Interaction Notes

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Coupling by Short Pulses

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ABSTRACT

Coupling by short pulses is studied in this report. The relationship between the real part of the forward scattering amplitude at the optical limit and the integral of the total cross section over all frequencies is re-derived with a new approach to confirm the relationship previously established. The case of transmission through a circular aperture is investigated in detail with a hope to gain some insight into how to exploit the relationship for studying short-pulse coupling.

PREFACE

The authors would like to thank Drs. Larry Warne, Ken Chen and Marv Morris of the Sandia National Laboratories, Albuquerque, New Mexico for their support and many helpful discussions. The work was performed for Purchasing Organization 3714.

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INTRODUCTION

The report is a follow-up investigation of a previous effort on the bounds of RF coupling by short and long waveforms [1]. While the investigation into step-function excitations has reached a definitive conclusion, the problem of bounding the coupling by impulses still remains a challenge.

In the next section we will formulate the problem from a fresh starting point without invoking the optical theorem and Hilbert transform. It is then shown that the new formulation is equivalent to the old one. In so doing additional insights into the problem are gained.

In the third section coupling through a circular hole is worked out in detail. The numerical results of Andrejewski supplemented by the Seshadri-Wu asymptotic formula [2] for electrically large hole are utilized to construct the transmission cross section for the entire frequency range. Then a plane wave of rectangular shape with various pulse widths is assumed to be normally incident on the hole. The problem is studied for short as well as for long pulses.

Two important appendices are included. One is on an evaluation of the real part of the forward scattering amplitude through a circular hole based on Millar's work [3]. The other is on the relationship of the real part of the forward scattering amplitude and the integral of the total cross section over frequency for a sphere based on published work in the literature [4]. The latter appendix on the sphere also serves the purpose of correcting a mistake in Reference [1].

FORMULATION

Figure 1 depicts a plane wave impinging on an object with an electric field given by

$$\mathbf{E}_{inc}(t,z) = \hat{x}\ E_o(t - z/c) \tag{1}$$

The total energy per hertz $(\tilde{w}_t)$, or power in the harmonic case, spent by the incident fields to create the induced current $\tilde{\mathbf{J}}$ on the object is given by

$$\begin{aligned} \tilde{w}_t &= \mathrm{Re}\int \tilde{\mathbf{J}} \cdot \tilde{\mathbf{E}}^*_{inc}\, dV' \\ &= \mathrm{Re}\left(\tilde{E}^*_o \int \tilde{J}_x\, e^{-ikz'} dV'\right) \end{aligned} \tag{2}$$

where $\tilde{E}_o\, e^{ikz'}$ is the Fourier transform of (1)

The far-zone scattered field can be written as

$$\tilde{\mathbf{E}}_{sc} = \tilde{\mathbf{A}}\ \frac{e^{ikr}}{4\pi r} \tag{3}$$

with

$$\tilde{\mathbf{A}} = -i\omega\mu\ \hat{r} \times \left(\hat{r} \times \int \tilde{\mathbf{J}}\ e^{-ik\hat{r}\cdot\mathbf{r}'} dV'\right) \tag{4}$$

Taking the x-component of $\tilde{\mathbf{A}}$ and settting $\hat{r} = \hat{z}$ (the forward scattering direction) one has from (4)

$$\tilde{A}_x = i\omega\mu \int \tilde{J}_x\ e^{-ikz'}\ dV' \tag{5}$$

Substitution of (5) in (2) gives

$$\tilde{w}_t = \mathrm{Re}\left(\tilde{E}^*_o\ \tilde{A}_x / i\omega\mu\right) \tag{6}$$

$$\mathbf{E}_{inc}(z,t) = \hat{x}\,E_o(t - z/c)$$

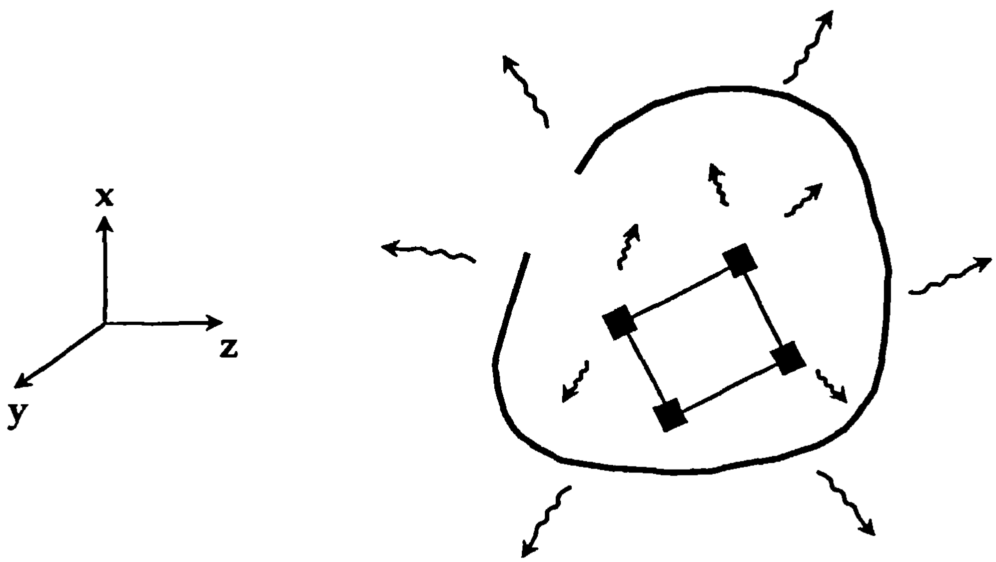

Figure 1. Plane wave impinging on a system populated with electronics.

Let $\tilde{S}$ be the forward scattering amplitude for incident electric field of unit amplitude defined as

$$\tilde{S} \equiv \frac{1}{ik} \tilde{A}_x / \tilde{E}_o \tag{7}$$

Then, (6) becomes

$$\tilde{w}_t = \frac{|\tilde{E}_o|^2}{Z_o} \operatorname{Re} \tilde{S} \equiv \frac{|\tilde{E}_o|^2}{Z_o} \sigma_t \tag{8}$$

where the total extinction cross section σ_t equal to Re $\tilde{S}$ is the same as would have been obtained by the optical theorem.

By virtue of the Parseval theorem the total energy W_t scattered and absorbed by the object is given by

$$W_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{w}_t \, d\omega = \frac{1}{Z_o} \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{E}_o|^2 \tilde{S} \, d\omega \tag{9}$$

In the case of an impulse incident wave, $\tilde{E}_o = 1$ and (9) becomes

$$W_t = \frac{1}{Z_o} \operatorname{Re} \frac{1}{2\pi} \int_{-R}^{R} \tilde{S} \, d\omega, \quad \text{for } R \to \infty \tag{10}$$

To evaluate the integral we will make use of the analyticity of $\tilde{S}$ in the upper ω-plane. Since $\tilde{A}_x$ is analytic in the upper ω-plane (according to the causality principle) and behaves as ω^2 near the origin of the complex ω-plane, $\tilde{S}$ as defined by (7) is free of singularities along the real axis and in the upper ω-plane. We then have

$$\int_{-R}^{R} \tilde{S} \, d\omega + \int_{C} \tilde{S} \, d\omega = 0 \tag{11}$$

(see Fig. 2). On C the leading term of $\tilde{S}$ is a constant σ_∞, the total cross section at the optical limit. Thus

$$\begin{aligned}\int_C \tilde{S}\, d\omega &= \int_C \sigma_\infty\, d\omega + c \int_C \frac{\tilde{A}_x}{i\omega}\, d\omega \\ &= -\int_{-R}^{R} \sigma_\infty\, d\omega + c\pi \left(\tilde{A}_x\right)_{\omega\to\infty}\end{aligned} \tag{12}$$

where the second step followed from the fact that $\oint \sigma_\infty\, d\omega = 0$ for any closed contour. Using (12) in (10) we finally arrive at

$$W_t = \frac{1}{2\pi} \int_{-R}^{R} \frac{\sigma_\infty}{Z_o}\, d\omega + \frac{c}{2Z_o} \left(-\mathrm{Re}\,\tilde{A}_x\right)_{\omega\to\infty} \tag{13a}$$

which is identical to the result obtained previously with the help of Hilbert transform [1], namely,

$$\int_o^\infty \left(\sigma_t - \sigma_\infty\right) d\omega = \frac{\pi c}{2} \left(-\mathrm{Re}\,\tilde{A}_x\right)_{\omega\to\infty} \tag{13b}$$

Equation (13) can be stated in physical terms for an impulse-like incident wave:

$$\begin{aligned}&\text{Energy Scattered + Energy absorbed} \\ &\quad = \text{Fluence} \times \text{Cross Section at Optical Limit} \\ &\quad\quad - \frac{c}{2Z_o} \times \text{Real Part of FSA at Optical Limit}\end{aligned} \tag{14}$$

where FSA = Forward Scattering Amplitude and fluence is defined as the time-integral of the incident Poynting vector. In the next section the problem of transmission through a

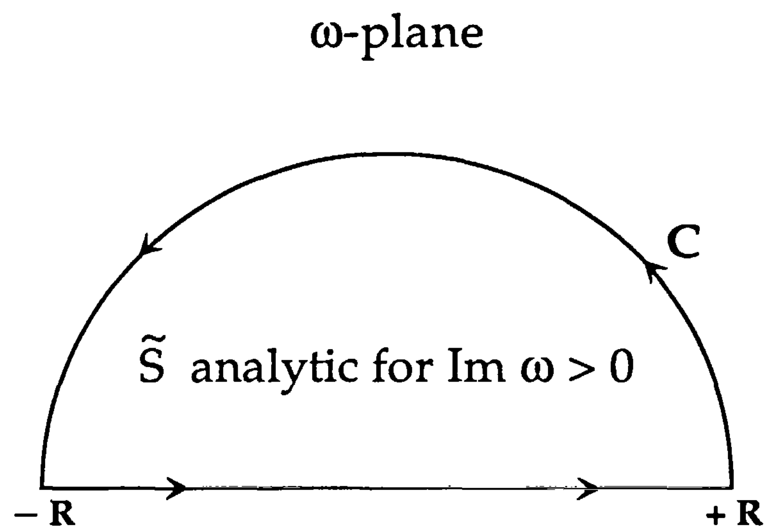

Figure 2. Contour for evaluating the integral $\int_{-R}^{R} \tilde{S}\, d\omega$.

circular aperture will be worked out numerically to see how well the first term on the right-hand side of (14) approximates the left-hand side.

A RECTANGULAR PULSE THROUGH A CIRCULAR HOLE

Consider a rectangular pulse of width T and unit amplitude for $\mathbf{E}_{inc}$ incident normally on a circular hole of radius a in an infinite ground plane. The phasor $\tilde{E}_o$ is then given by

$$\tilde{E}_o = \frac{2}{\omega} \sin\left(\frac{\omega T}{2}\right) e^{i\omega T/2}$$

and (15)

$$\left|\tilde{E}_o\right|^2 = \frac{4}{\omega^2} \sin^2(\omega T/2) = T^2 \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}$$

The transmitted energy W_{tr} through the hole is

$$W_{tr} = \frac{2}{Z_o} \int_o^\infty \left|\tilde{E}_o\right|^2 \sigma \, d\omega$$

$$= 4a^3 \varepsilon_o \int_o^\infty t(x) \frac{\sin^2(\tau x)}{x^2} dx \qquad (16)$$

where σ= transmission cross section, $t = \sigma/(\pi a^2)$, $\tau = cT/(2a)$, and $x = ka$. For $x \leq 10$, the Andrejewski's curve [2] for t was digitized and used to compute the integral (16). For $x > 10$ the following Seshadri-Wu formula [5] for t was used for numerical integration:

$$t(x) = 1 - \frac{1}{\sqrt{\pi}} \frac{1}{x^{3/2}} \sin\left(2x - \frac{\pi}{4}\right)$$

$$+ \frac{1}{x^2} \left[\frac{3}{4} + \frac{1}{2\pi} \sin\left\{2\left(2x - \frac{\pi}{4}\right)\right\}\right] \qquad (17)$$

$$- \frac{1}{\sqrt{\pi}} \frac{1}{x^{5/2}} \left[\frac{7}{4} \cos\left(2x - \frac{\pi}{4}\right) + \frac{1}{4\pi} \sin\left\{3\left(2x - \frac{\pi}{4}\right)\right\}\right]$$

Figure 3 shows the constructed transmission coefficient t (= $\sigma/\pi a^2$) all the way up to ka = 20. Figure 4 gives the normalized transmitted energy versus normalized pulse width (recall that the electric field of the pulse is unity). Figure 5 is for short pulses, the part of Figure 4 that rises rapidly.

Let us first examine Figure 4 for $\tau >> 1$ or $T >> 2\, a/c$, i.e., the case of long pulses compared to the transit time across the diameter of the hole. The numerical results say that

$$W_{tr} \to \frac{8}{3}\, a^3 \varepsilon_o, \quad \text{for } \tau >> 1 \qquad (18)$$

Recall that for a unit step-function electric field the theory predicts that W_{tr} should be given by [1]

$$W_{tr} = \mu_o \mathbf{H}_o \cdot \alpha_m \cdot \mathbf{H}_o - \varepsilon_o \mathbf{E}_o \cdot \alpha_e \cdot \mathbf{E}_o \qquad (19)$$

$$\to \frac{4}{3}\, \varepsilon_o\, a^3 \quad \text{for a circular hole and normal incidence} \qquad (20)$$

W_{tr} for a long pulse gives twice the transmitted energy for a step function because the energy spectrum for the long pulse is $4 \sin^2(\omega T/2)$ times the energy spectrum for the step function, as is evident from (15). When $T >> 2\, a/c$, the integral involving $\sin^2(\omega T/2)$ over all ω gives 1/2. Therefore the result in (18) should be twice the result in (20) and the theoretical result (19) is thus verified by means of a numerical example.

We now examine Figure 5 for $\tau < 1$ or $T < 2\, a/c$. Clearly we have, for $\tau \le 0.3$,

$$W_{tr} \doteq \frac{\pi\tau}{2} \times 4\, a^3 \varepsilon_o = \frac{\pi a^2 T}{Z_o} = \pi a^2 \cdot \int E_o H_o\, dt \qquad (21)$$

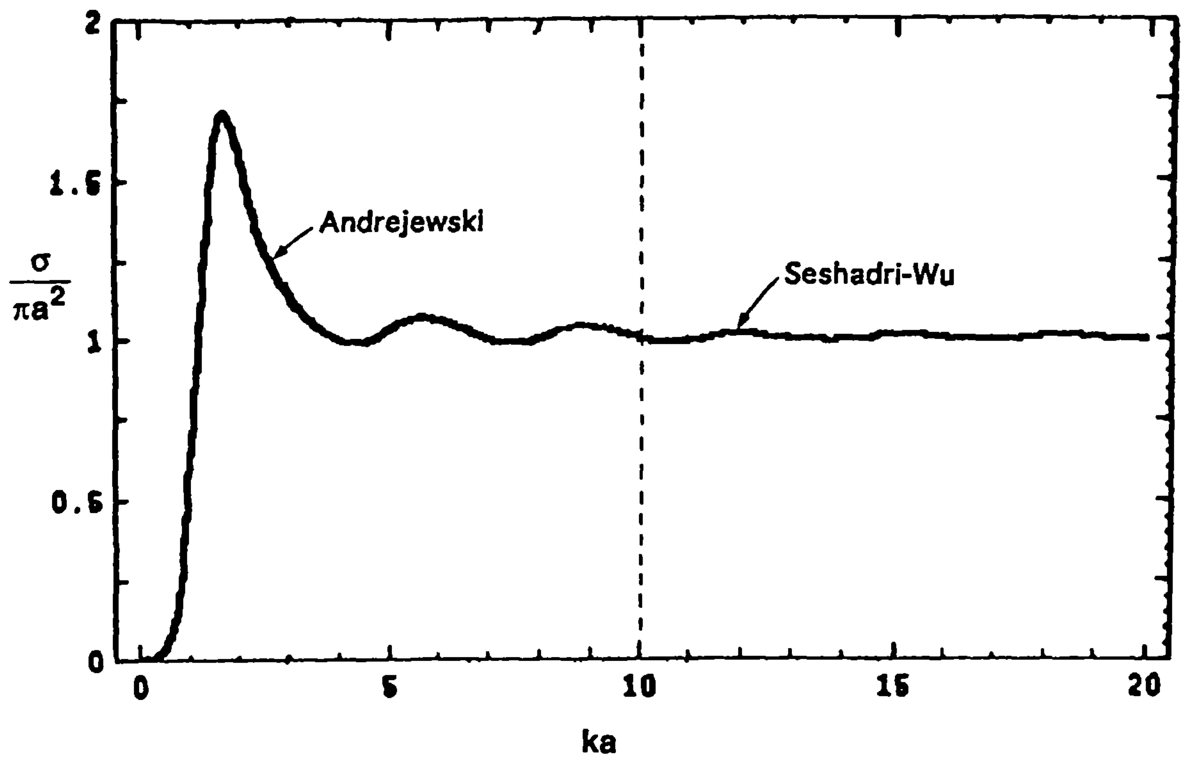

Figure 3. Transmission coefficient through a circular hole.

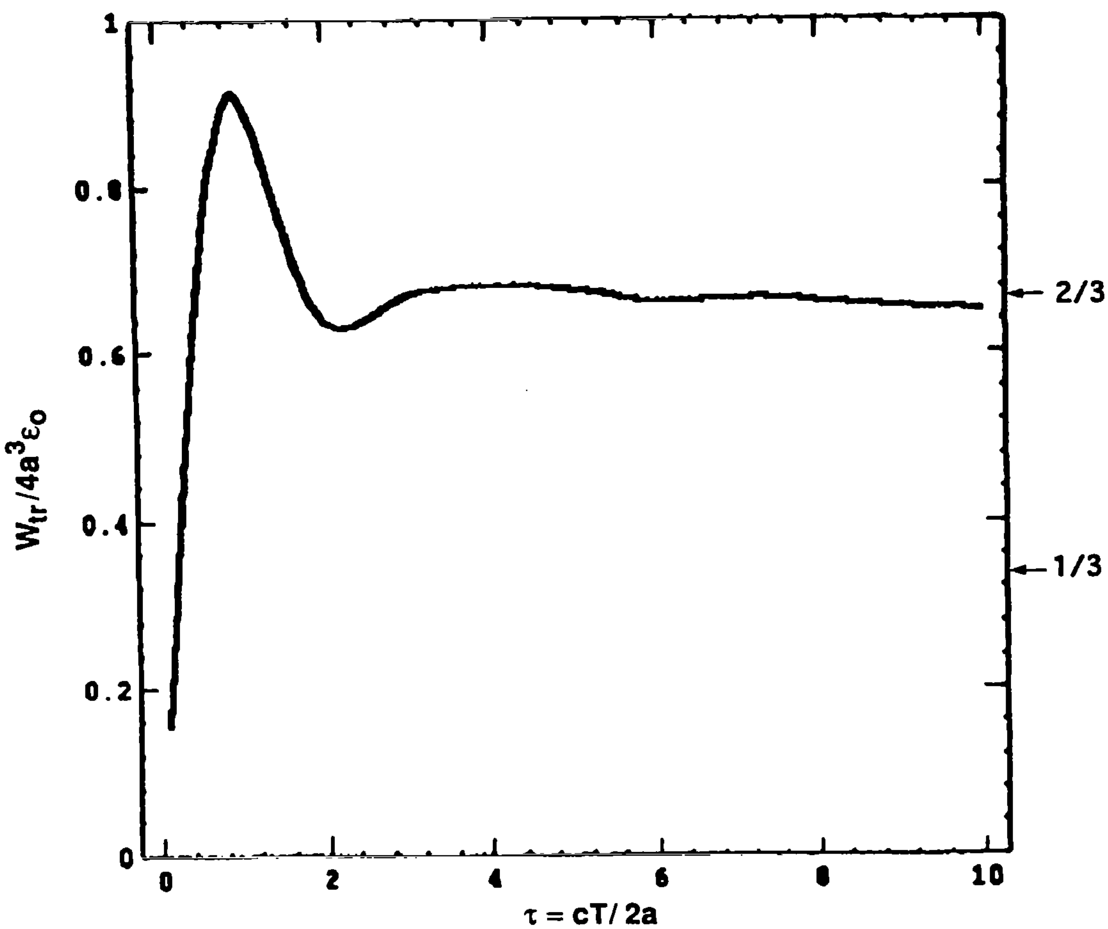

Figure 4. Normalized transmitted energy by a rectangular pulse whose electric field has unit amplitude.

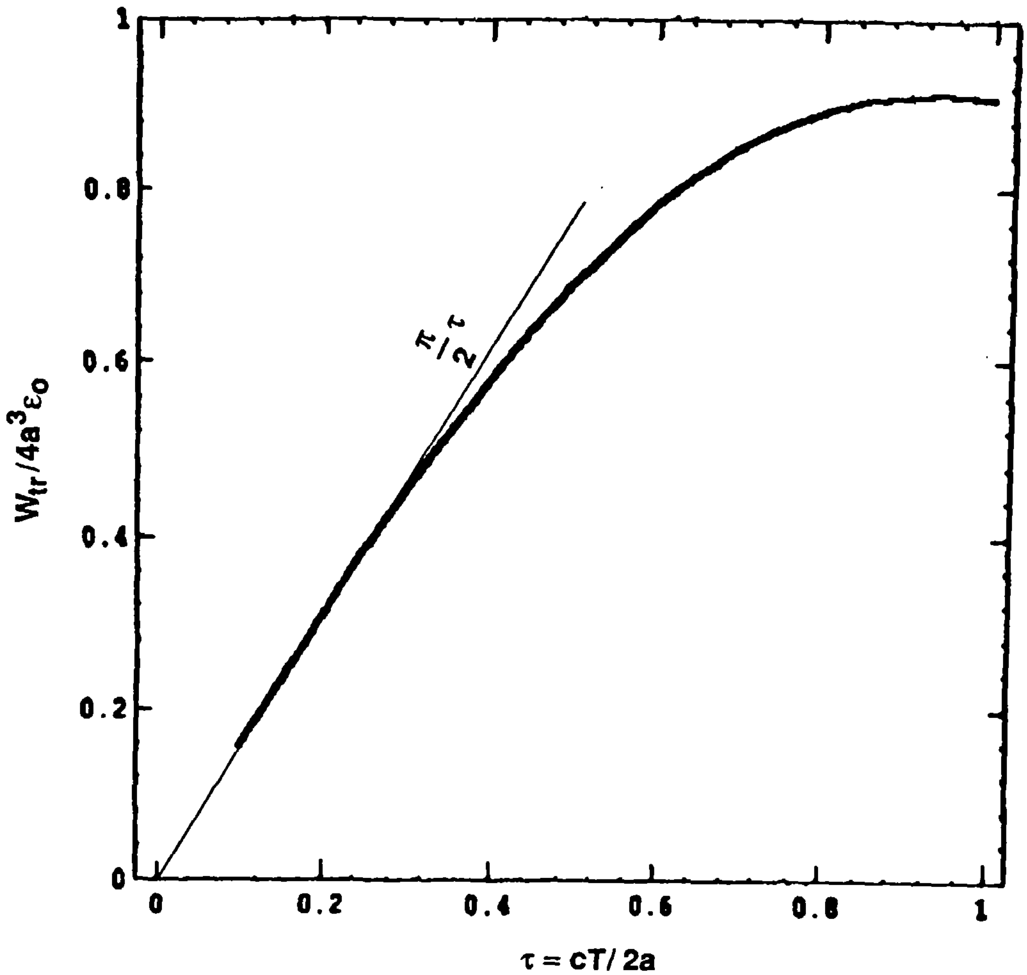

Figure 5. Normalized transmitted energy by a short pulse.

The last step followed from the assumption that $E_o = u(t) - u(t-T)$, where u is the unit step function. The integral $\int E_o H_o dt$ is called the fluence with dimension of joules per square meter. Thus, one may conclude that if the pulse is short enough compared to the transit time across the hole, the transmitted energy is well approximated by the incident fluence times the transmission cross section at the optical limit, which is πa^2 for the circular hole.

If E_o were a delta function, the fluence would be unbounded and W_{tr} would be infinite. Thus if one starts with the incident electric field as a delta function and if the optical cross section has a finite value, then the energy transmitted or scattered will be infinite. On the other hand, if one assumes the incident fluence to be finite, W_{tr} will also be finite even though E_o and H_o can be infinite provided that the product is integrable. Figure 6 is a plot of W_{tr} normalized with the fluence of the incident wave. The figure indicates that for a given fluence, the maximum transmission is for pulses as short as possible.

In Appendix A, we will demonstrate that the second (or correction) term in the right-hand side of (14) is zero for a circular hole. In appendix B, it is shown that this term for the sphere is small in comparison to the first term.

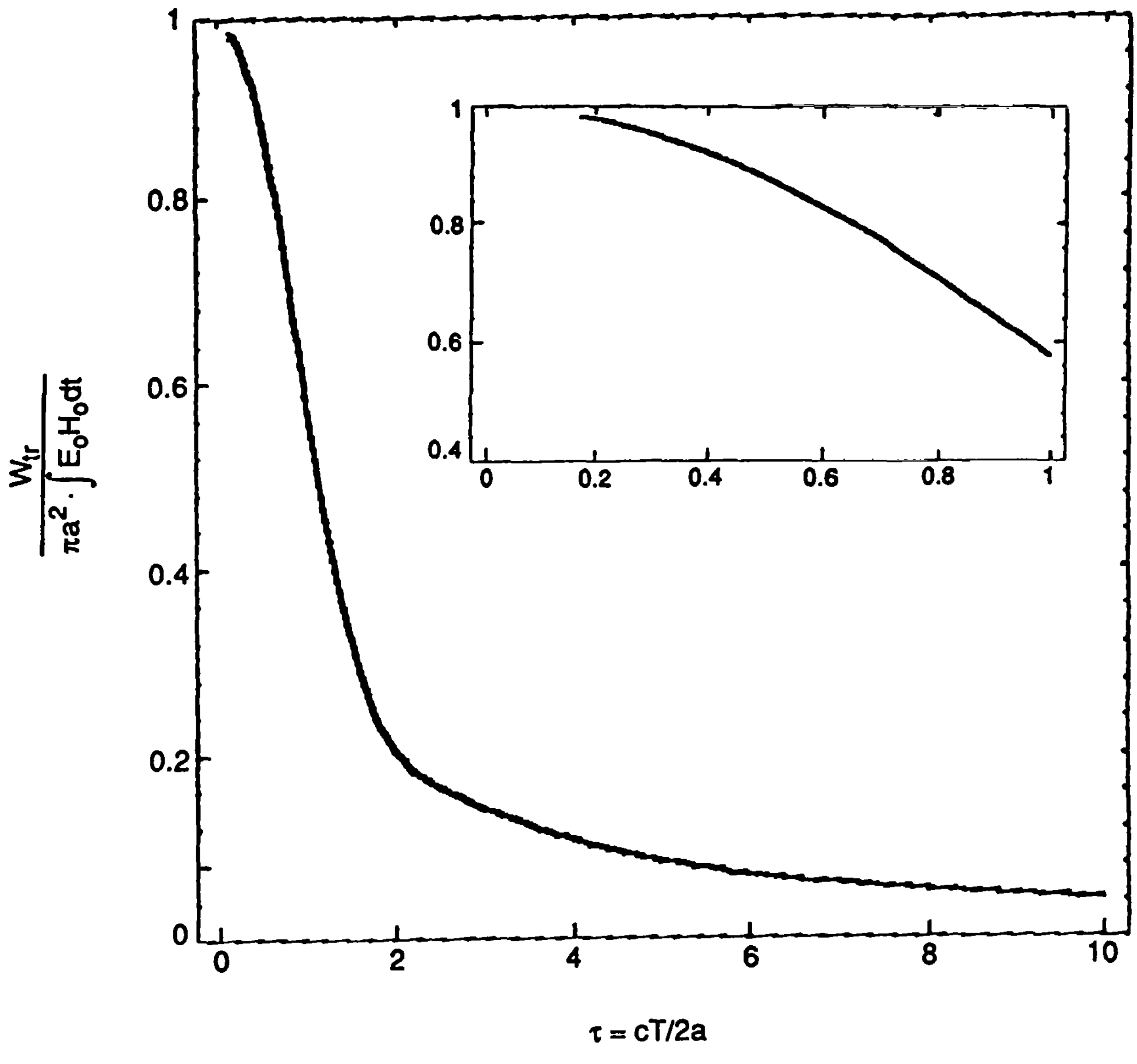

Figure 6. Transmitted energy normalized with respect to incident fluence of a rectangular pulse.

APPENDIX A

REAL PART OF FORWARD SCATTERING AMPLITUDE (FSA) OF A CIRCULAR HOLE AT THE OPTICAL LIMIT

In this appendix we will show that the real part of the forward scattering or transmission amplitude vanishes at the optical limit for the case of a circular aperture. To do this we will make use of the formulas given in the appendices of Reference 3. Using the same notations and time convention ($e^{j\omega t}$) as in [3] we have, for the transmitted field in the forward on-axis direction and same polarization of the incident field,

$$E_y = k^2 \Pi_y \tag{A·1}$$

where

$$\Pi_y = \frac{ja^2}{2k}\,\frac{e^{-jkz}}{z} + \frac{2\sqrt{a}}{\sqrt{\pi}\,k}\,e^{j3\pi/4}\,I_1 - \frac{\sqrt{a}}{2\sqrt{\pi}\,k^{3/2}}\,e^{j\pi/4}\,I_2 + 0\left(z^{-2}\right) \tag{A·2}$$

$$I_1 = -\frac{e^{-jkz}}{z}\left[\frac{a^{3/2}}{3}\sqrt{\pi}\;e^{-j\pi/4} + \frac{\sqrt{k}}{24}\pi a^2\,j\,e^{jka/2}\times \left(1+2j\,\frac{d}{d\eta} - \frac{d^2}{d\eta^2}\right)H_o^{(2)}(\eta)\right] \tag{A·3}$$

$$I_2 = -\frac{e^{-jkz}}{z}\left[\frac{j\pi a}{4}\,e^{jka/2} + \left(1+j\frac{d}{d\eta}\right)H_o^{(2)}(\eta)\right] \tag{A·4}$$

The argument η in the Hankel function is to be set equal to ka/2 after all the differentiations have been carried out.

We are seeking terms to the order of k^{-1} for I_1 and $k^{-1/2}$ for I_2 for large ka in Eqs. (A·3) and (A·4). Thus, we need the following asymptotic expressions for the Hankel function and its derivatives:

$$
\begin{aligned}
H_o^{(2)} &\sim \sqrt{\frac{2}{\pi\eta}}\; e^{-j(\eta-\pi/4)} \left[1+\frac{j}{8\eta}\right] \\
\frac{d}{d\eta} H_o^{(2)} &\sim \sqrt{\frac{2}{\pi\eta}}\; e^{-j(\eta-\pi/4)} \left[-j-\frac{3}{8\eta}\right] \\
\frac{d^2}{d\eta^2} H_o^{(2)} &\sim \sqrt{\frac{2}{\pi\eta}}\; e^{-j(\eta-\pi/4)} \left[-1+\frac{7j}{8\eta}\right]
\end{aligned}
\qquad (A\cdot 5)
$$

We now evaluate I_1 and I_2 given in (A·3) and (A·4) with the help of (A·5) and find

$$
\begin{aligned}
I_1 &\sim -\frac{e^{-jkz}}{z}\,\frac{a^{3/2}\sqrt{\pi}}{3}\left[e^{-j\pi/4} + j\,e^{j\pi/4}\left(1-\frac{3j}{4ka}\right)\right] \\
I_2 &\sim -\frac{e^{-jkz}}{z}\,\frac{j\sqrt{\pi a}}{\sqrt{k}}\; e^{j\pi/4}
\end{aligned}
\qquad (A\cdot 6)
$$

Hence,

$$
\Pi_y \sim \frac{ja^2}{2k}\,\frac{e^{-jkz}}{z}
$$

and, finally,

$$
E_z \sim \left[j2\pi ka^2 + 0\left(k^{-\alpha}\right)\right]\frac{e^{-jkz}}{4\pi z} \qquad (A\cdot 7)
$$

where $\alpha > 0$. Thus the forward scattering amplitude for a circular hole has no terms independent of k and its real part is zero for $k \to \infty$.

APPENDIX B

A SHORT PULSE SCATTERED AND ABSORBED BY A SPHERE

We will make use of the results given in [4] for the sphere to estimate the left and right hand sides of (13b). The high-frequency results are, in the notation of this report,

$$\frac{\sigma_t}{2\pi a^2} = 1 + 0.0660(ka)^{-2/3} + O\left[(ka)^{-4/3}\right] \tag{B·1}$$

$$\mathrm{Re}\,\tilde{A}_x = \frac{4\pi}{k}\,\mathrm{Re}\left[-S_2(\pi)\right] = -0.7182\,a\,(ka)^{1/3} + O\left[(ka)^{-1/3}\right] \tag{B·2}$$

where a is the radius of the sphere, and $\sigma_\infty = 2\pi a^2$. From (B · 1) one sees that the integral of (13b) is asymtotically equal to

$$\int (\sigma_t - \sigma_\infty)d\omega \sim 1.2441\,ca\,(ka)^{1/3} \tag{B·3}$$

for large ka. The right hand side of (13b) is, for large ka,

$$\frac{\pi c}{2}\left(-\mathrm{Re}\,\tilde{A}_x\right) \sim 1.1282\;ca\,(ka)^{1/3} \tag{B·4}$$

The difference in the coefficients of (B · 3) and (B · 4) is aboutr 10%, which is not bad in view of the approximations involved leading to (B· 1) and (B· 2). Thus, one may conclude that (13b) is validated by the sphere problem.

Let the incident electric field be a pulse of width T and amplitude T^{-1}. The corresponding energy density spectrum is

$$\frac{1}{Z_o}\left|\tilde{E}_o\right|^2 = \frac{1}{Z_o}\,\frac{\sin^2(\omega T/2)}{(\omega T/2)^2} \tag{B·5}$$

The limit $T \rightarrow 0$ gives the spectrum of an impulse. It is easy to see that the energy W_t scattered and absorbed by the sphere from such an incident pulse is

$$W_t = \frac{2\pi a^2}{Z_o}\frac{1}{T} + 1.1282\,\frac{ca}{Z_o}\left(\frac{2a}{cT}\right)^{1/3} + O\,(T^{1/3}) \qquad \text{(B· 6)}$$

Hence, for impulse-like incident waves W_t is contributed mainly by the first term of (14) and the correction term can be estimated by the real part of FSA at the optical limit.

REFERENCES

[1] K.S.H. Lee and F.C. Yang, "On the bounds of RF coupling by impulse and step-function waveforms," Interaction Notes, Note 491, September 30, 1992.

[2] R.W. P. King and T. T. Wu, The Scattering and Diffraction of Waves, Harvard University Press, 1959, pp. 126-127.

[3] R.F. Millar, "The diffraction of an electromagnetic wave by a circular aperture," Proc. IEE (London), Monograph No. 196R, September 1956.

[4] J.J. Bowman, et.al. (editors), Electromagnetic and Acoustic Scattering by Simple Shapes, North-Holland Publishing Co., New York, 1969, p. 414.

[5] S.R. Seshadri and T.T. Wu, "High-frequency diffraction of electromagnetic waves by a circular aperture in an infinite plane conducting screen," IRE Trans. Ant. and Propag. January 1960, pp.27-36.