Interaction Notes

Note 511

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Generalization of the BLT Equation

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Abstract

The response of a multiconductor transmission line network to electromagnetic excitation is described by the BLT equation. This paper generalizes the BLT equation to include nonuniform multiconductor transmission lines in two different ways. This allows one to model quite general electromagnetic systems in the topological formalism of the BLT equation.

1. Introduction

The BLT equation relates the waves leaving the junctions to the scattering matrices for the junctions and the propagation on and coupling to the tubes (individual multiconductor transmission lines) in a network of junctions and tubes [1, 2, 6]. This is being used to model various physical problems, including large cable networks such as appear in aircraft and other complex electronic systems [4, 9, 10, 12], and other electromagnetic structures (as in Marx generators that can be modelled as a network of MTLs (multiconductor transmission lines) [13].

In its original form the BLT equation is written for use with uniform MTLs, i.e., ones for which the per-unit-length impedance and admittance matrices are independent of position along each particular tube. In this case, there are two uncoupled vector waves propagating in opposite directions on each tube. In the present paper, this is generalized to NMTLs (nonuniform multiconductor transmission lines) for each tube in the network. The resulting NBLT equation has additional terms allowing for the coupling between the two waves on a tube. An alternate form is found by regarding the tube as just another junction with now two incoming waves, two outgoing waves, and two wave sources. This form of the BLT equation (say BLT2) can model very general problems.

2. Nonuniform tubes

For an N-wire transmission line (N conductors plus reference) we have the telegrapher equations

$$\begin{aligned}\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) &= -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right)+\left(\tilde{I}_n^{(s)'}(z,s)\right)\\ \frac{d}{dz}\left(\tilde{I}_n(z,s)\right) &= -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right)+\left(\tilde{V}_n^{(s)'}(z,s)\right)\end{aligned} \tag{2.1}$$

where

$$\begin{aligned} z &\equiv \text{position along the tube (real)}\\ \left(\tilde{V}_n(z,s)\right) &\equiv \text{voltage vector}\\ \left(\tilde{I}_n(z,s)\right) &\equiv \text{current vector (positive current in direction of increasing } z)\\ \left(\tilde{Z}'_{n,m}(z,s)\right) &\equiv \text{per-unit-length series impedance matrix}\\ \left(\tilde{Y}'_{n,m}(z,s)\right) &\equiv \text{per-unit-length shunt admittance matrix}\\ \left(\tilde{V}_n^{(s)'}(z,s)\right) &\equiv \text{per-unit-length series voltage source vector}\\ \left(\tilde{I}_n^{(s)'}(z,s)\right) &\equiv \text{per-unit-length shunt current source vector}\\ \sim &\equiv \text{Laplace transform (2-sided) over time, } t\\ s \equiv \Omega + j\omega &\equiv \text{Laplace-transform variable or complex frequency}\end{aligned} \tag{2.2}$$

This is illustrated by the diagram in fig. 2.1.

Here the vectors all have N components and the matrices are $N \times N$. In later use, N is subcripted (e.g., N_n) to indicate a wave on one of the various tubes (MTLs) in the network. We also have [1, 7, 8, 11, 17]

$$\begin{aligned}\left(\tilde{\gamma}_{n,m}(z,s)\right) &= \left[\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} \text{ (positive real (p.r.) square root)}\\ &\equiv \text{propagation matrix}\\ \left(\tilde{Z}_{c_{n,m}}(z,s)\right) &= \left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)=\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^{-1}\cdot\left(\tilde{Z}'_{n,m}(z,s)\right)\\ &=\left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{T} \equiv \text{characteristic impedance matrix}\\ \left(\tilde{Y}_{c_{n,m}}(z,s)\right) &= \left(\tilde{Z}_{c_{n,m}}(z,s)\right)^{-1} = \left(\tilde{Y}_{c_{n,m}}(z,s)\right)^{T}\\ &\equiv \text{characteristic admittance matrix}\end{aligned} \tag{2.3}$$

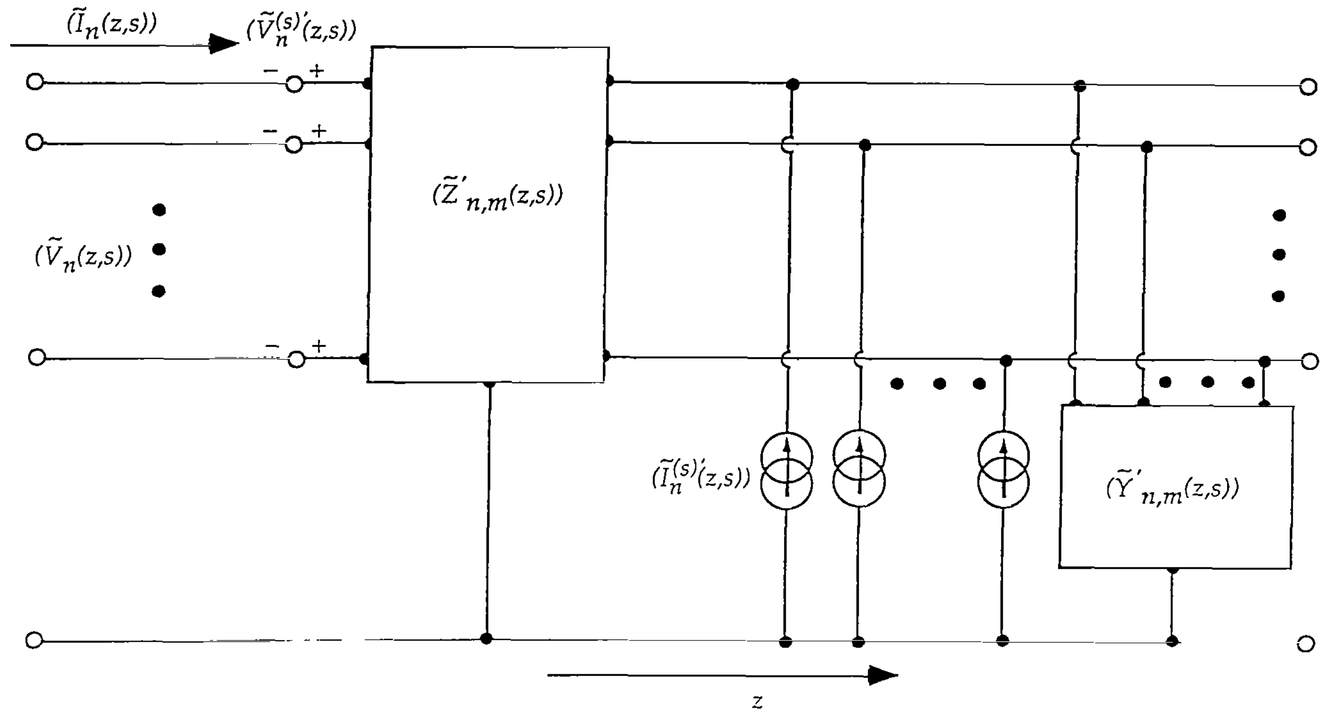

Fig. 2.1. Per-Unit-Length Equivalent Circuit of Multiconductor Transmission Line

where reciprocity has been assumed. Here these matrices can be functions of z , so let us also introduce

$$\left(\tilde{Z}_{n,m}(s)\right) = \left(\tilde{Y}_{n,m}(s)\right)^{-1} \equiv \left(\tilde{Z}_{n,m}(s)\right)^{T} \qquad \qquad \qquad (2.4)$$
$$\equiv \text{ normalizing impedance matrix}$$

which is independent of z. Combining the matrix telegrapher equations as

$$\frac{d}{dz}\begin{pmatrix} \left(V_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix} = -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}'_{n,m}(s)\right)\cdot\left(\tilde{Y}_{n,m}(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right)\end{pmatrix} \odot \begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right)\end{pmatrix} \qquad (2.5)$$

$$+\begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z,s)\right)\end{pmatrix}$$

gives a supermatrix ($2N \times 2N$) equation with supervectors ($2N$) for source and response. The form of this equation relies on the fact that the normalizing impedance matrix is independent of z.

Consider the first-order supermatrix ($2N \times 2N$) differential equation

$$\begin{aligned}
\frac{d}{dz}\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(\tilde{\xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \\
\left(\left(\tilde{\Xi}(z_0 z_0;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(1_{n,m}\right)_{\sigma,\sigma'}\right) \text{ (identity supermatrix)} \\
&= \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(1_{n,m}\right)\end{pmatrix} \\
1_{n,m} &= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m\end{cases} \\
0_{n,m} &= 0 \text{ for all } n,m
\end{aligned} \qquad (2.6)$$

The solution of this equation is called a matrizant [5, 14-16] (or supermatrizant if one prefers). It has the properties

$$\begin{aligned}
\left(\left(\tilde{\Xi}_{n,m}(z_1,z_2;s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\Xi}_{n,m}(z_2,z_3;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(\tilde{\Xi}_{n,m}(z_1,z_3;s)\right)_{\sigma,\sigma'}\right) \\
\left(\left(\tilde{\Xi}_{n,m}(z_0,z;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)^{-1}
\end{aligned} \qquad (2.7)$$

Applying this to (2.5) gives

$$\left(\left(\tilde{\xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \equiv -\begin{pmatrix} \left(0_{n,m}\right) & \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(0_{n,m}\right) \end{pmatrix}$$

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \end{pmatrix} = \left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z_0,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z_0,s)\right) \end{pmatrix}$$

$$+ \left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \odot \int_{z_0}^{z} \left(\left(\tilde{\Xi}_{n,m}(z',z_0;s)\right)_{\sigma,\sigma'}\right)^{-1} \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z',s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z',s)\right) \end{pmatrix} dz'$$

$$= \left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z_0,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n(z_0,s)\right) \end{pmatrix}$$

$$+ \int_{z_0}^{z} \left(\left(\tilde{\Xi}_{n,m}(z,z';s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z',s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{I}_n^{(s)'}(z',s)\right) \end{pmatrix} dz' \tag{2.8}$$

There are various cases that such a matrizant can be expressed in closed analytic form (using common special functions) as discussed in [11]. In any event the matrizant has representations in terms of a series of repeated integrals and the multiplicative integral. Note that since

$$tr\left(\left(\left(\tilde{\xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\right) = 0 \tag{2.9}$$

due to zero diagonal entries, then the matrizant is unimodular, i.e.,

$$\det\left(\left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)\right) = 1 \tag{2.10}$$

Noting that our choice of positive direction for increasing z on a particular tube is arbitrary (two choices being possible), let us define two choices as

$$\begin{aligned} z_r &\equiv \text{right increasing } z \\ z_\ell &\equiv \text{left increasing } z \\ z_r + z_\ell &= L \equiv \text{length of tube} \\ 0 \le z_r &\le L,\ 0 \le z_\ell \le L \end{aligned} \tag{2.11}$$

Then, defining a reflection supermatrix as

$$\left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right) \equiv \begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & -\left(1_{n,m}\right) \end{pmatrix} \tag{2.12}$$

we have

$$\begin{pmatrix} \left(\tilde{V}_n(z_\ell,s)\right)_\ell \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z_\ell,s)\right)_\ell \end{pmatrix} = \left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z_r,s)\right)_r \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z_\ell,s)\right)_r \end{pmatrix} \tag{2.13}$$

where the r and ℓ subscripts on the voltage and current vectors indicate the z coordinate to which these vectors are referenced. Remember that on reversing the direction of z the direction for positive current is also reversed.

By choice of z_0 as 0, and z as L, then (2.8) relates the voltages and currents at the ends of the tube to the sources along the tube. This still leaves the boundary conditions at the junctions at the ends of the tube. Since in the BLT equation these boundary conditions need to be expressed in terms of scattering matrices, then voltages and currents at the tube ends are not the desirable variables. A preferred form is in terms of combined-voltage waves defined by

$$\begin{aligned} \left(\tilde{V}_n(z,s)\right)_q &\equiv \left(\tilde{V}_n(z,s)\right) + q\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \\ \left(\tilde{V}_n^{(s)'}(z,s)\right)_q &\equiv \left(\tilde{V}_n^{(s)'}(z,s)\right) + q\left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z,s)\right) \\ q &\equiv \pm 1 \\ z &\equiv z_r \ \text{(right going reference for current)} \end{aligned} \tag{2.14}$$

which can be readily converted back to voltages and currents if desired. Note the use of the normalizing admittance matrix which is not in general the characteristic impedance matrix. As subscripts let us use + for right-going (increasing z) waves, and – for left-going waves in fig. 2.1. Note that here z is right going with the left pointing direction for current in the left-going wave accounted for by the minus sign in q. In (2.12) if we change from z_r to z_ℓ reference the roles of + and – waves interchange.

Converting from voltages and currents to waves (as in (2.14)) can be regarded as a change of basis described by the supermatrix transformation.

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right)_+ \\ \left(\tilde{V}_n(z,s)\right)_- \end{pmatrix} = \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right) \end{pmatrix}$$

$$\begin{aligned}
\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) &= \begin{pmatrix} \left(1_{n,m}\right) & \left(1_{n,m}\right) \\ \left(1_{n,m}\right) & -\left(1_{n,m}\right) \end{pmatrix} = \left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right) + \left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right) \\
\left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right) &= \begin{pmatrix} \left(0_{n,m}\right) & \left(1_{n,m}\right) \\ \left(1_{n,m}\right) & \left(0_{n,m}\right) \end{pmatrix} \\
\left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} &= \left(\left(R_{n,m}\right)_{\sigma,\sigma'}\right) \\
\left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} &= \left(\left(P_{n,m}\right)_{\sigma,\sigma'}\right) \\
\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} &= \frac{1}{2}\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)
\end{aligned} \tag{2.15}$$

The inverse transformation of course changes the waves back to voltages and currents. Similarly the sources transform as

$$\begin{pmatrix} \left(\tilde{V}_n^{(s)\prime}(z,s)\right)_+ \\ \left(\tilde{V}_n^{(s)\prime}(z,s)\right)_- \end{pmatrix} = \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)\prime}(z,s)\right) \\ \left(\tilde{Z}_{n,m}(s)\right) \cdot \left(\tilde{I}_n^{(s)\prime}(z,s)\right) \end{pmatrix} \tag{2.16}$$

Converting the matrizant to wave variables we have

$$\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &\equiv \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\Xi}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} \\
\left(\left(\tilde{\Gamma}_{n,m}(z_0,z_0;s)\right)_{\sigma,\sigma'}\right) &= \left(\left(1_{n,m}\right)_{\sigma,\sigma'}\right) \\
\frac{d}{dz}\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) &= -\left(\left(\tilde{\gamma}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right)
\end{aligned} \tag{2.17}$$

so that the transformed matrizant is also a matrizant. In this form the coefficient supermatrix is the propagation supermatrix

$$\begin{aligned}
&\left(\left(\tilde{\gamma}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right) = -\left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right) \odot \left(\left(\tilde{\xi}_{n,m}(z,s)\right)_{\sigma,\sigma'}\right) \odot \left(\left(Q_{n,m}\right)_{\sigma,\sigma'}\right)^{-1} = \\
&\frac{1}{2}\begin{pmatrix} -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right) - \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right) - \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) \\ -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right) + \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) & \left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}_{n,m}(s)\right) + \left(\tilde{Z}_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right) \end{pmatrix}
\end{aligned} \tag{2.18}$$

Note the sign reversals for the diagonal blocks and for the off-diagonal blocks of the propagation supermatrix. The new form of the matrizant (say, the propagation supermatrizant) in (2.17), being related to the previous form by a similarity transformation, is also unimodular as in (2.10). The notation is chosen in this form since the propagation supermatrix is a generalization of that used in [1]. This is seen by setting the normalizing impedance matrix in (2.4) equal to the characteristic impedance matrix in (2.3), assuming for present purposes that it is independent of z, giving

$$\left(\left(\tilde{\gamma}_{n,m}(s)\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} -\left(\tilde{\gamma}_{n,m}(s)\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(\tilde{\gamma}_{n,m}(s)\right) \end{pmatrix}$$
$$\left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) = \begin{pmatrix} e^{-\left(\tilde{\gamma}_{n,m}(s)\right)[z-z_0]} & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & e^{\left(\tilde{\gamma}_{n,m}(s)\right)[z-z_0]} \end{pmatrix} \tag{2.19}$$

with the + and − waves then separating.

With the propagation supermatrizant, the solution for the waves on the tube is

$$\begin{pmatrix} \left(\tilde{V}_n(z,s)\right)_+ \\ \left(\tilde{V}_n(z,s)\right)_- \end{pmatrix} = \left(\left(\tilde{\Gamma}_{n,m}(z,z_0;s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(z_0,s)\right)_+ \\ \left(\tilde{V}_n(z_0,s)\right)_- \end{pmatrix}$$
$$+\int_{z_0}^{z} \left(\left(\tilde{\Gamma}_{n,m}(z,z';s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z',s)\right)_+ \\ \left(\tilde{V}_n^{(s)'}(z',s)\right)_- \end{pmatrix} dz' \tag{2.20}$$

Since we will need the waves at the two ends of the tube we have

$$\begin{pmatrix} \left(\tilde{V}_n(L,s)\right)_+ \\ \left(\tilde{V}_n(L,s)\right)_- \end{pmatrix} = \left(\left(\tilde{\Gamma}_{n,m}(L,0;s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right)_+ \\ \left(\tilde{V}_n(0,s)\right)_- \end{pmatrix}$$
$$+\int_{0}^{L} \left(\left(\tilde{\Gamma}_{n,m}(L,z';s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z',s)\right)_+ \\ \left(\tilde{V}_n^{(s)'}(z',s)\right)_- \end{pmatrix} dz' \tag{2.21}$$

3. The NBLT Equation

In order to incorporate the waves and sources on a tube as in (2.21), define a matrix $(c_{u,v})$ (a topological matrix like others in [1]) with

$$c_{u,v} = \begin{cases} 1 \text{ if } u \text{ and } v \text{ are on the same tube and } u \neq v \\ 0 \text{ otherwise} \end{cases} \tag{3.1}$$

Here u and v are wave indices with

$$\begin{aligned} u,v &= 1, 2, \ldots, N_W \\ N_W &\equiv \text{number of waves on network} \\ &= 2N_T = \text{twice the number of tubes} \end{aligned} \tag{3.2}$$

Replace each $c_{u,v}$ by matrices

$$\left(C_{n,m}\right)_{u,v} = \begin{cases} \left(1_{n,m}\right)_{u,v} \; (\text{size } N_u \times N_u = N_v \times N_v) \text{ if } c_{u,v} = 1 \\ \left(0_{n,m}\right)_{u,v} \; (\text{size } N_u \times N_v) \text{ if } c_{u,v} = 0 \end{cases}$$

$N_u \equiv$ number of wires (N_u conductors plus reference) in tube with uth wave

$$\begin{aligned} &N_u = N_v \text{ if } c_{u,v} = 1 \\ &\left(\left(C_{n,m}\right)_{u,v}\right) = \left(\left(C_{n,m}\right)_{u,v}\right)^T \end{aligned} \tag{3.3}$$

Then we have (for later use)

$$\left(\tilde{V}_n(z_v, s)\right)_v = \sum_{v'=1}^{N_v} \left(C_{n,m}\right)_{u,v'} \cdot \left(\tilde{V}_n(z_v, s)\right)_{v'} \tag{3.4}$$

When $v' = v$ this term is picked out and associated with the index u. Here $z_{v'}$ is the coordinate (ranging from 0 to $L_{v'}$) for the v'th wave. Each tube has two waves, say the uth and vth , with

$$\begin{aligned} &z_u + z_v = L_u = L_v = \text{length of tube} \\ &0 \leq z_u \leq L_u \; , \; 0 \leq z_v \leq L_v \end{aligned} \tag{3.5}$$

which u and v paired this way being determined by $(c_{u,v})$.

The role of the junctions is summarized in the scattering supermatrix as

$$\left(\left(\tilde{V}_n(0,s)\right)_u\right) = \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \odot \left(\left(\tilde{V}_n(L_u, s)\right)_u\right)$$

$$\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \equiv \text{scattering supermatrix, size } \left(\sum_{u=1}^{N_W} N_u\right) \times \left(\sum_{u=1}^{N_W} N_u\right) \tag{3.6}$$

This relates the waves incident on the junctions, i.e., at the end of the tube $(z_u = L_u)$ for the uth wave, to the waves leaving the junctions, i.e., at the beginning of the tube $(z_u = 0)$ for the uth wave. At each junction, the coordinates are all zero for outgoing waves, but L_u for incoming waves, the same position on a tube being described by two different coordinates for outgoing and incoming waves as in (3.5). Note that (3.6) does not include sources since, for present purposes, all the sources are considered to be located on the tubes.

Let us substitute for the incoming waves in (3.6) in terms of outgoing waves on the same tube using the results of Section 2. In (2.21) relabel as

$$+ \to u \ , \ - \to v$$

$$\begin{pmatrix} \left(\tilde{V}_n(L_u,s)\right)_u \\ \left(\tilde{V}_n(0,s)\right)_v \end{pmatrix} = \left(\left(\tilde{\Gamma}_{n,m}(L_u,0;s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right)_u \\ \left(\tilde{V}_n(L_v,s)\right)_v \end{pmatrix}$$
$$+ \int_0^{L_u} \left(\left(\tilde{\Gamma}_{n,m}(L_u,z'_u;s)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix} \left(\tilde{V}_n^{(s)'}(z'_u,s)\right)_u \\ \left(\tilde{V}_n^{(s)'}(z'_v,s)\right)_v \end{pmatrix} dz'_u \tag{3.7}$$

The matrizant is here referenced to the z_u coordinates as indicated by the arguments Lu and z'_u The integration is over z_u, but could be changed to z_v, or to a mixed form with terms integrated with respect to z_u or z_v depending on which combined source is involved. Now consider the first vector block using the first matrix superrow in (3.7) as

$$\begin{aligned} \left(\tilde{V}_n(L_u,s)\right)_u &= \left(\tilde{\Gamma}_{n,m}(L_u,0;s)\right)_{1,1} \cdot \left(\tilde{V}_n(0,s)\right)_u \\ &+ \left(\tilde{\Gamma}_{n,m}(L_u,0;s)\right)_{1,2} \cdot \left(\tilde{V}_n(L_v,s)\right)_v \\ &+ \left(\tilde{V}_n^{(s)}(s)\right)_u \\ \left(\tilde{V}_n^{(s)}(s)\right)_u &\equiv \int_0^{L_u} \left(\tilde{\Gamma}_{n,m}(L_u,z'_u;s)\right)_{1,1} \cdot \left(\tilde{V}_n^{(s)'}(z'_u,s)\right)_u \, dz'_u \\ &+ \int_0^{L_u} \left(\tilde{\Gamma}_{n,m}(L_u,z'_u;s)\right)_{1,2} \cdot \left(\tilde{V}_n^{(s)'}(z'_v,s)\right)_v \, dz'_u \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&= \int_0^{L_u} \left[\left(\tilde{\Gamma}_{n,m}(L_u, z'_u; s) \right)_{1,1} + \left(\tilde{\Gamma}_{n,m}(L_u, z'_u) \right)_{1,2} \right] \cdot \left(\tilde{V}_n^{(s)'}(z'_u, s) \right) dz'_u \\
&+ \int_0^{L_u} \left[\left(\tilde{\Gamma}_{n,m}(L_u, z'_u; s) \right)_{1,1} - \left(\tilde{\Gamma}_{n,m}(L_u, z'_u) \right)_{1,2} \right] \cdot \left(\tilde{Z}_{n,m}(s) \right)_u \cdot \left(\tilde{I}_n^{(s)'}(z'_u, s) \right) dz'_u
\end{aligned}
$$

In the last form, the source vector is represented as integrals over the per-unit-length voltage and current source vectors, in case this is more convenient for computation. Note that the label u is still retained to indicate which tube is intended and what is the positive direction for current.

Substituting for the v-wave term in (3.8) via (3.4) and rearranging gives

$$
\begin{aligned}
&\sum_{v'=1}^{N_v} \left[\left(1_{n,m} \right)_{u,v'} - \left(\tilde{\Gamma}_{n,m}(L_u, 0; s) \right)_{1,2} \cdot \left(C_{n,m} \right)_{u,v'} \right] \cdot \left(\tilde{V}_n(L_{v'}, s) \right)_{v'} \\
&\qquad = \left(\tilde{\Gamma}_{n,m}(L_u, 0; s) \right)_{1,1} \cdot \left(\tilde{V}_n(0, s) \right)_u + \left(\tilde{V}_n^{(s)}(s) \right)_u
\end{aligned}
\tag{3.9}
$$

For convenience, define two supermatrices as

$$
\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}^{(1,1)}(L_u, 0; s) \right)_{u,v} \right) &\equiv \bigoplus_{u=1}^{N_W} \left(\tilde{\Gamma}_{n,m}(L_u, 0; s) \right)_{1,1} \\
\left(\left(\tilde{\Gamma}_{n,m}^{(1,2)}(L_u, 0; s) \right)_{u,v} \right) &\equiv \bigoplus_{u=1}^{N_W} \left(\tilde{\Gamma}_{n,m}(L_u, 0; s) \right)_{1,2}
\end{aligned}
\tag{3.10}
$$

where the direct sum constructs a block diagonal supermatrix from square ($N_u \times N_u$) matrices as

$$
\begin{aligned}
\left(\left(\tilde{\Gamma}_{n,m}^{(\sigma,\sigma')}(L_u, 0; s) \right)_{u,v} \right) &\equiv \bigoplus_{u=1}^{N_W} \left(\tilde{\Gamma}_{n,m}(L_u, 0; s) \right)_{\sigma,\sigma'} \\
&= \left(\tilde{\Gamma}_{n,m}(L_1, 0; s) \right)_{\sigma,\sigma'} \oplus \left(\tilde{\Gamma}_{n,m}(L_2, 0; s) \right)_{\sigma,\sigma'} \oplus \cdots \oplus \left(\tilde{\Gamma}_{n,m}(L_{N_W}, 0; s) \right)_{\sigma,\sigma'} \\
&\equiv \operatorname{diag} \left(\left(\tilde{\Gamma}_{n,m}(L_1, 0; s) \right)_{\sigma,\sigma'}, \left(\tilde{\Gamma}_{n,m}(L_2, 0; s) \right)_{\sigma,\sigma'}, \cdots, \left(\tilde{\Gamma}_{n,m}(L_{N_W}, 0; s) \right)_{\sigma,\sigma'} \right)
\end{aligned}
\tag{3.11}
$$

Aggregating over u in (3.9) to form supervectors and supermatrices (with dummy indices u, v) gives

$$
\begin{aligned}
&\left[\left(\left(1_{n,m} \right)_{u,v} \right) - \left(\left(\tilde{\Gamma}_{n,m}^{(1,2)}(L_u, 0; s) \right)_{u,v} \right) \odot \left(\left(C_{n,m} \right)_{u,v} \right) \right] \odot \left(\left(\tilde{V}_n(L_u, s) \right)_u \right) \\
&\qquad = \left(\left(\tilde{\Gamma}_{n,m}^{(1,1)}(L_u, 0; s) \right)_{u,v} \right) \odot \left(\left(\tilde{V}_n(0, s) \right)_u \right) + \left(\left(\tilde{V}_n^{(s)}(s) \right)_u \right)
\end{aligned}
\tag{3.12}
$$

with the source supervector having elementary vectors described in (3.8). Substituting in (3.6) for the incoming waves (at the L_u) in (3.12) gives

$$\left(\left(\tilde{V}_n(0,s)\right)_u\right) = \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \odot \left(\left(\tilde{V}_n(L_u,s)\right)_u\right)$$

$$= \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \odot \left[\left(\left(1_{n,m}\right)_{u,v}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(1,2)}(L_u,0;s)\right)_{u,v}\right) \odot \left(\left(C_{n,m}\right)_{u,v}\right)\right]^{-1} \tag{3.13}$$

$$\odot \left[\left(\left(\tilde{\Gamma}_{n,m}^{(1,1)}(L_u,0;s)\right)_{u,v}\right) \odot \left(\left(\tilde{V}_n(0,s)\right)_u\right) + \left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right)\right]$$

This is rearranged as

$$\boxed{\begin{aligned} &\left\{\left(\left(1_{n,m}\right)_{u,v}\right) - \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \odot \left[\left(\left(1_{n,m}\right)_{u,v}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(1,2)}(L_u,0;s)\right)_{u,v}\right) \odot \left(\left(C_{n,m}\right)_{u,v}\right)\right]^{-1}\right. \\ &\left.\odot \left(\left(\tilde{\Gamma}_{n,m}^{(1,1)}(L_u,0;s)\right)_{u,v}\right)\right\} \odot \left(\left(\tilde{V}_n(0,s)\right)_u\right) \\ &= \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \odot \left[\left(\left(1_{n,m}\right)_{u,v}\right) - \left(\left(\tilde{\Gamma}_{n,m}^{(1,2)}(L_u,0;s)\right)_{u,v}\right) \odot \left(\left(C_{n,m}\right)_{u,v}\right)\right]^{-1} \odot \left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right) \end{aligned}} \tag{3.14}$$

This is one form of the NBLT equation (nonuniform BLT equation, or nonuniform sandwich equation, or other identification of the letters as appropriate). As in the BLT equation in [1], this is cast in terms of outgoing waves at the junctions; incoming waves can also be used. As exhibited in (2.19), if the tubes are assumed uniform with the normalizing impedance matrices taken as the characteristic impedance matrices, then the supermatrices with (1,2) superscripts are all zero supermatrices, and the NBLT equation reduces to the BLT equation.

4. Tubes Treated as Junctions

In the foregoing the sources were derived from the per-unit-length sources along each tube. These were than "positioned" at the ends of the tubes to add to the waves leaving the tubes which are equivalently incoming at the junctions. As such, these sources do not appear in (3.6), the scattering equation for the junctions. If we model the tubes as junctions, then sources need to be included in the scatering equation as

$$\left(\left(\tilde{V}_n(s)\right)_u\right) = \left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right) \odot \left(\left(\tilde{V}_n(s)\right)_u\right) + \left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right) \tag{4.1}$$

Note that with no tube lengths appearing, all the waves are both outgoing from one junction and incoming into another. The sources are taken as adding to the outgoing waves, or equivalently are taken as just inside the junctions, as illustrated in fig. 4.1. This is consistent with modeling a tube as a junction where the sources are additive to the outgoing waves as in (3.7) and (3.8).

Modeling a tube as a junction, note that it is a special kind of junction with two vector ports, two incoming waves and two outgoing waves, as illustrated in fig. 4.2. Compared to (3.7) where there are two waves labelled u and v, but evaluated at two locations, there are now four waves: u_1 and v_1 at one end of the tube, and u_2 and v_2 at the other. Then (3.7) can be rewritten as

$$\begin{aligned}\begin{pmatrix}\left(\tilde{V}_n(s)\right)_{u_2}\\ \left(\tilde{V}_n(s)\right)_{v_2}\end{pmatrix} &= \left(\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}},0;s\right)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix}\left(\tilde{V}_n(s)\right)_{u_1}\\ \left(\tilde{V}_n(s)\right)_{v_1}\end{pmatrix}\\ &+ \int_0^{L_{u_{2,1}}} \left(\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}},z'_{u_0};s\right)\right)_{\sigma,\sigma'}\right) \odot \begin{pmatrix}\left(\tilde{V}_n^{(s)'}\left(z'_{u_{2,1}},s\right)\right)_{u_{2,1}}\\ \left(\tilde{V}_n^{(s)'}\left(z'_{v_{1,2}},s\right)\right)_{v_{1,2}}\end{pmatrix} dz'_{u_{2,1}}\end{aligned} \tag{4.2}$$

where the integration is along the $z_{u_{2,1}}$ coordinate which is the same direction as u_1 and u_2 in fig. 4.2. For clarity, the coordinates along the tube are now $z_{u_{2,1}}$ (from u_1 to u_2) and $z_{v_{1,2}}$ (from v_2 to v_1) so as not to be confused with the u and v wave indices used in this section. Rewriting (4.2) in terms of the blocks relating the v_1 outgoing wave to the u_1 and $v_2$incoming waves we have from the second superrow

$$\begin{aligned}\left(\tilde{V}_n(s)\right)_{v_2} &= \left(\tilde{\Gamma}_{n,m}\left(L_{2,1},0;s\right)\right)_{2,1} \cdot \left(\tilde{V}_n(s)\right)_{u_1} + \left(\tilde{\Gamma}_{n,m}\left(L_{2,1},0;s\right)\right)_{2,2} \cdot \left(\tilde{V}_n(s)\right)_{v_1}\\ &+ \int_0^{L_{u_{2,1}}} \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}},z'_{u_{2,1}};s\right)\right)_{2,1} \cdot \left(\tilde{V}_n^{(s)'}\left(z'_{u_{2,1}},s\right)\right)_{u_{2,1}} dz'_{u_{2,1}}\end{aligned} \tag{4.3}$$

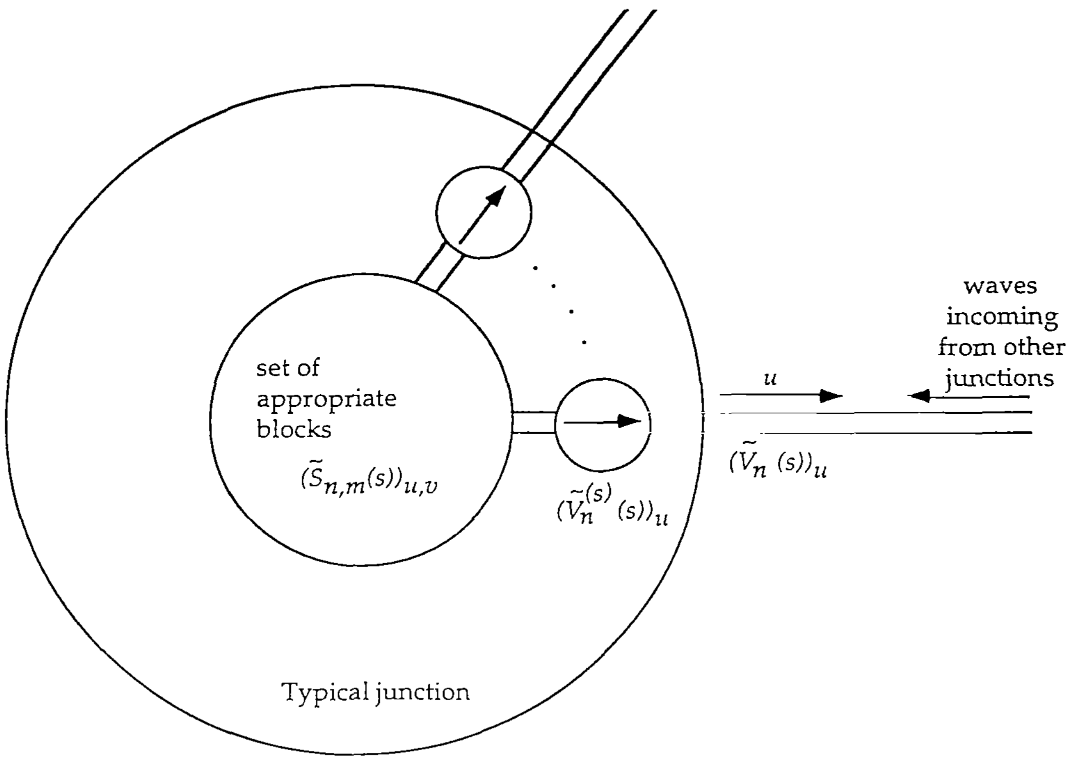

Fig. 4.1 Junction Model

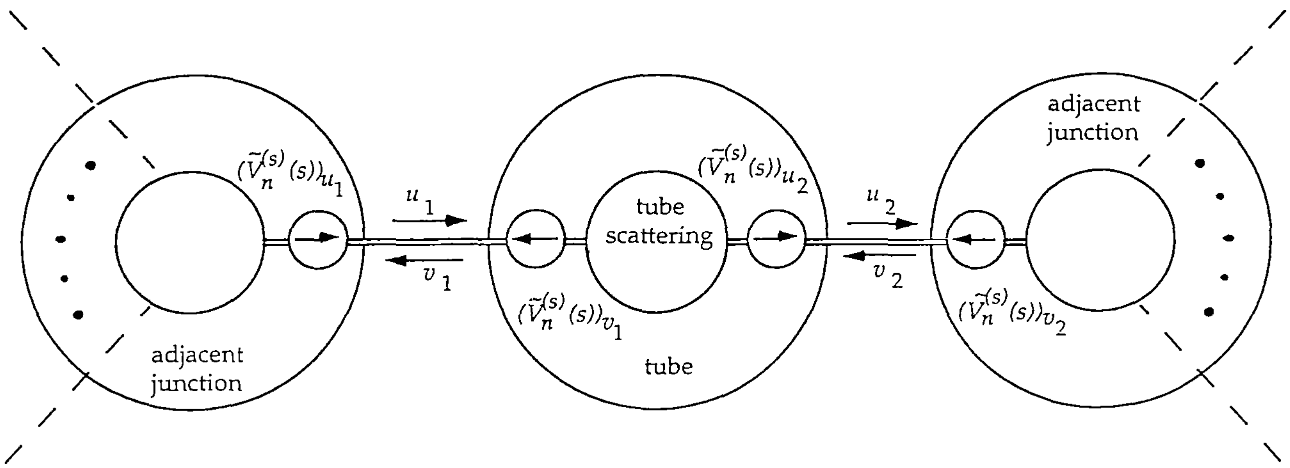

Fig. 4.2 Tube Modeled as Junction

$$+ \int_0^{L_{u_{2,1}}} \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,2} \cdot \left(\tilde{V}_n^{(s)'}\left(z'_{v_{1,2}}, s\right)\right)_{v_{1,2}} dz'_{u_{2,1}}$$

Considering v_1 as the output we have the scattering equation

$$\begin{aligned}\left(\tilde{V}_n(0,s)\right)_{v_1} = & \left(\tilde{S}_{n,m}(s)\right)_{v_1,u_1} \cdot \left(\tilde{V}_n(s)\right)_{u_1} \\ & + \left(\tilde{S}_{n,m}(s)\right)_{v_1,v_2} \cdot \left(\tilde{V}_n(s)\right)_{v_2} \\ & + \left(\tilde{V}_n^{(s)'}(s)\right)_{v_1}\end{aligned} \tag{4.4}$$

where the scattering matrices and sources are given by

$$\begin{aligned}\left(\tilde{S}_{n,m}(s)\right)_{v_1,u_1} &= -\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, 0; s\right)\right)_{2,2}^{-1} \cdot \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, 0; s\right)\right)_{2,1} \\ \left(\tilde{S}_{n,m}(s)\right)_{v_1,v_2} &= \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, 0; s\right)\right)_{2,2}^{-1} \\ \left(\tilde{V}_n^{(s)}(s)\right)_{v_1} &= -\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, 0; s\right)\right)_{2,2}^{-1} \cdot \left[\int_0^{L_{u_{2,1}}} \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,1} \cdot \left(\tilde{V}_n^{(s)'}\left(z'_{u_{2,1}}, s\right)\right)_{u_{2,1}} dz'_{u_{2,1}}\right. \\ &\qquad \left. + \int_0^{L_{u_{2,1}}} \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,2} \cdot \left(\tilde{V}_n^{(s)'}\left(z'_{v_{1,2}}, s\right)\right)_{v_{1,2}} dz'_{u_{2,1}}\right]\end{aligned} \tag{4.5}$$

$$\begin{aligned}= & -\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, 0; s\right)\right)_{2,2}^{-1} \\ & \cdot \left[\int_0^{L_{u_{2,1}}} \left[\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,1} + \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,2}\right] \cdot \left(\tilde{V}_n^{(s)'}\left(z'_{u_{2,1}}, s\right)\right) dz'_{u_{2,1}}\right. \\ & \left. + \int_0^{L_{u_{2,1}}} \left[\left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,1} - \left(\tilde{\Gamma}_{n,m}\left(L_{u_{2,1}}, z'_{u_{2,1}}; s\right)\right)_{2,2}\right] \cdot \left(\tilde{Z}_{n,m}(s)\right)_{u_{2,1}} \cdot \left(\tilde{I}_n^{(s)'}\left(z'_{u_{2,1}}, s\right)\right) dz'_{u_{2,1}}\right]\end{aligned}$$

Note that after the scattering matrices and source vectors are computed, the indices are relabeled with v_1, u_1, and v_2 assuming certain values of u and v in the overall network, these values of u and v being different from the earlier identification with waves on the tubes.

Having the scattering matrices and source vectors, then (4.1) is rearranged as

$$\left[\left(\left(1_{n,m}\right)_{u,v}\right)-\left(\left(\tilde{S}_{n,m}(s)\right)_{u,v}\right)\right] \odot \left(\left(\tilde{V}_n(s)\right)_u\right) = \left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right) \tag{4.6}$$

This is a very general form of the BLT equation, say BLT2, discussed in [2, 3]. Here the tubes have been replaced by junctions with sources configured according to outgoing waves (instead of incoming waves if one merely shrinks the tube lengths to zero).

Comparing this result to the NBLT form in (3.14), note that the NBLT form appears more complex. This is becuase of the interaction of the two waves on a nonuniform tube. Note, however, that the dimension (size) of the vectors and matrices is only helf that of those in BLT2. This is associated with the fact that N_W, the number of waves for the NBLT is twice the number of tubes, while for BLT2 it is four times the number of tubes due to the four waves used to reduce a tube to a junction (as in fig. 4.2).

5. Concluding Remarks

In this paper, two ways of including nonuniform tubes (NMTLs) in the BLT equation for a network have been considered. For the present discussion a normalizing impedance matrix which is independent of position along each tube has been assumed for simplicity. This is not an essential assumption. As discussed in [11], there are various ways to formulate the NMTL equations to arrive at the relations between the incoming waves, outgoing waves, and sources. These can be used in place of those in (3.8) to give the NBLT equation, or of those in (4.2) to give the BLT2 equation. The normalizing impedance matrix need not be the same at the two ends of the tube; one merely needs to use the appropriate impedance matrix to define the junction scattering at the corresponding tube end as in [1].

In the BLT2 form one is not restricted to MTL networks. The junctions can be used to model electrical networks and even cavities with appropriate care given to defining equivalent voltages and currents for the cavity ports. In the NBLT form, one need not consider only MTLs for tubes, but anything that connects to precisely two junctions. The number of equivalent voltages and currents need not be the same at each "end" of the tube. This gives two quite general formulations for complex electromagnetic systems.

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