Interaction Notes

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Target Symmetry and the Scattering Dyadic

Carl E. Baum

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Abstract

This paper considers the impact of reciprocity and geometrical symmetry of a target on the associated scattering dyadic. By orienting various rotation axes and reflection planes of the target in special ways with respect to both incidence and scattering directions, various symmetry-associated simplifications in the scattering dyadic can be made to occur. For special cases of back and forward scattering the various point symmetry groups are considered with preferred axes and planes now aligned according to the common axis defined by incidence and scattering directions, giving a rich structure to the scattering dyadic. For low frequencies (electrically small target) further simplifications occur, leading to further symmetries (including invariance to reversal of incidence and scattering directions) in the scattering dyadic.

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I. Introduction

In electromagnetic scattering one takes an incident plane wave

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_0 \tilde{f}(s) \vec{1}_p e^{-\tilde{\gamma} \vec{1}_i \cdot \vec{r}}, \vec{E}^{(inc)}(\vec{r},t) = E_0 \vec{1}_p f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_0}{Z_0} \tilde{f}(s) \vec{1}_i \times \vec{1}_p e^{-\tilde{\gamma} \vec{1}_i \cdot \vec{r}}, \vec{H}^{(inc)}(\vec{r},t) = \frac{E_0}{Z_0} \vec{1}_i \times \vec{1}_p f\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$\sim \equiv$ Laplace transform (two sided) over time

$s \equiv \Omega + j\omega \equiv$ Laplace – transform variable or complex frequency

$$c \equiv [\mu_0 \varepsilon_0]^{-\frac{1}{2}}, Z_0 \equiv \left[\frac{\mu_0}{\varepsilon_0}\right]^{-\frac{1}{2}}, \tilde{\gamma} = \frac{s}{c} \tag{1.1}$$

$\vec{1}_i \equiv$ direction of incidence

$\vec{1}_p \equiv$ polarization, $\vec{1}_i \cdot \vec{1}_p = 0$

and scatters it from a target (scatterer) contained in some volume V (here taken of finite linear dimensions) surrounded by a closed surface S as indicated in fig. 1.1. For convenience the coordinate center $\vec{r} = \vec{0}$ is appropriately centered near or inside the target. The scattered field (superscript sc) is given in the far-field limit (subscript f) by

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \tilde{\vec{E}}^{(inc)}\left(\vec{0}, s\right)$$

$$\vec{E}_f(\vec{r},t) = \frac{1}{4\pi r} \overleftrightarrow{\Lambda}\left(\vec{1}_o, \vec{1}_i; t\right) \overset{\circ}{\cdot} \vec{E}^{(inc)}\left(\vec{0}, t - \frac{r}{c}\right)$$

$$\vec{r} = r \vec{1}_o, \; \vec{1}_o \equiv \text{direction to observer}$$

$$\vec{H}_f(\vec{r},t) = \frac{1}{Z_0} \vec{1}_o \times \vec{E}_f(\vec{r},t), \vec{E}_f(\vec{r},t) \cdot \vec{1}_o = \vec{0}$$

$$\tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \equiv \text{scattering dyadic} \tag{1.2}$$

$$\overleftrightarrow{\Lambda}\left(\vec{1}_o, \vec{1}_i; t\right) \circ \equiv \text{scattering dyadic operator}$$

$\circ \equiv$ convolution with respect to time

It is the properties of the scattering dyadic (usually written in frequency domain, but sometimes taken as a temporal operator) that are of interest. This gives us all our information concerning the target (by hypothesis) if one is using this scattering to identify the target (as a type of aircraft, etc.).

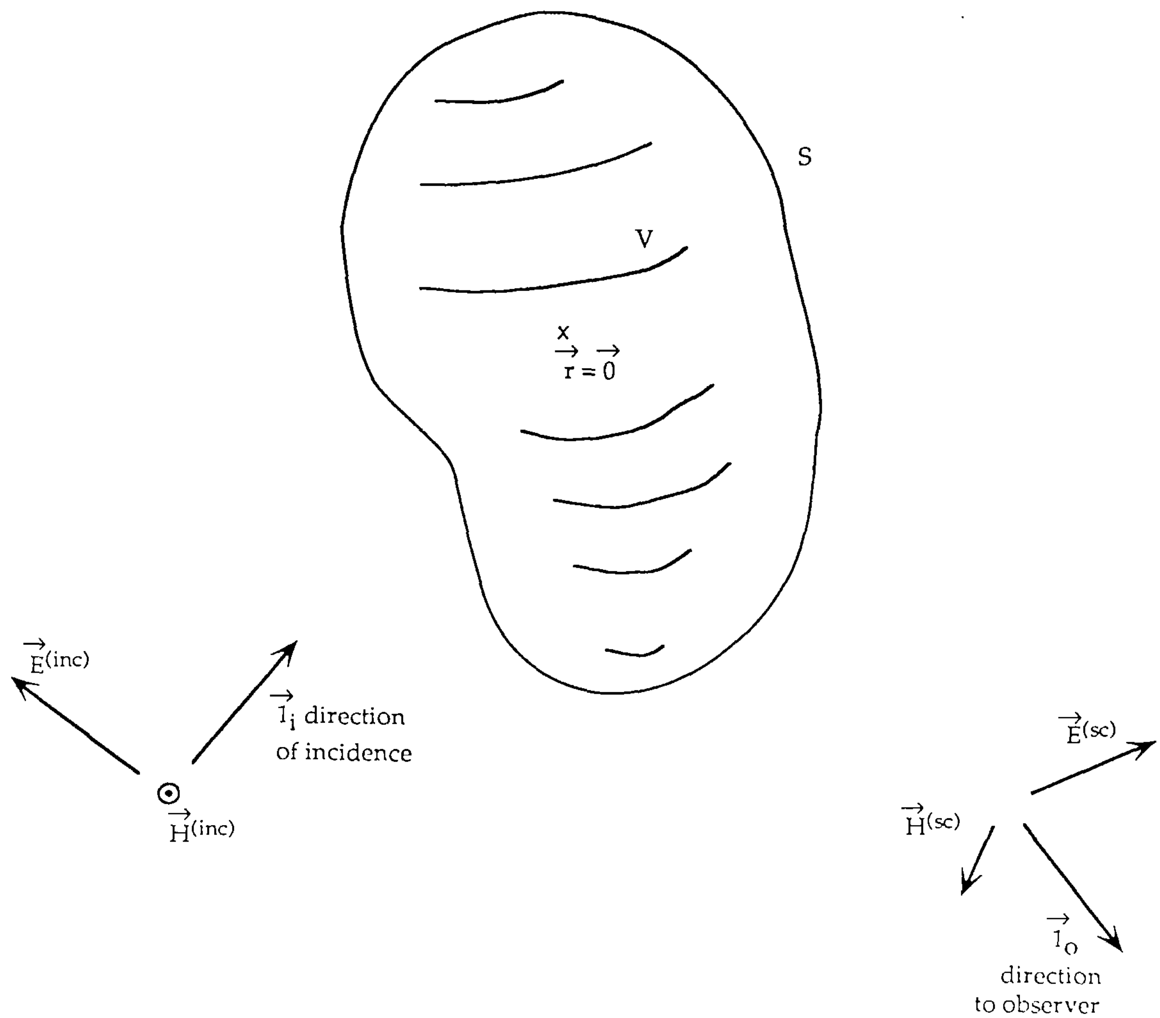


Figure 1.1. Scattering of Incident Plane Wave

Note that $\vec{E}^{(inc)}$ and $\vec{E}_f$ are perpendicular to $\vec{1}_i$ and $\vec{1}_o$ respectively, and thereby have only two components each with which to be concerned. However, except in the special cases of forward and back scattering these components are referred to different coordinates (being contained in different planes). As such there are four important components of the scattering dyadic due to the constraints

$$\tilde{\tilde{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \vec{1}_i = \vec{0} \ , \ \vec{1}_o \cdot \tilde{\tilde{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) = \vec{0} \tag{1.3}$$

In pairs, however, these four components are referred to two different planes. So it is useful to think of $\tilde{\tilde{\Lambda}}$ as a 3×3 dyadic subject to the above constraints. In another form this can be stated as

$$\begin{aligned} &\tilde{\tilde{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) = \overleftrightarrow{1}_o \cdot \tilde{\tilde{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \overleftrightarrow{1}_i \\ &\overleftrightarrow{1}_o \equiv \overleftrightarrow{1} - \vec{1}_o \vec{1}_o \equiv \text{ transverse identity with respect to } \vec{1}_o \\ &\overleftrightarrow{1}_i \equiv \overleftrightarrow{1} - \vec{1}_i \vec{1}_i \equiv \text{ transverse identity with respect to } \vec{1}_i \\ &\overleftrightarrow{1} \equiv \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \equiv \text{ identity (three dimensional)} \end{aligned} \tag{1.4}$$

For backscattering one often uses the radar coordinates $\vec{1}_h$ (horizontal, usually parallel to the local earth horizon) and $\vec{1}_v$ (vertical, except not strictly so for targets above the horizon) with the relations

$$\vec{1}_v \times \vec{1}_h = \vec{1}_i \ , \ \vec{1}_i \times \vec{1}_v = \vec{1}_h \ , \ \vec{1}_h \times \vec{1}_i = \vec{1}_v \tag{1.5}$$

Note that $\left(\vec{1}_h, \vec{1}_v, -\vec{1}_i\right)$ form a right-handed system. In backscattering

$$\vec{1}_o = -\vec{1}_i \tag{1.6}$$

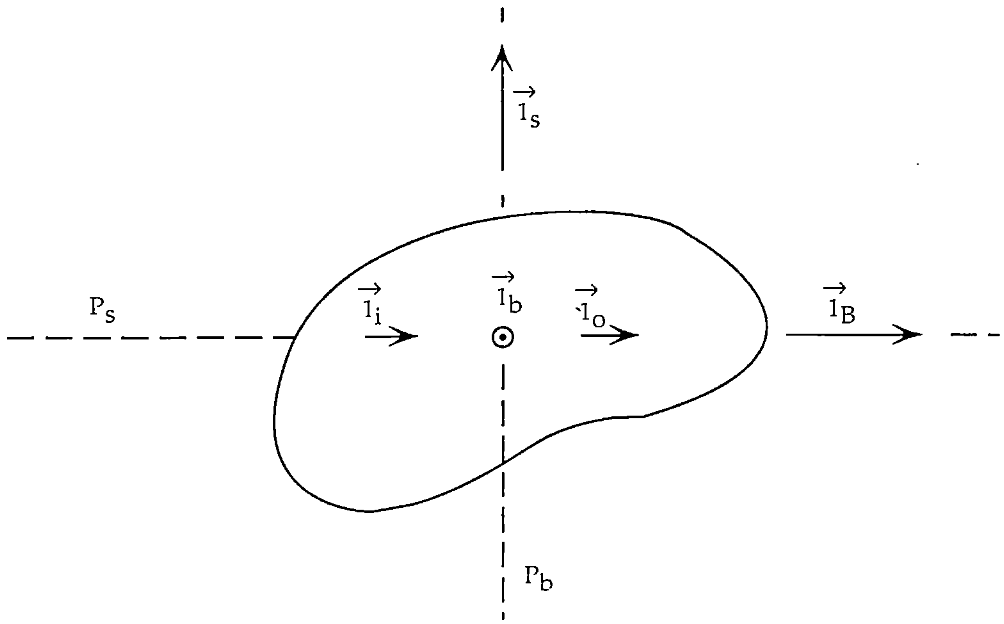
and $\vec{1}_h$ and $\vec{1}_v$ can be used for both transmitted and received (scattered) waves. In forward scattering

$$\vec{1}_o = \vec{1}_i \tag{1.7}$$

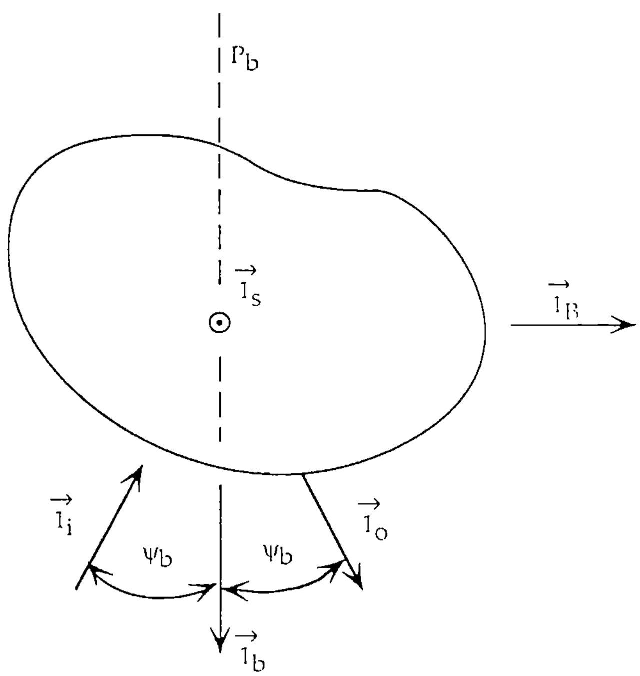
and $\vec{1}_h$ and $\vec{1}_v$ can still be used for both waves.

For $\vec{1}_o$ not so simply related to $\vec{1}_i$, one can establish coordinates using what are called the scattering plane P_s and bisectrix plane P_b [19]. As indicated in fig. 1.2, the scattering plane P_s is parallel to both $\vec{1}_i$ and $\vec{1}_o$. The bisectrix unit vector (direction) $\vec{1}_b$ lies between $-\vec{1}_i$ and $\vec{1}_o$ with equal angles ψ_b with

$$\vec{1}_b \cdot \vec{1}_o = -\vec{1}_b \cdot \vec{1}_i \quad , \quad \vec{1}_b \times \vec{1}_o = \vec{1}_b \times \vec{1}_i \tag{1.8}$$



A. Scattering and bisectrix planes perpendicular to page



B. Scattering plane parallel to page and bisectrix plane perpendicular to page

Figure 1.2. Coordinates for Scattering

The bisectrix plane P_b is parallel to $\vec{1}_b$ and perpendicular to P_s. For convenience construct unit vectors perpendicular to the two planes as

$$\begin{gathered} \vec{1}_s \perp P_s \quad , \quad \vec{1}_B \perp P_b \\ \vec{1}_b \times \vec{1}_B = \vec{1}_s \, , \, \vec{1}_B \times \vec{1}_s = \vec{1}_b \, , \, \vec{1}_s \times \vec{1}_b = \vec{1}_B \end{gathered} \tag{1.9}$$

Note the relations

$$\vec{1}_i \cdot \vec{1}_s = 0 = \vec{1}_o \cdot \vec{1}_s$$
$$\vec{1}_i \cdot \vec{1}_B = \vec{1}_o \cdot \vec{1}_B$$

With these definitions one could choose two different "horizontal" unit vectors $\vec{1}_h^{(i)}$ and $\vec{1}_h^{(o)}$ as parallel to P_s, and "vertical" unit vectors $\vec{1}_v^{(i)}$ and $\vec{1}_v^{(o)}$ as perpendicular to P_s (and parallel to P_b) with

$$\begin{gathered} \vec{1}_v^{(o)} \times \vec{1}_h^{(o)} = -\vec{1}_o \, , - \vec{1}_o \times \vec{1}_v^{(o)} = \vec{1}_h^{(o)} \, , - \vec{1}_h^{(o)} \times \vec{1}_o = \vec{1}_v^{(o)} \\ \vec{1}_h^{(o)} \cdot \vec{1}_b = -\vec{1}_h^{(i)} \cdot \vec{1}_b \, , \, \vec{1}_h^{(o)} \times \vec{1}_b = -\vec{1}_h^{(i)} \times \vec{1}_b \\ \vec{1}_v^{(o)} = \vec{1}_v^{(i)} \end{gathered} \tag{1.10}$$

where the i-superscript vectors are as in (1.5). Note the minus sign used with $\vec{1}_o$ in (1.10) to make $\vec{1}_h^{(o)}$ the same as $\vec{1}_h^{(i)}$ in backscattering, the usual radar convention.

In the case of backscattering

$$\vec{1}_o = -\vec{1}_i = \vec{1}_b \, , \, P_s \perp P_b \tag{1.11}$$

but there are an infinite number of possible pairs of P_s and P_s given by rotation about their common axis parallel to $\vec{1}_i$. In forward scattering

$$\vec{1}_o = \vec{1}_i \, , \, \vec{1}_b \cdot \vec{1}_i = 0 \, , \, P_s \perp P_b \tag{1.12}$$

and while P_b is oriented perpendicular to $\vec{1}_i$, P_s can be arbitrarily rotated about an axis parallel to $\vec{1}_i$. These two special cases will reappear with special symmetries.

II. Reciprocity

Assuming that the target is comprised of reciprocal media (symmetric constitutive-parameter matrices), then scattering reciprocity gives

$$\tilde{\tilde{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) = \tilde{\tilde{\Lambda}}^T\left(-\vec{1}_i, -\vec{1}_o; s\right) \tag{2.1}$$

This expresses the fact that if one interchanges the role of transmit and receive antennas in the scattering experiment one gets the same result. Note that the transpose expresses the interchange of the coordinates for transmission and reception.

In forward scattering we have

$$\vec{1}_o = \vec{1}_i \, , \; \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) \equiv \tilde{\tilde{\Lambda}}\left(\vec{1}_i, \vec{1}_i; s\right) = \tilde{\tilde{\Lambda}}^T\left(-\vec{1}_i, -\vec{1}_i; s\right) = \tilde{\tilde{\Lambda}}_f^T\left(-\vec{1}_i, s\right) \tag{2.2}$$

which expresses a relationship for direction reversal. In backscattering we have

$$\begin{gathered} \vec{1}_o = -\vec{1}_i \\ \tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) \equiv \tilde{\tilde{\Lambda}}\left(-\vec{1}_i, \vec{1}_i; s\right) = \tilde{\tilde{\Lambda}}^T\left(-\vec{1}_i, \vec{1}_i; s\right) = \tilde{\tilde{\Lambda}}_b^T\left(\vec{1}_i, s\right) \end{gathered} \tag{2.3}$$

which expresses the fact that $\tilde{\tilde{\Lambda}}_b$ is a complex symmetric dyadic and that $\overleftrightarrow{\Lambda}_b \circ$ is a real symmetric dyadic operator. In both forward and backscattering one can regard the dyadic as of size 2×2, being transverse to $\vec{1}_o$. In this form there are only four elements to consider. For backscattering this reduces to three elements since the off-diagonal elements are equal.

III. Symmetry Groups for Target

Let the target symmetry be expressed by a group G under which the target is invariant, given by

$$
\begin{aligned}
G &= \left\{(G)_\ell \,\middle|\, \ell = 1,2,\ldots,\ell_0\right\} \\
(G)_\ell &\equiv \text{ group element} \\
\ell_0 &\equiv \text{ group order (number of elements)} \\
(G)_\ell^{-1} &\in G\,,\,(1) \equiv \text{ identity } \in G \\
(G)_{\ell_1}(G)_{\ell_2} &\in G \text{ for all ordered pair of elements}
\end{aligned}
\tag{3.1}
$$

Let there be a set of 3×3 dyadics which form a representation of the group with

$$
(G)_\ell \to \overleftrightarrow{G}_\ell\,,\,(1) \to \overleftrightarrow{1} \tag{3.2}
$$

and group multiplication becoming the usual dot multiplication. For the point symmetry groups (rotations and reflections) with real coordinate transformations these dyadics are real and orthogonal with

$$
\begin{aligned}
\overleftrightarrow{G}_\ell^{-1} &= \overleftrightarrow{G}_\ell^{T} \\
\det\left(\overleftrightarrow{G}_\ell\right) &= \begin{cases} +1 \Rightarrow \text{ proper rotation} \\ -1 \Rightarrow \text{ improper rotation (includes a reflection)} \end{cases}
\end{aligned}
\tag{3.3}
$$

An even number of reflections gives a proper rotation.

The order of a group element, or its dyadic representation, is n_ℓ, the smallest integer (≥ 1) such that

$$
\overleftrightarrow{G}_\ell^{\,n_\ell} = \overleftrightarrow{1}\,,\,\frac{\ell_0}{n_\ell} = \text{ positive integer} \tag{3.4}
$$

The eigenvalues have the property

$$
\lambda_\beta^{n_\ell}\left(\overleftrightarrow{G}_\ell\right) = 1\,,\,\beta = 1,2,3 \tag{3.5}
$$

Combining with (3.3) gives

$$
\det\left(\overleftrightarrow{G}_\ell\right) = -1 \Rightarrow n_\ell = \text{ even} \tag{3.6}
$$

so improper rotations all have even periods. The group (cyclic) of order n_ℓ

$$G_\ell = \left\{ (G)_\ell^n \middle| n = 1, 2, \ldots, n_\ell \right\}$$
$$= \text{subgroup of } G \text{ of smallest order containing } (G)_\ell \tag{3.7}$$

is called the period of $(G)_\ell$.

A special case of interest for a group is one of order 2, called an involution [13,16]. If $(G)_\ell$ is an element of order 2 then its period is an involution as

$$\text{period of } (G)_\ell = \left\{ (1), (G)_\ell \right\} \tag{3.8}$$

In terms of the 3×3 dyadic representation then such a dyadic has the property

$$\overleftrightarrow{G}_\ell = \overleftrightarrow{G}_\ell^{-1} = \overleftrightarrow{G}_\ell^{T} \tag{3.9}$$

i.e., it is not only real and orthogonal, but also symmetric. Now any real, symmetric dyadic can be written as [4]

$$\overleftrightarrow{G}_\ell = \sum_{\beta=1}^{3} g_{\ell,\beta} \vec{1}_{\ell,\beta} \vec{1}_{\ell,\beta}$$
$$g_{\ell,\beta} \equiv \text{real eigenvalues}$$
$$\vec{1}_{\ell,\beta} \equiv \text{real eigenvectors} \tag{3.10}$$
$$= \text{real unit vectors giving spatial directions}$$
$$\vec{1}_{\ell,\beta_1} \cdot \vec{1}_{\ell,\beta_2} = 1_{\beta_1,\beta_2} \text{ (orthonormal)}$$

Furthermore, we have

$$\overleftrightarrow{G}_\ell^2 = \overleftrightarrow{1} = \sum_{\beta=1}^{3} g_{\ell,\beta}^2 \vec{1}_{\ell,\beta} \vec{1}_{\ell,\beta}$$
$$g_{\ell,\beta}^2 = 1 \, , \, g_{\ell,\beta} = \pm 1 \tag{3.11}$$

Rotate the Cartesian coordinates (x, y, z) to coincide with the $\vec{1}_\beta$ (all orthogonal, as required for such a correspondence). Then this involution dyadic gives combinations of reflections through the three coordinate planes (x = 0, y = 0, and z = 0) as the *only* possibilities. These can be categorized as all $g_{\ell,\beta} = 1 \Rightarrow$ identity $\overleftrightarrow{1}$ giving group order 1 (not an involution)

one $g_{\ell,\beta} = -1 \Rightarrow$ reflection $\overleftrightarrow{R}$ with respect to any plane (reflection symmetry R, improper rotation)

two $g_{\ell,\beta} = -1 \Rightarrow$ 2 - fold rotation axis $\overleftrightarrow{1}_\beta$ corresponding to the one $g_{\ell,\beta} = 1$ (3.12)

(rotation symmetry C_2, proper rotation)

all $g_{\ell,\beta} = -1 \Rightarrow$ inversion $-\overleftrightarrow{1}$ (inversion symmetry I, improper rotation)

Note that combinations of the above operations give groups with more than two elements.

IV. Target Symmetry

By a symmetric target is meant one that is invariant under a coordinate transformation via the group elements discussed in Section III. For each group element we have

$$\begin{aligned} \vec{r} &= x\vec{1}_x + y\vec{1}_y + z\vec{1}_z \\ \vec{r}^{(2)} &= \overleftrightarrow{G}_\ell \cdot \vec{r}^{(1)} \end{aligned} \tag{4.1}$$

Whatever of the target is at $\vec{r}^{(1)}$ is also $\vec{r}^{(2)}$, applying for all the elements of the symmetry-group representation. As in [10] transform the fields as

$$\begin{aligned} \vec{E}^{(2)}(\vec{r}^{(2)},t) &= \overleftrightarrow{G}_\ell \cdot \vec{E}^{(1)}(\vec{r}^{(1)},t) \\ \vec{H}^{(2)}(\vec{r}^{(2)},t) &= \pm\overleftrightarrow{G}_\ell \cdot \vec{H}^{(1)}(\vec{r}^{(1)},t) \end{aligned} \tag{4.2}$$

with + for proper rotations and - for improper rotations when dealing with magnetic parameters. This applies to all the fields and sources in the Maxwell equations and constitutive relations.

Applying this to the permittivity of the target gives

$$\begin{aligned} \tilde{\vec{D}}^{(2)}\left(\vec{r}^{(2)},s\right) &= \overleftrightarrow{G}_\ell \cdot \tilde{\vec{D}}^{(1)}\left(\vec{r}^{(1)},s\right) = \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\varepsilon}}\left(\vec{r}^{(1)},s\right) \cdot \overleftrightarrow{G}_\ell^T \cdot \overleftrightarrow{G}_\ell \cdot \tilde{\vec{E}}^{(1)}(\vec{r}^{(1)},s) \\ &= \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\varepsilon}}\left(\vec{r}^{(1)},s\right) \cdot \overleftrightarrow{G}_\ell^T \cdot \tilde{\vec{E}}^{(2)}(\vec{r}^{(2)},s) \\ &= \tilde{\overleftrightarrow{\varepsilon}}\left(\vec{r}^{(2)},s\right) \cdot \tilde{\vec{E}}^{(2)}(\vec{r}^{(2)},s) \end{aligned} \tag{4.3}$$

implying

$$\tilde{\overleftrightarrow{\varepsilon}}\left(\vec{r}^{(2)},s\right) = \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\varepsilon}}(\vec{r}^{(1)},s) \cdot \overleftrightarrow{G}_\ell^T \tag{4.4}$$

This applies similarly to the conductivity and permeability. So in a more general form we have

$$\begin{aligned} \tilde{\overleftrightarrow{\mu}}\left(\overleftrightarrow{G}_\ell \cdot \vec{r},s\right) &= \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\mu}}(\vec{r},s) \cdot \overleftrightarrow{G}_\ell^T \\ \tilde{\overleftrightarrow{\varepsilon}}\left(\overleftrightarrow{G}_\ell \cdot \vec{r},s\right) &= \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\mu}}(\vec{r},s) \cdot \overleftrightarrow{G}_\ell^T \\ \tilde{\overleftrightarrow{\sigma}}\left(\overleftrightarrow{G}_\ell \cdot \vec{r},s\right) &= \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\sigma}}(\vec{r},s) \cdot \overleftrightarrow{G}_\ell^T \end{aligned} \tag{4.5}$$

as the symmetry conditions of the target, noting that the above applies for all the $\overleftrightarrow{G}_\ell$ in the representation of the group G.

Applying (4.2) to the incident electric field gives

$$\vec{1}_i^{(2)} = \overleftrightarrow{G}_\ell \cdot \vec{1}_i^{(1)} \ , \ \vec{1}_p^{(2)} = \overleftrightarrow{G}_\ell \cdot \vec{1}_p^{(1)} \tag{4.6}$$

Similarly application to the far scattered field gives

$$\vec{1}_o^{(2)} = \overleftrightarrow{G}_\ell \cdot \vec{1}_o^{(1)} \tag{4.7}$$

Then transforming the scattering equation gives

$$\begin{aligned} \tilde{\vec{E}}_f^{(2)}(\vec{r}^{(2)},s) &= \overleftrightarrow{G}_\ell \cdot \tilde{\vec{E}}_f^{(1)}(\vec{r}^{(1)},s) = \frac{e^{-\gamma r}}{4\pi r} \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o^{(1)}, \vec{1}_i^{(1)}; s\right) \cdot \overleftrightarrow{G}_\ell^T \cdot \overleftrightarrow{G}_\ell \cdot \tilde{\vec{E}}_f^{(inc,1)}(\vec{0},s) \\ &= \frac{e^{-\gamma r}}{4\pi r} \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o^{(1)}, \vec{1}_i^{(1)}; s\right) \cdot \overleftrightarrow{G}_\ell^T \cdot \tilde{\vec{E}}^{(inc,2)}(\vec{0},s) \\ &= \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o^{(2)}, \vec{1}_i^{(2)}; s\right) \cdot \tilde{\vec{E}}^{(inc,2)}(\vec{0},s) \end{aligned} \tag{4.8}$$

implying

$$\tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o^{(2)}, \vec{1}_i^{(2)}; s\right) = \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o^{(1)}, \vec{1}_i^{(1)}; s\right) \cdot \overleftrightarrow{G}_\ell^T \tag{4.9}$$

So in a more general form we have

$$\tilde{\overleftrightarrow{\Lambda}}\left(\overleftrightarrow{G}_\ell \cdot \vec{1}_o, \overleftrightarrow{G}_\ell \cdot \vec{1}_i; s\right) = \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \overleftrightarrow{G}_\ell^T \tag{4.10}$$

as the symmetry conditions for the scattering dyadic.

V. Symmetry in General Bistatic Scattering

As discussed in [19] there are various transformations of the target which give a scattering dyadic simply related to that for the original target. These are:

a. rotation of the target by π about the bisectrix $\vec{1}_b$

b. reflection of the target with respect to the scattering plane P_S

c. reflection of the target with respect to the bisectrix plane P_b

The target need not have any of these symmetries; they still appear in the scattering dyadic.

Instead of transforming the target the above operations can be interpreted as transforming the coordinates, in particular $\vec{1}_i$ and $\vec{1}_o$. Define dyadics for the three operations above as:

$$
\begin{aligned}
\overleftrightarrow{C}_2^{(b)} &\equiv -\overleftrightarrow{1}_b + \vec{1}_b \vec{1}_b = -\overleftrightarrow{1} + 2\vec{1}_b \vec{1}_b \\
&\equiv \text{rotation by } \pi \text{ (2 - fold axis) about } \vec{1}_b \\
&= \text{two - dimensional inversion transverse to } \vec{1}_b \\
\overleftrightarrow{R}_s &\equiv \overleftrightarrow{1} - 2\vec{1}_s \vec{1}_s \\
&\equiv \text{reflection through } P_s \\
\overleftrightarrow{R}_B &\equiv \overleftrightarrow{1} - 2\vec{1}_B \vec{1}_B \\
&\equiv \text{reflection through } P_b
\end{aligned}
\tag{5.1}
$$

Each of these forms with the identity $\overleftrightarrow{1}$ an involution group. Noting that

$$
\begin{aligned}
\overleftrightarrow{R}_s \cdot \overleftrightarrow{R}_B &\equiv \overleftrightarrow{R}_B \cdot \overleftrightarrow{R}_s = \overleftrightarrow{1} - 2\vec{1}_s \vec{1}_s - 2\vec{1}_B \vec{1}_B \\
&= -\overleftrightarrow{1} + 2\vec{1}_B \vec{1}_B = \overleftrightarrow{C}_2^{(b)}
\end{aligned}
\tag{5.2}
$$

then the three dyads in (5.1) with the identity $\overleftrightarrow{1}$ form a representation of the $C_{2a}^{(b)}$ group ($C_2^{(b)}$ two-fold rotation axis $\vec{1}_b$, plus two axial symmetry planes P_S and P_b), a commutative group given by

$$
C_{2a}^{(b)} = \left\{(1), (R_s), (R_B), \left(C_2^{(b)}\right)_1\right\} = R_s \otimes R_B \tag{5.3}
$$

formed from the two involution groups

$$R_s = \{(1),(R_s)\}$$
$$R_b = \{(1),(R_b)\} \tag{5.4}$$

Apply these symmetry operations to the scattering dyadic. For a. we have

$$\overleftrightarrow{C}_2^{(b)} \cdot \vec{1}_i = -\vec{1}_o \ , \overleftrightarrow{C}_2^{(b)} \cdot \left(-\vec{1}_o\right) = \vec{1}_i$$
$$\tilde{\Lambda}\left(-\vec{1}_i, -\vec{1}_o; s\right) = \tilde{\Lambda}^T\left(\vec{1}_o, \vec{1}_i; s\right) = \tilde{\Lambda}\left(\overleftrightarrow{C}_2^{(b)} \cdot \vec{1}_o, \overleftrightarrow{C}_2^{(b)} \cdot \vec{1}_i; s\right) \tag{5.5}$$

where the reciprocity relation (2.1) has shown that this rotation of the coordinates has not reproduced the scattering dyadic but rather its transpose. Clearly a second application of this rotation gives back the original scattering dyadic (identity operation). For b. we have

$$\overleftrightarrow{R}_s \cdot \vec{1}_o = \vec{1}_o \ , \overleftrightarrow{R}_s \cdot \vec{1}_i = \vec{1}_i$$
$$\tilde{\Lambda}\left(\vec{1}_o, \vec{1}_i; s\right) = \tilde{\Lambda}\left(\overleftrightarrow{R}_s \cdot \vec{1}_o, \overleftrightarrow{R}_s \cdot \vec{1}_i; s\right) \tag{5.6}$$

For c. we have

$$\overleftrightarrow{R}_B \cdot \vec{1}_i = -\vec{1}_o \ , \overleftrightarrow{R}_B \cdot \left(-\vec{1}_o\right) = \vec{1}_i$$
$$\tilde{\Lambda}\left(-\vec{1}_i, -\vec{1}_o; s\right) = \tilde{\Lambda}^T\left(\vec{1}_o, \vec{1}_i; s\right) = \tilde{\Lambda}\left(\overleftrightarrow{R}_B \cdot \vec{1}_o, \overleftrightarrow{R}_B; s\right) \tag{5.7}$$

with the same form as (5.5). Note that these are not the same results in general as in (4.10) for geometrical symmetries in the target since they come from the reciprocity symmetry.

Going a step further let each of these symmetry dyadics individually be symmetry dyadics $\overleftrightarrow{G}_\ell$ for the target. Then use (4.10) in combination with the above results for each of these three involution dyadics. For a. we have

$$\tilde{\Lambda}\left(\overleftrightarrow{C}_2^{(b)} \cdot \vec{1}_o, \overleftrightarrow{C}_2^{(b)} \cdot \vec{1}_i; s\right) = \overleftrightarrow{C}_2^{(b)} \cdot \tilde{\Lambda}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \overleftrightarrow{C}_2^{(b)} = \tilde{\Lambda}^T\left(\vec{1}_o, \vec{1}_i; s\right) \tag{5.8}$$

For b. we have

$$\tilde{\Lambda}\left(\overleftrightarrow{R}_s \cdot \vec{1}_o, \overleftrightarrow{R}_s \cdot \vec{1}_i; s\right) = \overleftrightarrow{R}_s \cdot \tilde{\Lambda}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \overleftrightarrow{R}_s = \tilde{\Lambda}\left(\vec{1}_o, \vec{1}_i; s\right) \tag{5.9}$$

For c. we have

$$\tilde{\tilde{\Lambda}}\left(\overleftrightarrow{R}_B \cdot \vec{1}_o, \overleftrightarrow{R}_B \cdot \vec{1}_i; s\right) = \overleftrightarrow{R}_B \cdot \tilde{\tilde{\Lambda}}\left(\vec{1}_o, \vec{1}_i; s\right) \cdot \overleftrightarrow{R}_B = \tilde{\tilde{\Lambda}}^T\left(\vec{1}_o, \vec{1}_i; s\right) \tag{5.10}$$

This last case of a bisectrix symmetry plane (c.) is treated in some detail in [1, 17] in terms of symmetric and antisymmetric parts of the fields and associated natural modes. So, as the above results indicate, the particular geometrical symmetries combine with reciprocity to give more symmetry in the scattering dyadic.

Use of the h, v coordinates discussed in Section I is a natural choice for simplifying the representation of the scattering dyadic. In particular for b. (target symmetric with respect to the scattering plane) the incident field is a symmetric field when polarized parallel to $\vec{1}_h^{(i)}$ giving a far scattered field parallel to $\vec{1}_h^{(o)}$. When the incident field is polarized parallel to $\vec{1}_v^{(i)}$ the far scattered field is polarized parallel to $\vec{1}_v^{(o)} = \vec{1}_v^{(i)}$. This can be regarded as some kind of generalized diagonal form.

VI. Symmetry in Backscattering

Specializing to the important case of backscattering, recall from (2.3)

$$\vec{1}_o = -\vec{1}_i$$
$$\tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) \equiv \tilde{\tilde{\Lambda}}\left(-\vec{1}_i, \vec{1}_i; s\right) = \tilde{\tilde{\Lambda}}_b^T\left(\vec{1}_i, s\right) \tag{6.1}$$

The scattering dyadic can be regarded as a symmetric 2×2 dyadic or matrix by considering only coordinates (h, v) transverse to $\vec{1}_i$. Also, as discussed in Section 1, the scattering and bisectrix planes are not uniquely specified, but are any pair of orthogonal planes with $\vec{1}_i$ parallel to their common axis of intersection.

Consider an orthogonal (real) dyadic (matrix) of the form

$$\left(T_{h,m}\right) = \begin{pmatrix} T_{h,h} & T_{h,v} & 0 \\ T_{v,h} & T_{v,v} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \left(T_{n,m}\right)^{-1} = \left(T_{n,m}\right)^T \tag{6.2}$$

where the coordinates are arranged in a right-handed (h, v, o) coordinate system. Note that when operating on longitudinal vectors, no change is induced; i.e.,

$$\left(T_{n,m}\right) \cdot \vec{1}_i = \vec{1}_i \, , \left(T_{n,m}\right) \cdot \vec{1}_o = \vec{1}_o$$
$$\vec{1}_z = \vec{1}_o = -\vec{1}_i \tag{6.3}$$

Then for backscattering (4.10) becomes

$$\tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) = \left(T_{n,m}\right) \cdot \tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) \cdot \left(T_{n,m}\right)^T \tag{6.4}$$

provided $\left(T_{n,m}\right)$ is a symmetry of the target for the $\vec{1}_i$ of interest. Note that in the Cartesian coordinates appropriate to (6.2) the z coordinate has been chosen as a symmetry axis or parallel to a symmetry plane of the target.

Now apply the general symmetries noted in Section V. Since the bisectrix $\vec{1}_b = \vec{1}_o = -\vec{1}_i$, then for item a. we have a two-fold rotation axis as $\vec{1}_o$ or $-\vec{1}_i$. In two-dimensional form

$$C_2 = \{(1),(C_2)_1\}$$

$$(1) \rightarrow \overleftrightarrow{1}_i = \overleftrightarrow{1}_o \tag{6.5}$$

$$(C_2)_1 \rightarrow (C_{n,m}(\pi)) = \begin{pmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\overleftrightarrow{1}_i = -\overleftrightarrow{1}_o$$

In three-dimensional form rotations about $-\overrightarrow{1}_i$- are generalized as

$$(C_{n,m}(\phi)) = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftrightarrow{C}(\phi) \tag{6.6}$$

but for present purposes the two-dimensional form is adequate. Applying this to the scattering dyadic gives

$$\tilde{\overleftrightarrow{\Lambda}}_b(\overrightarrow{1}_i, s) = (C_{n,m}(\pi)) \cdot \tilde{\overleftrightarrow{\Lambda}}_b(\overrightarrow{1}_i, s) \cdot (C_{n,m}(\pi))^T \tag{6.7}$$

so the backscattering dyadic has C_2 symmetry, regardless of any such symmetry in the target with respect to the $\overrightarrow{1}_i$ axis.

Concerning items b. and c. dealing with reflection through P_s and P_b, these become any axial planes (containing $\overrightarrow{1}_i$). Let $\overleftrightarrow{R}_a$ represent any such reflection. Noting that the h, v coordinates can be arbitrarily rotated so that this reflection reverses only the h coordinate we have

$$\begin{aligned} \overleftrightarrow{R}_h \cdot \tilde{\overleftrightarrow{\Lambda}}_b(\overrightarrow{1}_i, s) \cdot \overleftrightarrow{R}_h &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\Lambda}_{h,h}(\overrightarrow{1}_i, s) & \tilde{\Lambda}_{h,v}(\overrightarrow{1}_i, s) \\ \tilde{\Lambda}_{v,h}(\overrightarrow{1}_i, s) & \tilde{\Lambda}_{v,v}(\overrightarrow{1}_i, s) \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\Lambda}_{h,h}(\overrightarrow{1}_i, s) & -\tilde{\Lambda}_{h,v}(\overrightarrow{1}_i, s) \\ -\tilde{\Lambda}_{v,h}(\overrightarrow{1}_i, s) & \tilde{\Lambda}_{v,v}(\overrightarrow{1}_i, s) \end{pmatrix} \\ \tilde{\Lambda}_{h,v}(\overrightarrow{1}_i, s) &= \tilde{\Lambda}_{v,h}(\overrightarrow{1}_i, s) \text{ (reciprocity)} \end{aligned} \tag{6.8}$$

which is a rather simple form for the transformation of the dyadic, but not an invariance. If the target has an axial symmetry plane and $\overleftrightarrow{R}_a$ is oriented to reflect through this plane, then we have

$$\tilde{\overleftrightarrow{\Lambda}}_b(\overrightarrow{1}_i, s) = \overleftrightarrow{R}_a \cdot \tilde{\overleftrightarrow{\Lambda}}_b(\overrightarrow{1}_i, s) \cdot \overleftrightarrow{R}_a = (-\overleftrightarrow{R}_a) \cdot \tilde{\overleftrightarrow{\Lambda}}_b(\overrightarrow{1}_i, s) \cdot (-\overleftrightarrow{R}_a) \tag{6.9}$$

as symmetries of the backscattering operator. To see the significance of this choose the (h, v) coordinates as in (6.8) to give

$$\overleftrightarrow{R}_h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \text{ reflection of } h \text{ coordinate}$$
$$\overleftrightarrow{R}_v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\overleftrightarrow{R}_h \equiv \text{ reflection of } v \text{ coordinate} \tag{6.10}$$

So one axial symmetry plane in the target ($\overleftrightarrow{R}_a$ reflection) gives two symmetry planes ($\overleftrightarrow{R}_a$ reflection and -$\overleftrightarrow{R}_a$ reflection) in the backscattering dyadic. Viewed another way, the C_2 symmetry in $\tilde{\tilde{\Lambda}}_b$ plus an axial symmetry plane (R_a symmetry) in the target give C_{2a} symmetry in $\tilde{\tilde{\Lambda}}_b$, a two-fold rotation axis with two axial symmetry planes. With this kind of symmetry (6.8) implies that with the orientation of one of the radar coordinates (either h or v) along this symmetry plane then $\tilde{\tilde{\Lambda}}_b$ is a diagonal dyadic (matrix), making this a convenient choice of coordinate axes.

For $\vec{1}_i$ as a higher-order (N-fold) rotation axis we have C_N symmetry

$$C_N = \{(C_N)_n | n = 1, 2, \ldots, N\}$$
$$(C_N)_n \equiv \text{ rotation by } \frac{2\pi n}{N} \text{ (positive } \phi \text{ direction)}$$
$$(C_N) = (C_N)_1^n \cdot (C_N)_N = (1)$$
$$(C_N)_n \to \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right) = \begin{pmatrix} \cos\left(\frac{2\pi n}{N}\right) & -\sin\left(\frac{2\pi n}{N}\right) \\ \sin\left(\frac{2\pi n}{N}\right) & \cos\left(\frac{2\pi n}{N}\right) \end{pmatrix}$$
$$= \left(C_{n,m}\left(\frac{2\pi}{N}\right)\right)^n \tag{6.11}$$
$$(C_N)_N \to \left(C_{n,m}(2\pi)\right) = \left(C_{n,m}(0)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \vec{1}_h \vec{1}_h + \vec{1}_v \vec{1}_v$$

This gives as a special case of (6.4)

$$\tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) = \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right) \cdot \tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) \cdot \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right)^T \tag{6.12}$$

This is the same equation as derived from a different procedure in [2, 11, 17], where it is shown that with reciprocity as in (6.1) the only solution for N ≥ 3 is

$$\tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right) = \tilde{\Lambda}_b\left(\vec{1}_i, s\right) \vec{1}_i \tag{6.13}$$

This form of scattering dyadic, proportional to the transverse identity, is invariant to all transverse rotations and reflections. Such a scattering dyadic then has $C_{\infty a}$ symmetry which is also labeled as O_2.

These point-symmetry results are summarized in table 6.1.

Table 6.1. Point Symmetry Groups (Rotation and Reflection) for Back-Scattering Dyadic for Reciprocal Target

Symmetry in Target (transverse to $\vec{1}_i$)	Form of $\tilde{\tilde{\Lambda}}_b$	Symmetry in $\tilde{\tilde{\Lambda}}_b$
C_1 (no symmetry)	$\tilde{\tilde{\Lambda}}_b = \tilde{\tilde{\Lambda}}_b^T$	C_2 (two-fold axis $\vec{1}_i$)
C_2 (two-fold axis $\vec{1}_i$	$\tilde{\tilde{\Lambda}}_b = \tilde{\tilde{\Lambda}}_b^T$	C_2 (two-fold axis $\vec{1}_i$)
R_a (single axial symmetry plane)	$\tilde{\tilde{\Lambda}}_b$ diagonal when referred to axial symmetry plane or perpendicular axial plane	C_{2a} (two-fold axis $\vec{1}_i$ with two axial symmetry planes)
C_N for $N \geq 3$ (N-fold axis $\vec{1}_i$)	$\tilde{\tilde{\Lambda}}_b \; \overleftrightarrow{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)

There are other kinds of target symmetries one can consider [9, 12]. One of interest here is continuous dilation symmetry (generalized conical symmetry [6, 7, 20], including cones, wedges, and half spaces. For a limited amount of time (the time of validity giving a partial symmetry) based on truncation of the target (to give a finite size) and arrival of multiple scattering at the observer (between tips for finite-length wedges, and tips and edges for finite-dimensioned half spaces) the scattering dyadic operator takes the form of one, two, or three kinds of terms. Each term contains a real symmetric 2x2 dyadic $\overleftrightarrow{K}_n\left(\vec{1}_i\right)$ times some order of temporal integration I_t^n (or power of 1/s in frequency domain). Each term can begin at different times, depending on the relative distances of tips and edges from the observer. As a real symmetric dyadic each of these $\overleftrightarrow{K}$ coefficient dyadics can always be diagonalized with real eigenvalues and real eigenvectors giving real and orthogonal spatial directions. The two eigenvectors specify two symmetry planes (except in the special case of equal eigenvalues). This is summarized in table 6.2.

Table 6.2. Generalized Conical Symmetry for Target and Back-Scattering Dyadic

Symmetry in Target	Form of $\overleftrightarrow{\Lambda}_b \circ$	Symmetry in $\overleftrightarrow{\Lambda}_b \circ$
continuous dilation (partial symmetry due to truncations)	$\overleftrightarrow{K}_n I_t^n$ $\overleftrightarrow{K}_n = \overleftrightarrow{K}_n^T$ (real) (number of values of n: simple cone: 1 finite-length wedge: 2 finite-dimensioned half space: 3)	C_{2a} for each term (two-fold axis $\overrightarrow{1}_i$ with two axial symmetry planes)

So here is a case of no point symmetry in the target, yet still giving additional point symmetry in the dyadic scattering operator, here expressed in time domain recognizing the limited time of validity (partial symmetry).

VII. Symmetry in Forward Scattering

For forward scattering we have

$$\vec{1}_o = \vec{1}_i \, , \, \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}_f^T\left(-\vec{1}_i, s\right) \tag{7.1}$$

but even with reciprocity in the target the scattering dyadic is not necessarily symmetric. Now the bisectrix plane P_b is perpendicular to $\vec{1}_i$, but the scattering plane P_s is any plane parallel to $\vec{1}_i$.

Consider first two-dimensional rotations and reflections in the transverse (bisectrix) plane as was done in Section VI for backscattering. As in (6.7), C_2 symmetry applies in forward scattering as well, due to the fact that the rotation dyadic is just $-\overleftrightarrow{1}_t$. For reflections, as in (6.8), let $\overleftrightarrow{R}_a$ represent reflection through some axial symmetry plane P_a (a scattering plane). Orient the h, v coordinates so that this reflection reverses only the h coordinate. Then the incident electric field can be chosen as symmetric, i.e., parallel to $\vec{1}_h$, producing a symmetric forward-scattered electric field also parallel to $\vec{1}_h$ [2, 17, 18]. Similarly, considering antisymmetric fields an electric field parallel to $\vec{1}_v$ forward scatters with this same polarization. So for this choice of coordinates the forward-scattering dyadic is *diagonal*. Applying (4.10) for reflection symmetry gives

$$\begin{aligned}
\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) &= \overleftrightarrow{R}_h \cdot \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) \cdot \overleftrightarrow{R}_h = \left(-\overleftrightarrow{R}_h\right) \cdot \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) \cdot \left(-\overleftrightarrow{R}_h\right) \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{\tilde{\Lambda}}_{f_{h,h}}\left(\vec{1}_i, s\right) & \tilde{\tilde{\Lambda}}_{f_{h,v}}\left(\vec{1}_i, s\right) \\ \tilde{\tilde{\Lambda}}_{f_{v,h}}\left(\vec{1}_i, s\right) & \tilde{\tilde{\Lambda}}_{f_{v,v}}\left(\vec{1}_i, s\right) \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\tilde{\Lambda}}_{f_{h,h}}\left(\vec{1}_i, s\right) & -\tilde{\tilde{\Lambda}}_{f_{h,v}}\left(\vec{1}_i, s\right) \\ -\tilde{\tilde{\Lambda}}_{f_{v,h}}\left(\vec{1}_i, s\right) & \tilde{\tilde{\Lambda}}_{f_{v,v}}\left(\vec{1}_i, s\right) \end{pmatrix} \\
\tilde{\tilde{\Lambda}}_{f_{h,v}}\left(\vec{1}_i, s\right) &= \vec{0} = \tilde{\tilde{\Lambda}}_{f_{v,h}}\left(\vec{1}_i, s\right)
\end{aligned} \tag{7.2}$$

confirming the diagonal nature of the dyadic as well as showing that the dyadic has two symmetry planes, the second one characterized by $\overleftrightarrow{R}_v$ which reflects the v coordinate. Now rotate the dyadic through any angle ϕ (about $\vec{1}_i$) as in (6.6) and take its transpose giving

$$\begin{aligned}
\left[\left(C_{n,m}(\phi)\right) \cdot \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) \cdot \left(C_{n,m}(\phi)\right)^T\right]^T &= \left(C_{n,m}(\phi)\right) \cdot \tilde{\tilde{\Lambda}}_f^T\left(\vec{1}_i, s\right) \cdot \left(C_{n,m}(\phi)\right)^T \\
&= \left(C_{n,m}(\phi)\right) \cdot \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) \cdot \left(C_{n,m}(\phi)\right)^T
\end{aligned} \tag{7.3}$$

since a diagonal dyadic is a special case of a symmetric dyadic. So we have in general for any choice of the h, v coordinates

$$\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}_f^T\left(\vec{1}_i, s\right) \tag{7.4}$$

if the target has an axial symmetry plane.

For $\vec{1}_i$ as a higher-order (N-fold) rotation axis given C_N symmetry in (6.11) and (6.12), now apply in forward scattering. However, with reciprocity not implying a symmetric $\tilde{\tilde{\Lambda}}$, then for $N \geq 3$ and following the procedure in [2, 17] we have

$$\begin{gathered}
\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) = \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right) \cdot \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) \cdot \left(C_{n,m}\left(\frac{2\pi n}{N}\right)\right)^T \\
\tilde{\tilde{\Lambda}}_{f_{1,2}}\left(\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}_{f_{2,2}}\left(\vec{1}_i, s\right) \\
\tilde{\tilde{\Lambda}}_{f_{1,2}}\left(\vec{1}_i, s\right) = -\tilde{\tilde{\Lambda}}_{f_{2,1}}\left(\vec{1}_i, s\right) \\
\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}_{f_{1,1}}\left(\vec{1}_i, s\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \tilde{\tilde{\Lambda}}_{f_{1,2}}\left(\vec{1}_i, s\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{gathered} \tag{7.5}$$

which is a transverse-identity part plus a rotation ($\pm\pi/2$) part. It is important that $N \geq 3$ since this assures that there is at least one n from n = 1, 2, 3,...,N such that $\sin(2\pi n/N) \neq 0$. This form of dyadic is invariant to any two-dimensional rotation so that it is characterized by the group C_∞ or O_2^+. Note that it is not invariant to axial reflections which can be seen by applying $\overleftrightarrow{R}_a$ to the above. If now the target has an axial symmetry plane (with C_N implying N such planes) then the target has C_{Na} symmetry and the forward scattering dyadic has $C_{\infty a}$ or O_2 symmetry with

$$\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.6}$$

Going on to symmetries that involve full 3x3 dyadics, let the target have inversion symmetry with group structure (an involution)

$$I = \{(1), (I)\}, \ (I)^2 = (1) \tag{7.7}$$

and dyadic representation

$$(1) \to \overleftrightarrow{1} \ , (I) \to -\overleftrightarrow{1} \tag{7.8}$$

Then (4.10) gives the general form

$$\tilde{\tilde{\Lambda}}\left(-\vec{1}_{o,s},-\vec{1}_i;s\right)=\tilde{\tilde{\Lambda}}\left(\vec{1}_o,\vec{1}_i;s\right) \tag{7.9}$$

In forward scattering we have

$$\vec{1}_o=\vec{1}_i\ ,\ \tilde{\tilde{\Lambda}}_f\left(-\vec{1}_i;s\right)=\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i;s\right) \tag{7.10}$$

Combining the above with reciprocity in (2.1) gives

$$\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i;s\right)=\tilde{\tilde{\Lambda}}_f^T\left(\vec{1}_i;s\right) \tag{7.11}$$

Thus inversion symmetry is sufficient to give a symmetric scattering dyadic for forward scattering for *all* $\vec{1}_i$, a result which applies in backscattering in (2.3) for targets with no particular geometric symmetry.

In forward scattering one can regard the scattering dyadic as a 2x2 matrix since incidence and scattering directions have the same axis. Since in (7.11) it is symmetric then only three of the four non-zero elements need to be considered. Inversion symmetry is rather general since it does not imply any rotation axes or symmetry planes in the target. There are many common shapes with inversion symmetry, e.g., rectangular parallelepipeds (bricks), finite-length straight wires, rectangular disks, circular loops, etc. Less common are strange shapes without symmetry planes and/or rotation axes.

If the target has one or more symmetry planes, and we restrict $\vec{1}_i$ to be perpendicular to one of these, then the result in (7.11) still holds for these particular forward-scattering directions. Describe this symmetry by the reflection group (transverse reflection plane, an involution)

$$R_t=\{(1),(R_t)\}\ ,\ (R_t)^2=(1) \tag{7.12}$$

and a dyadic representation

$$\begin{aligned}&(1)\rightarrow\overleftrightarrow{1}\ ,\ (R_t)\rightarrow\overleftrightarrow{R}_t=\overleftrightarrow{1}-2\,\vec{1}_i\vec{1}_i\\&\qquad\equiv\text{ reflection through a transverse (to }\vec{1}_i\text{) symmetry plane}\end{aligned} \tag{7.13}$$

this reflection is the same as $\overleftrightarrow{R}_B$ in (5.1) for the case of forward scattering, in which case P_b is perpendicular to $\vec{1}_i$. Reflection through a transverse plane is to be distinguished from reflection $\overleftrightarrow{R}_a$ through an axial plane.

Note that

$$\overleftrightarrow{R}_t \cdot \overrightarrow{1}_i = -\overrightarrow{1}_i, \overleftrightarrow{R}_t \cdot \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \cdot \overleftrightarrow{R}_t = \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \tag{7.14}$$

since $\tilde{\tilde{\Lambda}}$ now has no $\overrightarrow{1}_i$ components. Then (4.10) gives the same result as in (7.10), leading to the symmetric scattering dyadic as in (7.11), except that it applies only for particular $\overrightarrow{1}_i$.

Having a symmetric forward-scattering dyadic the results of Section VI for the symmetric back-scattering dyadic can now be directly applied. Adjoining the various symmetries transverse to $\overrightarrow{1}_i$ (two-dimensional) to the inversion symmetry I and reflection symmetry R_t discussed in this section, the same results are obtained as in table 6.1. The dilation symmetry in table 6.2 does *not* apply here since the scattering to the observer does not arrive first from cone tips and/or wedge edges, but rather from the entire scatterer (including truncations), all at the same time.

An additional point symmetry to be considered is rotation-reflection S_N. Keeping $\overrightarrow{1}_i$ as the rotation-reflection axis we have

$$\begin{aligned} &S_N = \left\{(S_N)_\ell \,|\, \ell = 1, 2, \ldots, N\right\} \\ &(S_N)_1 = (C_N)_1(R_t) = (R_t)(C_N)_1 \\ &(S_N)_\ell = (S_N)_1^\ell \\ &(S_N)_1^N = (1) \text{ for N even} \end{aligned} \tag{7.15}$$

Note that N even is the only interesting case since N odd requires that $(C_N)_1$ and (R_t) be separate symmetry elements giving C_{Nt} symmetry [14, 18]. Defining for N even

$$N' \equiv \frac{N}{2} \tag{7.16}$$

we have

$$\begin{aligned} &\overleftrightarrow{C}_N = \left(C_{n,m}\left(\frac{2\pi}{N}\right)\right) = \overleftrightarrow{C}\left(\frac{2\pi}{N}\right) \\ &\overleftrightarrow{C}_{N'} = \left(C_{n,m}\left(\frac{2\pi}{N'}\right)\right) = \overleftrightarrow{C}\left(\frac{2\pi}{N'}\right) = \overleftrightarrow{C}_N^2 \\ &\overleftrightarrow{S}_N = \overleftrightarrow{C}_N \cdot \overleftrightarrow{R}_t = \overleftrightarrow{R}_t \cdot \overleftrightarrow{C}_N \\ &\overleftrightarrow{S}_N^2 = \overleftrightarrow{C}_N^2 = \overleftrightarrow{C}_{N'} \end{aligned} \tag{7.17}$$

So $C_{N'}$ is a subgroup of S_N.

Applying $\overleftrightarrow{S}_N$ we have

$$\overleftrightarrow{S}_N \cdot \overrightarrow{1}_i = -\overrightarrow{1}_i$$

$$\begin{aligned}\tilde{\tilde{\Lambda}}_f\left(-\overrightarrow{1}_i, s\right) &= \overleftrightarrow{S}_N \cdot \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \cdot \overleftrightarrow{S}_N^T \\ &= \overleftrightarrow{C}_N \cdot \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \cdot \overleftrightarrow{C}_N^T \\ &= \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \text{ (reciprocity)}\end{aligned} \tag{7.18}$$

where the $\overleftrightarrow{R}_t$ part reverses $\overrightarrow{1}_i$ but does not operate on $\tilde{\tilde{\Lambda}}$ which has only transverse components. This operation can be carried out an odd number of times as

$$N'' \equiv \text{ odd integer}$$

$$\begin{aligned}\overleftrightarrow{S}_N^{N''} \cdot \overrightarrow{1}_i &= -\overrightarrow{1}_i \\ \tilde{\tilde{\Lambda}}_f\left(-\overrightarrow{1}_i, s\right) &= \overleftrightarrow{S}_N^{N''} \cdot \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \cdot \left(\overleftrightarrow{S}_N^{N''}\right)^T \\ &= \overleftrightarrow{C}_N^{N''} \cdot \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) \cdot \left(\overleftrightarrow{C}_N^{N''}\right)^T \\ &= \tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right)\end{aligned} \tag{7.19}$$

Now N is even, so consider the case that

$$\begin{aligned}&N'' = N' = \frac{N}{2}, N' = \text{odd} \\ &\overleftrightarrow{C}_N^{N'} = -\overleftrightarrow{1}_i \\ &\tilde{\tilde{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) = \tilde{\tilde{\Lambda}}_f^T\left(\overrightarrow{1}_i, s\right)\end{aligned} \tag{7.20}$$

giving a symmetric scattering dyadic, as with some cases previously discussed. Note that for $N' = 1$ we have

$$\begin{aligned}&S_2 = I \text{ (inversion)} \\ &\overleftrightarrow{S}_2 = \overleftrightarrow{C}_2 \cdot \overleftrightarrow{R}_t = -\overleftrightarrow{1}\end{aligned} \tag{7.21}$$

which is the case considered initially and does not have any particular rotation axis (i.e., is not related to $\overrightarrow{1}_i$). Other cases included in the above are $S_6, S_{10}, \ldots,$ i.e., $N = 2N'$ with N' odd.

The case of N' even (N a multiple of 4) is more complicated since, analogous to (7.13), we have

$$N'' = \text{ even integer } = 2N''$$

$$\overleftrightarrow{S}_N^{N''} \cdot \overrightarrow{1}_i = \overrightarrow{1}_i$$

$$\begin{aligned}\widetilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i,s\right) &= \overleftrightarrow{S}_N^{N''} \cdot \widetilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i,s\right) \cdot \left(\overleftrightarrow{S}_N^{N''}\right)^T \\ &= \overleftrightarrow{C}_N^{N''} \cdot \widetilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i,s\right) \cdot \left(\overleftrightarrow{C}_N^{N''}\right)^T \\ &= \overleftrightarrow{C}_{N'}^{N'''} \cdot \widetilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i,s\right) \cdot \left(\overleftrightarrow{C}_{N'}^{N'''}\right)^T\end{aligned} \tag{7.22}$$

but without the reversal of $\overrightarrow{1}_i$ to give the relation to the transpose scattering dyadic. This does exhibit, however, C_N and $C_{N'}$ as subgroups for the symmetry of the scattering dyadic. At this point let us note that for $N \geq 4$ (since N is even) the result in (7.5) with only two distinct elements for the forward-scattering dyadic applies.

Extending this rotation-reflection symmetry let us adjoin axial symmetry planes. Let any one of these be characterized by, say $\overleftrightarrow{R}_h$ as in (6.10), and align the h coordinate with this plane. Then letting the incident wave be symmetric or antisymmetric with respect to this plane (h and v polarization, respectively), the scattered fields including the forward-scattered fields will have the same symmetry properties. Hence, in this coordinate system the forward-scattering dyadic is diagonal (a special case of symmetric). Rotation about $\overrightarrow{1}_i$ (to any other choice of the transverse coordinates) of a symmetric dyadic gives another symmetric dyadic. Hence, adjunction of one or more axial symmetry planes makes

$$\widetilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i,s\right) = \widetilde{\overleftrightarrow{\Lambda}}_f^T\left(\overrightarrow{1}_i,s\right) \tag{7.23}$$

With a symmetric dyadic, then the result in (7.5) reduces to the transverse identity in (6.13) for back-scattering as

$$\widetilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i,s\right) = \widetilde{\Lambda}_f\left(\overrightarrow{1}_i,s\right)\overleftrightarrow{1}_i \quad \text{for } N \geq 4 \tag{7.24}$$

Note that adjunction of axial symmetry planes to S_N, giving S_{Na}, has $C_{N'a}$ as a subgroup and there are N' such axial symmetry planes. While S_N has N elements, S_{Na} has 2N elements and $C_{N'a}$ has N elements. These N' axial symmetry planes introduce N' secondary 2-fold rotation axes between these planes (diagonal) on the transverse rotation-reflection plane. This is dihedral symmetry $D_{N'}$ (N elements) with diagonal symmetry planes given $D_{N'd}$ symmetry (2N elements) [14, 18]. So we have

$$S_{Na} = D_{N'd} \tag{7.25}$$

as two equivalent labels for the same group, giving a symmetric dyadic as in (7.23) for which case the various results for backscattering also apply for forward scattering.

These point-symmetry results are summarized in table 7.1. Note that due to the relevance of three-dimensional rotation-reflections in the target symmetry the list is considerably longer than for back-scattering in table 6.1.

Table 7.1. Point Symmetry Groups (Rotation and Reflection) for Forward-Scattering Dyadic for Reciprocal Target

Symmetry in Target ($\vec{1}_i$ as reference axis)	Form of $\overleftrightarrow{\Lambda}_f$	Symmetry in $\overleftrightarrow{\Lambda}_f$
C_1 (no symmetry)	general 2x2	C_2 (two-fold axis $\vec{1}_i$)
C_2 (two-fold axis $\vec{1}_i$)	general 2x2	C_2 (two-fold axis $\vec{1}_i$)
R_a (single axial symmetry plane)	$\overleftrightarrow{\Lambda}_f = \overleftrightarrow{\Lambda}_f^T$ and diagonal when referred to axial symmetry plane or perpendicular axial plane	C_{2a} (two-fold axis $\vec{1}_i$ with two axial symmetry planes)
C_N for N ≥ 3 (N-fold axis $\vec{1}_i$)	$\tilde{\Lambda}_{f1,1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \tilde{\Lambda}_{f1,2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$C_\infty = O_2^+$ (continuous rotation axis $\vec{1}_i$)
C_{Na} for N ≥ 3 (N-fold axis $\vec{1}_i$ with N axial symmetry planes)	$\tilde{\Lambda}_f \overleftrightarrow{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)
I (inversion, no special axes or planes) R_t (symmetry plane $\perp \vec{1}_i$)	$\overleftrightarrow{\Lambda}_f = \overleftrightarrow{\Lambda}_f^T$	C_2 (two-fold axis $\vec{1}_i$)
C_{2t}	$\overleftrightarrow{\Lambda}_f = \overleftrightarrow{\Lambda}_f^T$	C_2 (two-fold axis $\vec{1}_i$)
C_{Nt} for N ≥ 3	$\tilde{\Lambda}_f \overleftrightarrow{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)
S_N (rotation-reflection, N (even) = $2N'$):		
N' odd	$\overleftrightarrow{\Lambda}_f = \overleftrightarrow{\Lambda}_f^T$	C_2 (two-fold axis $\vec{1}_i$)
odd $N' \geq 3$	$\tilde{\Lambda}_f \overleftrightarrow{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis 1_i with all axial planes as symmetry planes)
even $N' \geq 2$	$\tilde{\Lambda}_{f1,1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \tilde{\Lambda}_{f1,2}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$C_\infty = O_2^+$ (continuous rotation axis $\vec{1}_i$)
$S_{4a} = D_{2d}$ (dihedral with two diagonal (axial) symmetry planes)	$\overleftrightarrow{\Lambda}_f = \overleftrightarrow{\Lambda}_f^T$	C_{2a} (two-fold axis $\vec{1}_i$)
$D_{N'd}$ for $N' \geq 3$	$\tilde{\Lambda}_f \overleftrightarrow{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)

VIII. Symmetry in Low-Frequency Scattering

At low-frequencies for which the target is electrically small ($\lambda >>$ target dimensions), the scattering is dominated by the induced electric and magnetic dipole moments. The scattering dyadic then takes the form [5, 8]

$$\begin{aligned}
\tilde{\tilde{\Lambda}}\left(\vec{1}_i, \vec{1}_o; s\right) &= \tilde{\tilde{\Lambda}}^T\left(-\vec{1}_o, -\vec{1}_i; s\right) \text{ (reciprocity)} \\
&= \tilde{\gamma}^2\left[-\vec{1}_o \cdot \tilde{\tilde{P}}(s) \cdot \vec{1}_i + \vec{1}_o \times \tilde{\tilde{M}}(s) \times \vec{1}_i\right] \quad \text{as } s \to 0 \\
\tilde{\tilde{P}}(s) &= \tilde{\tilde{P}}^T(s) \equiv \text{ electric - polarizability dyadic} \\
\tilde{\tilde{M}}(s) &= \tilde{\tilde{M}}^T(s) \equiv \text{magnetic - polarizability dyadic}
\end{aligned} \tag{8.1}$$

These polarizabilities have dimensions of volume (m^3) and, with the incident electric and magnetic fields, give the induced electric and magnetic dipole moments. Reciprocity makes these dyadics symmetric. For perfectly conducting targets they are also frequency independent.

For convenience express the cross-projects in dyadic/dot-product form [3]. Consider an arbitrary real unit vector $\vec{1}_a$ and form

$$\begin{aligned}
\overleftrightarrow{\tau}_a &\equiv \vec{1}_a \times \overleftrightarrow{1} = \vec{1}_a \times \overleftrightarrow{1}_a \\
\overleftrightarrow{1}_a &\equiv \overleftrightarrow{1} - \vec{1}_a \vec{1}_a \equiv \text{ identity transverse to } \vec{1}_a
\end{aligned} \tag{8.2}$$

This is the most general skew symmetric dyadic [15] with the relations

$$\overleftrightarrow{1} \times \vec{1}_a = -\overleftrightarrow{\tau}_a = \overleftrightarrow{\tau}_{-a} = \overleftrightarrow{\tau}_a^{\,T}, \overleftrightarrow{\tau}_a^{\,2} = -\overleftrightarrow{1}_i \tag{8.3}$$

where the subscript -a refers to replacing $\vec{1}_a$ by - $\vec{1}_a$. This dyadic can be described as a rotation by $\pi/2$ of the coordinates transverse to $\vec{1}_a$ around $\vec{1}_a$ in a right-handed sense. The longitudinal coordinate (in the direction of $\vec{1}_a$) is multiplied by zero. To illustrate the form of this dyadic consider the right-handed coordinates based on $\vec{1}_h, \vec{1}_v, -\vec{1}_i$ satisfying (1.5). In this case

$$\overleftrightarrow{\tau}_{-i} = -\vec{1}_i \times \overleftrightarrow{1}_i = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\overleftrightarrow{\tau}_i = \overleftrightarrow{\tau}_i^{\,T} \tag{8.4}$$

gives a positive sense of rotation in the h, v plane in the usual radar coordinates. This is equivalent to $\left(C_{n,m}(\pi/2)\right)$ if only transverse coordinates are used. However, in three dimensions this is not an orthogonal dyadic, unless the (3,3) position is changed from 0 to 1 as in (6.6) for $\overleftrightarrow{C}(\pi/2)$. Similarly for $\overrightarrow{1}_o$ we have

$$\overleftrightarrow{\tau}_o = \overrightarrow{1}_o \times \overleftrightarrow{1}_o = -\overleftrightarrow{\tau}_{-o} = -\overleftrightarrow{\tau}_o^T \tag{8.5}$$

Now rewrite (8.1) as

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}\left(\overrightarrow{1}_o, \overrightarrow{1}_i; s\right) &= \tilde{\overleftrightarrow{\Lambda}}^T\left(-\overrightarrow{1}_i, -\overrightarrow{1}_o; s\right) \\ &= \tilde{\gamma}^2\left[-\overleftrightarrow{1}_o \cdot \tilde{\overleftrightarrow{P}}(s) \cdot \overleftrightarrow{1}_i - \overleftrightarrow{\tau}_o \cdot \tilde{\overleftrightarrow{M}}(s) \cdot \overleftrightarrow{\tau}_i\right] \text{ as } s \to 0\end{aligned} \tag{8.6}$$

Interchanging $\overrightarrow{1}_o$ and $\overrightarrow{1}_i$ we have

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}\left(\overrightarrow{1}_i, \overrightarrow{1}_o; s\right) &= \tilde{\gamma}^2\left[-\overleftrightarrow{1}_i \cdot \tilde{\overleftrightarrow{P}}(s) \cdot \overleftrightarrow{1}_o - \overleftrightarrow{\tau}_i \cdot \tilde{\overleftrightarrow{M}}(s) \cdot \overleftrightarrow{\tau}_o\right] \text{ as } s \to 0 \\ \tilde{\overleftrightarrow{\Lambda}}^T\left(\overrightarrow{1}_i, \overrightarrow{1}_o; s\right) &= \tilde{\gamma}^2\left[-\overleftrightarrow{1}_i \cdot \tilde{\overleftrightarrow{P}}^T(s) \cdot \overleftrightarrow{1}_i - \overleftrightarrow{\tau}_o^T \cdot \tilde{\overleftrightarrow{M}}^T(s) \cdot \overleftrightarrow{\tau}_i\right] \text{ as } s \to 0 \\ &= \tilde{\gamma}^2\left[-\overleftrightarrow{1}_o \cdot \tilde{\overleftrightarrow{P}}(s) \cdot \overleftrightarrow{1}_i - \overleftrightarrow{\tau}_o \cdot \tilde{\overleftrightarrow{M}}(s) \cdot \overleftrightarrow{\tau}_i\right] \text{ as } s \to 0 \\ &= \tilde{\overleftrightarrow{\Lambda}}^T\left(-\overrightarrow{1}_i, -\overrightarrow{1}_o; s\right) \text{ as } s \to 0\end{aligned} \tag{8.7}$$

so that we have the additional invariance (symmetry)

$$\tilde{\overleftrightarrow{\Lambda}}\left(\overrightarrow{1}_i, \overrightarrow{1}_o; s\right) = \tilde{\overleftrightarrow{\Lambda}}\left(-\overrightarrow{1}_i, -\overrightarrow{1}_o; s\right) \text{ as } s \to 0 \tag{8.8}$$

Note that this result, using the leading term (dipoles) for low frequencies, has only been shown to be valid in a low-frequency asymptotic sense, the error being thus far associated with the next higher-order (quadrupole) terms.

Applying this result to back and forward scattering gives

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}_b\left(\overrightarrow{1}_i, s\right) &= \tilde{\overleftrightarrow{\Lambda}}_b\left(-\overrightarrow{1}_i, s\right) = \tilde{\overleftrightarrow{\Lambda}}_b^T\left(\overrightarrow{1}_i, s\right) \\ \tilde{\overleftrightarrow{\Lambda}}_f\left(\overrightarrow{1}_i, s\right) &= \tilde{\overleftrightarrow{\Lambda}}_f\left(-\overrightarrow{1}_i, s\right) = \tilde{\overleftrightarrow{\Lambda}}_f^T\left(\overrightarrow{1}_i, s\right) \text{ as } s \to 0\end{aligned} \tag{8.9}$$

For low frequencies the back- and forward-scattering dyadics are both symmetric and invariant to direction reversal.

With both $\tilde{\tilde{\Lambda}}_b$ and $\tilde{\tilde{\Lambda}}_f$ symmetric, then the results of Section VI for point symmetries as in table 6.1 apply to both. In addition, the fact that the target is electrically small means that the incident electric and magnetic fields are uniform (quasistatic) in the vicinity of the target, so the front and back of the target are seen "simultaneously." This is also exhibited in (8.6) where only the dipole terms are retained.

Writing $\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i, s\right)$ for both $\tilde{\tilde{\Lambda}}_b\left(\vec{1}_i, s\right)$ and $\tilde{\tilde{\Lambda}}_f\left(\vec{1}_i, s\right)$ we have, in the electrically small regime,

$$\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}_i\left(-\vec{1}_i, s\right) = \tilde{\tilde{\Lambda}}^T\left(\vec{1}_i, s\right) \tag{8.10}$$

where we can take the dipole terms in (8.1) and (8.6) to *define* this case and give exact equality in (8.10). From (8.6) we have

$$\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i, s\right) = \gamma^2\left[-\vec{1}_i \cdot \tilde{\tilde{P}}(s) \cdot \vec{1}_i \pm \vec{\tau}_i \cdot \tilde{\tilde{M}}(s) \cdot \vec{\tau}_i\right]$$
$$\pm \Rightarrow \begin{cases} + \text{ for backscattering} \\ - \text{ for forward scattering} \end{cases} \tag{8.11}$$

Now the scattering dyadic is in a form in which the results of [10] can be used. In that paper the symmetries of the magnetic polarizability dyadic are analyzed. These also apply to the electric polarizability dyadic by appropriate interchange of electric and magnetic parameters. In accordance with (4.5) the same symmetries are taken for both electric and magnetic parameters so that $\tilde{\tilde{P}}(s)$ and $\tilde{\tilde{M}}(s)$ have the same symmetries.

Write $\tilde{\tilde{D}}(s)$ as a general dipole polarizability dyadic, applying to both electric and magnetic polarizabilities. Then table 9.1 of [10] can be written as table 8.1 here. Note that the axes (z, etc.) here are aligned according to the target and are not in general $\vec{1}_i$ unless so selected for a particular case.

Concerning rotations and axial reflections with respect to $\vec{1}_i$ as the symmetry axis (two-dimensional transformations), table 6.1 is applicable. Including transverse reflections $\tilde{\tilde{R}}_t$ changes the results from the previous results for back and forward scattering somewhat. Both inversion I and transverse reflection R_t symmetry do not add to the symmetries in $\tilde{\tilde{\Lambda}}_d$ as they did for $\tilde{\tilde{\Lambda}}_f$ in Section VII because $\tilde{\tilde{\Lambda}}_d$ is already symmetric. The discussion can then jump ahead to rotation-reflection symmetry in (7.15) through (7.17).

Table 8.1. Decomposition of Polarizability Dyadic According to Target Point Symmetries.

Form of $\tilde{\tilde{D}}(s)$	Symmetry Type (Groups)
$\tilde{D}_z(s)\vec{1}_z\vec{1}_z + \tilde{\tilde{D}}_t(s)$ $\left(\tilde{\tilde{D}}_t(s)\cdot\vec{1}_z = \vec{0}\right)$	R_z (single symmetry plane) C_2 (2-fold rotation axis)
$\tilde{D}_z(s)\vec{1}_z\vec{1}_z + \tilde{D}_x(s)\vec{1}_x\vec{1}_x + \tilde{D}_y(s)\vec{1}_y\vec{1}_y$	$C_{2a} = R_x \otimes R_y$ D_2 (three 2-fold rotation axes)
$\tilde{D}_z(s)\vec{1}_z\vec{1}_z + \tilde{D}_t(s)\vec{1}_z$ $\left(\vec{1}_z = \vec{1} - \vec{1}_z\vec{1}_z \Rightarrow \text{double degeneracy}\right)$	C_N for N ≥ 3 (N-fold rotation axis) S_N for N even and N ≥ 4 (N-fold rotation-reflection axis) D_{2d} (three 2-fold rotation axes plus diagonal symmetry planes)
$\tilde{D}(s)\vec{1}$ $\left(\vec{1} \Rightarrow \text{triple degeneracy}\right)$	O_3 (generalized sphere) T, O, Y (regular polyhedra)

For our symmetric dyadic we now have

$$\begin{aligned}\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i,s\right) &= \tilde{\tilde{\Lambda}}_i\left(-\vec{1}_i,s\right) = \vec{\vec{S}}_N^n\cdot\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i,s\right)\cdot\vec{\vec{S}}_N^n \\ &= \vec{\vec{C}}_N^n\cdot\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i,s\right)\cdot\vec{\vec{C}}_N^n \\ &= \tilde{\tilde{\Lambda}}^T\left(\vec{1}_i,s\right) \\ \vec{\vec{S}}_N^n\cdot\vec{1}_i &= (-1)^n\vec{1}_i \ ,\ n = 1,2,\ldots,N\end{aligned} \tag{8.12}$$

The invariance to $\vec{1}_i$ reversal avoids the effect of $\vec{\vec{S}}_N$ on $\vec{1}_i$. The symmetry of the dyadic then makes

$$\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i,s\right) = \tilde{\Lambda}_i\left(\vec{1}_i,s\right)\vec{1}_i \text{ for } N \geq 4 \ (N \text{ being even}) \tag{8.13}$$

which is $C_{\infty a}$ or O_2 symmetry. This result can also be found using table 8.1 noting that for $\vec{1}_z = \vec{1}_i$ the transverse part of the polarizability dyadics have the form $\tilde{D}_t(s)\vec{1}_i$. So for low frequencies the results for rotation-reflection symmetry are considerably simpler than those in Section VII for general forward scattering.

Referring to table 8.1 we also have dihedral symmetry to consider. Here we see that D_2 with three 2-fold rotation axes (one taken as - $\vec{1}_i$ and the other two perpendicular to - $\vec{1}_i$ and each other) is sufficient to give two axial symmetry planes in the polarizabilities. Note in (8.11) that $\overleftrightarrow{\tau}_i$ is a $-\pi/2$ rotation, merely rotating the axial symmetry planes into each other. So D_2 results in C_{2a} symmetry in $\tilde{\tilde{\Lambda}}_i$. Going a step further D_{2d} symmetry makes the transverse part of the polarizabilities proportional to $\overleftrightarrow{1}_z$ so that (8.11) takes the simple form

$$\begin{aligned}\tilde{\tilde{\Lambda}}_i\left(\vec{1}_i,s\right) &= \gamma^2\left[-\tilde{P}_t(s)\mp\tilde{M}_t(s)\right]\overleftrightarrow{1}_i \\ &= \tilde{\Lambda}_i(s)\overleftrightarrow{1}_i\end{aligned} \tag{8.14}$$

which is $C_{\infty a}$ or O_2 symmetry.

Finally, there are the symmetry groups which give $\tilde{D}(s)\overleftrightarrow{1}$, namely O_3 (invariance to all rotations and reflections), T (tetrahedral), O (octahedral), and Y (icosahedral) symmetries. This also gives $C_{\infty a}$ or O_2 as in (8.14) with the additional invariance of being independent of $\vec{1}_i$. One can think of this as O_3 symmetry. Remember that T, O, and Y give this result for only low frequencies for which the induced dipole moments dominate the scattering.

These point-symmetry results are summarized in table 8.2. These low-frequency results can be compared to tables 6.1 and 7.1 for general frequencies.

Table 8.2. Point-Symmetry Groups (Rotation and Reflection) for Low-Frequency Back- and Forward-Scattering Dyadics

Symmetry in Target ($\vec{1}_i$ as reference axis)	Form of $\tilde{\tilde{\Lambda}}_i$	Symmetry in $\tilde{\tilde{\Lambda}}_i$
C_1 (no symmetry)	$\tilde{\tilde{\Lambda}}_i = \tilde{\tilde{\Lambda}}_i^T$	C_2 (two-fold axis $\vec{1}_i$)
C_2 (two-fold axis $\vec{1}_i$)	$\tilde{\tilde{\Lambda}}_i = \tilde{\tilde{\Lambda}}_i^T$	C_2 (two-fold axis $\vec{1}_i$)
R_a (single axial symmetry plane)	$\tilde{\tilde{\Lambda}}_i$ diagonal when referred to axial symmetry plane or perpendicular axial plane	C_{2a} (two-fold axis $\vec{1}_i$ with two axial symmetry planes)
C_N for N ≥ 3	$\tilde{\Lambda}_i \vec{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)
S_N for N ≥ 4 (rotation-reflection, N even)	$\tilde{\Lambda}_i \vec{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)
D_2 (dihedral, three rotation axes at right angles)	$\vec{\Lambda}_i$ diagonal when referred to transverse rotation axes	C_{2a} (two-fold axis $\vec{1}_i$ with two axial symmetry planes)
D_{2d} (dihedral, two transverse rotation axes and two axial diagonal symmetry planes)	$\tilde{\Lambda}_i \vec{1}_i$	$C_{\infty a} = O_2$ (continuous rotation axis $\vec{1}_i$ with all axial planes as symmetry planes)
O_3 (all rotations and reflections) T (tetrahedral) O (octahedral) Y (icosahedral) (orientation of rotation axes arbitrary with respect to $\vec{1}_i$)	$\tilde{\Lambda}_i \vec{1}_i$ independent of $\vec{1}_i$	O_3 (all rotations and reflections of both polarization and $\vec{1}_i$)

IX. Concluding Remarks

As shown by the various cases considered here, reciprocity is fundamental to the symmetry properties of the scattering dyadic, including both back and forward scattering. when combined with geometrical symmetries in the target this gives yet higher order symmetries in the scattering dyadic as summarized in the tables in Sections VI through VIII. There are also symmetries in general bistatic scattering if the target has a symmetry plane aligned with the scattering plane or bisectrix plane as discussed in Section V. For low-frequency scattering (electrically small target) there is also the general bistatic symmetry in (8.8) of invariance to direction reversal for both incidence and scattering directions.

Another related topic concerns the impact of symmetry on the eigenmodes and natural modes. As discussed in [1, 2, 11, 17] there is an interesting case of 2-fold degeneracy for a body of revolution with axial symmetry planes ($C_{\infty a} = O_2$ symmetry). For backscattering, the rotation axis and the observer ($\vec{1}_i$) determine a symmetry plane. The symmetry plane makes the various residue vectors for the poles align parallel (symmetric) or perpendicular (antisymmetric) to the plane, thereby determining the target orientation. The 2-fold degeneracy for modes with Fourier-series terms cos(m ϕ') and sin(m ϕ') and $m \geq 1$ (ϕ' around rotation axis) give rise to two natural modes and residue vectors (one parallel and the other perpendicular to the plane) for each natural frequency s_α. Other types of symmetry (such as C_N for $N \geq 3$) can also lead to such 2-fold modal degeneracy. Such degeneracy is treated in detail for the magnetic polarizability dyadic (and by simple substitution the electric polarizability dyadic) in [10]. There various point symmetries gave rise to 2- and 3-fold degeneracies. Noting that only dipole terms are treated there, and that the scattering at higher frequencies (target not electrically small) involves in general all multipoles, then there can be various kinds of modal degeneracies associated with the various point symmetries. These considerations are beyond the scope of the present paper.

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