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The Magnetic Polarizability Dyadic and Point Symmetry

Carl E. Baum
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Abstract

In low-frequency target identification one approach involves quasi-magnetostatic illumination for determining the magnetic polarizability dyadic. For highly conducting permeable targets one can expand this in terms of a set of natural frequencies and modes as well as other limiting terms for identification. The targets of interest may have various point symmetries (rotation and reflection). Such symmetries simplify the forms that this dyadic and the associated modes can take, thereby aiding in the identification of a particular target from a library of targets.

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I. Introduction

A recent paper [3] has considered the quasi-magnetostatic scattering from highly conducting bodies in terms of the induced magnetic dipole moment via the magnetic polarizability dyadic. Dimensions of interest are assumed small compared to wavelengths or skin depths in the external medium. Thus we are concerned with near-field scattering in which the scattered magnetic field decays as r^{-3}. However, we avoid locations too close to the object so that we can neglect higher order magnetic multipole moments. The incident magnetic field is quasi-magnetostatic from some set of source coils, ideally to give three orthogonal directions of incident magnetic field. The scattered magnetic field is measured with the effects of the scattered electric field suppressed so that we can ignore the effects of inhomogeneities in the electric constitutive parameters (conductivity and permitivity) of the external medium. The permeability of the external medium is assumed to be that of free space, μ_0 (or at least similarly uniform and isotropic).

The object of interest (or target) is characterized by dyadic permeability $\overleftrightarrow{\mu}(\vec{r})$ and conductivity $\overleftrightarrow{\sigma}(\vec{r})$. The permeability is assumed negligible due to the high conductivity (e.g. metal) of the target. The target assumed reciprocal, i.e.

$$\overleftrightarrow{\mu}^T(\vec{r}) = \overleftrightarrow{\mu}(\vec{r}) \; , \; \overleftrightarrow{\sigma}^T(\vec{r}) = \overleftrightarrow{\sigma}(\vec{r}) \tag{1.1}$$

One might also let these be frequency dependent, but as shown in [3] all the natural frequencies s_α are shown to be negative real in the s $(= \Omega + j\omega)$ or complex-frequency plane in the above case of real symmetric matrices with appropriate positive-semi-definite restrictions. In addition the vectors in the dyadic expansion of the resulting magnetic polarizability are real.

Summarizing from [3], we have

$$\tilde{\overleftrightarrow{M}}(s) = \tilde{\overleftrightarrow{M}}(\infty) + \sum_\alpha M_\alpha \vec{M}_\alpha \vec{M}_\alpha [s - s_\alpha]^{-1}$$

$$\frac{1}{s}\tilde{\overleftrightarrow{M}}(s) = \frac{1}{s}\tilde{\overleftrightarrow{M}}(0) + \sum_\alpha \frac{M_\alpha}{s_\alpha} \vec{M}_\alpha \vec{M}_\alpha [s - s_\alpha]^{-1}$$

$$\vec{M}_\alpha \cdot \vec{M}_\alpha = 1 \; , \; \vec{M}_\alpha \equiv \text{real unit vector for } \alpha\text{th mode}$$

$$M_\alpha = \text{real scalar}$$

$$s_\alpha < 0 \;\; \text{(all negative real natural frequencies)}$$

$$\tilde{\overleftrightarrow{M}}(\infty) = \sum_{v=1}^{3} M_v^{(\infty)} \vec{M}_v^{(\infty)} \vec{M}_v^{(\infty)}$$

$$\vec{M}_v^{(\infty)} \equiv \text{real eigenvectors (three)}$$

$$\vec{M}_{v_1}^{(\infty)} \cdot \vec{M}_{v_2}^{(\infty)} = 1_{v_1,v_2} \text{ (orthonormal)}$$

$$M_v^{(\infty)} \equiv \text{real eigenvalues (non positive, not necessarily distinct)}$$

$$\tilde{\overleftrightarrow{M}}(0) = \sum_{v=1}^{3} M_v^{(0)} \vec{M}_v^{(0)} \vec{M}_v^{(0)}$$

$$\vec{M}_v^{(0)} \equiv \text{real eigenvectors (three)}$$

$$\vec{M}_{v_1}^{(0)} \cdot \vec{M}_{v_2}^{(0)} = 1_{v_1,v_2} \text{ (orthonormal)}$$

$$M_v^{(0)} \equiv \text{real eigenvalues (non negative, not necessarily distinct)} \tag{1.2}$$

The DC (zero-frequency) polarizability applies only to permeable scatterers characterized by $\overleftrightarrow{\mu}(\vec{r})$ different from $\mu_o \overleftrightarrow{1}$ (free space, or more generally the external medium). Note that reciprocity is already included as the dyadics are all symmetric, i.e.,

$$\tilde{\overleftrightarrow{M}}^T_{(s)} = \tilde{\overleftrightarrow{M}}(s) \tag{1.3}$$

This particular kind of symmetry is in the medium properties and is independent of the shape of the target.

The expansions in (1.2) are useful for target identification based on the signatures inherent in them [4, 5]. There are the aspect-independent natural frequencies s_α which can be compared to those in an appropriate target library. The M_α , $M_v^{(\infty)}$, and $M_v^{(0)}$ are also aspect independent. Furthermore, while the unit vectors in (1.2) are aspect dependent (rotating with the target) the dot products of these pairwise are all aspect independent, i.e.

$$\left.\begin{array}{c} \vec{M}_{\alpha_1} \cdot \vec{M}_{\alpha_2} \\ \vec{M}_\alpha \cdot \vec{M}_v^{(\infty)} \\ \vec{M}_\alpha \cdot \vec{M}_v^{(0)} \\ \vec{M}_{v_1}^{(\infty)} \cdot \vec{M}_{v_2}^{(0)} \end{array}\right\} \equiv \text{aspect independent parameters} \tag{1.4}$$

Note in (1.2) that the $\vec{M}_v^{(\infty)}$ are orthonormal as are the $\vec{M}_v^{(0)}$. Having identified the target, of course the various unit vectors can be used to orient the target (measure aspect). Note that the unit vectors have a sign ambiguity (±), however, since they appear as dyadic products with themselves.

The organization of these various unit vectors (and hence the modes) according to the symmetries of the target is the subject of this paper. As will become clearer these symmetries will make various of the dot products in (1.4) zero or one so that the associated unit vectors are orthogonal. This is associated with various axes and planes of symmetry.

II. Symmetric Scatterers

For present purposes a symmetric scatterer is one which is invariant under one or more point symmetry operations, i.e. rotations and reflections. These and other types of symmetry are discussed in an electromagnetic context in [6]. Fundamental to such considerations is the concept of a group

$$
\begin{aligned}
G &= \{(G)_\ell \,|\, \ell\, 1, 2, \ldots, \ell_o\} \\
(G)_\ell &\equiv \text{group element} \\
\ell_o &\equiv \text{group order} \\
(G)_\ell^{-1} &\in G\ , \quad (1) \equiv \text{identity} \in G \\
(G)_{\ell_1}(G)_{\ell_2} &\in G \ \text{ for all ordered pairs of elements}
\end{aligned}
\tag{2.1}
$$

Associated with each group there are one or more matrix representations (square d × d matrices where d is the dimension of the representation) with group multiplication becoming the usual dot multiplication. So we can substitute

$$
(G)_\ell \to \overleftrightarrow{G}_\ell\ , \quad (1) \to \overleftrightarrow{1} = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z
\tag{2.2}
$$

where the matrices are here taken as dyadics appropriate for three-dimensional Euclidean space. For finite groups (finite order) such representations are unitary. For present purposes with real coordinate transformations these dyadics are real and orthogonal with

$$
\overleftrightarrow{G}_\ell^{-1} = \overleftrightarrow{G}_\ell^{T}
\tag{2.3}
$$

For infinite groups our present concern is with O_3 (the group of all rotations and reflection in 3 dimensions) and O_3^+ (the group of proper rotations (no reflections)). In these cases the form in (2.3) still applies with the distinction

$$
\det(\overleftrightarrow{G}_\ell) = \begin{cases} +1 \Rightarrow \text{proper rotation (no reflection)} \\ -1 \Rightarrow \text{improper rotation (includes a reflection)} \end{cases}
\tag{2.4}
$$

Note that an even number of reflections takes the form of a proper rotation.

Now for our target to be symmetric we mean that it is invariant under transformation by the matrix elements. For each group element the coordinates transform as

$$
\begin{aligned}
\vec{r} &= x\,\vec{1}_x + y\,\vec{1}_y + z\,\vec{1}_z \\
\vec{r}^{(2)} &= \vec{G}_\ell \cdot \vec{r}^{(1)}
\end{aligned}
\tag{2.5}
$$

Under this rotation/reflection operation the target is the same as before the operation. In terms of target shape this is a fairly simple concept. However, it goes further in that what was at $\vec{r}^{(1)}$ is also present at $\vec{r}^{(2)}$ which for scalar constitutive parameters means

$$\mu(\vec{r}^{(2)}) = \mu(\vec{r}^{(1)}) \ , \ \sigma(\vec{r}^{(2)}) = \sigma(\vec{r}^{(1)}) \ , \ \varepsilon(\vec{r}^{(2)}) = \varepsilon(\vec{r}^{(1)}) \tag{2.6}$$

For dyadic constitutive parameters we have

$$\begin{aligned} &\overleftrightarrow{\mu}(\vec{r}^{(2)}) = \overleftrightarrow{G}_\ell \cdot \overleftrightarrow{\mu}(\vec{r}^{(1)}) \cdot \overleftrightarrow{G}_\ell^T \ , \ \overleftrightarrow{\sigma}(\vec{r}^{(2)}) = \overleftrightarrow{G}_\ell \cdot \overleftrightarrow{\sigma}(\vec{r}^{(1)}) \cdot \overleftrightarrow{G}_\ell^T \ , \\ &\overleftrightarrow{\varepsilon}(\vec{r}^{(2)}) = \overleftrightarrow{G}_\ell \cdot \overleftrightarrow{\varepsilon}(\vec{r}^{(1)}) \cdot \overleftrightarrow{G}_\ell^T \end{aligned} \tag{2.7}$$

This form is readily obtained by dot-multiplying these dyadics into associated fields (e.g. $\vec{H}$) to produce other fields (e.g. $\vec{B}$) and then transforming these vector fields as in (2.5). An alternate way to view this transformation is to approximately synthesize an anisotropic medium by a locally inhomogeneous isotropic medium consisting of rods or sheets of material to control the medium properties in certain directions. In any event (2.7) can be viewed as a definition of a target symmetric (invariant) under operation by $\overleftrightarrow{G}_\ell$ (noting that permittivity is unimportant for present considerations).

Of course, the group G contains ℓ_0 elements, so when we ascribe G as the group describing the symmetries of the target we mean that the relations in (2.7) apply for *all* the $\overleftrightarrow{G}_\ell$. If $\vec{E}(\vec{r},t)$, $\vec{H}(\vec{r},t)$ is some solution of the Maxwell equations

$$\begin{aligned} \nabla \times \vec{E}(\vec{r},t) &= -\overleftrightarrow{\mu}(\vec{r}) \cdot \frac{\partial}{\partial t} \vec{H}(\vec{r},t) - \vec{J}_h(\vec{r},t) \\ \nabla \times \vec{H}(\vec{r},t) &= \left[\overleftrightarrow{\sigma}(\vec{r}) + \overleftrightarrow{\varepsilon}(\vec{r}) \frac{\partial}{\partial t}\right] \cdot \vec{E} + \vec{J}_e(\vec{r},t) \end{aligned} \tag{2.8}$$

$\vec{J}_e$, $\vec{J}_h$ ≡ electric and magnetic source current densities

then the symmetry operation gives

$$\begin{aligned} \vec{E}^{(2)}(\vec{r}^{(2)}, t) &= \overleftrightarrow{G}_\ell \cdot \vec{E}^{(1)}(\vec{r}^{(1)}, t) \\ \vec{H}^{(2)}(\vec{r}^{(2)}, t) &= \pm \overleftrightarrow{G}_\ell \cdot \vec{H}^{(1)}(\vec{r}^{(1)}, t) \end{aligned} \tag{2.9}$$

As another solution with sources similarly transformed. Note the ± with magnetic parameters, + for proper rotations and − for improper rotations (including a reflection). This can be interpreted as proper rotations commuting with $\nabla\times$, but improper rotations anticommuting with $\nabla\times$. Reflections involve changing coordinate signs and are associated with time reversal.

The order of a group element is n_ℓ, the smallest integer (≥1) such that

$$\begin{aligned} (G)_\ell^{n_\ell} &= (1) \\ \frac{\ell_o}{n_\ell} &= \text{positive integer} \end{aligned} \tag{2.10}$$

There is a group (cyclic) of order n_ℓ called the *period* of $(G)_\ell$ as

$$\begin{aligned} G_\ell &= \left\{(G)_\ell^n \middle| n = 1, 2, \ldots, n_\ell\right\} \\ &= \text{subgroup of } G \text{ of smallest order containing } (G)_\ell \end{aligned} \tag{2.11}$$

So n_ℓ applications of $\overleftrightarrow{G}_\ell$ in (2.5) and (2.7) brings the target back to its original configuration, and

$$\overleftrightarrow{G}_\ell^{\,n_\ell} = \overleftrightarrow{1} \tag{2.12}$$

As such $\overleftrightarrow{G}_\ell$ can be thought of as an n_ℓth root of $\overleftrightarrow{1}$, and all its eigenvalues have the property

$$\lambda_n^{n_\ell}(\overleftrightarrow{G}_\ell) = 1 \tag{2.13}$$

i.e. are roots of unity. From (2.4) and (2.12) we then have

$$\det(\overleftrightarrow{G}_\ell) = -1 \Rightarrow n_\ell = \text{even} \tag{2.14}$$

so improper rotations all have even periods.

III. Symmetry and Eigenmodes

There are various integral equations used to describe electromagnetic scattering. In the context of our quasi-magnetostatic case one can include inhomogeneous anisotropic conductivity and permeability as discussed in [3]. For present purposes consider the integral equation with the form

$$\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s)\ ;\ \tilde{\vec{J}}(\vec{r}\,',s) \right\rangle = \tilde{\vec{E}}^{(inc)}(\vec{r},s) \tag{3.1}$$

This is the form of an E-field or impedance integral equation with integration over the domain of the target (surface and/or volume integration denoted by < , > with integration over the common (prime) coordinates. While the incident field is here taken as electric, this could include the incident magnetic field via the curl or as a pair of coupled integral equations. The response has the form of a current density $\vec{J}$ but could include a magnetic current density as well.

Use (3.1) to define an eigenmode equation as

$$\begin{aligned} &\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s)\ ;\ \tilde{\vec{j}}_{\beta}(\vec{r}\,',s) \right\rangle = \tilde{Z}_{\beta}(s)\,\tilde{\vec{j}}_{\beta}(\vec{r},s) \\ &\tilde{\vec{j}}_{\beta}(\vec{r},s) \equiv \beta\text{th eigenmode} \\ &\tilde{Z}_{\beta}(s) \equiv \beta\text{th eigenvalue} \end{aligned} \tag{3.2}$$

These eigenmode parameters are functions of the complex frequency s . A related set of modes is the natural modes with associated natural frequencies as

$$\left\langle \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s_{\alpha})\ ;\ \tilde{\vec{j}}_{\alpha}(\vec{r}\,') \right\rangle = \vec{0} \tag{3.3}$$

which are related to the former parameters by [9]

$$\begin{aligned} \alpha &= (\beta,\beta') \\ \tilde{Z}_{\beta}\left(s_{\beta,\beta'}\right) &= 0 \\ s_{\beta,\beta'} \equiv s_{\alpha} &= \beta'\text{th root of } \beta\text{th eigenvalue} \\ \vec{j}_{\alpha}(\vec{r}) &= \tilde{\vec{j}}_{\beta}(\vec{r},s_{\alpha}) \text{ (times an arbitrary scaling constant, if desired)} \end{aligned} \tag{3.4}$$

For present purposes we can deal with the eigenmodes, noting how they are sorted by the symmetry applies to the natural modes and natural frequencies as well.

Now consider the degeneracy of the eigenmodes, i.e., for a given eigenvalue there can be more than one eigenmode [7]. Noting that the $\tilde{Z}_\beta(s)$ are functions of frequency, for some values of s two of these may become equal; this is referred to as "accidental degeneracy" in which it is the symmetry of the target that gives two or more eigenmodes the same eigenvalue (here in a frequency-independent sense). This subject has been considered in the context of symmetries in quantum mechanics [7]. Here it is adapted to electromagnetic scattering, noting the different parameters involved (complex frequency instead of energy, etc.).

Now take our general group element in dyadic form from (2.2) and operate on (3.2) to give

$$\left\langle \overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{Z}}(\vec{r},\vec{r}\,';s) \cdot \overleftrightarrow{G}_\ell^T \; ; \; \overleftrightarrow{G}_\ell \cdot \tilde{\vec{j}}_\beta(\vec{r}\,',s) \right\rangle = \tilde{Z}_\beta(s)\, \overleftrightarrow{G}_\ell \cdot \tilde{\vec{j}}_\beta(\vec{r}\,',s) \tag{3.5}$$

From which we find that $\overleftrightarrow{G}_\ell \cdot \tilde{\vec{j}}_\beta(\vec{r},s)$ is also an eigenmode associated with the same eigenvalue $\tilde{Z}_\beta(s)$. The kernel $\tilde{\overleftrightarrow{Z}}$ has been transformed as $\overleftrightarrow{G}_\ell \cdot \tilde{\overleftrightarrow{Z}} \cdot \overleftrightarrow{G}_\ell^T$, but this is just the transformation in (2.7) under which the target is invariant (by hypothesis). One can also view this in a more geometrical or physical way by merely transforming the coordinates as in (2.5) and noting that in the new $\vec{r}^{\,(2)}$ coordinate system the problem is the same and admits the same solutions.

Extending this we find that

$$\overleftrightarrow{G}_\ell^{\,n} \cdot \tilde{\vec{j}}(\vec{r},s) \equiv \text{eigenmodes for } n = 0,1,\ldots,n_\ell - 1 \tag{3.6}$$

However, these n_ℓ solutions are not necessarily linearly independent. Note that any linear combination of the above is also an eigenmode corresponding to the eigenvalue $\tilde{Z}_\beta(s)$. What we need are m_β linearly independent eigenmodes with $1 \le m_\beta \le n_\ell$ which span the space of these eigenmodes. As has been developed in a quantum-mechanical context [7, 8], m_β is given by the dimension (say d_β) of the associated *irreducible representation* [6] of the group. While our dyadics have dimension 3 the irreducible representations also include lower dimensions. While one can approach the problem from this mathematical point of view, here a more physical point of view is pursued. In any event note that our eigenmodes can now be written as

$$\begin{aligned} \tilde{\vec{j}}_\beta^{\,(m)}(\vec{r},s) &\equiv m\text{th independent eigenmode associated with } \tilde{Z}_\beta(s) \\ m &= 1,\ldots,m_\beta \end{aligned} \tag{3.7}$$

Similarly for natural modes we have

$$\vec{j}^{(m)}_{\beta,\beta'}(\vec{r}) \equiv m\text{th independent natural mode associated with } s_{\beta,\beta'}$$
$$m = 1,\ldots,m_{\beta,\beta'} \tag{3.8}$$

As discussed in [2] one can in some cases divide the eigenmodes according to E and H modes, but this will not be used for present purposes.

IV. Reflection Symmetry

Start first with the case of symmetry planes, i.e. reflection symmetry

$$R \equiv \{(1), (R)\}\ , \ (R)^2 = (1) \tag{4.1}$$

This is an example of a two-element or involution group. In the usual Cartesian system let

$$\begin{aligned}
\overleftrightarrow{R}_x &\equiv \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftrightarrow{1} - 2\,\vec{1}_x\,\vec{1}_x \\
\overleftrightarrow{R}_y &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \overleftrightarrow{1} - 2\,\vec{1}_y\,\vec{1}_y \\
\overleftrightarrow{R}_z &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \overleftrightarrow{1} - 2\,\vec{1}_z\,\vec{1}_z \\
\overleftrightarrow{1} &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \vec{1}_x\,\vec{1}_x + \vec{1}_y\,\vec{1}_y + \vec{1}_z\,\vec{1}_z
\end{aligned} \tag{4.2}$$

corresponding to reflection of the coordinates indicated by subscript.

Consider a single symmetry plane which is defined to be the xy (or $z = 0$) plane. As discussed in previous papers [6] all electromagnetic parameters can be decomposed into two kinds laballed symmetric (sy) and antisymmetric (as) with the properties

$$\begin{aligned}
\vec{r}^{(2)} &= \overleftrightarrow{R}_z \cdot \vec{r}^{(1)} \\
\vec{E}_{\substack{sy\\as}}(\vec{r}^{(2)},t) &= \pm \overleftrightarrow{R}_z \cdot \vec{E}_{\substack{sy\\as}}(\vec{r}^{(1)},t) \\
\vec{H}_{\substack{sy\\as}}(\vec{r}^{(2)},t) &= \mp \overleftrightarrow{R}_z \cdot \vec{H}_{\substack{sy\\as}}(\vec{r}^{(1)},t)
\end{aligned} \tag{4.3}$$

where the upper sign is for symmetric fields and the lower sign is for antisymmetric fields. Note the opposite symmetry properties for electric and magnetic parameters, this extending to potentials, currents, etc. In particular the eigenmodes can be written this way, i.e.

$$\begin{aligned}
\vec{\tilde{j}}_{\beta}(\vec{r},s) &= \vec{\tilde{j}}_{n,\substack{sy\\as}}(\vec{r},s) = \pm \overleftrightarrow{R}_z \cdot \vec{\tilde{j}}_{n,\substack{sy\\as}}(\vec{r},s) \\
\tilde{Z}_{\beta}(s) &= \tilde{Z}_{n,\substack{sy\\as}}(s)
\end{aligned} \tag{4.4}$$

where in general the symmetric and antisymmetric eigenvalues are distinct. In particle physics this separation into two kinds of modes or wave functions is referred to as parity (+ or -), especially as applied to full coordinate inversion I.

Our interest in these currents concerns their contribution to the induced magnetic dipole moment given by

$$\tilde{\vec{m}}_\beta(s) = \frac{1}{2}\int_V \vec{r} \times \tilde{\vec{J}}_\beta(\vec{r},s)\,dV \tag{4.5}$$

This formula includes the electric-current part. As discussed in [3] there is also a magnetic-current-density part due to the permeability with the magnetic field. However, for present purposes of symmetry the form in (4.5) is sufficient. Figure 4.1 shows spatial distributions appropriate to these two kinds of eigenmodes. Here we consider only modes which give a magnetic dipole moment, i.e. such that the integral in (4.5) is non zero. With $z = 0$ as the symmetry plane we have symmetric modal currents which are parallel to this plane on the plane and have equal parallel components but opposite normal components off the plane as given by (4.4). Then (4.5) immediately gives $\tilde{\vec{m}}_{n,sy}$ parallel to the z axis. So we can write

$$\tilde{\vec{m}}_{n,sy}(s) = \tilde{m}_{n,sy}\,\vec{1}_z \tag{4.6}$$

Similarly, antisymmetric modal currents have equal normal components but opposite parallel components off the plane. Then the integral in (4.5) implies

$$\tilde{\vec{m}}_{n,as}(s) \cdot \vec{1}_z = 0 \tag{4.7}$$

i.e., the antisymmetric magnetic moment has two components parallel to the symmetry plane.

Interpreting this in terms of polarizability, consider a uniform incident magnetic field parallel to the z axis. This couples only to symmetric modes and produces a magnetic dipole moment as in (4.6). If the incident magnetic field is parallel to the symmetry plane then a dipole moment as in (4.7) is produced. Then the magnetic polarizability dyadic in (1.2) has the various modes separated as

$$\tilde{\overleftrightarrow{M}}(s) = \tilde{\overleftrightarrow{M}}(\infty) + \sum_{\alpha_z} M_{\alpha_z}^{(x)}\,\vec{1}_z\,\vec{1}_z\left[s - s_{\alpha_z}\right]^{-1} + \sum_{\alpha_t} M_{\alpha_t}^{(t)}\,\vec{M}_{\alpha_t}^{(t)}\,\vec{M}_{\alpha_t}^{(t)}\left[s - s_{\alpha_t}\right]^{-1}$$

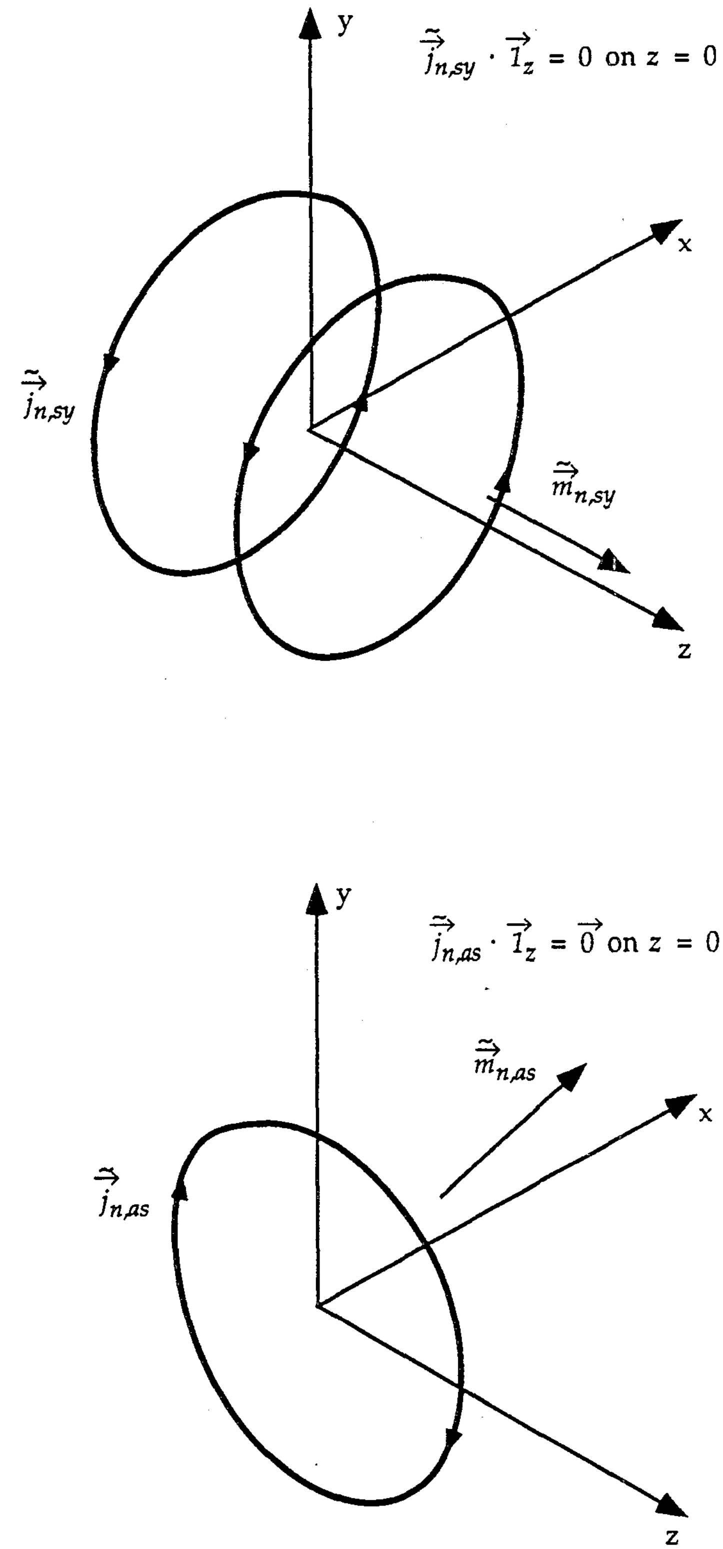

Figure 4.1. Electric-Current-Density Eigenmodes in Presense of Symmetry Plane

$\alpha_z \equiv$ index for modal unit vectors parallel to $\vec{1}_z$ (symmetric modes)

$\alpha_t \equiv$ index for modal unit vectors perpendicular (or transverse) to $\vec{1}_z$ (antisymmetric modes)

$$
\begin{aligned}
\vec{M}^{(t)}_{\alpha_t} \cdot \vec{1}_z &= 0 \\
\overset{\leftrightarrow}{\tilde{M}}(\infty) &= M_z^{(\infty)} \vec{1}_z \vec{1}_z + \sum_{v=1}^{2} M_v^{(\infty)} \vec{M}_v^{(\infty)} \vec{M}_v^{(\infty)} \\
\vec{M}_v(\infty) \cdot \vec{1}_z &= 0 \text{ for } v = 1,2 \\
\overset{\leftrightarrow}{\tilde{M}}(0) &= M_z^{(0)} \vec{1}_z \vec{1}_z + \sum_{v=1}^{2} M_v^{(0)} \vec{M}_v^{(0)} \vec{M}_v^{(0)} \\
\vec{M}_v^{(0)} \cdot \vec{1}_z &= 0 \text{ for } v = 1,2
\end{aligned}
\tag{4.8}
$$

Thus all the dyadics separate into a parallel (or (z,z)) component and perpendicular ((x,x), (y,y), and $(x,y) = (y,x)$) components. Other than accidental degeneracies all the natural frequencies and modal unit vectors separate into parallel and perpendicular (to $\vec{1}_z$) components. The dot products of the various modal vectors in (1.4) simplify in that (x,x) products are all unity, and (x,z) and (y,z) products are all zero. Since the symmetry plane is a property of the target and is a function of the aspect of the target (with respect to some observer's coordinate system) the orientation of $\pm\vec{1}_z$ is readily obtained in trying to determine the target aspect.

Now extend these results to the case of two perpendicular symmetry planes which we take as the $x = 0$ and $y = 0$ planes. These intersect on the z axis which becomes a two-fold rotation axis, but rotations will be discussed later. So now we have a symmetry group with four elements as

$$
\begin{aligned}
R_x \otimes R_y &= \{\overset{\leftrightarrow}{1}, \overset{\leftrightarrow}{R}_x, \overset{\leftrightarrow}{R}_y, \overset{\leftrightarrow}{R}_x \cdot \overset{\leftrightarrow}{R}_y = \overset{\leftrightarrow}{R}_y \cdot \overset{\leftrightarrow}{R}_x\} \\
&= C_{2a}
\end{aligned}
\tag{4.9}
$$

So now apply the previous results for one symmetry plane twice to see what happens. In (4.8) not only do the $(x,x$ terms separate out, but so also do the (y,y) terms, leaving us with a complete decomposion according to the Cartesian axes as

$$
\begin{aligned}
\overset{\leftrightarrow}{\tilde{M}}(s) = \overset{\leftrightarrow}{\tilde{M}}(\infty) &+ \sum_{\alpha_x} M_{\alpha_x}^{(x)} \vec{1}_x \vec{1}_x \left[s - s_{\alpha_x}\right]^{-1} \\
&+ \sum_{\alpha_y} M_{\alpha_y}^{(y)} \vec{1}_y \vec{1}_y \left[s - s_{\alpha_y}\right]^{-1} + \sum_{\alpha_z} M_{\alpha_z}^{(z)} \vec{1}_z \vec{1}_z \left[s - s_{\alpha_z}\right]^{-1}
\end{aligned}
$$

$$\overleftrightarrow{M}(\infty) = M_x^{(\infty)} \vec{1}_x \vec{1}_x + M_y^{(\infty)} \vec{1}_y \vec{1}_y + M_z^{(\infty)} \vec{1}_z \vec{1}_z$$
$$\overleftrightarrow{M}(0) = M_x^{(0)} \vec{1}_x \vec{1}_x + M_y^{(0)} \vec{1}_y \vec{1}_y + M_z^{(0)} \vec{1}_z \vec{1}_z \tag{4.10}$$

Now all the dot products of the modal vectors in (1.4) are unity for the same coordinate index, and zero for mixed coordinate indices.

Note that all we have assumed here is two perpendicular symmetry planes, not three. A special case of a target shape with such decomposition is a uniform isotropic rectangular parallelepiped, but more general shapes, including nonuniform and anisotropic media, are also possible.

If one has three orthogonal symmetry planes then the target also has what is called inversion symmetry I. Invension is simply represented by $-\overleftrightarrow{1}$ so that $\vec{r} \to -\vec{r}$. This operation is an improper rotation and can be constructed by three reflections (an odd number) using the dyadics in (4.2).

V. Single Rotation Axis

Now assume that there is an N-fold rotation axis which we take as the z axis by the usual convention. This defines the group

$$
\begin{aligned}
C_N &\equiv \left\{(C_N)_\ell \,\middle|\, \ell = 1, 2, \ldots, N\right\} \\
(C_N)_\ell &\equiv \text{rotation by } \phi_\ell = \frac{2\pi\ell}{N} = (C_N)_1^\ell \\
(C_N)_1^N &= (C_N)_N = (1)
\end{aligned}
\tag{5.1}
$$

For such rotations it is convenient to use the usual (Ψ, ϕ, z) cylindrical coordinate system with

$$
x = \Psi\cos(\phi)\,, \quad y = \Psi\sin(\phi) \tag{5.2}
$$

This has a 2×2 matrix representation

$$
\begin{aligned}
\left(C_{n,m}(\phi_\ell)\right) &= \begin{pmatrix} \cos(\phi_\ell) & -\sin(\phi_\ell) \\ \sin(\phi_\ell) & \cos(\phi_\ell) \end{pmatrix} = \exp\left(\phi_\ell \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right) \\
\left(C_{n,m}(0)\right) &= \left(C_{n,m}(2\pi)\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
\tag{5.3}
$$

as well as a scalar form with complex rotation $e^{j\phi_\ell}$. Note that under this transformation both Ψ and z remain unchanged, and the 2×2 matrices operate only on the (x,y) coordinates. If desired a 3×3 form can be written with a one in the (z,z) position and zeros in the four additional new positions.

Suppose there is an incident quasi-magnetostatic field parallel to the z axis. This excites some of the eigenmodes $\vec{j}_\beta(\vec{r},s)$, each giving a magnetic dipole moment $\overleftrightarrow{\tilde{m}}_\beta(s)$. Now rotate the coordinates $\phi \to \phi + \phi_\ell$ which is multiplication by $\left(C_{n,m}(\phi_1)\right)$. The z component of $\overleftrightarrow{\tilde{m}}_\beta(s)$ is unchanged, but the transverse (x,y) coordinates are changed provided $N \geq 2$. However, the target is unchanged by this transformation by hypothesis, so $\overleftrightarrow{\tilde{m}}_\beta(s)$ *cannot* change. Hence, in this case

$$
\overleftrightarrow{\tilde{m}}_\beta(s) = \tilde{m}_\beta(s)\,\vec{1}_z \tag{5.4}
$$

i.e., the transverse components must be zero. Similarly (by reciprocity) x and y directed incident magnetic fields produce no z-directed magnetic dipole moment. So the magnetic polarizability dyadic takes the same form as in (4.8). This separation of the z-axis modes from the transverse (x,y) modes merely requires C_N symmetry for $N \geq 2$. Note that no symmetry planes are required or implied by this symmetry. Note that for C_2 symmetry the adjunction of axial symmetry planes (R_x and R_y) gives C_{2a} which is isomorphic with (4.9) giving the form in (4.10).

This analysis can be carried further for the case that $N \geq 3$. As derived in [1, 10] for on-axis incidence there is no depolarization of the backscattered field in this case. It matters not whether one thinks of the electric field or magnetic field. However, reciprocity is important and is assumed in the present paper. Noting that the (z,z) part of the magnetic polarizability separates out (like in (4.8)), one now only needs to consider a uniform incident magnetic field perpendicular to the z axis which induces a dipole moment perpendicular to the z axis. Using (5.3) for rotations one first scatters (field to dipole moment) then rotates (by amount ϕ_ℓ), and second rotates then scatters. The physical requirement that these give the same result means that these two operations commute. This implies that if $\sin(\phi_\ell) \neq 0$ the induced magnetic dipole moment (transverse) is always parallel to the transverse incident magnetic field. The magnetic polarizability dyadic then takes the form

$$
\begin{aligned}
\tilde{\overleftrightarrow{M}}(s) &= \tilde{\overleftrightarrow{M}}(\infty) + \sum_{\alpha_z} M_{\alpha_z}^{(z)} \vec{1}_z \vec{1}_z \left[s - s_{\alpha_z}\right]^{-1} \\
&\quad + \sum_{\alpha_t} M_{\alpha_t}^{(t)} \overleftrightarrow{1}_z \left[s - s_{\alpha_t}\right]^{-1} \\
\overleftrightarrow{1}_z &\equiv \overleftrightarrow{1} - \vec{1}_z \vec{1}_z = \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y = \vec{1}_\Psi \vec{1}_\Psi + \vec{1}_\phi \vec{1}_\phi \\
&\equiv \text{transverse (to } \vec{1}_z\text{) identity} \\
\alpha_z &\equiv \text{index for modal unit vectors parallel to } \vec{1}_z \\
\alpha_t &\equiv \text{index for modal unit vectors perpendicular to } \vec{1}_z \text{ (transverse modes)} \\
\tilde{\overleftrightarrow{M}}(\infty) &= M_z^{(\infty)} \vec{1}_z \vec{1}_z + M_t^{(\infty)} \overleftrightarrow{1}_z \\
\tilde{\overleftrightarrow{M}}(0) &= M_z^{(0)} \vec{1}_i \vec{1}_i + M_t^{(0)} \overleftrightarrow{1}_z
\end{aligned}
\tag{5.5}
$$

Note that the transverse part has a common coefficient for both (x,x) and (y,y) components (with no (x,y) = (y,x) component). This is a case of modal degeneracy as discussed previously. There are two modes associated with each s_{α_t}. Together they span the two-dimensional (x,y) space. This result also applies to continuous rotation symmetry C_∞ and to adjunction of N axial and/or one transverse symmetry planes. However, all that is required is C_N for $N \geq 3$.

For completeness there is also rotation-reflection symmetry S_N given by

$$
\begin{aligned}
S_N &= \left\{ (S_N)_\ell \middle| \ell = 1, 2, \dots, N \right\} \\
(S_N)_1 &= (C_N)_1 (R_t) = (R_t)(C_N)_1 \\
(S_N)_\ell &= (S_N)_1^\ell = (C_N)_1^\ell (R_t)^\ell = (C_N)_\ell (R_t)^\ell
\end{aligned}
\tag{5.6}
$$

where the subscript t indicates a plane (say the $z = 0$ plane) transverse (perpendicular) to the rotation axis (the z axis). For N odd this is equivalent to C_{Nt} [8] with (R_t) and $(C_N)_1$ as independent symmetry elements.

The new interesting case has N even so that

$$(S_N)_N = (S_N)_1^N = (C_N)_N \; (R_t)^N = (R_t)^N = (1) \tag{5.7}$$

Then $C_{N'}$ is a subgroup with

$$\begin{aligned} N' &= \frac{N}{2} \\ (C_{N'})_1 &= (S_N)_1^2 = (C_N)_1^2 \end{aligned} \tag{5.8}$$

The target then has an N'-fold rotation axis. For S_2 the combination of $(C_1)_1$ and (R_t) are the same as inversion symmetry I. The operation (I) of inversion is represented by $-\overleftrightarrow{1}$ so that $\vec{r} \to -\vec{r}$. While this is equivalent to $(S_2)_1$ or three reflections in mutually perpendicular planes, it does not imply that the target has any symmetry plane or rotation axis. For S_2 the choice of rotation-reflection axis is arbitrary as long as it contains $\vec{r} = \vec{0}$. As such the previous decompositions of the magnetic-polarizability dyadic according to certain planes and/or axes do not apply.

Of greater interest is S_4. Since C_2 is a subgroup of the decomposition of the magnetic-polarizability dyadic in (4.8) is applicable. This implies that a transverse (x and y component) uniform incident magnetic field induces only a transverse magnetic dipole moment. Note that under operation of $(S_4)_1$ there is no z component of the magnetic field or magnetic dipole moment to be changed in any way. Transverse components remain transverse. Then as in the previous case of C_N for $N \geq 3$ [1, 10] let us first scatter (field to dipole moment) and then rotate (by amount $\pi/2$) and reflect, and second rotate and reflect and then scatter. Require that these operations commute (S_N being a commutative group) so that the results are the same regardless of order. This implies (since $\sin(\pi/2) = 1$) that the induced magnetic dipole moment (transverse) is always parallel to the transverse incident magnetic field (i.e. no depolarization). The magnetic polarizability dyadic then takes the form in (5.5). For S_N with N even and $N \geq 6$, then $C_{N'}$ with $N' \geq 3$ as a subgroup and the results in (5.5) also apply.

VI. Dihedral Symmetry

One can adjoin N 2-fold rotation axes at right angles to the principal N-fold axis to form dihedral symmetry D_N, a group with $2N$ elements. Noting that C_N is a subgroup, then the results of Section V apply.

Considering first D_2, note that C_2 separates out the z-coordinate part of the magnetic polarizability dyadic (like (4.8)). However, there are two more 2-fold axes (which can be taken as the x and y axes) which do the same. So the modes separate according to all three coordinate axes giving the same results as in (4.10). Note, however, that D_2 need not have symmetry planes; the axes are the 2-fold rotation axes. If one adjoins symmetry planes P_1 and P_2 containing the z axis then C_{2a} is a subgroup. These planes do not contain the secondary C_2 axes, i.e., $\vec{1}_x$ and $\vec{1}_y$ but are oriented symmetrically between these axes giving D_{2d} for the complete group [6]. Call the normals to these planes $\vec{1}_{P_1}$ and $\vec{1}_{P_2}$ with

$$\vec{1}_{P_1} = \frac{1}{\sqrt{2}}\left[\vec{1}_x + \vec{1}_y\right], \quad \vec{1}_{P_2} = \frac{1}{\sqrt{2}}\left[-\vec{1}_x + \vec{1}_y\right] \tag{6.1}$$

By our previous results concerning reflection symmetry, the magnetic polarizability tensor takes the form in (4.10) with separate terms for $\vec{1}_{P_1}$ and $\vec{1}_{P_2}$. However, the same must hold for the $\vec{1}_x$ and $\vec{1}_y$ decomposition of the transverse-to-z part. This gives the case of double degeneracy of the transverse part of the magnetic polarizability dyadic in (5.5) with

$$\overleftrightarrow{1}_z = \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y = \vec{1}_{P_1} \vec{1}_{P_1} + \vec{1}_{P_2} \vec{1}_{P_2} \tag{6.2}$$

One could also adjoin a transverse-to-z symmetry plane giving D_{2t} symmetry, but the normal to this plane, being $\vec{1}_z$ which already gives one of the axes, gives the form in (4.10) discussed under D_2 above.

Next considering D_N for $N \geq 3$, note that C_N is a subgroup, giving immediately the form in (5.5) with z and transverse (doubly degenerate) parts. Adjoining symmetry planes to give D_{Nd} and D_{Nt} does not extend this further.

VII. Orthogonal Symmetry in Three Dimensions

The full orthogonal group in three dimensions O_3 allows all rotations and reflections which keep r invariant, where we have the usual spherical coordinates (r, θ, ϕ) with

$$\Psi = r\sin(\theta)\ ,\ \ z = r\cos(\theta) \tag{7.1}$$

The only point to remain immobile under all the group operations is $\vec{r} = \vec{0}$. The other groups being here are all subgroups of O_3 with a finite number of elements, whereas the order of O_3 is infinite and includes infinitessimal (continuous) rotations.

From a physical point of view every axis through the origin is a rotation axis of every order. Considering C_N for $N \geq 3$ for this axis, the representation of the magnetic polarizability dyadic in (5.5) applies. However, this applies to all other axes besides the z-axis, so all axes must give the same results. The magnetic polarizability dyadic then takes the form

$$\begin{gathered}\tilde{\overleftrightarrow{M}}(s) = \left[\tilde{M}(s) + \sum_{\alpha} M_{\alpha}[s-s_{\alpha}]^{-1}\right]\overleftrightarrow{1} \\ \tilde{\overleftrightarrow{M}}(\infty) = \tilde{M}(\infty)\ \overleftrightarrow{1}\ ,\ \ \tilde{\overleftrightarrow{M}}(0) = \tilde{M}(0)\ \overleftrightarrow{1}\end{gathered} \tag{7.2}$$

where the modes are now triply degenerate. Note that no particular axes are preferred in this case and $\theta = 0$ is chosen arbitrarily.

This result is consistent with the well-known solution for the uniform and isotropic conducting permeable sphere of radius a [3]. However, the form in (7.2) applies more generally. Full O_3 symmetry allows the constitutive parameters as in (2.7) to take the form

$$\begin{aligned}\overleftrightarrow{\mu}(\vec{r}) &= \mu_{r,r}(r)\ \vec{1}_r\vec{1}_r + \mu_{t,t}(r)\ \overleftrightarrow{1}_r \\ \overleftrightarrow{\sigma}(\vec{r}) &= \sigma_{r,r}(r)\ \vec{1}_r\vec{1}_r + \sigma_{t,t}(r)\ \overleftrightarrow{1}_r \\ \overleftrightarrow{\varepsilon}(\vec{r}) &= \varepsilon_{r,r}(r)\ \vec{1}_r\vec{1}_r + \varepsilon_{t,t}(r)\ \overleftrightarrow{1}_r \\ \overleftrightarrow{1}_r &\equiv \overleftrightarrow{1} - \vec{1}_r\vec{1}_r \equiv \text{transverse (to } \vec{1}_r\text{) identity}\end{aligned} \tag{7.3}$$

Note that the radial and transverse parts are independent of θ and ϕ. For our present purposes (7.3) can be considered as defining a generalized sphere.

VIII. Regular Polyhedral Symmetry

Consider groups having more than one N-fold axis for $N \geq 3$. These are known to be the tetrahedral T, octahedral O, and icosahedral Y groups [8]. The tetrahedral group has four 3-fold axes. As C_3 is a subgroup, then consider one such axis as the z axis giving the form for the magnetic polarizability dyadic in (5.5). However, this is the result with respect to all four axes. Any three of these axial unit vectors are sufficient to span three dimensional space and the applicable form for the magnetic polarizability dyadic is the isotropic one in (7.2).

Octahedral symmetry has four 3-fold axes, three 4-fold axes, and four 2-fold axes. The cube and regular octahedron are examples of such symmetry. Icosahedral symmetry has six 5-fold axes, ten 3-fold axes, and fifteen 2-fold axes. The dodecahedron and icosahedron are examples of such symmetry. For both O and Y the rotation axes are also sufficient to enforce the isotropic form for the magnetic polarizability dyadic as in (7.2).

IX. Concluding Remarks

So now we have a progression of simplicity in the form of the magnetic polarizability dyadic as the symmetry is increased from no symmetry other than reciprocity up to O_3 for all rotations and reflections with respect to a common point $(\vec{r} = \vec{0})$. This is summarized in Table 9.1 which shows the progressive simplification in the sense of fewer and fewer independent terms associated with various axes. Due to reciprocity $\tilde{\overleftrightarrow{M}}(s)$ has at most six independent components. The symmetries move this first in the direction of frequency-independent diagonalization, and then reduction of the number of independent diagonal terms with associated increasing modal degeneracy.

Referring back to earlier sections, we have the forms that the SEM pole terms, as well as $\tilde{\overleftrightarrow{M}}(0)$ and $\tilde{\overleftrightarrow{M}}(\infty)$ take, these all being special cases of the terms in Table 9.1. Note that the various unit vectors for the modes have dot products (relative direction cosines) as in (1.4) which become 0 or 1 as the symmetry is increased making various of these vectors align parallel and perpendicular to the various directions given by the symmetry. In the case of modal degeneracy there is some choice available in assigning these modal directions, but they can be conveniently assigned as perpendicular and parallel to common coordinate axes.

In this paper the concern has been with the induced magnetic dipole moment, the lowest order magnetic multipole. If one considers magnetic quadrupoles, etc. the situation becomes more complicated. So the present results apply to situations in which one is not too close to the target in terms of target dimensions.

For various problems of interest it is the quasi-magnetostatic response (referred to the external medium) of a target that is of interest. For cases in which one is concerned with the quasi-electrostatic response it is the induced electric dipole moment and associated electric polarizability dyadic that is of interest. The decomposition of the electric polarizability dyadic according to the target point symmetries takes the same forms for the same symmetries as discussed here for the magnetic polarizability dyadic.

Form of $\overset{\leftrightarrow}{\tilde{M}}(s)$	Symmetry Types (Groups)
$\tilde{M}_z(s)\, \vec{1}_z \vec{1}_z + \overset{\leftrightarrow}{\tilde{M}}_t(s)$ $\left(\overset{\leftrightarrow}{\tilde{M}}_t(s) \cdot \vec{1}_z = \vec{0} \right)$	R_z (single symmetry plane) C_2 (2-fold rotation axis)
$\tilde{M}_z(s)\, \vec{1}_z \vec{1}_z + \tilde{M}_x(s)\, \vec{1}_x \vec{1}_x + \tilde{M}_y(s)\, \vec{1}_y \vec{1}_y$	$C_{2a} = R_x \otimes R_y$ (two axial symmetry planes) D_2 (three 2-fold rotation axes)
$\tilde{M}_z(s)\, \vec{1}_z \vec{1}_z + \tilde{M}_t(s)\, \overset{\leftrightarrow}{1}_z$ $\left(\overset{\leftrightarrow}{1}_z = \overset{\leftrightarrow}{1} - \vec{1}_z \vec{1}_z \Rightarrow \text{double degeneracy} \right)$	C_N for $N \geq 3$ (N-fold rotation axis) S_N for N even and $N \geq 4$ $(N-\text{fold rotation}-\text{reflection axis})$ D_{2d} (three 2-fold rotation axes plus diagonal symmetry planes)
$\tilde{M}(s)\, \overset{\leftrightarrow}{1}$ $(\overset{\leftrightarrow}{1} \Rightarrow$ triple degeneracy)	O_3 (generalized sphere) T, O, Y (regular polyhedra)

Table 9.1 Decomposition of Magnetic Polarizability Dyadic According to Target Point Symmetries

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1. C. E. Baum, SEM Backscattering, Interaction Note 476, July 1989.

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4. C. E. Baum, Signature-Based Target Identification, Interaction Note 500, January 1994.

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6. C. E. Baum, and H. N. Kritikos, Symmetry in Electromagnetics, Physics Note 2, December 1990.