Interaction Notes

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Low-Frequency Near-Field Magnetic Scattering from Highly, But Not Perfectly, Conducting Bodies

Carl E. Baum

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Abstract

This paper considers the properties of the quasi-magnetostatic scattering from permeable and highly conducting scatterers in terms of the magnetic-polarizability dyadic. This is treated in terms of the singularity expansion method (SEM). The natural frequencies are negative real and serve as identifiers of target type. The symmetric dyads which characterize the residues of the first order poles are comprised of real vectors which give the target aspect (orientation in space).

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Contents

I. Introduction

In geophysical exploration for large underground ore deposits low-frequency magnetic fields, including pulses with large low-frequency components, are often used [42, 44, 49]. Such techniques depend on the interaction of the magnetic field with the permeability μ_2 and/or high conductivity σ_2 of the target. Devices for detection of such targets are often referred to as "metal detectors." They can also be used for smaller targets, say with dimensions less than a meter, provided the distance of the target from the detector (typically depth of burial) is not too large compared to the target dimensions.

There are various configurations of loops near (above) the surface of the ground which are suitable for this purpose. Figure 1.1 gives an example of such a device, consisting of four loops just above the ground surface with magnetic-dipole moments $\vec{m}_n^{(s)}$ oriented parallel to the z axis. The superscript "s" indicates source parameters. With the four sources located at

$$\left(x^{(s)}, y^{(s)}, z^{(s)}\right) = \left(\pm\ell, \pm\ell, 0_+\right) \tag{1.1}$$

then various symmetry properties can be exploited. Let the four source magnetic moments be characterized by

$$\vec{m}_n^{(s)} = \pm m_0 \vec{1}_z \tag{1.2}$$

i.e., all of the same magnitude but with various sign combinations. Considering the orientation of the magnetic field on the z axis for negative z we have in symbolic form

$$\left(\vec{m}_1^{(s)}, \vec{m}_2^{(s)}, \vec{m}_3^{(s)}, \vec{m}_4^{(s)}\right) = \begin{cases} (+,+,+,+) m_0 \vec{1}_z \Rightarrow \vec{H}_{inc} = H_{inc} \vec{1}_z \\ (+,-,-,+) m_0 \vec{1}_z \Rightarrow \vec{H}_{inc} = H_{inc} \vec{1}_x \\ (+,+,-,-) m_0 \vec{1}_z \Rightarrow \vec{H}_{inc} = H_{inc} \vec{1}_y \\ (+,-,+,-) m_0 \vec{1}_z \Rightarrow \vec{H}_{inc} = \vec{0} \end{cases} \tag{1.3}$$

with the first three cases giving three orthogonal axes of incident magnetic field $\vec{H}_{inc}$ on the symmetry axis. There are other symmetric magnetic-dipole arrays (e.g., three loops spaced 120° apart) which can also be used to give three-axis illumination.

Let the buried target be located (centered) at $\vec{r} = \vec{r}_0$. For $\vec{r}_0$ near the negative z axis the illumination conditions in (1.3) are appropriate. Of course, one first has to locate the target (find $\vec{r}_0$). This can be done by this or other loop arrays, basically by measuring the magnetic-dipole fields scattered

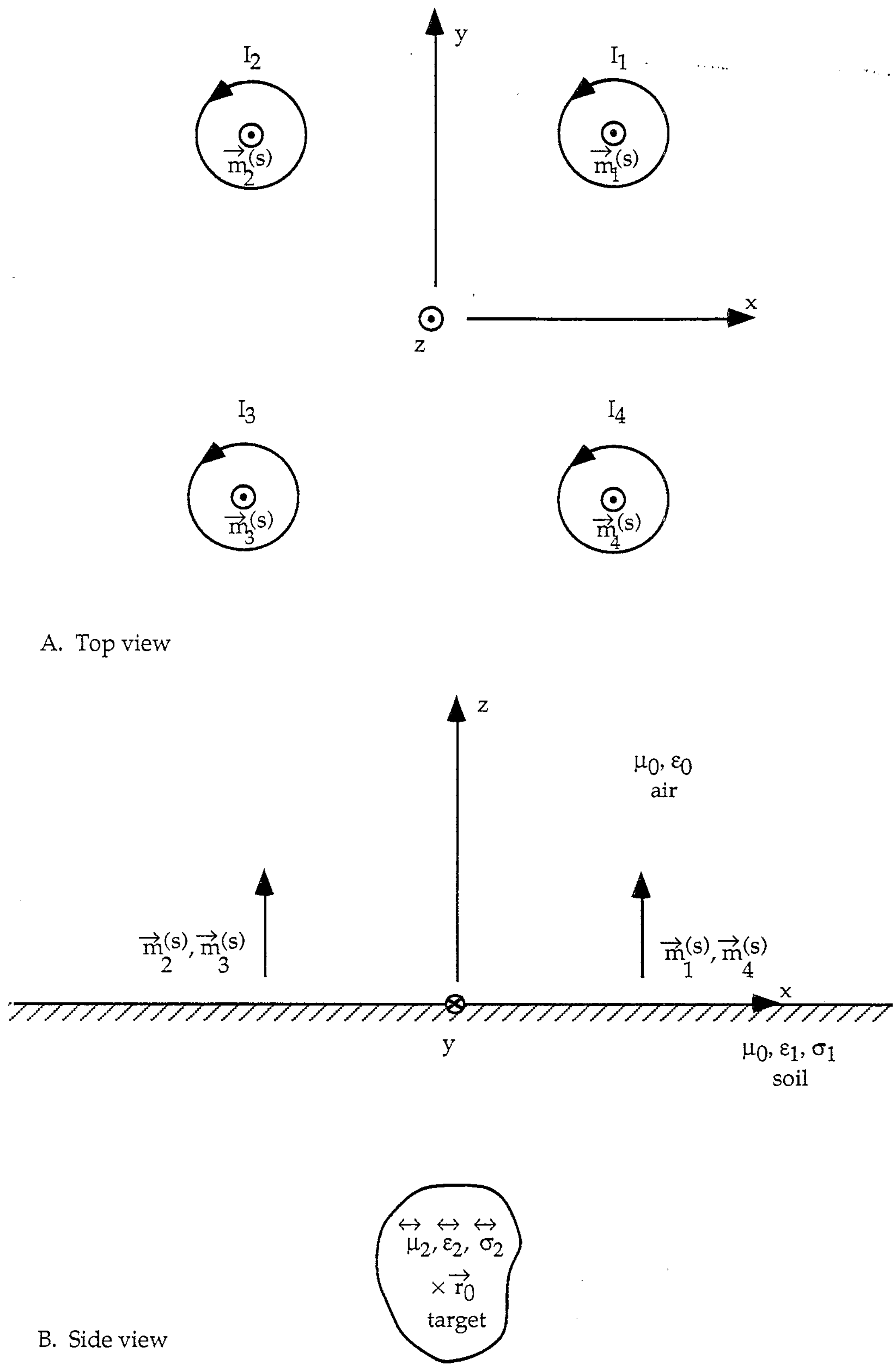

Fig. 1.1. Buried Target Illuminated by Array of Magnetic Dipoles (Loops) Above Ground Surface

by the target at a sufficient number of locations at the ground surface. Then the source array, such as in fig. 1.1, can be moved over the target to achieve the desired illumination conditions. By measurement of the induced magnetic dipole moments under the orthogonal illumination conditions, one can then infer the magnetic polarizability dyadic of the target. Note that one can measure the horizontal components of the magnetic field at the ground surface, and that these components will not include the incident field from the source array provided the soil permeability is μ_0 (the same as air) and the frequency is low enough that the radian wavelength (skin depth) in the soil is large compared to distances of interest in the near field of the source array.

II. Approximations Appropriate to Low-Frequency Near-Field Magnetic Scattering

A previous paper [12] has considered the propagation of electromagnetic fields (approximated as a plane wave) down to a buried target and scattering from the target to another location. Let the soil be considered for the moment as a uniform isotropic medium with constitutive parameters

$$\begin{aligned} \mu_0 &\equiv \text{permeability (free space)} \\ \varepsilon_1 &\equiv \text{permittivity} \\ \sigma_1 &\equiv \text{conductivity} \end{aligned} \tag{2.1}$$

This gives

$$\begin{aligned} \tilde{\gamma}_1(s) &= \left[s\mu_0(\sigma_1 + s\varepsilon_1)\right]^{\frac{1}{2}} \equiv \text{ propagation constant} \\ \tilde{Z}_1(s) &= \left[\frac{s\mu_0}{\sigma_1 + s\varepsilon_1}\right]^{\frac{1}{2}} \equiv \text{ wave impedance} \\ s &\equiv \Omega + j\omega \equiv \text{ complex frequency (two - sided - Laplace - transform variable)} \\ \sim &\equiv \text{ Laplace transform (two - sided)} \end{aligned} \tag{2.2}$$

For present purposes the constitutive parameters are taken as frequency independent which will not be a significant restriction under the conditions discussed later.

For the incident field consider a plane wave as

$$\begin{aligned} \tilde{\vec{E}}^{(inc)}(\vec{r},s) &= \tilde{Z}_1(s)H_0\tilde{f}(s)\vec{1}_p e^{-\tilde{\gamma}_1\vec{1}\cdot[\vec{r}-\vec{r}_0]} \\ \tilde{\vec{H}}^{(inc)}(\vec{r},s) &= H_0\tilde{f}(s)\vec{1}_1\times\vec{1}_p e^{-\tilde{\gamma}_1\vec{1}\cdot[\vec{r}-\vec{r}_0]} \\ \vec{1}_1 &\equiv \text{ direction of incidence} \\ \vec{1}_p &\equiv \text{ electric polarization} \\ \vec{1}_1\times\vec{1}_p &\equiv \text{ magnetic polarization} \\ f(t) &\equiv \text{ incident waveform} \end{aligned} \tag{2.3}$$

Note the inclusion of $\vec{r}_0$, the target location, to effectively shift the coordinate center and normalize the field to arrival at this position. The scaling constant H_0 is chosen for the incident magnetic field at $\vec{r}_0$ which will be important in our considerations.

Defining

$$a \equiv \text{characteristic dimension (size) of the target}$$
$$r \equiv \left| \vec{r} - \vec{r}_0 \right| \tag{2.4}$$

then for sufficiently low frequencies that

$$\left| \tilde{\gamma}_1(s) \right| a << 1 \tag{2.5}$$

(i.e., the target is electrically small in the external medium) and sufficiently large distances that

$$r >> a \tag{2.6}$$

the scattered field is dominated by the induced electric and magnetic dipole moments. Under these conditions, as discussed in [12] the far scattered field is given by

$$\left| \tilde{\gamma}_1(s) \right| f >> 1$$
$$\tilde{\vec{E}}^{(sc)}(\vec{r}, s) \simeq \tilde{\vec{E}}_f(\vec{r}, s) = \frac{e^{-\tilde{\gamma}_1(s) r}}{4\pi r} \mu_0 s^2 \overleftrightarrow{1}_r \cdot \left[-\tilde{\vec{p}}(s) + \frac{1}{c} \vec{1}_r \times \tilde{\vec{m}}(s) \right]$$
$$\vec{1}_r \equiv \frac{\vec{r} - \vec{r}_0}{r} \tag{2.7}$$
$$\overleftrightarrow{1}_r \equiv \overleftrightarrow{1} - \vec{1}_r \vec{1}_r \text{ (dyadic transverse to direction of scattering)}$$
$$\overleftrightarrow{1} \equiv \vec{1}_x \vec{1}_x + \vec{1}_y \vec{1}_y + \vec{1}_z \vec{1}_z \text{ (identity dyadic)}$$

where the dipole moments are related to the incident fields by polarizability dyadics as

$$\tilde{\vec{p}}(s) = \varepsilon_1 \overleftrightarrow{\tilde{P}}(s) \cdot \tilde{\vec{E}}^{(inc)}(\vec{r}_0, s) = \varepsilon_1 \tilde{Z}_1(s) H_0 \tilde{f}(s) \overleftrightarrow{\tilde{P}}(s) \cdot \vec{1}_p$$
$$\tilde{\vec{m}}(s) = \overleftrightarrow{\tilde{M}}(s) \cdot \tilde{\vec{H}}^{(inc)}(\vec{r}_0, s) = H_0 \tilde{f}(s) \overleftrightarrow{\tilde{M}}(s) \cdot \left[\vec{1}_1 \times \vec{1}_p \right] \tag{2.8}$$

These dyadics are in general frequency dependent as will be discussed later, even for low frequencies restricted as in (2.5).

Assume that all the media constitutive parameters are reciprocal. In the case that any of these are dyadic this means that they are symmetric, i.e., in the case of the target

$$\overleftrightarrow{\mu}_2(\vec{r}) = \overleftrightarrow{\mu}_2^T(\vec{r}) \ , \ \overleftrightarrow{\varepsilon}_2(\vec{r}) = \overleftrightarrow{\varepsilon}_2^T(\vec{r}) \ , \ \overleftrightarrow{\sigma}_2(\vec{r}) = \overleftrightarrow{\sigma}_2^T(\vec{r}) \tag{2.9}$$

Then the low-frequency far-field scattering is given by

$$\tilde{\overleftrightarrow{E}}_f(\vec{r},s) = \frac{e^{-\tilde{\gamma}_1(s)r}}{4\pi r}\tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_r,\vec{1}_1;s\right)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_0,s)$$

$$\tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_r,\vec{1}_1;s\right) = \tilde{\gamma}_1^2(s)\left[-\overleftrightarrow{1}_r\cdot\tilde{\overleftrightarrow{P}}(s)\cdot\overleftrightarrow{1}_1 + \vec{1}_r\times\tilde{\overleftrightarrow{M}}(s)\times\vec{1}_r\right] \tag{2.10}$$

$$= \tilde{\overleftrightarrow{\Lambda}}\left(-\vec{1}_1,-\vec{1}_r;s\right) \qquad \text{(reciprocity)}$$

$$\overleftrightarrow{1}_1 \equiv \overleftrightarrow{1} - \vec{1}_1\vec{1}_1 \qquad \text{(dyadic transverse to direction of incidence)}$$

In backscattering we have

$$\vec{1}_r = \vec{1}_1$$

$$\tilde{\overleftrightarrow{\Lambda}}_b\left(\vec{1}_1,s\right) = \overleftrightarrow{\Lambda}\left(-\vec{1}_1,\vec{1}_1;s\right) = \overleftrightarrow{\Lambda}_b^T\left(\vec{1}_1,s\right) \tag{2.11}$$

From this and varying $\vec{1}_1$ over 4π steradians we find that the polarizability dyadics are also symmetric due to reciprocity, i.e.,

$$\tilde{\overleftrightarrow{P}}(s) = \tilde{\overleftrightarrow{P}}^T(s) \quad , \quad \tilde{\overleftrightarrow{M}}(s) = \tilde{\overleftrightarrow{M}}^T(s) \tag{2.12}$$

Consider now the case that the near field for which

$$\left|\tilde{\gamma}_1(s)\right| r << 1 \tag{2.13}$$

is of interest, but still within the restriction of (2.6). Then we have the more general formulas for the dipole scattered fields for $s \to 0$ as [1]

$$\tilde{\vec{E}}^{(sc)}(\vec{r},s) = \frac{1}{4\pi\varepsilon_1 r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right]\cdot\tilde{\vec{p}}(s) + \frac{\mu_0 s}{4\pi r^2}\vec{1}_r\times\tilde{\vec{m}}(s)$$

$$= \frac{1}{4\pi r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right]\cdot\tilde{\overleftrightarrow{P}}(s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_0,s) + \frac{\mu_0 s}{4\pi r^2}\vec{1}_r\times\tilde{\overleftrightarrow{M}}(s)\cdot\tilde{\vec{H}}^{(inc)}(\vec{r}_0,s)$$

$$\tilde{\vec{H}}^{(sc)}(\vec{r},s) = -\frac{s}{4\pi r^2}\vec{1}_r\times\tilde{\vec{p}}(s) + \frac{1}{4\pi r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right]\cdot\tilde{\vec{m}}(s) \tag{2.14}$$

$$= -\frac{\varepsilon_1 s}{4\pi r^2}\vec{1}_r\times\tilde{\overleftrightarrow{P}}(s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_0,s) + \frac{1}{4\pi r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right]\cdot\tilde{\overleftrightarrow{M}}(s)\cdot\tilde{\vec{H}}^{(inc)}(\vec{r}_0,s)$$

where only leading terms associated with the two incident fields have been retained.

The foregoing scattered near fields are for a general combination of incident fields, including a plane wave as in (2.3). Suppose, however, that one emphasizes the incident magnetic field, such as by being in the near field of an illuminator as in fig. 1.1. In such a case the incident electric field is proportional to $s\tilde{\vec{H}}^{(inc)}$, and the scattered magnetic field in (2.10) has terms proportional to $\tilde{\overleftrightarrow{M}} \cdot \tilde{\vec{H}}^{(inc)}$ and $s\tilde{\overleftrightarrow{P}} \cdot \tilde{\vec{E}}^{(inc)}$, the latter term being negligible compared to the former provided $s^2\tilde{\overleftrightarrow{P}}$ can be neglected compared to $\tilde{\overleftrightarrow{M}}$ for low frequencies of interest. So provided one measures the scattered magnetic field (instead of the electric field), then the target is characterized by the magnetic-polarizability dyadic which enters as

$$\tilde{\vec{H}}^{(sc)}(\vec{r},s) = \frac{1}{4\pi r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \cdot \tilde{\overleftrightarrow{M}}(s) \cdot \tilde{\vec{H}}^{(inc)}(\vec{r}_0,s) \quad \text{as } s \to 0 \tag{2.15}$$

One reason to consider a low-frequency magnetic scattering is its insensitivity to ε_1 and σ_1 of the surrounding soil, even if this permittivity and/or conductivity are inhomogeneous and/or anisotropic (i.e., $\overleftrightarrow{\varepsilon}_1(\vec{r})$ and $\overleftrightarrow{\sigma}_1(\vec{r})$). The restriction (2.13) assures that the local magnetic field is not perturbed by such variations provided that the conductivity especially is not locally too large. Furthermore, if the soil permeability is μ_0 (free space) to a good approximation, then all components of the magnetic field are continuous through the earth/air interface and the interface has no significant influence on the magnetic fields (incident and scattered).

Within the foregoing restrictions we have defined a quasi-magnetostatic problem in terms of near fields for both incident and scattered fields, with the scattered field dominated by the magnetic polarizability dyadic via (2.15). Furthermore, the dielectric and conducting properties of the surrounding media (soil and air) being insignificant, one can analyze the problem as though the target were in free space, i.e., a uniform isotropic medium of permeability μ_0. For what follows we can now drop the subscript 2 for the target constitutive parameters.

III. Magnetic-Polarizability Dyadic at Zero Frequency

With our target comprised of reciprocal media and embedded in a medium of permeability μ_0 (effectively free space), we then need to consider the properties of $\tilde{\overleftrightarrow{M}}(s)$. At zero frequency we have

$$\begin{aligned}\tilde{\overleftrightarrow{M}}(0) &= \tilde{\overleftrightarrow{M}}^T(0) = \text{ real dyadic} \\ &= \sum_{\upsilon=1}^{3} M_\upsilon^{(0)} \vec{M}_\upsilon^{(0)} \vec{M}_\upsilon^{(0)}\end{aligned} \tag{3.1}$$

Since a real symmetric dyadic is always diagonalizable with real eigenvalues and a complete set (here 3) of eigenvectors [10], we have

$$\begin{aligned} M_\upsilon^{(0)} &\equiv \text{ real eigenvalues (non negative) (not necessarily distinct)} \\ \vec{M}_\upsilon^{(0)} &\equiv \text{ real eigenvectors (three)} \\ \vec{M}_{\upsilon_1}^{(0)} \cdot \vec{M}_{\upsilon_2}^{(0)} &= 1_{\upsilon_1,\upsilon_2} \text{ (orthonormal)} \end{aligned} \tag{3.2}$$

The real eigenvectors give real directions in 3-dimensional space and, as such, can be regarded as principal axes. There are exceptions in that if the eigenvalues are degenerate (as in the case of a sphere) the diagonalization is not unique, i.e., linear combinations of eigenvectors corresponding to degenerate (equal) eigenvalues are also eigenvectors.

A special case has zero DC magnetic polarizability, i.e.,

$$\begin{aligned} \tilde{\overleftrightarrow{M}}(0) &= \overleftrightarrow{0} \\ M_\upsilon^{(0)} &= 0 \text{ for } \upsilon = 1,2,3 \end{aligned} \tag{3.3}$$

To understand this case let us consider the constitutive parameters of the target

$$\begin{aligned} \overleftrightarrow{\mu}(\vec{r}) &\equiv \text{ permeability} \\ \overleftrightarrow{\varepsilon}(\vec{r}) &\equiv \text{ permittivity} \\ \overleftrightarrow{\sigma}(\vec{r}) &\equiv \text{ conductivity} \end{aligned} \tag{3.4}$$

For the low-frequencies of interest (as discussed in section II) let us neglect $s\overleftrightarrow{\varepsilon}$ compared to $\overleftrightarrow{\sigma}$, i.e., let conduction current density dominate he displacement current density in the target. Furthermore, let $\overleftrightarrow{\sigma}(\vec{r})$ be bounded everywhere in the target. Then at zero frequency (DC) an incident static magnetic field will not be affected by the conductivity. Only the permeability $\overleftrightarrow{\mu}(\overrightarrow{r})$ can interact with the static incident field to produce a scattered magnetic field and thereby give a zero-frequency magnetic polarizability.

If

$$\overleftrightarrow{\mu}(\overrightarrow{r}) = \mu_0 \overleftrightarrow{1} \tag{3.5}$$

so that the target has the same permeability as the background medium, then the static incident magnetic field is not perturbed by the target. There is no scattering and (3.5) is then a sufficient condition for (3.3). Stated another way, a non-magnetic target has no DC magnetic polarizability. This property is a target discriminant. The presence of a non-zero magnetic-polarizability dyadic implies a target which includes magnetic materials (e.g., iron).

If one had the case that the target conductivity were infinite, then the magnetic field would not penetrate the target. This also leads to a magnetic-polarizability dyadic of the form in (3.1), except that the eigenvalues are now negative. The static polarizability tensors (electric and magnetic) for perfectly conducting bodies are tabulated for various shapes in [45]. For the case of permeable bodies with eigenvalues of $\overleftrightarrow{\mu}$ greater than μ_0, the concentration of the magnetic field by the target gives positive eigenvalues to the DC magnetic polarizability.

IV. SEM Representation of the Magnetic-Polarizability Dyadic

Now consider the target as a scatterer located in free space, i.e., the soil is replaced by air. For the frequencies and fields of interest as discussed in Section II this is a reasonable approximation. Let us then expand the response in the form of the singularity expansion method (SEM) in terms of the singularities in the complex s plane. Early considerations [4, 5, 40] concerned the currents induced on objects, but more recent investigations have concentrated on scattered far fields [8, 9, 32, 50].

Consider plane-wave incidence with $\tilde{\gamma}_1(s)$ and $\tilde{Z}_1(s)$ in (2.3) replaced by

$$\tilde{\gamma} = \frac{s}{c} \quad , \quad Z_0 = \left[\frac{\mu_0}{\varepsilon_0}\right]^{\frac{1}{2}}$$
$$c = \left[\mu_0 \varepsilon_0\right]^{-\frac{1}{2}} \tag{4.1}$$

Current density or surface-current density on the object can then be written in the form

$$\begin{aligned}
\tilde{\vec{J}}(\vec{r}, s) = Z_0 H_0 \sum_{\alpha, n_\alpha} \tilde{f}(s_\alpha) \eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) \vec{j}_\alpha(\vec{r}) [s - s_\alpha]^{-n_\alpha} \\
&+ \text{possible entire function} \\
&+ \text{singularities of } \tilde{f}(s)
\end{aligned}$$
$$\begin{aligned}
s_\alpha &\equiv \text{natural frequencies or pole locations (aspect and position independent)} \\
\vec{j}_\alpha(\vec{r}) &\equiv \text{natural modes (aspect independent)} \\
\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) &\equiv \text{coupling coefficient (class 1 form) (contains aspect information)} \\
n_\alpha &\equiv \text{pole order (positive integer)}
\end{aligned} \tag{4.2}$$

While one typically encounters targets with only first-order poles (from various examples of perfectly conducting targets) there have been discovered special cases of lossy scatterers with second order poles and higher orders are in principle possible [2, 7, 41].

Here we have assumed that the target has finite linear dimensions so that there are no branch contributions in the delta-function response. For perfectly conducting objects this is established in [4] using a Moment-Method approximation of an integral equation and in [5] directly from the H-field integral equation. For imperfectly conducting targets, such as in fig. 1.1, this is established in [6], again from an integral-equation representation of the scattering. This is established under the sufficient conditions

$$\tilde{\mu}(\vec{r},s)\,,\,\tilde{\sigma}(\vec{r},s)\,,\,\tilde{\varepsilon}(\vec{r},s)\,,\,\frac{\nabla\tilde{\mu}(\vec{r},s)}{\tilde{\mu}(\vec{r},s)}\,,\,\frac{\nabla[\tilde{\sigma}(\vec{r},s)+s\tilde{\varepsilon}(\vec{r},s)]}{\tilde{\sigma}(\vec{r},s)+s\tilde{\varepsilon}(\vec{r},s)} \tag{4.3}$$
$$\equiv \text{entire functions of } s \text{ (analytic in entire } s \text{ plane)}$$

For present purposes we have assumed that the constitutive parameters are frequency independent, clearly meeting the above conditions. However, the target of interest may have dyadic constitutive parameters. This should not introduce an essential difficulty since one can view a dyadic conductivity, for example, as a macroscopic description of what is only an inhomogeneous medium on a small scale with say sheets or wires of conducting material of various conductivities going in various directions, and similarly for permeability and permittivity.

It is instructive and will help in later sections to take an appropriate set of integral equations and look at their properties from a discretized matrix point of view, the approach taken in [4]. The Maxwell equations in the target are

$$\begin{aligned}
\nabla\times\tilde{\vec{E}}(\vec{r},s) &= -s\,\tilde{\vec{B}}(\vec{r},s) = -s\overleftrightarrow{\mu}(\vec{r})\cdot\tilde{\vec{H}}(\vec{r},s) \\
\nabla\times\tilde{\vec{H}}(\vec{r},s) &= \overleftrightarrow{\sigma}(\vec{r})\cdot\tilde{\vec{E}}(\vec{r},s)+s\,\tilde{\vec{D}}(\vec{r},s) = \left[\overleftrightarrow{\sigma}(\vec{r})+s\overleftrightarrow{\varepsilon}(\vec{r})\right]\cdot\tilde{\vec{E}}(\vec{r},s)
\end{aligned} \tag{4.4}$$

The fields have incident and scattered parts as

$$\begin{aligned}
\tilde{\vec{E}}(\vec{r},s) &= \tilde{\vec{E}}^{(inc)}(\vec{r},s)+\tilde{\vec{E}}^{(sc)}(\vec{r},s) \\
\tilde{\vec{H}}(\vec{r},s) &= \tilde{\vec{H}}^{(inc)}(\vec{r},s)+\tilde{\vec{H}}^{(sc)}(\vec{r},s)
\end{aligned} \tag{4.5}$$

where the incident field is a free-space plane wave as in (2.3) and (4.1). Then define electric and magnetic current densities by

$$\begin{aligned}
\tilde{\vec{J}}_e(\vec{r},s) &= \left[\overleftrightarrow{\sigma}(\vec{r})+s\left[\overleftrightarrow{\varepsilon}(\vec{r})-\varepsilon_0\overleftrightarrow{1}\right]\right]\cdot\tilde{\vec{E}}(\vec{r},s) \\
\tilde{\vec{J}}_h(\vec{r},s) &= s\left[\overleftrightarrow{\mu}(\vec{r})-\mu_0\overleftrightarrow{1}\right]\cdot\tilde{\vec{H}}(\vec{r},s)
\end{aligned} \tag{4.6}$$

These include both electric and magnetic polarization current densities, i.e., everything not included in ε_0 and μ_0, the free-space parameters. From the divergence of (4.4) we have the auxiliary relationships (continuity equations)

$$\nabla \cdot \tilde{\vec{B}}(\vec{r},s) = 0 = \nabla \cdot \tilde{\vec{J}}_h(\vec{r},s) + s\mu_0 \nabla \cdot \tilde{\vec{H}}(\vec{r},s)$$

$$\nabla \cdot \left[\left[\vec{\vec{\sigma}}(\vec{r}) + s\vec{\vec{\varepsilon}}(\vec{r}) \right] \cdot \tilde{\vec{E}}(\vec{r},s) \right] = 0 = \nabla \cdot \tilde{\vec{J}}_e(\vec{r},s) + s\varepsilon_0 \nabla \cdot \tilde{\vec{E}}(\vec{r},s) \tag{4.7}$$

$$\nabla \cdot \tilde{\vec{J}}_h(\vec{r},s) = -s\mu_0 \nabla \cdot \tilde{\vec{H}}(\vec{r},s) \equiv -s\tilde{\rho}_h(\vec{r},s)$$

$$\nabla \cdot \tilde{\vec{J}}_e(\vec{r},s) = -s\varepsilon_0 \nabla \cdot \tilde{\vec{E}}(\vec{r},s) \equiv -s\tilde{\rho}_e(\vec{r},s)$$

One can integrate over these current densities using a scalar free-space Green's function to form electric and magnetic vector potentials and, similarly, for the scalar potentials. This gives a pair of coupled integral equations over the spatial domain V of the target. This is discussed in [6, 28, 31, 33, 38, 47] where divergence and gradient operators appear inside the volume integrals. Surface integrals also appear at surfaces where the material parameters are discontinuous, such as at the surfaces of metallic conductors. It we regard the constitutive parameters as differentiable functions of position, this merely makes one ascribe some small distance to vary the parameters between the desired values. The scattered fields are related to the potentials via

$$\tilde{\vec{E}}^{(sc)}(\vec{r},s) = -\nabla \tilde{\Phi}_e^{(sc)}(\vec{r},s) - s\tilde{\vec{A}}_e^{(sc)}(\vec{r},s) - \frac{1}{\varepsilon_0} \nabla \times \tilde{\vec{A}}_h^{(sc)}(\vec{r},s)$$

$$\tilde{\vec{H}}^{(sc)}(\vec{r},s) = -\nabla \tilde{\Phi}_m^{(sc)}(\vec{r},s) - s\tilde{\vec{A}}_h^{(sc)}(\vec{r},s) + \frac{1}{\mu_0} \nabla \times \tilde{\vec{A}}_e^{(sc)}(\vec{r},s) \tag{4.8}$$

and the potentials are related to the current densities via

$$\tilde{\Phi}_e^{(sc)}(\vec{r},s) = -\frac{1}{s\varepsilon_0} \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';s), \nabla' \cdot \tilde{\vec{J}}_e(\vec{r}\,',s) \right\rangle$$

$$\tilde{\Phi}_h^{(sc)}(\vec{r},s) = -\frac{1}{s\mu_0} \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';s), \nabla' \cdot \tilde{\vec{J}}_h(\vec{r}\,',s) \right\rangle$$

$$\tilde{A}_e^{(sc)}(\vec{r},s) = \mu_0 \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';s), \tilde{\vec{J}}_e(\vec{r}\,',s) \right\rangle \tag{4.9}$$

$$\tilde{A}_h^{(sc)}(\vec{r},s) = \varepsilon_0 \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';s), \tilde{\vec{J}}_h(\vec{r}\,',s) \right\rangle$$

with integration over the common coordinates ($\vec{r}\,'$ here) over the volume V of the target denoted by the symmetric product. This leads to the integral equations

$$
\begin{aligned}
\tilde{\vec{E}}^{(sc)}(\vec{r},s) &= \frac{1}{s\varepsilon_0}\left\langle \nabla\tilde{G}_0(\vec{r},\vec{r}\,';s), \nabla'\cdot\tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle \\
&\quad -s\mu_0\left\langle \tilde{G}_0(\vec{r},\vec{r}\,';s), \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle - \left\langle \nabla\tilde{G}_0(\vec{r},\vec{r}\,';s) \mathop{\times}_{;} \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
\tilde{\vec{H}}^{(sc)}(\vec{r},s) &= \frac{1}{s\mu_0}\left\langle \nabla\tilde{G}_0(\vec{r},\vec{r}\,';s), \nabla'\cdot\tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle \\
&\quad -s\varepsilon_0\left\langle \tilde{G}_0(\vec{r},\vec{r}\,';s), \tilde{\vec{J}}_h(\vec{r}\,',s)\right\rangle + \left\langle \nabla\tilde{G}_0(\vec{r},\vec{r}\,';s) \mathop{\times}_{;} \tilde{\vec{J}}_e(\vec{r}\,',s)\right\rangle
\end{aligned}
\tag{4.10}
$$

where the current densities are given in terms of the fields (incident plus scattered) in (4.6).

Various forms of the free-space Green function are given by [7]

$$
\begin{aligned}
\tilde{G}_0(\vec{r},\vec{r}\,';s) &= \frac{e^{-\zeta}}{4\pi R} \\
\nabla\tilde{G}_0(\vec{r},\vec{r}\,';s) &= -\nabla'\tilde{G}_0(\vec{r},\vec{r}\,';s) = \vec{1}_R\frac{\gamma^2}{4\pi}\left[-\zeta^{-2}-\zeta^{-1}\right]e^{-\zeta} \\
\tilde{\overleftrightarrow{G}}_0(\vec{r},\vec{r}\,';s) &= \left[\overleftrightarrow{1}-\tilde{\gamma}^{-2}\nabla\nabla\right]\tilde{G}_0(\vec{r},\vec{r}\,';s) \\
&= \frac{\tilde{\gamma}}{4\pi}\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\vec{1}_R\vec{1}_R \right. \\
&\qquad \left. +\left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\left[\overleftrightarrow{1}-\vec{1}_R\vec{1}_R\right]\right\} \\
&\qquad +\frac{1}{3\gamma^2}\delta(\vec{r}-\vec{r}\,')\overleftrightarrow{1} \\
R &= \left|\vec{r}-\vec{r}\,'\right| \quad , \quad \zeta = \gamma R \\
\vec{1}_R &= \frac{\vec{r}-\vec{r}\,'}{R}
\end{aligned}
\tag{4.11}
$$

The first two forms are used in (4.10). The third is the dyadic Green function which can be used to give an alternate form of the integral equations involving integrals over the current densities. In this case the singularity at $\vec{r} = \vec{r}\,'$ needs to be specially handled. The form above uses a spherical volume for the principal value integral [19]; other more general forms are also available [27]. For present purposes let us use (4.10) for which the singularities are integrable.

Now rearrange (4.10) into the form

$$\tilde{\vec{E}}^{(sc)}(\vec{r},s) - X_E\left(\tilde{\vec{E}}^{(sc)}(\vec{r},s), \tilde{\vec{H}}^{(sc)}(\vec{r},s)\right) = X_E\left(\tilde{\vec{E}}^{(inc)}(\vec{r},s), \tilde{\vec{H}}^{(inc)}(\vec{r},s)\right)$$
$$\tilde{\vec{H}}^{(sc)}(\vec{r},s) - X_H\left(\tilde{\vec{E}}^{(sc)}(\vec{r},s), \tilde{\vec{H}}^{(sc)}(\vec{r},s)\right) = X_H\left(\tilde{\vec{E}}^{(inc)}(\vec{r},s), \tilde{\vec{H}}^{(inc)}(\vec{r},s)\right) \tag{4.12}$$

where X_E and X_H are linear integro-differential operators from (4.10) with substitutions from (4.5) through (4.7). Now consider (4.12) from a matrix point of view. Let us subdivide the target into a large number N of zones [28, 31]. Then in each zone there are 3 components each of electric and magnetic fields giving 6N field variables to consider. Write this vector as $\left(\tilde{F}_n(s)\right)$ where n = 1, 2, ..., 6N. Then (4.12) takes the approximate moment-method form

$$\left[\left(1_{n,m}\right) - \left(\tilde{X}_{n,m}(s)\right)\right] \cdot \left(\tilde{F}_n^{(sc)}(s)\right) = \left(\tilde{X}_{n,m}(s)\right) \cdot \left(\tilde{F}_n^{(inc)}(s)\right) \tag{4.13}$$

Here $\left(\tilde{X}_{n,m}(s)\right)$ is the discretized form (6N × 6N) of the two operators in (4.12). The matrix elements $\tilde{X}_{n,m}(s)$ come from the constitutive parameters (which are frequency independent), divergence operators which involve differences from adjacent zones, and $\tilde{G}_0$ and $\nabla\tilde{G}_0$ in (4.11). Note that these two Green's functions are analytic functions for s, even at $s = 0$. The matrix elements are then entire functions of s (no singularities in the finite s plane). They include exponential as well as polynomial terms in s.

The formal solution to (4.13) is

$$\left(\tilde{F}_n^{(sc)}(s)\right) = \left[\left(1_{n,m}\right) - \left(\tilde{X}_{n,m}(s)\right)\right]^{-1} \cdot \left(\tilde{X}_{n,m}(s)\right) \cdot \left(\tilde{F}_n^{(inc)}(s)\right) \tag{4.14}$$

Since the matrix elements are entire functions there are no branch contributions from the target. The only singularities in the finite s plane are poles at the natural frequencies given by

$$\det\left[\left(1_{n,m}\right) - \left(\tilde{X}_{n,m}(s_\alpha)\right)\right] = 0 \tag{4.15}$$

The determinant involves sums and products of the matrix elements, including thereby exponentials and polynomials in s and, hence, is itself an entire function, but specifically an entire function with zeros. As such (4.13) can be written as an SEM expansion of the same form as in (4.2), where now $\left(\tilde{F}_n^{(sc)}(s)\right)$ is the response and $\left(\tilde{F}_n^{(inc)}(s)\right)$ is the forcing function. Note that while we may generally expect the poles to be of first order there may be exceptions. Causality assures that poles (with non-zero coupling coefficients) all lie in the left half s-plane. The real character of the time-domain response assures that poles not on the negative real s axis come in complex conjugate pairs.

Since by (4.6) the electric and magnetic current densities are related to the incident and scattered fields by polynomials in s, then the singularity expansion of these also has the same form as (4.2). For present purposes it is the induced magnetic dipole moment which is of particular interest, this having the form

$$\tilde{\vec{m}}(s) = \tilde{\vec{m}}_e(s) + \tilde{\vec{m}}_h(s) \tag{4.16}$$

The part from the electric current density is the usual form

$$\tilde{\vec{m}}_e(s) = \frac{1}{2}\int_V \vec{r}' \times \tilde{\vec{J}}_e(\vec{r}', s)dV' \tag{4.17}$$

The magnetic current density contributes a term dual to the usual electric-dipole term as [34]

$$\begin{aligned} \tilde{\vec{m}}_h(s) &= \frac{1}{\mu_0}\int_V \vec{r}\tilde{\rho}_h(\vec{r}', s)dV' = \int_V \vec{r}\nabla' \cdot \tilde{\vec{H}}(\vec{r}', s)dV' = -\frac{1}{s\mu_0}\int_V \vec{r}\nabla' \cdot \tilde{\vec{J}}_h(\vec{r}', s)dV' \\ &= \frac{1}{s\mu_0}\int_V \tilde{\vec{J}}_h(\vec{r}', s)dV' = \int_V \left[\frac{\overleftrightarrow{\mu}(\vec{r}')}{\mu_0} - \overleftrightarrow{1}\right] \cdot \tilde{\vec{H}}(\vec{r}', s)dV' \end{aligned} \tag{4.18}$$

The last form coming from integration by parts, noting that there is no magnetic current density passing through the surface of V. One can find this in various ways such as by the fields or potentials as in (4.8) and (4.9), noting the dual form of the expressions. The important point to note is that there are terms for both the circulating electric currents and the permeability-induced magnetization of the medium. Of course, one can ascribe the latter term to some other equivalent electric current density [34, 48], but the foregoing is the standard form.

Since the magnetic dipole moment is a spatial integral over the electric and magnetic current densities, which in turn are both representable by the form in (4.2), then $\tilde{\vec{m}}(s)$ can be so represented. For this purpose one sets the incident magnetic field to any convenient orientation and time history. Using three orthogonal axes for the incident magnetic field, then (2.8) and (2.1) allow one to write an SEM expansion of the magnetic polarizability dyadic as

$$\begin{aligned} \tilde{\overleftrightarrow{M}}(s) &= \sum_{\alpha, n_\alpha} M_\alpha \vec{M}_\alpha \vec{M}_\alpha [s - s_\alpha]^{-n_\alpha} \\ &\quad + \text{possible entire function} \end{aligned} \tag{4.19}$$

$$\vec{M}_\alpha \cdot \vec{M}_\alpha = 1 \text{ (normalization)}$$

At the present stage in the development the poles include complex conjugate pairs, so the vectors (natural modes) $\vec{M}_\alpha$ may include complex ones. The poles may possibly include ones of multiple order. Reciprocity is incorporated in the symmetric dyads in the sum. Note the similarity of the form to the static case in (3.1) which is chosen for convenience. The role of the coupling coefficient is taken by M_α as a scaling factor and $\vec{M}_\alpha$ for magnetic-polarization dependence. There can be cases of degeneracy as in the case of symmetric targets (e.g., spherical symmetry); in such cases there can be more than one $\vec{M}_\alpha$ for a given s_α, for which allowance is easily made.

V. Special Treatment of Body of Revolution

Let us now consider the case of a body of revolution discussed in [26, 44]. Let the z axis be the axis of revolution in a cylindrical (Ψ, ϕ, z) coordinate system. With

$$\tilde{\vec{H}}^{(inc)}(s) = H_0 \tilde{f}(s) \vec{1}_z \tag{5.1}$$

and restrictions to low frequencies (as discussed previously) so that the wavelength/skin-depth outside the target is large compared to the target, then the exterior quasi-static condition implies that the electric current density has only a ϕ component, i.e., $\tilde{J}_{e_\phi}(\Psi, z; s)$, independent of ϕ. Then visualizing this current density as some number N_ℓ of current carrying loops, each with current $\tilde{I}_n(s)$, one sets up a circuit problem. Each loop has a resistance R_n, a self inductance $L_{n,n}$, and mutual inductances $L_{n,m}$ $(n \neq m)$ between loops. Each loop has a voltage source $\tilde{V}_n(s)$ given by

$$\begin{aligned} &\tilde{V}_n(s) = s\mu_0 H_0 \tilde{f}(s) \pi \Psi_n^2 \\ &\quad \Psi_n \equiv \text{ radius of nth loop} \end{aligned} \tag{5.2}$$

This leads to a matrix equation of the form

$$\begin{aligned} &\left[\left(R_{n,m}\right) + s\left(L_{n,m}\right)\right] \cdot \left(\tilde{I}_n(s)\right) = \left(\tilde{V}_n(s)\right) \\ &\qquad \left(R_{n,m}\right) = \begin{pmatrix} R_1 & & & O \\ & R_2 & & \\ & & \ddots & \\ O & & & R_{N_\ell} \end{pmatrix} \end{aligned} \tag{5.3}$$

Note that the permittivity has been assumed negligible so as not to introduce a capacitive term.

Note that for this derivation the permeability has been taken as μ_0, being that of the surrounding medium. This gives the simple form to the source voltages in (5.2). However, the general form of (5.3) applies a little more generally. The rotation symmetry is preserved even for dyadic constitutive parameters of the target provided that when expressed in terms of cylindrical coordinates including the unit vectors $\left(\vec{1}_\Psi, \vec{1}_\phi, \vec{1}_z\right)$, they are independent of ϕ. This is the general case of C_∞ symmetry [13]. There is, however, another difficulty in that for $\overleftrightarrow{\mu} \neq \mu_0 \overleftrightarrow{1}$ there is also the magnetic current density to be included as in (4.6).

Looking at the general form of (5.3) we have an expression for the natural frequencies as

$$\det\left[\left(R_{n,m}\right) + s_\alpha\left(L_{n,m}\right)\right] = 0 \tag{5.4}$$

More generally one can invert the matrix coefficient and solve for the numerical current vector as a matrix times (dot product) the numerical voltage vector. This matrix has elements which are ratios of polynomials in s. This fact alone, however, is not sufficient to exclude second and higher order poles, nor to give only real natural frequencies. There is, however, the canonical example of the sphere in [21, 48] and Appendix E, as well as practical experience [18] to indicate that the natural frequencies are real and the pole are first order, at least for some cases of interest.

This problem can be overcome by appeal to certain physical properties which can be imposed on the matrices. First recast (5.3) in time domain as

$$\left[\left(R_{n,m}\right)+\left(L_{n,m}\right)\frac{d}{dt}\right]\cdot\left(I_n(t)\right)=\left(V_n(t)\right) \tag{5.5}$$

$$\left.\begin{array}{l}\left(V_n(t)\right)=\left(0_n\right) \text{ for } t<0 \\ \left(I_n(t)\right)=\left(0_n\right) \text{ for } t<0\end{array}\right\} \text{ zero initial conditions}$$

This has the solution via the well-known matrizant (discussed in Appendix B)

$$\begin{aligned}\left(I_n(t)\right)&=\int_{0-}^{t} e^{-\left(L_{n,m}\right)^{-1}\cdot\left(R_{n,m}\right)[t-t']}\cdot\left(L_{n,m}\right)^{-1}\cdot\left(V_n(t')\right)u(t-t')dt' \\ &=\left[e^{-\left(L_{n,m}\right)^{-1}\cdot\left(R_{n,m}\right)t}u(t)\right]\circ\left[\left(L_{n,m}\right)^{-1}\cdot\left(V_n(t)\right)\right]\end{aligned} \tag{5.6}$$

$\circ \equiv$ convolution with respect to time

Now, as discussed in Appendix A, a sufficient condition that $\left(L_{n,m}\right)^{-1}\cdot\left(R_{n,m}\right)$ have a complete set $\left(N_\ell\right)$ of eigenvalues (all real)

$$\begin{aligned}&\left(L_{n,m}\right)^{-1}\cdot\left(R_{n,m}\right)\cdot\left(r_n\right)_\alpha=\chi_\alpha\left(r_n\right)_\alpha \\ &\left(\ell_n\right)_\alpha\cdot\left(L_{n,m}\right)^{-1}\cdot\left(R_{n,m}\right)=\chi_\alpha\left(r_n\right)_\beta \\ &\qquad\left(\ell_n\right)_{\alpha_1}\cdot\left(r_n\right)_{\alpha_2}=1_{\alpha_1,\alpha_2}\end{aligned} \tag{5.7}$$

is that both matrices in the product be real symmetric and that one of the matrices be positive definite. Reciprocity insures

$$
\begin{aligned}
\left(L_{n,m}\right)^T &= \left(L_{n,m}\right) \\
\left(\left(L_{n,m}\right)^{-1}\right)^T &= \left(L_{n,m}\right)^{-1} \\
\left(R_{n,m}\right)^T &= \left(R_{n,m}\right)
\end{aligned}
\tag{5.8}
$$

If we require all the R_n (diagonal values) to be strictly positive (implying bounded σ^{-1} or at least integrable σ^{-1} for each loop) then $\left(R_{n,m}\right)$ is a positive definite matrix, thereby meeting the condition. Note that implicitly $\left(L_{n,m}\right)^{-1}$ must exist. For real physical problems $\left(L_{n,m}\right)$ is also positive definite (and hence also $\left(L_{n,m}\right)^{-1}$ is positive definite) based on energy considerations in that any set of currents not identically zero must generate magnetic fields with non-zero energy (square integral overspace). This implies that all the eigenvalues χ_α are positive.

With this sufficient condition we can now write.

$$
\begin{aligned}
\left(L_{n,m}\right)^{-1} \cdot \left(R_{n,m}\right) &= \sum_{\alpha=1}^{N_\ell} \chi_\alpha \left(r_n\right)_\alpha \left(\ell_n\right)_\alpha \\
e^{-\left(L_{n,m}\right)^{-1} \cdot \left(R_{n,m}\right) t} u(t) &= u(t) \sum_{\alpha=1}^{N_\ell} e^{-\chi_\alpha t} \left(r_n\right)_\alpha \left(\ell_n\right)_\alpha \\
&= u(t) \sum_{\alpha=1}^{N_\ell} e^{s_\alpha t} \left(r_n\right)_\alpha \left(\ell_n\right)_\alpha \\
s_\alpha &= -\chi_\alpha \quad (\text{all } s_\alpha < 0)
\end{aligned}
\tag{5.9}
$$

In complex-frequency domain one can now rewrite (5.6) as

$$
\left(\tilde{I}_n(s)\right) = \left\{ \sum_{\alpha=1}^{N_\ell} \left[s - s_\alpha\right]^{-1} \left(r_n\right)_\alpha \left(\ell_n\right)_\alpha \right\} \cdot \left(L_{n,m}\right)^{-1} \cdot \left(\tilde{V}_n(s)\right) \tag{5.10}
$$

showing that the $\left(r_n\right)_\alpha$ are the natural modes for the currents. Note that the physical conditions previously imposed on the matrices now have given all negative-real s_α and all first-order poles in the explicit solution above. Note also that eigenvalues in time domain correspond to natural frequencies.

This body-of-revolution case is instructive in that a simple picture of mutually-coupled resistive loops describes the situation with axial incident magnetic field. One would like to extend this to more general shapes and directions for the incident magnetic field, and to include the more general permeability $\overleftrightarrow{\mu} \neq \mu_0 \overleftrightarrow{1}$ which gives a magnetic-current-density part as well.

VI. LR-Network Approach to Finite-Size Permeable Conducting Bodies

Let us take a different approach and consider a circuit model based on the differential Maxwell equations (4.4). This is like replacing the spatial derivatives by finite differences, but constructed in terms of lumped circuit elements.

The general form that this takes is described in [14-17, 20]. There is more than one type (duals) which can be used and the reader is referred to the original papers for the rather elaborate circuit diagrams. While one can use finite differences in an arbitrary orthogonal curvilinear coordinate system (u_1, u_2, u_3), together with the scale factors (h_1, h_2, h_3) of the transformation in computing the element values, for present purposes one can use simple Cartesian coordinates with finite differences $(\Delta x, \Delta y, \Delta z)$ which may even be taken as equal. In its basic form there are parallel conductances (proportional to σ) and capacitances (proportional to ε) with a voltage across this proportional to the electric field $\vec{E}$ in the direction of concern (one for each of three directions). There are similarly inductances (proportional to μ) carrying currents proportional to the magnetic field $\vec{H}$. Note now that there are electric currents $\left(\nabla \times \tilde{\vec{H}} \text{ or } \tilde{\vec{J}}_e + s\varepsilon_0 \tilde{\vec{E}}\right)$ through the conductances and capacitances, and magnetic currents $\left(\nabla \times \tilde{\vec{E}} \text{ or } \tilde{\vec{J}}_h + s\mu_0 \tilde{\vec{H}}\right)$ as voltages across the inductances. So both types are included in this model, including the free-space background contributions (defining what might be called total current densities in each case).

The derivation of these equivalent circuits has been for scalar $\mu(\vec{r}), \varepsilon(\vec{r})$, and $\sigma(\vec{r})$. This can be readily generalized to dyadic $\overleftrightarrow{\mu}(\vec{r})$, $\overleftrightarrow{\varepsilon}(\vec{r})$, and $\overleftrightarrow{\sigma}(\vec{r})$. Since these are real symmetric dyadics they can always be diagonalized with real eigenvectors. If the eigenvectors (principal axes) are common to all the constitutive parameters, then a coordinate rotation can align these with the coordinate axes (at least locally) thereby making the eigenvalues correspond to the element values in the coordinate directions. A more general approach is to consider an anisotropic medium as a special kind of inhomogeneous medium as mentioned in Section II. On a small scale (compared to radian wavelengths and skin depths in the medium) one replaces the original medium by idealized sheets and/or rods of material running in various directions to give the desired directional variation of the various constitutive parameters. Considering an even smaller scale for the size of the coordinate finite differences, then on this scale the medium is isotropic (modeling the sheets and/or rods). One can even smooth the transition on going through the surfaces of sheets and rods to adjacent ones to avoid discontinuities in the network modeling.

Another form of network that has been introduced goes by the name of transmission-line matrix (TLM) [22-25, 29, 30]. In this form the medium is represented for each $(\Delta x, \Delta y, \Delta z)$ by a network of transmission lines, including in some of its versions lumped elements as well (such as for the conductivity). Of course, transmission lines can also be represented by sets of lumped elements, as is well known. So for our present purposes the TLM form is consistent with the first form of lumped-element network discussed above.

Now introduce a low-frequency approximation for the assumed highly conducting target, i.e.,

$$\left\|\overleftrightarrow{\sigma}(\overrightarrow{r})\right\| >> \left\|s\overleftrightarrow{\varepsilon}(\overrightarrow{r})\right\| = |s|\left\|\overleftrightarrow{\varepsilon}(\overrightarrow{r})\right\| \tag{6.1}$$

where norms are used to generalize complex magnitude to dyadics. The point is that displacement currents can be neglected compared to conduction currents, thereby removing all capacitive elements from the lumped element representation of the target. This leaves an LR network to represent the target. For a given non-zero cell size (Δx, etc.) and finite target size a, the number of elements here is finite.

There is still the question of the external medium in this circuit representation. From (2.5) the target is assumed electrically small in terms of radian wavelengths and skin depths in the external medium, leading to replacement of the external medium by free space. Being electrically small, the target does not radiate significant power at frequencies of interest. Losses can be approximated as being entirely internal to the target. As discussed in Appendix C one can replace the incident fields by a set of sources on a surface surrounding the target. Furthermore, the scattering problem can be decomposed into quasi-magnetostatic and quasi-electrostatic parts. Let us restrict our attention to the quasi-magnetostatic part for which the sources are electric-surface-current-density sources proportional to the incident (quasistatic or approximately uniform) magnetic field. These sources go into the circuit representation as incremental sources in each cell corresponding to the source surface. For this quasi-magnetostatic problem the only field of significance outside the source surface is magnetic, including the scattered magnetic field and a magnetic field associated with the current sources (the magnetic-current-density sources having been removed). This quasi-static magnetic field is derivable as the gradient of a scalar potential satisfying the Laplace equation. It falls off as r^{-3} for large r, which can be truncated at some large r, not accounting for only a negligible part of the magnetic energy. The circuit representation for this includes only inductors, and the truncation at finite r makes the number of elements finite.

With now a network consisting of a finite number of inductors and capacitors (plus current sources), let us appeal to circuit theory. In [36 (chapter 2)] and other texts the properties of LR networks are derived. Considering any driving point impedance or admittance (say in response to one of the current sources for the source surface), these have poles and zeros (alternating) on the negative real s axis, and only there. Furthermore, these poles and zeros are all first order and the coefficients (residues) of the admittance poles are all positive real, and of the impedance poles are all negative real except for non-negative real coefficients of terms proportional to s^0 and s^1 as $s \to \infty$. Furthermore, transfer impedances (giving voltages at other positions in the network in response to a current source) have simple poles at the same frequencies (natural frequencies) as the driving point admittances (with real residues, but not only negative residues). So we can conclude that our LR-network representation of the scatterer gives:

$$
\begin{aligned}
&1. \quad s_\alpha < 0 \ \text{(all natural frequencies negative real)} \\
&2. \quad \text{all poles first order with real residues}
\end{aligned}
\tag{6.2}
$$

One can arrive at the same result by considering the matrix equation of the LR network as

$$
\left[\left(R_{n,m}\right)+s\left(L_{n,m}\right)\right]\cdot\left[\left(\tilde{I}_n(s)\right)+\left(\tilde{I}_n^{(s)}(s)\right)\right]=\left(0_n\right) \tag{6.3}
$$

based on the loop-parameter form used in circuit theory [35]. Here the $\left(\tilde{I}_n(s)\right)$ are the loop currents with $\left(\tilde{I}_n^{(s)}(s)\right)$ as the source currents which are present in the loops including current sources from circuit cells on the source surface. Rewriting we have

$$
\begin{aligned}
\left[\left(R_{n,m}\right)+s\left(L_{n,m}\right)\right]\cdot\left(\tilde{I}_n(s)\right) &= -\left[\left(R_{n,m}\right)+s\left(L_{n,m}\right)\right]\cdot\left(\tilde{I}_n^{(s)}(s)\right) \\
&\equiv \left(\tilde{V}_n(s)\right) \\
\left[\left(R_{n,m}\right)+\left(L_{n,m}\right)\frac{d}{dt}\right]\cdot\left(I_n(t)\right) &= \left(V_n(t)\right)
\end{aligned}
\tag{6.4}
$$

Here the $\tilde{V}_n(s)$ can have zeros at various frequencies, but no poles if the $\tilde{I}_n^{(s)}(s)$ (and hence the incident magnetic field) have no poles. So the poles of $\left(\tilde{I}_n(s)\right)$ (and hence of $\tilde{\tilde{M}}(s)$) come from the network natural frequencies which solve the same form of equation as in (5.4). In fact, now (6.4) can be solved by the same techniques used to solve (5.5) and give the same result as (5.10) with negative real s_α, only first order poles, and real residue dyadics (real right and left eigenvectors). Again this relies on the positive definiteness of $\left(L_{n,m}\right)$ and $\left(R_{n,m}\right)$.

VII. Integral-Equation Considerations

Neglecting displacement currents for the quasi-magnetostatic case also simplifies the integral equations (4.10). First we have

$$
\begin{aligned}
&\tilde{\vec{J}}_e(\vec{r},s) = \vec{\vec{\sigma}}(\vec{r}) \cdot \tilde{\vec{E}}(\vec{r},s) \\
&\nabla \cdot \tilde{\vec{J}}_e(\vec{r},s) = \nabla \cdot \left[\nabla \times \tilde{\vec{H}}(\vec{r},s) \right] = \vec{0} \\
&\tilde{\vec{J}}_h(\vec{r},s) = s\left[\vec{\vec{\mu}}(\vec{r}) - \mu_0 \vec{\vec{1}} \right] \cdot \tilde{\vec{H}}(\vec{r},s) \\
&\nabla \cdot \tilde{\vec{J}}_h(\vec{r},s) = s\nabla \cdot \tilde{\vec{B}}(\vec{r},s) - s\mu_0 \nabla \cdot \tilde{\vec{H}}(\vec{r},s) \\
&\qquad = -s\mu_0 \nabla \cdot \tilde{\vec{H}}(\vec{r},s) = s\nabla \cdot \left[\left[\vec{\vec{\mu}}(\vec{r}) - \mu_0 \vec{\vec{1}} \right] \cdot \tilde{\vec{H}}(\vec{r},s) \right]
\end{aligned}
\tag{7.1}
$$

The Green's functions can also be replaced by their low-frequency (quasi-static) forms from (4.11) as

$$
\begin{aligned}
&\tilde{G}_0(\vec{r},\vec{r}\,';0) = \frac{1}{4\pi R} \\
&\nabla \tilde{G}_0(\vec{r},\vec{r}\,';0) = -\frac{1}{4\pi R^2} \vec{1}_R \\
&R \equiv \left| \vec{r} - \vec{r}\,' \right| \\
&\vec{1}_R \equiv \frac{\vec{r} - \vec{r}\,'}{R}
\end{aligned}
\tag{7.2}
$$

Keeping leading terms for low frequency (4.10) becomes

$$
\begin{aligned}
\tilde{\vec{E}}^{(sc)}(\vec{r},s) = &-s\mu_0 \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';0), \vec{\vec{\sigma}}(\vec{r}\,') \cdot \tilde{\vec{E}}(\vec{r}\,',s) \right\rangle \\
&+ s\left\langle \nabla \tilde{G}_0(\vec{r},\vec{r}\,';0) \overset{\times}{,} \left[\vec{\vec{\mu}}(\vec{r}\,') - \mu_0 \vec{\vec{1}} \right] \cdot \tilde{\vec{H}}(\vec{r}\,',s) \right\rangle \\
\tilde{\vec{H}}^{(sc)}(\vec{r},s) = &\frac{1}{\mu_0} \left\langle \nabla \tilde{G}_0(\vec{r},\vec{r}\,';0), \nabla' \cdot \left[\left[\vec{\vec{\mu}}(\vec{r}\,') - \mu_0 \vec{\vec{1}} \right] \cdot \tilde{\vec{H}}(\vec{r}\,',s) \right] \right\rangle \\
&+ \left\langle \nabla \tilde{G}_0(\vec{r},\vec{r}\,';0) \overset{\times}{,} \vec{\vec{\mu}}(\vec{r}\,') \cdot \tilde{\vec{H}}(\vec{r}\,',s) \right\rangle
\end{aligned}
\tag{7.3}
$$

Note that outside V the above does not describe the scattered electric field because of the neglect of the displacement current in the first term of (4.10). It is only where $\overleftrightarrow{\sigma}(\vec{r})$ is dominant that we are concerned with the electric field. Note also that the scattered electric field is down by a factor of s from both the electric current density and the magnetic field. In (7.3) one can now replace the fields by the incident and scattered parts, the incident parts giving the source terms. As discussed in Appendix C, for the quasi-magnetostatic problem one can have the small part of the incident electric field associated with the incident magnetic field. This is also represented by a set of equivalent surface-electric-current-density sources on a surface surrounding the scatterer.

As a special case consider a non-permeable scatterer, i.e.,

$$\begin{aligned}
\overleftrightarrow{\mu} &= \mu_0 \overleftrightarrow{1} \\
\tilde{\vec{J}}_h(\vec{r},s) &= \vec{0} \\
\nabla \cdot \tilde{\vec{H}}(\vec{r},s) &= 0
\end{aligned} \tag{7.4}$$

This gives a simpler set of coupled integral equations as

$$\begin{aligned}
\tilde{\vec{E}}^{(sc)}(\vec{r},s) &= -s\mu_0 \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';0), \overleftrightarrow{\sigma}(\vec{r}\,') \cdot \tilde{\vec{E}}(\vec{r}\,',s) \right\rangle \\
\tilde{\vec{H}}^{(sc)}(\vec{r},s) &= \left\langle \nabla \tilde{G}_0(\vec{r},\vec{r}\,';s) \overset{\times}{,} \overleftrightarrow{\sigma}(\vec{r}\,') \cdot \tilde{\vec{E}}(\vec{r}\,',s) \right\rangle
\end{aligned} \tag{7.5}$$

These are uncoupled in the sense that the first equation can be solved for the electric field or electric current density. This solution can then be inserted into the second equation.

An instructive application of the first of (7.5) is casting it in a form similar to that discussed in Section V. Rewrite this as

$$\begin{aligned}
\tilde{\vec{E}}(\vec{r},s) + s\mu_0 \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';0), \overleftrightarrow{\sigma}(\vec{r}\,') \cdot \tilde{\vec{E}}(\vec{r}\,',s) \right\rangle &= \tilde{\vec{E}}^{(inc)}(\vec{r},s) \\
\overleftrightarrow{\sigma}^{-1}(\vec{r}) \cdot \tilde{\vec{J}}_e(\vec{r},s) + s\mu_0 \left\langle \tilde{G}_0(\vec{r},\vec{r}\,';0), \tilde{\vec{J}}_e(\vec{r}\,',s) \right\rangle &= \tilde{\vec{E}}^{(inc)}(\vec{r},s) \\
\left\langle \overleftrightarrow{\sigma}^{-1}(\vec{r}\,')\delta(\vec{r}-\vec{r}\,') + s\mu_0 \tilde{G}_0(\vec{r},\vec{r}\,';0); \tilde{\vec{J}}_e(\vec{r}\,',s) \right\rangle &= \tilde{\vec{E}}^{(inc)}(\vec{r},s)
\end{aligned} \tag{7.6}$$

This is an operator equation of the same form as the matrix equation (5.3). Consider the first term of the operator, namely $\overleftrightarrow{\sigma}^{-1}(\vec{r}\,')\delta(\vec{r}-\vec{r}\,')$. This has the form of a resistance operator and, like that resistance matrix $\left(R_{n,m}\right)$, it is diagonal due to $\delta(\vec{r}-\vec{r}\,')$. It is, of course, real and symmetric and is positive definite provided $\overleftrightarrow{\sigma}^{-1}(\vec{r}\,')$ is bounded and has positive eigenvalues (with exceptional points allowed if integrability is included in these conditions). The second term, $s\mu_0\tilde{G}(\vec{r},\vec{r}\,';0)$ is clearly s times something which is real (and positive) and symmetric, and takes the form of an inductance operator.

If one discretizes (7.6), then one can construct resistance and inductance matrices of the same form as in Section V. Following the same procedures from Appendices A and B for the diagonalization, the solution takes the same form as (5.6) and (5.10) with the same properties. A more detailed analysis could consider the integral equation as such, noting the real symmetric operators as special cases of Hermitian operators for which a spectral theory exists [43].

VIII. Concluding Remarks

The quasi-magnetostatic scattering problem is characterized by a magnetic polarizability dyadic which can now be written from (4.19) as

$$\tilde{\overleftrightarrow{M}}(s) = \tilde{\overleftrightarrow{M}}(\infty) + \sum_{\alpha} M_\alpha \overrightarrow{M}_\alpha \overrightarrow{M}_\alpha [s - s_\alpha]^{-1}$$

$$\overrightarrow{M}_\alpha \cdot \overrightarrow{M}_\alpha = 1 \ , \ \overrightarrow{M}_\alpha = \text{ real vector}$$

$$M_\alpha = \text{ real scalar} \tag{8.1}$$

$$s_\alpha < 0 \ \text{ (all negative real natural frequencies)}$$

where the results of Sections V through VII are now incorporated. Note the similarity of form to that for the sphere in Appendix E. In the case of the sphere there is a degeneracy of the natural frequencies/modes so that the dyad $\overrightarrow{M}_\alpha \overrightarrow{M}_\alpha$ is replaced by the identity dyadic

$$\overleftrightarrow{1} = \overrightarrow{1}_x \overrightarrow{1}_x + \overrightarrow{1}_y \overrightarrow{1}_y + \overrightarrow{1}_z \overrightarrow{1}_z \tag{8.2}$$

which is the sum of three dyads. Note that the entire function is just the constant term $\tilde{\overleftrightarrow{M}}(\infty)$ which is just the magnetic-polarizability dyadic of a perfectly conducting target which we can then also write as [45]

$$\tilde{\overleftrightarrow{M}}(\infty) = \sum_{\nu=1}^{3} M_\nu^{(\infty)} \overrightarrow{M}_\nu^{(\infty)} \overrightarrow{M}_\nu^{(\infty)}$$

$$M_\nu^{(\infty)} \equiv \text{ real eigenvalues (non positive) (not necessarily distinct)}$$

$$\overrightarrow{M}_\nu^{(\infty)} \equiv \text{ real eigenvectors (three)} \tag{8.3}$$

$$\overrightarrow{M}_{\nu_1}^{(\infty)} \cdot \overrightarrow{M}_{\nu_2}^{(\infty)} = 1_{\nu_1,\nu_2} \ \text{ (orthonormal)}$$

The case of degenerate modes in (8.1) is handled by having more than one $\overrightarrow{M}_\alpha$ (an orthonormal set) associated with a particular s_α. A common case of this is bodies of revolution with two modes for one natural frequency (applying to the incident magnetic field perpendicular to the axis of revolution).

In time domain, the δ-function response is given by

$$\overleftrightarrow{M}(t) = \overleftrightarrow{M}(\infty)\delta(t) + \left\{ \sum_{\alpha} M_\alpha \overrightarrow{M}_\alpha \overrightarrow{M}_\alpha e^{s_\alpha t} \right\} u(t) \tag{8.4}$$

The step-function response is given by

$$\int_{-\infty}^{t} \overleftrightarrow{M}(t')dt' = \left\{ \overleftrightarrow{M}(0) + \sum_{\alpha} \frac{M_\alpha}{s_\alpha} \vec{M}_\alpha \vec{M}_\alpha e^{s_\alpha t} \right\} u(t)$$

$$\frac{\tilde{\overleftrightarrow{M}}(s)}{s} = \frac{\tilde{\overleftrightarrow{M}}(0)}{s} + \sum_{\alpha} \frac{M_\alpha}{s_\alpha} \vec{M}_\alpha \vec{M}_\alpha [s - s_\alpha]^{-1} \tag{8.5}$$

As discussed in Section III, the constant term $\overleftrightarrow{M}(0)$ is the DC magnetic-polarizability dyadic as

$$\tilde{\overleftrightarrow{M}}(0) = \sum_{\nu=1}^{3} M_\nu^{(0)} \vec{M}_\nu^{(0)} \vec{M}_\nu^{(0)}$$

$$M_\nu^{(0)} \equiv \text{real eigenvalues (non negative) (not necessarily distinct)} \tag{8.6}$$

$$\vec{M}_\nu^{(0)} \equiv \text{real eigenvectors (three)}$$

$$\vec{M}_{\nu_1}^{(0)} \cdot \vec{M}_{\nu_2}^{(0)} = 1_{\nu_1,\nu_2} \quad \text{(orthonormal)}$$

This DC polarizability applies only to permeable scatterers characterized by $\overleftrightarrow{\mu}(\vec{r})$ different from $\mu_0 \overleftrightarrow{1}$.

Assuming that one has located a buried target at some position $\vec{r}_0$ (as in fig. 1.1) by some quasi-static magnetic field or other technique, one would next like to identify this target and even determine its orientation in the ground. Assume that one has some set of target types (a target library) that describes the potential target types of interest. The nth target type consists of a set of (practically) identical targets. As such it has its own set of quasi-magnetostatic natural frequencies s_α. In general there will be a different set of natural frequencies for the mth target type for m $\neq$ n. Processing the scattering data to determine the s_α set (albeit practically truncated to a finite set) will in principle (with sufficient accuracy in the presence of noise) allow one to identify which target type is present. Note that if a target type is present but not included in the library, the identification procedure will ideally merely state that the target type is not included in the library.

Associated with each s_α there is one or more eigenvalues M_α. Knowing the target location $\vec{r}_0$ one can use the magnetic-dipole fields (as in (2.15)) to estimate the M_α. Since these are characteristic of the target type they can also aid in the identification. A special case is the eigenvalues $M_\nu^{(0)}$ for zero frequency which are non-zero (positive) only for permeable targets. The presence of such terms in the step response is a target discriminant in that it restricts the target types to a subset of the library, namely the permeable target types. Conversely, the absence of such terms leads to the subset of non-permeable target types. Another special case is the eigenvalues $M_V^{(\infty)}$ for high frequency which appear in the δ-function response. These are also characteristic of the target type.

Having identified the target type, consider now the natural-mode vectors $\vec{M}_\alpha$ for the magnetic polarizability. As the target is rotated in space the real $\vec{M}_\alpha$ are rotated with it. The $\vec{M}_\alpha$ then can be used to determine the target aspect (orientation). Note, however, that these enter via the dyadic product $\vec{M}_\alpha \, \vec{M}_\alpha$ so that the $\vec{M}_\alpha$ have a sign ambiguity ($-\vec{M}_\alpha$ being equally acceptable). This leads to an orientation ambiguity. As an example, consider a long slender body of revolution. Certain of the $\vec{M}_\alpha$ are associated with the incident magnetic field parallel to the rotation axis. For an oriented body with front and back ends one can determine the orientation of the axis in space, but not which end is front and which back. Similar comments apply to the $\vec{M}_V^{(0)}$ and $\vec{M}_V^{(\infty)}$. Besides their use in determining target orientation, these vectors can also help in the target identification because their relative orientation (one vector to another, noting sign ambiguity) is fixed for a given target type, all these vectors rotating together as the target is rotated. This might be considered as some kind of consistency check.

Summarizing, we have the following properties:

1. $\tilde{\overleftrightarrow{M}}(0) : \begin{cases} \text{presence} \Rightarrow \text{ permeable target type} \\ \text{absence} \Rightarrow \text{non - permeable target type} \end{cases}$

2. s_α set: identifies target type from target library

3. M_α set, $M_V^{(0)}, M_V^{(\infty)}$: also identifies target type from target library

4. $\vec{M}_\alpha$ set, $\vec{M}_V^{(0)}, \vec{M}_V^{(\infty)}$: determines target aspect (noting sign ambiguity), also aids in identification

Appendix A: Diagonalization of Product of Two Hermitian or Real-Symmetric Matrices

For a Hermitian matrix (N×N)

$$
\begin{aligned}
&\left(H_{n,m}\right)=\left(H_{n,m}\right)^{\dagger} \equiv \text{ Hermitian matrix} \\
&\qquad \dagger \equiv \text{adjoint} = T\,* \\
&\qquad T \equiv \text{ transpose} \\
&\qquad * \equiv \text{ conjugate} \\
&H_{n,m} = H_{m,n}^{*}
\end{aligned}
\tag{A.1}
$$

it is well known (e.g., [37]) that it can always be diagonalized as

$$
\begin{aligned}
&\left(H_{n,m}\right)=\sum_{\beta=1}^{N} H_{\beta}\left(x_n\right)_{\beta}\left(x_n\right)_{\beta}^{*} \\
&\left(H_{n,m}\right)\cdot\left(x_n\right)_{\beta} = H_{\beta}\left(x_n\right)_{\beta} \quad \text{(right eigenvectors)} \\
&\left(x_n\right)_{\beta}^{*}\cdot\left(H_{n,m}\right) = H_{\beta}\left(x_n\right)_{\beta}^{*} \quad \text{(left eigenvectors)} \\
&\left(x_n\right)_{\beta_1}\cdot\left(x_n\right)_{\beta_2}^{*} = 1_{\beta_1,\beta_2} \quad \text{(biorthonormal)} \\
&\qquad H_{\beta} = \text{ real eigenvalues}
\end{aligned}
\tag{A.2}
$$

where the eigenvalues need not be distinct. There is always a complete set of eigenvectors.

A real symmetric matrix is a special case as

$$
\begin{aligned}
&\left(h_{n,m}\right)=\left(h_{n,m}\right)^{T} \equiv \text{ real symmetric matrix} \\
&h_{n,m} = h_{m,n} \equiv \text{ real number}
\end{aligned}
\tag{A.3}
$$

for which we have not only real eigenvalues from (A.2), but can construct a set of real eigenvectors [10] as

$$
\begin{aligned}
&\left(h_{n,m}\right)=\sum_{\beta=1}^{N} h_{\beta}\left(x_n\right)_{\beta}\left(x_n\right)_{\beta} \\
&\qquad h_{\beta} \equiv \text{ real eigenvalues} \\
&\qquad \left(x_n\right)_{\beta} \equiv \text{ real eigenvectors (both left and right)} \\
&\left(x_n\right)_{\beta_1}\cdot\left(x_n\right)_{\beta_1} = 1_{\beta_1,\beta_2} \quad \text{(orthonormal)}
\end{aligned}
\tag{A.4}
$$

The real eigenvectors are readily constructed from any complex ones that may be found (e.g., in cases of degenerate eigenvalues) by noting that real and imaginary parts separately satisfy the eigenvector equation.

Diagonalizable matrices (or matrices of simple structure in [37]), when arguments of functions, can be expressed in terms of the function of the eigenvalues in the dyadic forms as in (A.2) and (A.4). In particular, the square root can be written as

$$\begin{aligned} \left(H_{n,m}\right)^{\frac{1}{2}} &= \sum_{\beta=1}^{N} H_{\beta}^{\frac{1}{2}} \left(x_n\right)_{\beta} \left(x_n\right)_{\beta}^{*} \\ \left(h_{n,m}\right)^{\frac{1}{2}} &= \sum_{\beta=1}^{N} h_{\beta}^{\frac{1}{2}} \left(x_n\right)_{\beta} \left(x_n\right)_{\beta} \end{aligned} \tag{A.5}$$

For these matrices to be respectively Hermitian or real symmetric a necessary and sufficient condition is

$$\begin{aligned} &H_{\beta}^{\frac{1}{2}} = \text{real} \;,\; h_{\beta}^{\frac{1}{2}} = \text{real} \\ &H_{\beta} \geq 0 \quad,\quad h_{\beta} \geq 0 \\ &\qquad \text{for all } \beta = 1,2,\ldots,N \end{aligned} \tag{A.6}$$

A positive definite matrix has all eigenvalues strictly positive, while a positive semidefinite matrix has all eigenvalues ≥ 0. Applying this to a Hermitian or real symmetric matrix, note that the eigenvalues are already real. We merely need to constrain

$$\begin{aligned} &H_{\beta} \begin{cases} > 0 \Rightarrow \text{ positive definite} \\ \geq 0 \Rightarrow \text{ positive semidefinite} \end{cases} \\ &\qquad \text{for all } \beta = 1,2,\ldots,N \end{aligned} \tag{A.7}$$

and similarly for the h_{β}. If desired one can also make the square-root matrices positive definite (or positive semidefinite) by choosing the positive (or non-negative) square roots of the eigenvalues.

Now consider two Hermitian matrices (N×N), either or both of which may be real symmetric, and consider the diagonalization of the product as

$$\begin{aligned} \left(X_{n,m}^{(1)}\right) \cdot \left(X_{n,m}^{(2)}\right) \cdot \left(r_n\right)_{\beta} &= X_{\beta} \left(r_n\right)_{\beta} \\ \left(\ell_n\right)_{\beta} \cdot \left(X_{n,m}^{(1)}\right) \cdot \left(X_{n,m}^{(2)}\right) &= X_{\beta} \left(\ell_n\right)_{\beta} \end{aligned} \tag{A.8}$$

Can this be diagonalized with a complete set of N eigenvectors (right and left)?

A sufficient condition for this diagonalizability is that at least one of these matrices be positive definite. Consider this as the first one in which case (A.8) can be manipulated into the form (using the fact that the first matrix is nonsingular because positive definite)

$$\left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} \cdot \left(X_{n,m}^{(2)}\right) \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} \cdot \left[\left(X_{n,m}^{(1)}\right)^{-\frac{1}{2}} \cdot (r_n)_\beta\right] = X_\beta \left[\left(X_{n,m}^{(1)}\right)^{-\frac{1}{2}} \cdot (r_n)_\beta\right]$$

$$\left[(\ell_n)_\beta \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}}\right] \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} \cdot \left(X_{n,m}^{(2)}\right) \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} = X_\beta \left[(\ell_n)_\beta \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}}\right]$$

$$\left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} \cdot \left(X_{n,m}^{(2)}\right) \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} \equiv \text{Hermitian matrix (or even real symmetric matrix if both matrices are real symmetric)}$$

$$\left(X_{n,m}^{(1)}\right)^{-\frac{1}{2}} \cdot (r_n)_\beta = (r_n)_\beta \cdot \left(X_{n,m}^{(1)}\right)^{-\frac{1}{2}} \equiv \text{right eigenvectors} \tag{A.9}$$

$$(\ell_n)_\beta \cdot \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} = \left(X_{n,m}^{(1)}\right)^{\frac{1}{2}} \cdot (\ell_n)_\beta \equiv \text{left eigenvectors}$$

$$X_\beta \equiv \text{real eigenvalues}$$

Alternately one can let the second matrix be positive definite and follow a similar procedure.

For the special case that both matrices are real symmetric, then the right and left eigenvectors in (A.9) can be taken as real, making both the $(r_n)_\beta$ and $(\ell_n)_\beta$ real.

The above can be generalized to the case that one of the Hermitian (or real-symmetric) matrices is negative definite (i.e., all eigenvalues < 0). In this case multiply each of the two matrices by -1, leaving the product unchanged. The negative definite matrix is now replaced by a positive-definite Hermitian (or real-symmetric) matrix, and the above result applies.

In summary:

1. The product of two Hermitian matrices can be always diagonalized with real eigenvalues if one of the matrices is positive definite or negative definite.

2. If both matrices are real symmetric the eigenvectors can also be taken as real.

Appendix B. First-Order Linear Matrix Differential Equation

Consider the matrix differential equation

$$\left[\left(A_{n,m}\right)+\left(B_{n,m}\right)\frac{d}{dt}\right]\cdot\left(a_n(t)\right)=\left(b_n(t)\right)$$

$$\left(A_{n,m}\right),\left(B_{n,m}\right)\equiv N\times N \text{ constant matrices (real)} \tag{B.1}$$

$$\left(a_n(t)\right)\equiv \text{ N - component response vector (real)}$$

$$\left(b_n(t)\right)\equiv \text{ N - component source vector (real)}$$

with a boundary condition as $\left(a_n(t_0)\right)$ specified. Provided $\left(B_{n,m}\right)$ is non-singular this can be rewritten as

$$\left[\left(B_{n,m}\right)^{-1}\cdot\left(A_{n,m}\right)+\left(1_{n,m}\right)\frac{d}{dt}\right]\cdot\left(a_n(t)\right)=\left(B_{n,m}\right)^{-1}\cdot\left(b_n(t)\right) \tag{B.2}$$

From this we can write the homogeneous matrix equation

$$\left[\left(B_{n,m}\right)^{-1}\cdot\left(A_{n,m}\right)+\left(1_{n,m}\right)\frac{d}{dt}\right]\cdot\left(\Phi_{n,m}(t,t_0)\right)=\left(0_{n,m}\right)$$

$$\left(\Phi_{n,m}(t,t_0)\right)=\left(1_{n,m}\right) \text{ (boundary condition)} \tag{B.3}$$

$$\left(\Phi_{n,m}(t,t_0)\right)\equiv \text{ matrizant}$$

where t_0 is often taken as zero for convenience.

This has the well-known solution [39]

$$\left(\Phi_{n,m}(t,t_0)\right)=e^{-\left(B_{n,m}\right)^{-1}\cdot\left(A_{n,m}\right)[t-t_0]} \tag{B.4}$$

which is used to solve (B.2) and hence (B.1) as

$$\begin{aligned}\left(a_n(t)\right)&=e^{-\left(B_{n,m}\right)^{-1}\cdot\left(A_{n,m}\right)[t-t_0]}\cdot\left(a_n(t_0)\right)\\&+\int_{t_0}^{t} e^{-\left(B_{n,m}\right)^{-1}\cdot\left(A_{n,m}\right)[t-t']}\cdot\left(B_{n,m}\right)^{-1}\cdot\left(b_n(t')\right)dt'\end{aligned} \tag{B.5}$$

Now, to meet the conditions in Appendix A, additionally require

$$\left.\begin{aligned}\left(A_{n,m}\right)&=\left(A_{n,m}\right)^T\\\left(B_{n,m}\right)&=\left(B_{n,m}\right)^T\end{aligned}\right\} \text{ real symmetric} \tag{B.6}$$

with at least one of these positive definite. Then we have the diagonalization

$$\begin{aligned}
\left(B_{n,m}\right)^{-1} \cdot \left(A_{n,m}\right) &= \sum_{\beta=1}^{N} \chi_\beta (r_n)_\beta (\ell_n)_\beta \\
\chi_\beta &\equiv \text{ real eigenvalues} \\
(r_n)_\beta &\equiv \text{ real right eigenvectors} \\
(\ell_n)_\beta &\equiv \text{ real left eigenvectors} \\
(r_n)_{\beta_1} \cdot (\ell_n)_{\beta_2} &= 1_{\beta_1,\beta_2} \quad \text{(biorthonormal)}
\end{aligned} \tag{B.7}$$

Note that if both matrices are positive definite, then so is the product above and the eigenvalues are all strictly positive. If $\left(A_{n,m}\right)$ is only positive semidefinite then the eigenvalues are all non negative. Such conditions are appropriate for passive physical problems.

The matrizant is now

$$\left(\Phi_{n,m}(t,t_0)\right) = \sum_{\beta=1}^{N} e^{-\chi_\beta [t-t_0]} (r_n)_\beta (\ell_n)_\beta \tag{B.8}$$

Thus the case of real symmetric matrices with $\left(B_{n,m}\right)$ non singular and at least one of the matrices positive definite leads to a solution involving simple exponential functions with real time constants (non increasing for passive physical problems) and real coefficients.

For convenience set

$$\begin{aligned}
t_0 &= 0_- \quad \text{(i.e., just before } t = 0) \\
(a_n(0_-)) &= (0_n) \quad \text{(zero initial conditions)}
\end{aligned} \tag{B.9}$$

giving

$$(a_n(t)) = \int_{0_-}^{t} \left[\sum_{\beta=1}^{N} e^{-\chi_\beta [t-t']} (r_n)_\beta (\ell_n)_\beta \right] \cdot \left(B_{n,m}\right)^{-1} \cdot (b_n(t'))dt' \tag{B.10}$$

Next taking the forcing vector function as a delta function at t = 0, we have

$$(b_n(t)) = (b_n)\delta(t)$$

$$(a_n(t)) = \left[\sum_{\beta=1}^{N} e^{-\chi_\beta t}(r_n)_\beta(\ell_n)_\beta\right]\cdot\left(B_{n,m}\right)^{-1}\cdot(b_n)u(t) \tag{B.11}$$

In terms of the Laplace transform this is

$$(\tilde{a}_n(s)) = \left[\sum_{\beta=1}^{N}\left[s+\chi_\beta\right]^{-1}(r_n)_\beta(\ell_n)_\beta\right]\cdot\left(B_{n,m}\right)^{-1}\cdot(b_n) \tag{B.12}$$

which is characterized by only simple (first-order) poles with real natural frequencies $-\chi_\beta$ (non positive for passive physical problems) and real coefficients. Other forms of excitation can be used in (B.10) which is just a convolution, which in complex frequency domain is

$$(\tilde{a}_n(s)) = \left[\sum_{\beta=1}^{N}\left[s+\chi_\beta\right]^{-1}(r_n)_\beta(\ell_n)_\beta\right]\cdot\left(B_{n,m}\right)^{-1}\cdot\left(\tilde{b}_n(s)\right) \tag{B.13}$$

The equivalence principle can be used to replace an incident field throughout all 3-dimensional space by one inside some volume (V_+ in fig. C.1) plus a set of equivalent sources on the boundary surface (S_+) [3, 13]. Consider that there is some incident field which we write in combined form

$$\vec{E}_q^{(inc)}(\vec{r},t) = \vec{E}^{(inc)}(\vec{r},t) + jqZ_0\,\vec{H}^{(inc)}(\vec{r},t) \qquad \text{(C.1)}$$

$$q = \pm 1 \text{ (separation index)}$$

This can be incident from ∞, such as a plane wave, or can come from some sources outside of S_+. Note that this incident wave is defined to exist in a free space medium with usual constitutive parameters (μ_0, ε_0).

This incident field can be replaced outside S_+ by a set of equivalent sources on S_+ as

$$\vec{J}_{S_q}^{(inc)}(\vec{r}_{S_+},t) = j\frac{q}{Z_0}\vec{1}_{S_+}(\vec{r}_{S_+}) \times \vec{E}_q^{(inc)}(\vec{r}_{S_+},t)$$

$$\equiv \text{ equivalent combined surface current density on } S_+$$

$$= \vec{J}_{S_e}^{(inc)}(\vec{r},t) + j\frac{q}{Z_0}\vec{J}_{S_h}^{(inc)}(\vec{r},t)$$

$$\text{(C.2)}$$

$$\vec{J}_{S_e}(\vec{r},t) = -\vec{1}_{S_+}(\vec{r}_{S_+}) \times \vec{H}^{(inc)}(\vec{r}_{S_+},t)$$

$$\equiv \text{ equivalent electric surface current density } (A/m) \text{ on } S_+$$

$$\vec{J}_{S_h}(\vec{r},t) = \vec{1}_{S_+}(\vec{r}_{S_+}) \times \vec{E}^{(inc)}(\vec{r}_{S_+},t)$$

$$\equiv \text{ equivalent electric surface current density } (V/m) \text{ on } S_+$$

$$\vec{1}_{S_+}(\vec{r}_{S_+}) \equiv \text{ outward pointing normal to } S_+$$

$$\vec{r}_{S_+} \equiv \text{ coordinates (position) on } S_+$$

With this set of equivalent sources (instead of those for the original incident wave) we have

$$\vec{E}_q^{(inc)}(\vec{r},t) = \begin{cases} \text{original } \vec{E}_q^{(inc)}(\vec{r},t) \text{ for } \vec{r} \in V_+ \\ \vec{0} \text{ for } \vec{r} \in V_+ \cup S_+ \end{cases} \qquad \text{(C.3)}$$

Now introduce the scatterer characterized by some general (even dyadic) constitutive parameters $\left(\overleftrightarrow{\mu}(\vec{r}), \overleftrightarrow{\varepsilon}(\vec{r}), \overleftrightarrow{\sigma}(\vec{r})\right)$ within the volume V of the scatterer where

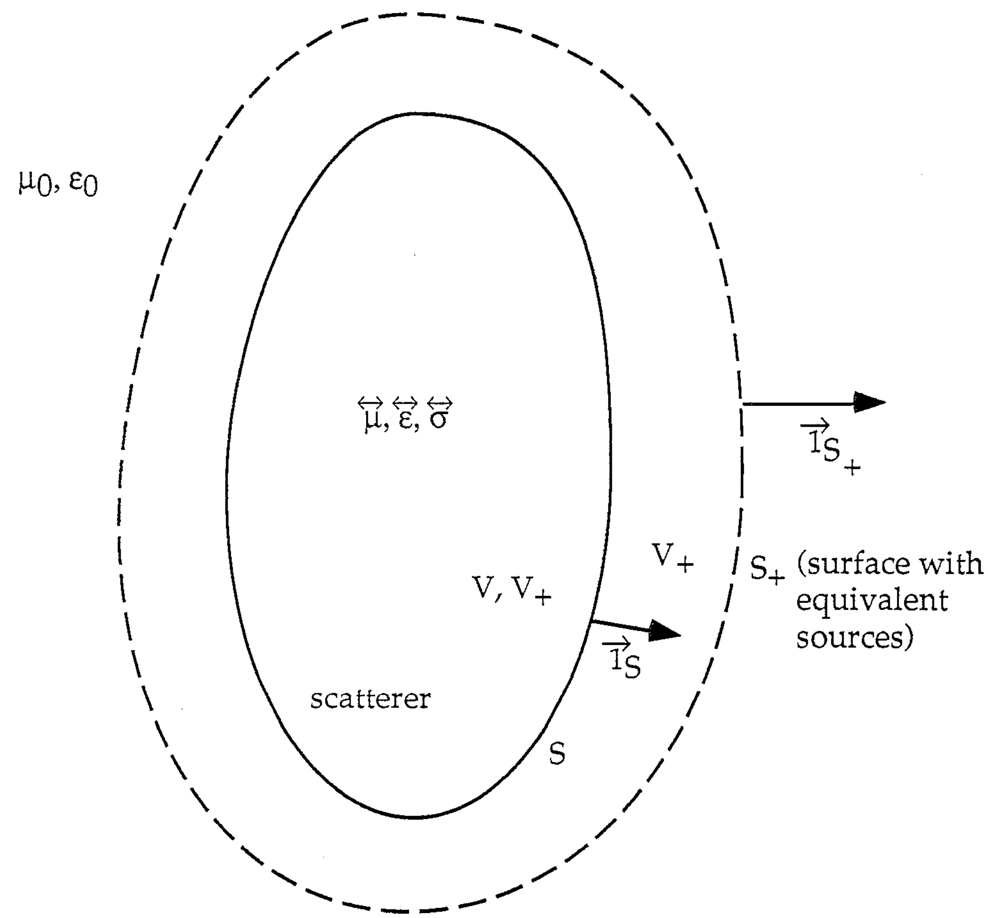

Fig. C.1. Scatterer Contained within Volume with Equivalent Sources on Boundary

$$V_{+} \supset [V \cup S] \tag{C.4}$$

This introduces a scattered field in response to the incident field with a total field as

$$\vec{E}_q(\vec{r},t) = \vec{E}_q^{(inc)}(\vec{r},t) + \vec{E}_q^{(sc)}(\vec{r},t) \tag{C.5}$$

Note that outside S the scattered field is in a free-space medium, the details of what happens inside V not being of interest in this Appendix.

Replacing the original incident field outside S_+ by the equivalent-surface-current-density sources on S_+, the incident field in V_+ remains unchanged and the scattered field remains the same throughout all space. Outside S_+ the scattered field is the only field.

Now consider the case of low frequencies such as discussed in Section II. Let

$$\begin{aligned} &\left|\frac{s}{c}\right| r << 1 \\ &r >> a \\ &a \equiv \text{ characteristic dimension of } S \text{ (scatterer)} \\ &r \equiv \text{ distance from } \vec{r}_0\text{, some reference point in scatterer} \end{aligned} \tag{C.6}$$

where r is some distance beyond which we can neglect the scattered fields due to r^{-3} dependence (electric- and magnetic-dipole static fields). Then outside S one can think of the scattered fields in a quasi-static sense.

One can separate the scattering into electric and magnetic parts by separately applying the two kinds of equivalent sources in (C.2). Associate with the incident magnetic field there is an equivalent electric surface current density on S_+, and conversely for the incident electric field. Then set the equivalent magnetic surface current density to zero and consider the solution of the Maxwell equations

$$\begin{aligned} &\nabla \times \tilde{\vec{E}}(\vec{r},s) = -s\overleftrightarrow{\mu}(\vec{r}) \cdot \tilde{\vec{H}}(\vec{r},s) \\ &\nabla \times \tilde{\vec{H}}(\vec{r},s) = \begin{cases} \overleftrightarrow{\sigma}(\vec{r}) \cdot \tilde{\vec{E}}(\vec{r},s) \text{ in } V \\ \tilde{\vec{J}}_{s_e}^{(inc)}(\vec{r}_{s_+},s)\,\delta\left((\vec{r}-\vec{r}_{s_+})\cdot \vec{1}_{s_+}\right) \text{ (one-dimensional delta function on going through } S_+) \end{cases} \end{aligned} \tag{C.7}$$

where the displacement current density has been neglected (as being small compared to conduction current density in V, and not important to the magnetic field outside S due to (C.6)). Furthermore, one can consider only that part of the equivalent surface-electric-current density source for which

$$\nabla_s \cdot \vec{J}_s^{(inc)}(\vec{r}_{s_+}) = 0 \tag{C.8}$$

so that any small residual part giving a surface electric charge density and associated electric field (noting factor of $1/s$) is removed. If one wishes one can include the divergence term (via loops) from the surface magnetic current density in the sources in (C.7).

An alternate approach is simply to define a divergenceless surface electric current density on S_+ which gives a uniform quasi-static magnetic field (incident field) in V_+. Various finite-size geometries (coils) give this, the simplest being perhaps a spherical coil giving a surface electric current density in the ϕ direction with a $\sin(\theta)$ distribution in an appropriate spherical coordinate system. By reorienting the z axis in this spherical coordinate system any particular quasi-static incident magnetic field $H_0 \, \vec{1}_1 \times \vec{1}_p$ (as in (2.3)) can be achieved. This approach is also appropriate for experimental realization.

Aside from the discontinuity at S_+ due to the equivalent source the magnetic field can now be represented outside S by

$$\begin{aligned} \vec{H}(\vec{r},t) &= -\nabla\Phi_h(\vec{r},t) \\ \nabla^2\Phi_h(\vec{r},t) &= 0 \end{aligned} \tag{C.9}$$

Note that $\overleftrightarrow{\mu}(\vec{r}) = \mu_0 \overleftrightarrow{1}$ outside S. This defines a quasi-magnetostatic scattering problem.

Similarly one can look at the response to the equivalent magnetic surface current density to define a quasi-electrostatic scattering problem. The total scattering is the sum of the scattering for the two parts. Relating this to the discussion in Section II, note the dominance of the distant scattering (both near and far fields) by induced electric and magnetic polarizability dyadics, each multiplying the incident electric and magnetic fields respectively. If, for example, one were to consider an incident plane wave (as in (2.3)), and then reverse the electric polarization and direction of incidence, leaving the magnetic polarization unchanged, the induced electric dipole moment would reverse sign but the induced magnetic dipole moment would remain unchanged. Consider then two oppositely propagating plane waves with incident electric fields summing to zero at $\vec{r}_0$. This gives no electric dipole moment, but twice the magnetic dipole moment.

The two oppositely incident plane waves with canceling electric fields at $\vec{r}_0$ can be thought of as another way to define the quasi-magnetostatic problem in (C.7). Within the approximation (C.6), zero incident electric field at $\vec{r}_0$ implies zero electric field on S_+, and hence zero equivalent magnetic surface current density. One can have other kinds of incident waves to give this property. For example, in the spherical vector wave functions one can choose the term corresponding to the magnetic dipole of the multipole expansion and take both incoming and outgoing waves to define an acceptable electric current density on S_+, say chosen as a sphere.

This quasi-magnetostatic problem then gives a way to separate out the magnetic part in low-frequency scattering. The quasi-electrostatic part can similarly be constructed with due care to the role of the permittivity and conductivity. While S_+ has been defined outside of and enclosing S, this is to avoid problems with the fields at S, the scatterer boundary. One can now let $S_+ \rightarrow S$ so that the equivalent sources are at the boundary of the scatterer, as convenient.

Appendix D. Scattering from Penetrable Sphere

Consider a sphere of radius a as in fig. D.1 with uniform, isotropic constitutive parameters μ_2, ε_2, and σ_2. The external medium is characterized by μ_1, ε_1, and σ_1. There is an incident wave which we take as incident along the z axis and polarized in the $\vec{1}_x$ direction (without loss of generality due to the symmetry). These give the propagation constants and wave impedances of the two media as

$$\begin{aligned}
&\tilde{\gamma}_1(s) = \left[s\mu_1(\sigma_1 + s\varepsilon_1)\right]^{\frac{1}{2}} \quad , \quad \tilde{\gamma}_2(s) = \left[s\mu_2(\sigma_2 + s\varepsilon_2)\right]^{\frac{1}{2}} \\
&\tilde{Z}_1(s) = \left[\frac{s\mu_1}{\sigma_1 + s\varepsilon_1}\right]^{\frac{1}{2}} \quad , \quad \tilde{Z}_2(s) = \left[\frac{s\mu_2}{\sigma_2 + s\varepsilon_2}\right]^{\frac{1}{2}}
\end{aligned} \tag{D.1}$$

For use elsewhere, when normalization is in terms of the incident magnetic field we have

$$E_0 \equiv \tilde{Z}_1(s) H_0 \tag{D.2}$$

for the incident plane wave.

From [11] we have the various spherical wave functions. The incident field takes the form

$$\begin{aligned}
\tilde{\vec{E}}^{(inc)}(\vec{r},s) &= E_0 \tilde{f}(s) \vec{1}_x e^{-\tilde{\gamma}_1 z} \\
&= E_0 \tilde{f}(s) \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \left[-\vec{M}^{(1)}_{n,1,o}(\tilde{\gamma}_1 \vec{r}) + \vec{N}^{(1)}_{n,1,e}(\tilde{\gamma}_1 \vec{r})\right] \\
\tilde{\vec{H}}^{(inc)}(\vec{r},s) &= \frac{E_0}{\tilde{Z}_1(s)} \vec{1}_y e^{-\tilde{\gamma}_1 z} \\
&= \frac{E_0}{\tilde{Z}_1(s)} \tilde{f}(s) \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \left[\vec{M}^{(1)}_{n,1,e}(\tilde{\gamma}_1 \vec{r}) + \vec{N}_{n,1,o}(\tilde{\gamma}_1 \vec{r})\right]
\end{aligned} \tag{D.3}$$

Expand the scattered fields in the exterior medium as

$$\begin{aligned}
\tilde{\vec{E}}^{(sc)}(\vec{r},s) &= E_0 \tilde{f}(s) \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \left[-a_n \vec{M}^{(2)}_{n,1,o}(\tilde{\gamma}_1 \vec{r}) + b_n \vec{N}^{(2)}_{n,1,e}(\tilde{\gamma}_1 \vec{r})\right] \\
\tilde{\vec{H}}^{(sc)}(\vec{r},s) &= \frac{E_0}{\tilde{Z}_1(s)} \tilde{f}(s) \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \left[b_n \vec{M}^{(2)}_{n,1,e}(\tilde{\gamma}_1 \vec{r}) + a_n \vec{N}^{(2)}_{n,1,o}(\tilde{\gamma}_1 \vec{r})\right]
\end{aligned} \tag{D.4}$$

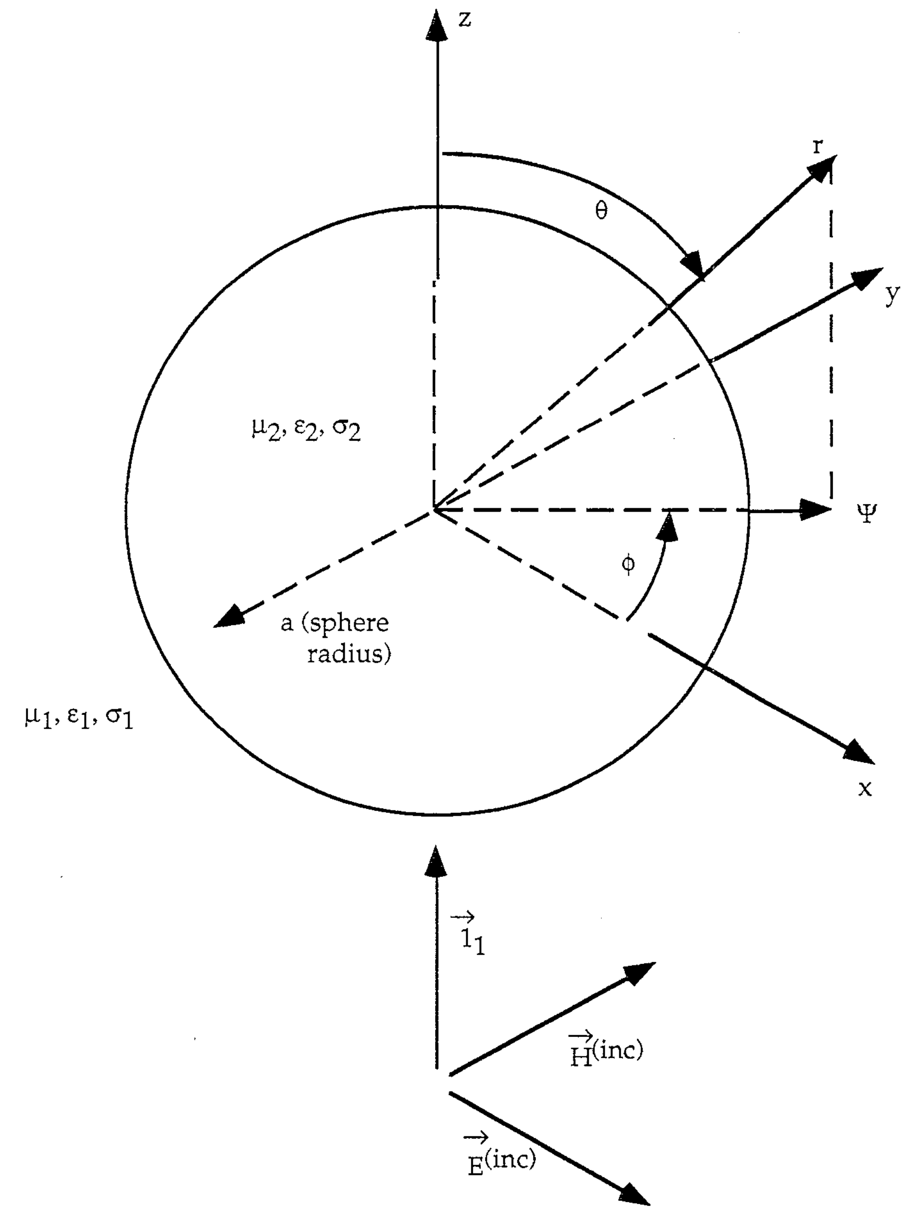

Fig. D.1. Scattering from Penetrable Spheres and interior to the sphere (medium 2) as

$$\vec{\tilde{E}}^{(2)}(\vec{r},s) = E_0 \tilde{f}(s) \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \left[-c_n \vec{M}^{(1)}_{n,1,o}(\tilde{\gamma}_2 \vec{r}) + d_n \vec{N}^{(1)}_{n,1,e}(\tilde{\gamma}_2 \vec{r}) \right]$$

$$\vec{\tilde{H}}^{(2)}(\vec{r},s) = \frac{E_0}{\tilde{Z}_2(s)} \tilde{f}(s) \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)} \left[d_n \vec{M}^{(1)}_{n,1,e}(\tilde{\gamma}_2 \vec{r}) + c_n \vec{N}^{(1)}_{n,1,o}(\tilde{\gamma}_2 \vec{r}) \right] \tag{D.5}$$

where

$$\vec{M}^{(\ell)}_{n,m,v}(\gamma\vec{r}) = f_n^{(\ell)}(\gamma r)\vec{R}_{n,m,v}(\theta,\phi) = -\frac{1}{\gamma}\nabla \times \vec{N}^{(\ell)}_{n,m,v}(\gamma\vec{r})$$

$$\vec{N}^{(\ell)}_{n,m,v}(\gamma r) = n(n+1)\frac{f_n^{(\ell)}(\gamma r)}{\gamma r}\vec{P}_{n,m,v}(\theta,\phi) + \frac{\left[\gamma r f_n^{(\ell)}(\gamma r)\right]'}{\gamma r}\vec{Q}_{n,m,v}(\theta,\phi)$$

$$= \frac{1}{\gamma}\nabla \times \vec{M}^{(\ell)}_{n,m,v}(\gamma\vec{r})$$

$$f_n^{(1)}(\gamma r) = i_n(\gamma r) \qquad \text{for waves analytic at } \gamma r = 0 \text{ (including waves incident from } \infty)$$

$$f_n^{(2)}(\gamma r) = k_n(\gamma r) \qquad \text{for outgoing waves}$$

$$Y_{n,m,v}(\theta,\phi) = P_n^{(m)}(\cos(\theta)) \begin{cases} \cos(m\phi) \text{ for } v = e \quad \text{(even)} \\ \sin(m\phi) \text{ for } v = o \quad \text{(odd)} \end{cases} \tag{D.6}$$

$$\vec{P}_{n,m,v}(\theta,\phi) = Y_{n,m,v}(\theta,\phi)\vec{1}_r$$

$$\vec{Q}_{n,m,v}(\theta,\phi) = \vec{1}_\theta \frac{\partial}{\partial\theta}Y_{n,m,v}(\theta,\phi) + \vec{1}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}Y_{n,m,v}(\theta,\phi)$$

$$= \vec{1}_r \times \vec{R}_{n,m,v}(\theta,\phi)$$

$$\vec{R}_{n,m,v}(\theta,\phi) = \vec{1}_\theta \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}Y_{n,m}(\theta,\phi) - \vec{1}_\phi \frac{\partial}{\partial\theta}Y_{n,m,v}(\theta,\phi)$$

$$= -\vec{1}_r \times \vec{Q}_{n,m,v}(\theta,\phi)$$

where a prime indicates differentiation with respect to the argument of the spherical Bessel function.

Matching the tangential electric field on r = a gives

$$i_n(\tilde{\gamma}_1 a) + a_n k_n(\tilde{\gamma}_1 a) = c_n i_n(\tilde{\gamma}_2 a)$$

$$\frac{\left[\tilde{\gamma}_1 a\, i_n(\tilde{\gamma}_1 a)\right]'}{\tilde{\gamma}_1 a} + b_n \frac{\left[\tilde{\gamma}_1 a\, k_n(\tilde{\gamma}_1 a)\right]'}{\tilde{\gamma}_1 a} = d_n \frac{\left[\tilde{\gamma}_2 a\, i_n(\tilde{\gamma}_2 a)\right]'}{\tilde{\gamma}_2 a} \tag{D.7}$$

Matching the tangential magnetic field on r = a gives

$$\frac{1}{\tilde{Z}_1(s)}\left\{i_n(\tilde{\gamma}_1 a)+b_n k_n(\tilde{\gamma}_1 a)\right\}=\frac{1}{\tilde{Z}_2(s)}d_n i_n(\tilde{\gamma}_2 a)$$

$$\frac{1}{\tilde{Z}_1(s)}\left\{\frac{\left[\tilde{\gamma}_1 a i_n(\tilde{\gamma}_1 a)\right]'}{\tilde{\gamma}_1 a}+a_n\frac{\left[\tilde{\gamma}_1 a k_n(\tilde{\gamma}_1 a)\right]'}{\tilde{\gamma}_1 a}\right\}=\frac{1}{\tilde{Z}_2(s)}c_n\frac{\left[\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)\right]'}{\tilde{\gamma}_2 a} \tag{D.8}$$

We need to solve for the a_n and b_n coefficients for the scattered field. These are readily found as

$$a_n=-\left\{i_n(\tilde{\gamma}_1 a)-\frac{\tilde{Z}_2(s)}{\tilde{Z}_1(s)}\frac{\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)}{\left[\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)\right]'}\frac{\left[\tilde{\gamma}_1 a i_n(\gamma_1 a)\right]'}{\tilde{\gamma}_1 a}\right\}$$

$$\left\{k_n(\gamma_1 a)-\frac{\tilde{Z}_1(s)}{\tilde{Z}_2(s)}\frac{\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)}{\left[\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)\right]'}\frac{\left[\tilde{\gamma}_1 a k_n(\gamma_1 a)\right]'}{\tilde{\gamma}_1 a}\right\}^{-1}$$

$$\tag{D.9}$$

$$b_n=-\left\{i_n(\gamma_1 a)-\frac{\tilde{Z}_1(s)}{\tilde{Z}_2(s)}\frac{\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)}{\left[\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)\right]'}\frac{\left[\tilde{\gamma}_1 a i_n(\tilde{\gamma}_1 a)\right]'}{\tilde{\gamma}_1 a}\right\}$$

$$\left\{k_n(\gamma_1 a)-\frac{\tilde{Z}_1(s)}{\tilde{Z}_2(s)}\frac{\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)}{\left[\tilde{\gamma}_2 a i_n(\tilde{\gamma}_2 a)\right]'}\frac{\left[\tilde{\gamma}_1 a k_n(\tilde{\gamma}_1 a)\right]'}{\tilde{\gamma}_1 a}\right\}^{-1}$$

The far-field scattering has the form

$$\tilde{\vec{E}}_f(\vec{r},s)=\frac{e^{-\tilde{\gamma}_1(s)r}}{4\pi r}\tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_r,\vec{1}_z;s\right)\cdot\tilde{\vec{E}}^{(inc)}(\vec{0},s) \tag{D.10}$$

For backscattering the scattering dyadic takes the form

$$\vec{1}_r=-\vec{1}_z \quad , \quad \theta=\pi$$

$$\tilde{\overleftrightarrow{\Lambda}}_b\left(\vec{1}_z,s\right)=\tilde{\overleftrightarrow{\Lambda}}\left(-\vec{1}_z,\vec{1}_z;s\right) \tag{D.11}$$

Due to the rotation symmetry of the scatterer about the z axis, the backscattering dyadic takes the form of a function of frequency times $\overleftrightarrow{1}_z$ (no depolarization) where

$$\overleftrightarrow{1}_z=\overleftrightarrow{1}-\vec{1}_z\vec{1}_z \tag{D.12}$$

Expanding the scattered electric field for large r gives

$$k_n(\tilde{\gamma}_1 r) = \frac{e^{-\tilde{\gamma}_1 r}}{\tilde{\gamma}_1 r}\left[1 + O\left((\tilde{\gamma}_1 r)^{-1}\right)\right] \quad \text{as } \tilde{\gamma}_1 r \to \infty$$
$$\frac{\left[\tilde{\gamma}_1 r k_n(\tilde{\gamma}_1 r)\right]'}{\tilde{\gamma}_1 r} = -\frac{e^{-\tilde{\gamma}_1 r}}{\tilde{\gamma}_1 r}\left[1 + O\left((\tilde{\gamma}_1 r)^{-1}\right)\right] \quad \text{as } \tilde{\gamma}_1 r \to \infty \tag{D.13}$$

This gives the far scattered electric field as

$$\tilde{\vec{E}}_f(\vec{r}, s) = -E_o \tilde{f}(s) \frac{e^{-\tilde{\gamma}_1 r}}{\tilde{\gamma}_1 r} \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n(n+1)}\left[a_n \vec{R}_{n,1,o}(\theta,\phi) + b_n \vec{Q}_{n,1,e}(\theta,\phi)\right] \tag{D.14}$$

Specializing this to backscattering $(\theta = \pi)$ we have

$$P_n^{(1)}(\cos(\theta))\Big|_{\theta=\pi} = 0 \quad , \quad \frac{P_n^{(1)}(\cos(\theta))}{\sin(\theta)}\Bigg|_{\theta=\pi} = (-1)^n \frac{n(n+1)}{2}$$
$$\frac{d}{d\theta} P_n^{(1)}(\cos(\theta))\Big|_{\theta=\pi} = (-1)^{n+1} \frac{n(n+1)}{2}$$
$$\vec{Q}_{n,m,e}(\pi,\phi) = (-1)^n \frac{n(n+1)}{2}\left[-\cos(\phi)\vec{1}_\theta - \sin(\phi)\vec{1}_\phi\right]$$
$$= (-1)^n \frac{n(n+1)}{2}\vec{1}_x$$
$$\vec{R}_{n,m,o}(\pi,\phi) = (-1)^n \frac{n(n+1)}{2}\left[\cos(\phi)\vec{1}_\theta + \sin(\phi)\vec{1}_\phi\right]$$
$$= (-1)^{n+1} \frac{n(n+1)}{2}\vec{1}_x \tag{D.15}$$

The far backscattered electric field is then

$$\tilde{\vec{E}}_f(-z\vec{1}_z, s) = \vec{1}_x \; E_0 \tilde{f}(s) \frac{e^{-\tilde{\gamma}_1 r}}{\tilde{\gamma}_1 r} \sum_{n=1}^{\infty} \frac{2n+1}{2}\left[a_n - b_n\right] \tag{D.16}$$

Noting the rotation symmetry (C_∞) with respect to the z axis, the backscattering dyadic is

$$\tilde{\overleftrightarrow{\Lambda}}_b\left(\vec{1}_z, s\right) = \overleftrightarrow{1}_z \frac{2\pi}{\tilde{\gamma}_1} \sum_{n=1}^{\infty} (2n+1)\left[a_n - b_n\right] \tag{D.17}$$

Due to the continuous rotation symmetry in three dimensions (O_3^+) the direction of incident $\vec{1}_z$ can be replaced by another, say $\vec{1}_1$ with associated transverse dyadic.

A well-known special case is the perfectly conducting sphere for which we can set

$$
\begin{aligned}
&\tilde{Z}_2(s) = 0 \\
&a_n = -\frac{i_n(\tilde{\gamma}_1 a)}{k_n(\tilde{\gamma}_1 a)} \\
&b_n = -\frac{\left[\tilde{\gamma}_1 a i_n(\tilde{\gamma}_1 a)\right]'}{\left[\tilde{\gamma}_1 a k_n(\tilde{\gamma}_1 a)\right]'} \\
&\tilde{\overleftrightarrow{\Lambda}}_b\left(\overrightarrow{1}_z, s\right) = \overleftrightarrow{1}_z 2\pi a \sum_{n=1}^{\infty} \frac{(2n+1)}{\gamma_1 a}\left\{\frac{\left[\tilde{\gamma}_1 a i_n(\tilde{\gamma}_1 a)\right]'}{\left[\tilde{\gamma}_1 a k_n(\tilde{\gamma}_1 a)\right]'} - \frac{i_n(\tilde{\gamma}_1 a)}{k_n(\tilde{\gamma}_1 a)}\right\} \\
&\qquad = \overleftrightarrow{1}_z 2\pi a \sum_{n=1}^{\infty} (2n+1)\left[(\tilde{\gamma}_1 a)^2 k_n(\tilde{\gamma}_1 a)\left[\tilde{\gamma}_1 a k_n(\tilde{\gamma}_1 a)\right]'\right]^{-1}
\end{aligned}
\tag{D.18}
$$

which agrees with previous results [11, 46].

Appendix E. Quasi-Magnetostatic Scattering from Highly Conducting Permeable Sphere

For the sphere problem in Appendix D, now consider low frequencies (as in Section II), for which (for bounded r) we have

$$\begin{gathered}\tilde{\gamma}_1 r \to 0 \quad , \quad \tilde{\gamma}_1 a \to 0 \\ \frac{\tilde{Z}_2(s)}{\tilde{Z}_1(s)} = \frac{\tilde{\gamma}_1 s \mu_2}{\tilde{\gamma}_2 s \mu_1} = \frac{\tilde{\gamma}_1 a}{\tilde{\gamma}_2 a}\mu_r \to 0 \quad , \quad \mu_r \equiv \frac{\mu_2}{\mu_1}\end{gathered} \tag{E.1}$$

while radian wavelengths or skin depths in the penetrable sphere are comparable to a. Then from (D.2) through (D.4) the magnetic fields of interest are

$$\begin{gathered}\tilde{\vec{H}}^{(inc)}(\vec{r},s) = H_0 \tilde{f}(s)\vec{1}_y \\ \tilde{\vec{H}}^{(sc)}(\vec{r},s) = H_0 \tilde{f}(s)\left(-\frac{3}{2}\right)\left[b_1 \vec{M}^{(2)}_{1,1,e}\left((\tilde{\gamma}_1 \vec{r})\right) + a_1 \vec{N}^{(2)}_{1,1,o}\left((\tilde{\gamma}_1 \vec{r})\right)\right]\end{gathered} \tag{E.2}$$

where we have just kept the dipole (n = 1) terms in the scattered magnetic field, the higher order multipoles being negligible at these low frequencies. The magnetic-dipole term is the one now to be exhibited.

Expanding the spherical Bessel functions gives

$$\begin{gathered}i_1(\tilde{\gamma}_1 r) \to \frac{\tilde{\gamma}_1 r}{3} \quad , \quad k_1(\tilde{\gamma}_1 r) \to (\tilde{\gamma}_1 r)^{-2} \\ \left[\tilde{\gamma}_1 r i_1(\tilde{\gamma}_1 r)\right]' \to \frac{2}{3}\tilde{\gamma}_1 r \quad , \quad \left[\tilde{\gamma}_1 r k_1(\tilde{\gamma}_1 r)\right]' \to -(\tilde{\gamma}_1 r)^{-2} \\ \text{as } \tilde{\gamma}_1 r \to 0\end{gathered} \tag{E.3}$$

Applying this to the vector wave functions gives

$$\begin{gathered}\vec{M}^{(2)}_{1,1,e}(\tilde{\gamma}_1 r) = k_1(\tilde{\gamma}_1 r)\vec{R}_{1,1,e}(\theta,\phi) \to (\gamma_1 r)^{-2}\vec{R}_{1,1,e}(\theta,\phi) \\ \vec{N}^{(2)}_{1,1,o}(\tilde{\gamma}_1 r) = \frac{2k_1(\tilde{\gamma}_1 r)}{\tilde{\gamma}_1 r}\vec{P}_{1,1,o}(\theta,\phi) + \frac{\left[\tilde{\gamma}_1 r k_1(\tilde{\gamma}_1 r)\right]'}{\tilde{\gamma}_1 r}\vec{Q}_{1,1,o}(\theta,\phi) \\ \to 2(\tilde{\gamma}_1 r)^{-3}\vec{P}_{1,1,o}(\theta,\phi) - (\tilde{\gamma}_1 r)^{-3}\vec{Q}_{1,1,o}(\theta,\phi) \\ \text{as } \tilde{\gamma}_1 r \to 0\end{gathered} \tag{E.4}$$

Here we can identify the r^{-3} dependence of the magnetic dipole, the other term giving the near magnetic field of an electric dipole.

So the scattered magnetic field is

$$\tilde{\vec{H}}^{(sc)}(\vec{r},s) = H_0 \tilde{f}(s) \frac{3}{2} a_1 (\tilde{\gamma}_1 r)^{-3} \left[-2\vec{P}_{1,1,o}(\theta,\phi) + \vec{Q}_{1,1,o}(\theta,\phi) \right]$$
$$\text{as } (\tilde{\gamma}_1 r) \to 0 \tag{E.5}$$

The vector spherical harmonics are rewritten as

$$\begin{aligned} -2\vec{P}_{1,1,o}(\theta,\phi) &= \vec{Q}_{1,1,o}(\theta,\phi) \\ &= 2\sin(\theta)\sin(\phi)\,\vec{1}_r - \cos(\theta)\sin(\phi)\,\vec{1}_\theta - \cos(\phi)\,\vec{1}_\phi \\ &= 3\sin(\theta)\sin(\phi)\,\vec{1}_r - \vec{1}_y \\ &= 3\,\vec{1}_r\vec{1}_r \cdot \vec{1}_y - \vec{1}_y \\ &= \left[3\,\vec{1}_r\vec{1}_r - \overleftrightarrow{1} \right] \cdot \vec{1}_y \end{aligned} \tag{E.6}$$

This allows us to identify the magnetic polarizability dyadic from (2.15), with coordinate origin now at $z = 0$ as

$$\begin{aligned} &\tilde{\vec{H}}^{(sc)}(\vec{r},s) = \frac{1}{4\pi r^3} \left[3\,\vec{1}_r\vec{1}_r - \overleftrightarrow{1} \right] \cdot \tilde{\overleftrightarrow{M}}(s) \cdot \tilde{\vec{H}}^{(inc)}(\vec{0},s) \\ &\text{as } s \to 0 \\ &\tilde{\overleftrightarrow{M}}(s) = \overleftrightarrow{1}\,\tilde{M}(s) \\ &\tilde{M}(s) = 6\pi\gamma_1^{-3} a_1 \end{aligned} \tag{E.7}$$

From (D.9) we have for our quasi-magnetostatic case

$$\begin{aligned} a_1 &= -\left\{ \frac{1}{3}\tilde{\gamma}_1 a - \frac{2}{3}\tilde{\gamma}_1 a \mu_r \frac{i_1(\tilde{\gamma}_2 a)}{[\tilde{\gamma}_2 a i_1(\tilde{\gamma}_2 a)]'} \right\} \left\{ (\tilde{\gamma}_1 a)^{-2} + (\tilde{\gamma}_1 a)^{-2} \mu_r \frac{i_1(\tilde{\gamma}_2 a)}{[\tilde{\gamma}_2 a i_1(\tilde{\gamma}_2 a)]'} \right\}^{-1} \\ \tilde{M}(s) &= 2\pi a^3 \frac{2\mu_r i_1(\tilde{\gamma}_2 a) - [\tilde{\gamma}_2 a i_1(\tilde{\gamma}_2 a)]'}{\mu_r i_1(\tilde{\gamma}_2 a) + [\tilde{\gamma}_2 a i_1(\gamma_2 a)]'} \\ &= 2\pi a^3 \frac{[2\mu_r - 1] i_1(\tilde{\gamma}_2 a) - \gamma_2 a i_1'(\gamma_2 a)}{[\mu_r + 1] i_1(\tilde{\gamma}_2 a) + \gamma_2 a i_1'(\gamma_2 a)} \\ &= 2\pi a^3 \left\{ \frac{3\mu r i_1(\gamma_2 a)}{\mu_r i_1(\tilde{\gamma}_2 a) + [\tilde{\gamma}_2 a i_1(\gamma_2 a)]'} - 1 \right\} \end{aligned} \tag{E.8}$$

There are limiting cases of interest. In the low-frequency limit we have

$$\tilde{M}(0) = 4\pi a^3 \frac{\mu_r - 1}{\mu_r + 2} \begin{cases} \to 0 & \text{for } \mu_r \to 1 \\ \to 4\pi a^3 & \text{for } \mu_r \to \infty \end{cases} \tag{E.9}$$

which is in agreement with the result for non-permeable scatterers ($\mu_r = 1$) discussed in Section III. For high frequencies the sphere behaves as though it were perfectly conducting for which we have [11]

$$\begin{aligned} i_1(\tilde{\gamma}_2 a) &= \frac{e^{\tilde{\gamma}_2 a}}{2\tilde{\gamma}_2 a}\left[1 + O\left((\tilde{\gamma}_2 a)^{-1}\right)\right] \text{ in RHP} \\ \left[\tilde{\gamma}_2 a i_1(\tilde{\gamma}_2 a)\right]' &= \frac{e^{\tilde{\gamma}_2 a}}{2}\left[1 + O\left((\tilde{\gamma}_2 a)^{-1}\right)\right] \text{ in RHP} \\ \tilde{M}(s) &= -2\pi a^3 \text{ as } \sigma \to \infty \text{ (or } \tilde{\gamma}_2 \to \infty) \end{aligned} \tag{E.10}$$

giving a negative polarizability as expected.

Expressing the spherical Bessel function by hyperbolic functions as [4]

$$\begin{aligned} i_1(\xi) &= \frac{\cosh(\xi)}{\xi} - \frac{\sinh(\xi)}{\xi^2} \\ \left[\xi i_1(\xi)\right]' &= -\frac{\cosh(\xi)}{\xi} + \sinh(\xi)\left[1 + \xi^{-2}\right] \\ \xi &\equiv \gamma_2 a \end{aligned} \tag{E.11}$$

the magnetic polarizability is written as

$$\begin{aligned} \tilde{M}(s) &= 2\pi a^3 \frac{2\mu_r[\xi\cosh(\xi) - \sinh(\xi)] + \xi\cosh(\xi) - \sinh(\xi)\left[1 + \xi^2\right]}{\mu_r[\xi\cosh(\xi) - \sinh(\xi)] - \xi\cosh(\xi) + \sinh(\xi)\left[1 + \xi^2\right]} \\ &= 2\pi a^3 \frac{[2\mu_r + 1]\xi\cosh(\xi) - \left[2\mu_r + 1 + \xi^2\right]\sinh(\xi)}{[\mu_r - 1]\xi\cosh(\xi) - \left[\mu_r - 1 - \xi^2\right]\sinh(\xi)} \\ &= 2\pi a^3 \frac{[2\mu_r + 1]\xi\coth(\xi) - 2\mu_r - 1 - \xi^2}{[\mu_r - 1]\xi\coth(\xi) - \mu_r + 1 + \xi^2} \end{aligned} \tag{E.12}$$

The natural frequencies are given by the roots of the denominator (excluding $\xi = 0$) as

$$
\begin{aligned}
&\xi_\alpha = [s_\alpha \mu_2 \sigma_2]^{\frac{1}{2}} a \\
&s_\alpha \mu_2 \sigma_2 a^2 = \xi_\alpha^2 \\
&\coth(\xi_\alpha) = \frac{\mu_r - 1 - \xi_\alpha^2}{[\mu_r - 1]\xi_\alpha} = \xi_\alpha^{-1} - \frac{\xi_\alpha}{\mu_r - 1} \\
&\cot(j\xi_\alpha) = \frac{1}{j\xi_\alpha} + \frac{j\xi_\alpha}{\mu_r - 1}
\end{aligned}
\tag{E.13}
$$

This is known to have all imaginary roots ξ_α [21, 48], implying

$$
s_\alpha = \frac{\xi_\alpha^2}{\mu_2 \sigma_2 a^2} < 0 \quad \text{for all } \alpha \tag{E.14}
$$

so that all the natural frequencies are negative real. Note from (E.13) how the natural frequencies scale, proportional to $\left[\mu_2 \sigma_2 a^2\right]^{-1}$, which shows the diffusion character of the problem. However, ξ_α^2 also involves the scattered magnetic field in the external medium through μ_r, altering this simple scaling somewhat.

The magnetic polarizability is, of course, an operator in time domain. For a step-function incident magnetic field one can use

$$
\int_{-\infty}^{t} M(t')dt' = 2\pi a^3 \left\{ 2\frac{\mu_r - 1}{\mu_r + 2} - 6\mu_r \sum_\alpha \frac{e^{s_\alpha t}}{[\mu_r + 2][\mu_r - 1] - \xi_\alpha^2} \right\} u(t) \tag{E.15}
$$

Note the positive term from $\widetilde{M}(0)$ which gives the late-time response. This is reversed at early time by the decaying exponentials, all with negative residues. The delta-function response is just

$$
M(t) = 2\pi a^3 \left\{ -\delta(t) - 6\mu_r \left\{ \sum_\alpha \frac{s_\alpha e^{s_\alpha t}}{[\mu_r + 2][\mu_r - 1] - \xi_\alpha^2} \right\} u(t) \right\} \tag{E.16}
$$

where the delta-function part is related to the high-frequency limit in (E.10). These response are tabulated and displayed in [21, 44, 49].

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