gamma_o)\right\rangle = 0 \qquad Z_{\beta'} \neq Z_\beta\ . \tag{23}$$

Assuming that the eigenmodes form a complete set of modes which span the domain of the integral operator, along with the normalization

$$\left\langle \vec{j}_{\beta'}(\vec{r},\gamma_o)\ ;\ \vec{j}_\beta(\vec{r},\gamma_o)\right\rangle = \delta_{\beta',\beta} \tag{24}$$

then

$$\vec{F}(\vec{r},\gamma_o)=\sum_{\beta}\left\langle\vec{j}_{\beta}(\vec{r},\gamma_o)\ ;\ \vec{F}(\vec{r},\gamma_o)\right\rangle\vec{j}_{\beta}(\vec{r},\gamma_o) \tag{25}$$

where $\vec{F}$ is an arbitrary vector function in the domain of the operator. Further properties follow immediately as

$$\vec{I}\,\delta(\vec{r}-\vec{r}')=\sum_{\beta}\vec{j}_{\beta}(\vec{r},\gamma_o)\vec{j}_{\beta}(\vec{r}',\gamma_o) \tag{26}$$

$$\tilde{Z}^{v}(\vec{r}|\vec{r}',\gamma_o)=\sum_{\beta}Z_{\beta}^{v}(\gamma_o)\vec{j}_{\beta}(\vec{r},\gamma_o)\vec{j}_{\beta}(\vec{r}',\gamma_o) \tag{27}$$

$$Z_{\beta}(\gamma_o)=\left\langle\vec{j}_{\beta}(\vec{r},\gamma_o)\ ;\ \tilde{Z}(\vec{r}|\vec{r}',\gamma_o)\ ;\ \vec{j}_{\beta}(\vec{r}',\gamma_o)\right\rangle \tag{28}$$

$$\frac{d}{d\gamma_j}Z_{\beta}(\gamma_o)=\left\langle\vec{j}_{\beta}(\vec{r},\gamma_o)\ ;\ \frac{d}{d\gamma_j}\tilde{Z}(\vec{r}|\vec{r}',\gamma_o)\ ;\ \vec{j}_{\beta}(\vec{r}',\gamma_o)\right\rangle \tag{29}$$

$$\left\langle\vec{j}_{\beta}(\vec{r},\gamma_o)\ ;\ \frac{d}{d\gamma_j}\vec{j}_{\beta}(\vec{r}',\gamma_o)\right\rangle=0 \tag{30}$$

where $\dfrac{\partial}{\partial\gamma_j}$ indicates differentiation with respect to the j-th region propagation constant.

The solution of IE (10) is

$$\vec{J}(\vec{r},\gamma_o,\gamma)=\sum_{\beta}\frac{1}{Z_{\beta}(\gamma_o,\gamma)}\left\langle\vec{j}_{\beta}(\vec{r}',\gamma_o)\ ;\ \vec{E}^{inc}(\vec{r}',\gamma_o)\right\rangle\vec{j}_{\beta}(\vec{r},\gamma_o) \tag{31}$$

where Z_{β} comes from (19) with $Z_{\beta}^{u},\vec{j}_{\beta}$ the solutions of (18). The field scattered by the object is

$$\vec{E}^s(\vec{r},\gamma_o,\gamma) = -\left\langle \vec{\vec{Z}}^e(\vec{r}|\vec{r}',\gamma_o) \; ; \; \vec{J}(\vec{r}) \right\rangle$$

$$= -\sum_\beta \frac{1}{Z_\beta(\gamma_o,\gamma)} \left\langle \vec{j}_\beta(\vec{r},\gamma_o) \; ; \; \vec{E}^{inc}(\vec{r},\gamma_o) \right\rangle \left\langle \vec{\vec{Z}}^e(\vec{r}|\vec{r}',\gamma_o) \; ; \; \vec{j}_\beta(\vec{r}',\gamma_o) \right\rangle \tag{32}$$

$$= -\sum_\beta \frac{s\mu_0}{Z_\beta(\gamma_o,\gamma)} \left\langle \vec{j}_\beta(\vec{r},\gamma_o) \; ; \; \vec{E}^{inc}(\vec{r},\gamma_o) \right\rangle \left\langle \vec{\vec{g}}(\vec{r}|\vec{r}',\gamma_o) \; ; \; \vec{j}_\beta(\vec{r}',\gamma_o) \right\rangle$$

where the last relation is valid for fields outside of the object since $\vec{r} \neq \vec{r}'$.

It should be noted that in formulating the eigenvalue problem (16), and in subsequent expansions, e.g., (25), it is assumed that eigenmodes exist, and that they form a complete set of modes which span the domain of the integral operator. Since the kernel (and associated operator) are complex symmetric but not Hermitian, this is not guaranteed to always be correct.

For additional insight, it is helpful to consider the matrix form of the integral equation, such as would be obtained by employing a method of moments (MoM) solution. This reduces the IE to an NXN matrix equation. A matrix is said to be of simple structure if it is similar to a diagonal matrix, i.e., if it is diagonalizable. If the MoM matrix has N distinct eigenvalues, then the matrix is of simple structure and the eigenmode expansion is valid. This follows from the fact that eigenmodes corresponding to distinct eigenvalues are linearly independent, and so the N linearly independent eigenmodes would form a complete set. Even if the eigenvalues are not all distinct, an N-element linearly independent set may exist, although it is not guaranteed for a matrix lacking sufficient special properties. An important case of degenerate eigenvalues would occur at a branch point in the complex plane, where two or more eigenvalues coalesce. It has been conjectured that branch points may always be present when sufficient object symmetry is lacking [11]. The occurrence of branch points may invalidate the eigenmode expansion at certain points in the complex plane, although at other complex frequencies the expansion would remain complete. The eigenmode expansion would remain valid even at points of eigenvalue degeneracy if the geometric multiplicity of the eigenvalue is equal to its algebraic multiplicity, resulting in N linearly independent eigenmodes. It is assumed here that the matrix representation of the IE is of simple structure, such that the eigenmode expansion is valid, as was assumed in [1]. If this is not true, a generalized expansion may still be obtained involving root vectors [10], although this would considerably complicate the analysis. Some of these issues are summarized in [12], and further discussion can be found in [1], [13]. Lastly, it should be noted that for one of the examples used in [11] to illustrate the occurrence of eigenvalue branch points, the non-uniform transmission line, the geometric multiplicity was equal to the algebraic multiplicity for the cases checked, validating the EEM for that example. Perhaps this will be the case for most physically realistic problems of interest, although it would be premature to make that conjecture at this point.

IV. Conclusion

In this note the EEM previously developed is applied to dielectric bodies immersed in inhomogeneous media. It is found that for the case of a homogeneous, isotropic object, eigenmodes can be found which only depend upon the space external to the body, and the shape of the body. Eigenvalues for a specific interior medium are related to the set of interior-independent eigenvalues by a simple shift factor. This allows the general study of objects which may be considered as belonging to a certain class, which is defined by object shape and background environment, independent of the specific medium comprising the object.

Appendix A. EEM for Inhomogeneous Anisotropic Objects

In this section, the general inhomogeneous, anisotropic volume IE (10) will be considered, along with its eigenvalue equation. Rather than defining left and right eigenmodes as in (13), it will be found that eigenmodes that are in some sense independent of the interior region permittivity can be found for a wider class of objects than the isotropic, homogeneous case considered earlier. The most general class of objects which would share this common set of eigenmodes can be inhomogeneous and anisotropic, but must be connected to one another by a simple scalar relationship.

The appropriate IE to consider is

$$\left\langle \tilde{\tilde{Z}}^{e}(\vec{r}|\vec{r}',\gamma_o)+\tilde{\tilde{Z}}_d(\gamma_l,\tilde{\tilde{\gamma}})\,\delta(\vec{r}-\vec{r}')\,;\vec{J}(\vec{r}',\gamma_o,\tilde{\tilde{\gamma}})\right\rangle=\vec{E}^{inc}(\vec{r},\gamma_o) \tag{A1}$$

where

$$\tilde{\tilde{Z}}_d(\gamma_l,\tilde{\tilde{\gamma}})=\left(\tilde{\tilde{d}}_\gamma(\vec{r})-\frac{\gamma_1}{Z_1}\tilde{\tilde{I}}\right)^{-1} \tag{A2}$$

with $\tilde{\tilde{d}}_\gamma(\vec{r})=\tilde{\tilde{\gamma}}(\vec{r})\cdot\tilde{\tilde{Z}}^{-1}(\vec{r})$. Let the dyadic for the interior region be written as

$$\tilde{\tilde{d}}_\gamma(\vec{r})=k\,\tilde{\tilde{d}}(\vec{r})+\frac{\gamma_1}{Z_1}\tilde{\tilde{I}} \tag{A3}$$

where $\tilde{\tilde{d}}(\vec{r})$ is similar to a diagonal matrix, $\tilde{\tilde{d}}(\vec{r})\sim diag\{f_1(\vec{r}),f_2(\vec{r}),f_3(\vec{r})\}$ with $f_n>0$, which can generally be assumed to be invertible for a real material. The IE becomes

$$\left\langle \tilde{\tilde{Z}}^{e}(\vec{r}\,|\,\vec{r}',\gamma_o)+k^{-1}\tilde{\tilde{d}}^{-1}(\vec{r}')\,\delta(\vec{r}-\vec{r}')\,;\vec{J}(\vec{r}')\right\rangle=\vec{E}^{inc}(\vec{r},\gamma_o) \tag{A4}$$

which can be written as

$$\begin{aligned}&\Big\langle \tilde{\tilde{Z}}^{e}(\vec{r}\,|\,\vec{r}',\gamma_o)\cdot\left[\tilde{\tilde{d}}^{1/2}(\vec{r}')\cdot\tilde{\tilde{d}}^{-1/2}(\vec{r}')\right]\\ &\quad+k^{-1}\left[\tilde{\tilde{d}}^{-1/2}(\vec{r}')\cdot\tilde{\tilde{d}}^{1/2}(\vec{r}')\right]\cdot\left[\tilde{\tilde{d}}^{-1/2}(\vec{r}')\cdot\tilde{\tilde{d}}^{-1/2}(\vec{r}')\right]\delta(\vec{r}-\vec{r}')\,;\vec{J}(\vec{r}')\Big\rangle=\vec{E}^{inc}(\vec{r},\gamma_o)\end{aligned} \tag{A5}$$

Applying $\tilde{\tilde{d}}^{1/2}(\vec{r})$ to (A5) results in

$$\left\langle \tilde{\tilde{Z}}^{e\prime}(\vec{r}\,|\,\vec{r}',\gamma_o)+k^{-1}\tilde{\tilde{I}}\,\delta(\vec{r}-\vec{r}')\,;\vec{J}'(\vec{r}')\right\rangle=\vec{E}^{inc\prime}(\vec{r},\gamma_o) \tag{A6}$$

where

$$\tilde{Z}^{e\prime}(\vec{r}\,|\,\vec{r}\,',\gamma_o) \equiv \tilde{d}^{1/2}(\vec{r})\cdot\tilde{Z}^{e}(\vec{r}\,|\,\vec{r}\,',\gamma_o)\cdot\tilde{d}^{1/2}(\vec{r}\,')$$
$$\vec{J}'(\vec{r}) \equiv \tilde{d}^{-1/2}(\vec{r})\cdot\vec{J}(\vec{r}) \tag{A7}$$
$$\vec{E}^{inc\prime}(\vec{r}) \equiv \tilde{d}^{1/2}(\vec{r})\cdot\vec{E}^{inc}(\vec{r})$$

This leads to the eigenvalue equation

$$\left\langle \tilde{Z}^{e\prime}(\vec{r}\,|\,\vec{r}\,',\gamma_o) + k^{-1}\tilde{I}\,\delta(\vec{r}-\vec{r}\,')\,;\vec{j}_\beta{}'(\vec{r}\,',\gamma_o)\right\rangle = z_\beta{}'(\gamma_o,k)\,\vec{j}_\beta{}'(\vec{r},\gamma_o) \tag{A8}$$

where the eigenmodes are independent of k as in (16). Thus, the same set of eigenmodes can be used for any interior region for a fixed anisotropic spatial dependance $\tilde{d}(\vec{r})$.

Appendix B. Electric Dyadic Green's Function

For convenience, the specific forms of the electric dyadic Green's function are presented for a homogeneous space, as well as the general form for a layered media environment. In all cases, isotropic media characterized by $\epsilon(\vec{r}),\mu_0$ will be considered.

I. Homogeneous Space: Spatial Form

For a material characterized by ϵ,μ_0, the appropriate equations are

$$\nabla\times\vec{E}(\vec{r}) = -s\mu_0\vec{H}(\vec{r})$$
$$\nabla\times\vec{H}(\vec{r}) = s\epsilon\vec{E}(\vec{r})+\vec{J}(\vec{r}) \tag{B1}$$
$$\nabla\cdot\vec{J}(\vec{r})+s\rho(\vec{r})=0$$

The divergence of the first of (B1) yields $\nabla\cdot\mu_0\vec{H}(\vec{r})=0$, from which the magnetic vector potential is defined as

$$\mu_0\vec{H}(\vec{r}) = \nabla\times\vec{A}(\vec{r}). \tag{B2}$$

Substitution back into the first of (B1) yields $\nabla\times[\vec{E}(\vec{r})+s\vec{A}(\vec{r})]=0$ leading to the scaler potential

$$\vec{E}(\vec{r}) = -s\vec{A}(\vec{r})-\nabla\phi(\vec{r}). \tag{B3}$$

Inserting the fields in terms of the potentials into the second of (B1) and invoking the Lorentz condition $\nabla\cdot\vec{A}(\vec{r}) = -s\mu_0\epsilon\,\phi(\vec{r})$ leads to

$$(\nabla^2-\gamma^2)\vec{A}(\vec{r}) = -\mu_0\vec{J}(\vec{r}). \tag{B4}$$

The solution of the above is

$$\vec{A}(\vec{r}) = \mu_0 \lim_{\delta\to 0}\int_{V-V_\delta}\vec{J}(\vec{r})\,G(\vec{r}|\vec{r}')\,dV' \tag{B5}$$

where

$$G(\vec{r}|\vec{r}') = \frac{e^{-\gamma|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \tag{B6}$$

is the free-space Green's function and V_δ is a small exclusion volume surrounding the point $\vec{r}=\vec{r}'$. The exclusion volume, having some specific shape and position, has functional dependence on the observation point $\vec{r}$, where δ is the maximum chord of the exclusion volume. The volume within the exclusion region vanishes while maintaining the original shape of the volume [7].

That (B5) is the proper solution of the wave equation can be seen by application of an appropriate Green's theorem in solving (B4), where some conditions must be imposed on terms in the Green's theorem, such as continuity and continuity of derivatives. The limiting form of (B5) is often omitted, since (B5) converges to a unique value independent of the shape of V_δ, as long as the current is piecewise continuous [14]. Special care must be taken when operating on (B5), especially when performing differentiation [7].

Once $\vec{A}(\vec{r})$ is determined, the electric field can be obtained from either of the following,

$$\vec{E}(\vec{r}) = \frac{\nabla\times\nabla\times\vec{A}(\vec{r})}{s\mu_0\epsilon} - \frac{\vec{J}(\vec{r})}{s\epsilon}$$

$$\vec{E}(\vec{r}) = -s\left[1 - \frac{\nabla\nabla\cdot}{\gamma^2}\right]\vec{A}(\vec{r}) \qquad \text{(B7)}$$

One appealing method of obtaining a dyadic Green's function is to carefully interchange the $\nabla\times\nabla\times$ operator in the first of (B7) with the volume integral operator in (B5) using a three dimensional form of Leibniz's rule for the curl operator [15]. This leads to

$$\vec{E}(\vec{r}) = -s\mu_0 \lim_{\delta\to 0} \int_{V-V_\delta} \vec{g}(\vec{r}|\vec{r}')\cdot\vec{J}(\vec{r}')dV' - \frac{\vec{L}_\delta\cdot\vec{J}(\vec{r})}{s\epsilon} \qquad \text{(B8)}$$

where

$$\vec{g}(\vec{r}|\vec{r}') = \left[\vec{I} - \frac{\nabla\nabla}{\gamma^2}\right]G(\vec{r}|\vec{r}') \qquad \text{(B9)}$$

For formal manipulations, (B8) can be written as an integral over all space as

$$\vec{E}(\vec{r}) = -\int_V \vec{Z}^e(\vec{r}|\vec{r}')\cdot\vec{J}(\vec{r}')dV'$$

$$= -\langle\vec{Z}^e(\vec{r}|\vec{r}');\vec{J}(\vec{r}')\rangle \qquad \text{(B10)}$$

where

$$\tilde{Z}^e(\bar{r}|\bar{r}') = s\mu_0\left\{P.V.\,\tilde{g}(\bar{r}|\bar{r}') + \frac{\tilde{L}_\delta\,\delta(\bar{r}-\bar{r}')}{\gamma^2}\right\} \tag{B11}$$

The notation P.V. indicates that the term about to follow should be integrated in the limiting sense as

$$\int_V P.V.\; F(\bar{r}')dV' \equiv \lim_{\delta\to 0} \int_{V-V_\delta} F(\bar{r}')dV' \,. \tag{B12}$$

II. Homogeneous Space: Spectral Form

It is often convenient to express the Green's dyadic in the spectral domain. This is especially so for the case for layered media environments, which naturally prompt Fourier transformation on the two transverse coordinates. For homogeneous space, one of several forms for the homogeneous space scalar Green's function is [16]

$$G(\bar{r}|\bar{r}') = \frac{e^{-\gamma|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} = \iint_{-\infty}^{\infty} \frac{e^{jk_x(x-x')}e^{jk_y(y-y')}e^{-p|z-z'|}}{2(2\pi)^2 p}\,dk_x\,dk_y \tag{B13}$$

where $p = \sqrt{k_x^2 + k_y^2 + \gamma^2}$. The singularity at $\bar{r} = \bar{r}'$ of the space domain form is manifested by non-convergence of the spectral integral for $z = z'$. Since the singular point is excluded by the principal value integration, the spectral integral is convergent, allowing the interchange of the spectral integral and the differential operator in (B9), resulting in

$$\begin{aligned}
\tilde{g}(\bar{r}|\bar{r}') &= \left(\tilde{I} - \frac{\nabla\nabla}{\gamma^2}\right)\iint_{-\infty}^{\infty} \frac{e^{jk_x(x-x')}e^{jk_y(y-y')}e^{-p|z-z'|}}{2(2\pi)^2 p}\,dk_x\,dk_y \\
&= \iint_{-\infty}^{\infty} \left(\tilde{I} - \frac{\nabla\nabla}{\gamma^2}\right)\frac{e^{jk_x(x-x')}e^{jk_y(y-y')}e^{-p|z-z'|}}{2(2\pi)^2 p}\,dk_x\,dk_y \\
&= \iint_{-\infty}^{\infty} \left(\tilde{I} - \frac{\bar{k}\bar{k}}{\gamma^2}\right)\frac{e^{jk_x(x-x')}e^{jk_y(y-y')}e^{-p|z-z'|}}{2(2\pi)^2 p}\,dk_x\,dk_y
\end{aligned} \tag{B14}$$

where $\bar{k} = \hat{x}\,jk_x + \hat{y}\,jk_y + \hat{z}\,sgn(z'-z)p$. It may be convenient to write the Green's dyadic as

$$\vec{\vec{Z}}^e(\vec{r}|\vec{r}') = s\mu_0 \left\{ P.V.\, \vec{\vec{g}}(\vec{r}|\vec{r}') + \frac{\vec{\vec{L}}_\delta \delta(\vec{r}-\vec{r}')}{\gamma^2} \right\}$$
$$= \frac{s\mu_0}{(2\pi)^2} \int\int_{-\infty}^{\infty} \left\{ \left(\vec{\vec{I}} - \frac{\vec{k}\vec{k}}{\gamma^2} \right) \frac{e^{-p|z-z'|}}{2p} + \frac{\vec{\vec{L}}_\delta \delta(z-z')}{\gamma^2} \right\} e^{jk_x(x-x')} e^{jk_y(y-y')} \, dk_x \, dk_y \quad \text{(B15)}$$

For this geometry, the depolarizing dyad is $\vec{\vec{L}}_\delta = \hat{z}\hat{z}$. In the first term in the integrand of (B15) the P.V. notation is omitted since the absolute value naturally decomposes the problem into two regions, $z<z', z>z'$, thereby implicitly enforcing a "slice" type principle volume [17] of vanishing thickness.

III. Inhomogeneous Space: Planarly Layered Media

For media external to the object as depicted in Fig. 1, it is convenient to use scattering superposition to determine the Green's function [16]. Attention will be focused on the special case of source and observation point in the same layer, which provides the relevant kernel for the IE of interest. Let $\vec{A} = \vec{A}^p + \vec{A}^s$ where the first term is the principle part (for an unbounded region containing the source) and the second term is the scattered part, such that

$$(\nabla^2 - \gamma^2)\vec{A}^p(\vec{r}) = -\mu_0 \vec{J}(\vec{r})$$
$$(\nabla^2 - \gamma^2)\vec{A}^s(\vec{r}) = 0 \quad \text{(B16)}$$

The total potential $\vec{A}$ satisfies the appropriate boundary conditions at the interfaces. Defining the transform pair

$$\vec{A}(k_x, k_y, z) = \int\int_{-\infty}^{\infty} \vec{A}(x,y,z)\, e^{-jk_x(x-x')} e^{-jk_y(y-y')} dx\, dy$$
$$\vec{A}(x,y,z) = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} \vec{A}(k_x, k_y, z) e^{jk_x(x-x')} e^{jk_y(y-y')} dk_x\, dk_y \quad \text{(B17)}$$

and substitution of the second of (B17) into (B16) allows for the determination of $\vec{A}^s(k_x, k_y, z)$, with $\vec{A}^p$ determined as in Sec. B.II. This leads to the form

$$\vec{A}(\vec{r}) = \mu_0 \int_V [\vec{\vec{g}}^p(\vec{r}|\vec{r}') + \vec{\vec{g}}^s(\vec{r}|\vec{r}')] \cdot \vec{J}(\vec{r}')\, dV' \quad \text{(B18)}$$

where the P.V. type of integration is implicitly implied for the principle term, and

$$\bar{\bar{g}}^p(\bar{r}|\bar{r}') = \bar{\bar{I}}\, G(\bar{r}|\bar{r}')$$

$$\bar{\bar{g}}^s(\bar{r}|\bar{r}') = (\hat{x}\hat{x}+\hat{y}\hat{y})g_t^s(\bar{r}|\bar{r}') + \hat{z}\hat{z}g_n^s(\bar{r}|\bar{r}') + \left(\hat{z}\hat{x}\frac{d}{dx} + \hat{z}\hat{y}\frac{d}{dy}\right)g_c^s(\bar{r}|\bar{r}') \quad \text{(B19)}$$

In the above, $G(\bar{r}|\bar{r}')$ is given by (B13) and

$$\begin{Bmatrix} g_t^s(\bar{r}|\bar{r}') \\ g_n^s(\bar{r}|\bar{r}') \\ g_c^s(\bar{r}|\bar{r}') \end{Bmatrix} = \int\!\!\int_{-\infty}^{\infty} \begin{Bmatrix} R_t^s(k_x,k_y,z,z') \\ R_n^s(k_x,k_y,z,z') \\ R_c^s(k_x,k_y,z,z') \end{Bmatrix} \frac{e^{jk_x(x-x')}e^{jk_y(y-y')}}{2(2\pi)^2 p}\, dk_x\, dk_y \quad \text{(B20)}$$

The Green's dyadic is then obtained as

$$\bar{\bar{Z}}^e(\bar{r}|\bar{r}') = s\mu_0\left\{\left(1 - \frac{\nabla\nabla\cdot}{\gamma^2}\right)\left[P.V.\,\bar{\bar{g}}^p(\bar{r}|\bar{r}') + \bar{\bar{g}}^s(\bar{r}|\bar{r}')\right] + \frac{\bar{\bar{L}}_\delta\,\delta(\bar{r}-\bar{r}')}{\gamma^2}\right\}$$

$$= \frac{s\mu_0}{(2\pi)^2}\int\!\!\int_{-\infty}^{\infty}\left\{\left(1 - \frac{\nabla_z\nabla_z\cdot}{\gamma^2}\right)\frac{\left[\bar{\bar{I}}e^{-p|z-z'|} + (\hat{x}\hat{x}+\hat{y}\hat{y})R_t^s + \hat{z}\hat{z}R_n^s + [\hat{z}\hat{x}(jk_x) + \hat{z}\hat{y}(jk_y)]R_c^s\right]}{2p}\right. \quad \text{(B21)}$$

$$\left. + \frac{\bar{\bar{L}}_\delta\,\delta(z-z')}{\gamma^2}\right\} e^{jk_x(x-x')}e^{jk_y(y-y')}\, dk_x\, dk_y$$

where $\bar{\bar{L}}_\delta = \hat{z}\hat{z}$ and $\nabla_z = \hat{x}(jk_x) + \hat{y}(jk_y) + \hat{z}\dfrac{\partial}{\partial z}$. The above definition allows the formulation of IE's for dielectric objects in layered media in a simple and consistent way. The relevant IE's for three, two, and one-dimensional dielectric objects in layered media are, respectively,

$$\left\langle \bar{\bar{Z}}^e(\bar{r}|\bar{r}') + Z_d\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}')\,;\bar{J}(\bar{r}')\right\rangle = \bar{E}^{inc}(\bar{r})$$

$$\left\langle \bar{\bar{Z}}^e(\bar{\rho}|\bar{\rho}') + Z_d\bar{\bar{I}}\,\delta(\bar{\rho}-\bar{\rho}')\,;\bar{J}(\bar{\rho}')\right\rangle = \bar{E}^{inc}(\bar{\rho},k_x) \quad \text{(B22)}$$

$$\left\langle \bar{\bar{Z}}^e(z|z') + Z_d\bar{\bar{I}}\,\delta(z-z')\,;\bar{J}(z')\right\rangle = \bar{E}^{inc}(z,k_x,k_z)$$

where the dependance on the spectral parameters in the second and third of (B22) is included on the rhs to indicate the appropriate one and two-dimensional Fourier transformation, respectively. In (B22) $\bar{\bar{Z}}^e(\bar{r}|\bar{r}')$ is given by (B21), and the two other kernels follow from Fourier transformation of (B21) once and twice, respectively. The above formulas and corresponding discussion are presented here for generality, and to provide a foundation for the one-dimensional example presented in Appendix C.

Appendix C. Eigenmode Expansion Applied to Infinite Slab

As an example of the EEM method described here, consider the problem of an infinite dielectric slab immersed in homogeneous space, as depicted in Fig. C1. The appropriate IE is the third of (B22)

$$\left\langle \tilde{Z}^{e}(z|z',\gamma_1) + Z_d(\gamma_1,\gamma_2)\tilde{I}\,\delta(z-z')\,;\vec{J}(z',\gamma_1,\gamma_2)\right\rangle = \vec{E}^{inc}(z,\gamma_1) \tag{C1}$$

with

$$\tilde{Z}^{e}(z|z') = s\mu_0\left\{\left(1-\frac{\nabla_z\nabla_z\cdot}{\gamma^2}\right)\frac{\left[\tilde{I}e^{-p|z-z'|}\right]}{2p} + \frac{\hat{z}\hat{z}\,\delta(z-z')}{\gamma^2}\right\} \tag{C2}$$

since $R_t^s = R_n^s = R_c^s = 0$ for the homogeneous background environment. If the incident field is

$$\vec{E}^{inc}(\vec{r}) = \hat{y}\,e^{-j(k_x x + k_z z)} \tag{C3}$$

with $jk_x = \gamma_1 \sin\theta_i$, $jk_z = \gamma_1\cos\theta_i$, the correct transform domain field for the rhs of (C1) is $\hat{y}e^{-p_1 z}$ with $p_1 = \sqrt{k_x^2 + k_y^2 + \gamma_1^2} = \gamma_1\cos\theta_i$. Integral equation (C1) becomes

$$\left\langle Z_{yy}(z|z',\gamma_1) + Z_d(\gamma_1,\gamma_2)\,\delta(z-z')\,;J_y(z',\gamma_1,\gamma_2)\right\rangle = e^{-p_1 z} \tag{C4}$$

with

$$Z_{yy}(z|z',\gamma_1) = s\mu_0\frac{e^{-p_1|z-z'|}}{2p_1} \tag{C5}$$

The relevant eigenvalue problem is

$$\left\langle \frac{e^{-p_1|z-z'|}}{2p_1}\,;j_\beta(z',\gamma_1)\right\rangle = Z_\beta^u(\gamma_1)\,j_\beta(z,\gamma_1) \tag{C6}$$

with actual eigenvalues for the object given by

$$\begin{aligned} Z_\beta(\gamma_1,\gamma_2) &= s\mu_0 Z_\beta^u(\gamma_1) + Z_d(\gamma_1,\gamma_2) \\ &= s\mu_0 Z_\beta^u(\gamma_1) + \left(\frac{\gamma_2}{Z_2} - \frac{\gamma_1}{Z_1}\right)^{-1} \end{aligned} \tag{C7}$$

The kernel of (C6) is the traditional transmission line kernel, which can be derived from transmission line theory. The IE (C6) can be solved analytically by the method described in [18],[19]. Taking two

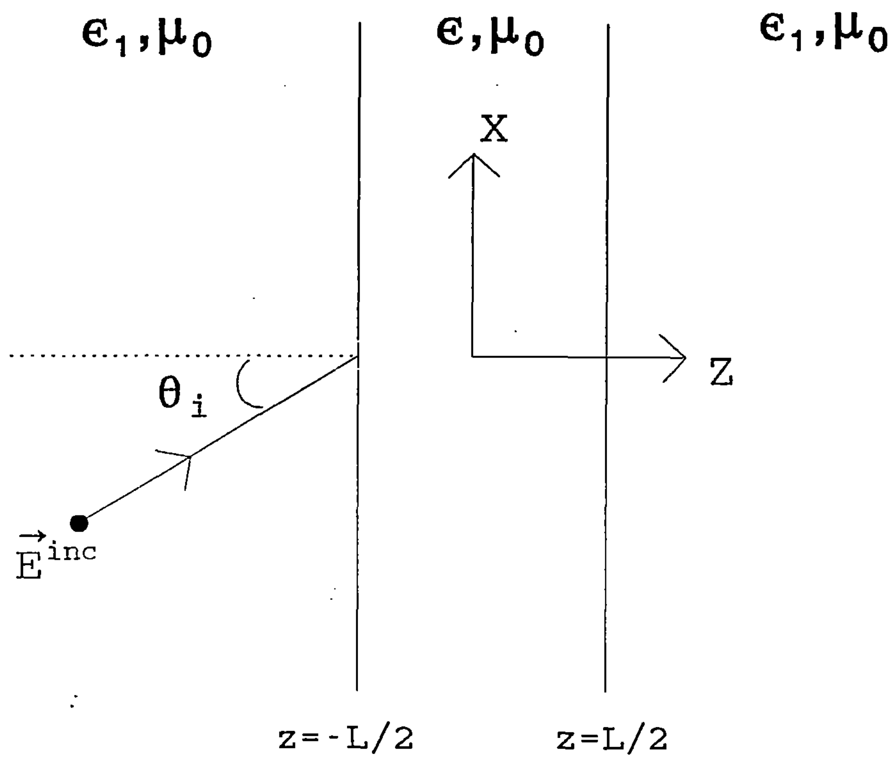

Fig. C1. Plane wave incident on infinite dielectric slab.

derivatives of (C6), with regard to the fact that the integration implicitly omits the singular point $z=z'$, we obtain

$$\left(\frac{d^2}{dz^2} - \gamma_\beta^2\right) j_\beta(z) = 0 \tag{C8}$$

where $\gamma_\beta^2 = p_1^2 - \frac{1}{z_\beta^u}$. Boundary conditions for (C8) can be obtained by applying one derivative to (C6), leading to

$$\begin{aligned} j_\beta'(z) - p_1 j_\beta(z)\Big|_{z=-\frac{L}{2}} &= 0 \\ j_\beta'(z) + p_1 j_\beta(z)\Big|_{z=\frac{L}{2}} &= 0 \end{aligned} \tag{C9}$$

The solution is

$$j_\beta(z) = e^{\gamma_\beta z} - e^{-jn\pi} e^{-\gamma_\beta z} \tag{C10}$$

where γ_β satisfies

$$\gamma_\beta = \frac{j\beta\pi}{L} + \frac{1}{L}\ln\left(\frac{p_1 - \gamma_\beta}{p_1 + \gamma_\beta}\right). \tag{C11}$$

In [18], various properties of the modes were determined, such as symmetry of the eigenvalues, orthogonality of the eigenmodes, and completeness of the modes, thereby justifying the eigenmode expansion which follows. It was also shown that the nullspace of the integral operator in (C6) is empty, which must be true based upon physical reasons.

Based upon an eigenmode expansion, the induced current in the slab is, from (31),

$$J_y(z) = \sum_\beta \frac{1}{N_\beta} \frac{1}{Z_\beta} \left\langle j_\beta(z) \, ; e^{-p_1 z} \right\rangle j_\beta(z) \tag{C12}$$

where

$$N_\beta = \left\langle j_\beta(z) \, ; j_\beta(z) \right\rangle \tag{C13}$$

and

$$Z_\beta = s\mu_0 \frac{1}{p_1^2 - \gamma_\beta^2} + Z_d(\gamma_1, \gamma_2) \ . \tag{C14}$$

The backscattered field (z<-L/2) is, from (32),

$$\begin{aligned} E_y^s(z) &= -\left\langle Z_{yy}(z|z') \, ; J_y(z') \right\rangle \\ &= -\sum_\beta \frac{1}{N_\beta} \frac{1}{Z_\beta} \frac{s\mu_0}{2p_1} \left\langle j_\beta(z) \, ; e^{-p_1 z} \right\rangle^2 e^{p_1 z} \end{aligned} \tag{C15}$$

As a check, the reflection coefficient at the slab interface is

$$\Gamma = \left. \frac{E_y^s}{E_y^{inc}} \right|_{z=-L/2} = -\sum_\beta \frac{1}{N_\beta} \frac{1}{Z_\beta} \frac{s\mu_0}{2p_1} \left\langle j_\beta(z) \, ; e^{-p_1 z} \right\rangle^2 e^{-p_1 L} \tag{C16}$$

Numerical results based upon (C16) were compared to standard reflection coefficient formulas to verify the formulation presented here.

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