Interaction Notes

Note 497

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Transforms of Frequency Spectra in Target Identification

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Abstract

This paper considers the use of filter and wavelet transforms on the frequency spectrum of transient/broad-band electromagnetic scattering. The multiresolution in frequency thereby attainable can be used to analyze frequency-like target signatures. This applies to global and substructure target features. An interesting special case concerns a linear array of scatterers which can be approximately considered as giving a set of poles on the $j\omega$ axis of the complex frequency plane. The associated frequencies are aspect dependent.

I. Introduction

A recent paper [2] considers the use of window Laplace/Fourier transforms (WLT) and wavelet transforms (WT) for processing time-domain waveforms f(t) that one may encounter in the scattered signals from targets. There the concern was in resolving target substructures such as might be used in a scattering center representation of a target for identification purposes. The symmetry properties of these substructures (partial symmetries) were found to be carried over into the symmetry properties of the 2 x 2 dyadic scattering operator corresponding to the particular substructures.

In summary form we have the Laplace/Fourier transform

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st} dt \equiv LT[f(t)] \equiv \text{2-sided Laplace (or Fourier) transform}$$

$$f(t) = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st} ds$$

$Br \equiv$ Bromwich contour along $\mathrm{Re}[s] = \Omega_{Br}$ from $\Omega_{Br} - j\infty$ to $\Omega_{Br} + j\infty$ in strip of convergence

$$s \equiv \Omega + j\infty \equiv \text{complex frequency} \tag{1.1}$$

$$\mathrm{f(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega)\, e^{j\omega t}\, d\omega \ \text{(for } j\omega \text{ axis in strip of convergence)}$$

with the Parseval-like relation

$$\int_{-\infty}^{\infty} f_1(t)\ f_2(t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s)\ \tilde{f}_2(-s)\, ds = \frac{1}{2\pi j} \int_{-\infty}^{\infty} \tilde{f}_1(j\omega)\ \tilde{f}_2(-j\omega)\, d\omega \tag{1.2}$$

and the convolution formulas

$$LT[f_1(t)\ f_2(t)] = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s')\ \tilde{f}_2(s-s')\, ds' = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(j\omega')\ \tilde{f}_2(j\omega - j\omega')\, d\omega'$$

$$LT\left[\int_{-\infty}^{\infty} f_1(t')\ f_2(t-t')\, dt'\right] = \tilde{f}_1(s)\,\tilde{f}_2(s) \tag{1.3}$$

By multiresolution [7] is meant the analysis of a signal on various different scales of resolution. For temporal multiresolution we have

$$\frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) e^{-st} \equiv \text{triwave kernel}$$

$$\frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) \equiv \text{window or wavelet} \tag{1.4}$$

$$t_1 > 0\,,\; t_o \text{ real}$$

Here t_1 is a scaling time which adjusts the width of the window in time, this width being of the order of t_1, depending on the specific window function g chosen. Noting the frequency form of this window as

$$LT\left[\frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right)\right] = e^{-st_o} \underset{\tau\to\Sigma}{LT}\left[g(\tau)\right] = e^{-st_o}\, \tilde{g}(\Sigma) \tag{1.5}$$

we have the triwave transform as

$$TT[f(t)] \equiv \tilde{f}(s,t_o,t_1) \equiv \int_{-\infty}^{\infty} f(t)\frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) e^{-st} dt$$

$$= \frac{e^{-st_o}}{2\pi j}\int_{Br} \tilde{f}(s')\;\; g(st_1 - s't_1)\, e^{s't_o}\, dt_o \tag{1.6}$$

$$= \frac{e^{-j\omega t_o}}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(j\omega')\, g(j\omega t_1 - j\omega' t_1)\, e^{j\omega' t_o} d\omega'$$

So in its temporal form one multiplies by a window function which emphasizes times near t_0 with a resolution t_1. Furthermore the transform has an alternate representation as a convolution in frequency domain.

This triwave transform has special cases as the window Laplace/Fourier transform

$$\tilde{f}(s,t_o) = \tilde{f}(s,t_o,t_1) \equiv WLT[f(t)] \tag{1.7}$$

where the variation of t_1 is suppressed, and the wavelet transform

$$\hat{f}(t_o,t_1) = \tilde{f}(0,t_o,t_1) \equiv WLT[f(t)] \tag{1.8}$$

where complex frequency s is removed, except that it can still be expressed as an integral over s', ω' as in (1.6). The concept of phase space for the WLT where one plots $\left|\tilde{f}(j\omega,\; t_o)\right|$ with ω as the vertical axis and t_0 as the horizontal axis, can be extended to the WT by regarding $1/t_1$ as an equivalent frequency for the vertical axis as indicated in fig. 1.1A. With $f(t)$ and $g(t)$ taken as real valued (for real t) we can note that $\hat{f}(t_o, t_1)$ is real valued. As indicated in fig. 1.1A, temporal multiresolution concerns some set of values t_o, which for decreasing t_1 (increasing $1/t_1$) have interesting events which are analyzed on multiple t_1 scales. Frequency multiresolution as indicated in fig. 1.1B concerns the frequency form of the wavelet transform to be discussed later.

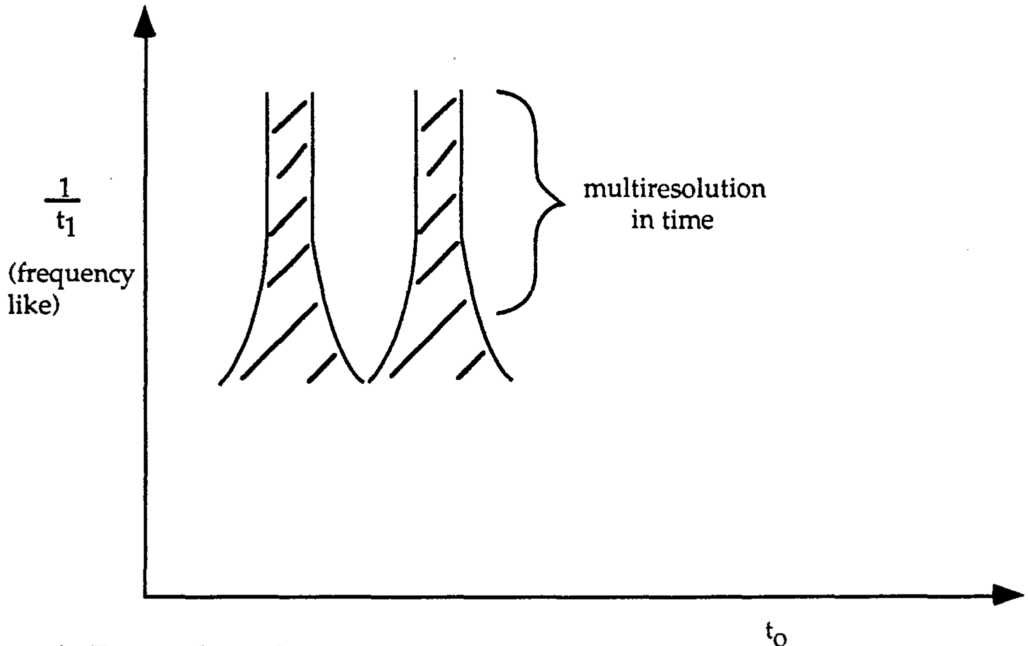
An important distinction between the WLT and WT is in the inversion of the two transforms. The window Laplace/Fourier transform has

$$\begin{aligned} f(t) &= \left\{ \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \right\}^{-1} \frac{1}{2\pi j} \int_{-\infty}^{\infty} \int_{Br} \tilde{f}(s, t_o, t_1)\, g\left(\frac{t - t_o}{t_1} \right) e^{st}\, ds\, dt_o \\ &= \left\{ \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \right\}^{-1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{Br}^{\infty} \tilde{f}(j\omega, t_o, t_1)\, g\left(\frac{t - t_o}{t_1} \right) e^{j\omega t}\, d\omega\, dt_o \end{aligned} \tag{1.9}$$

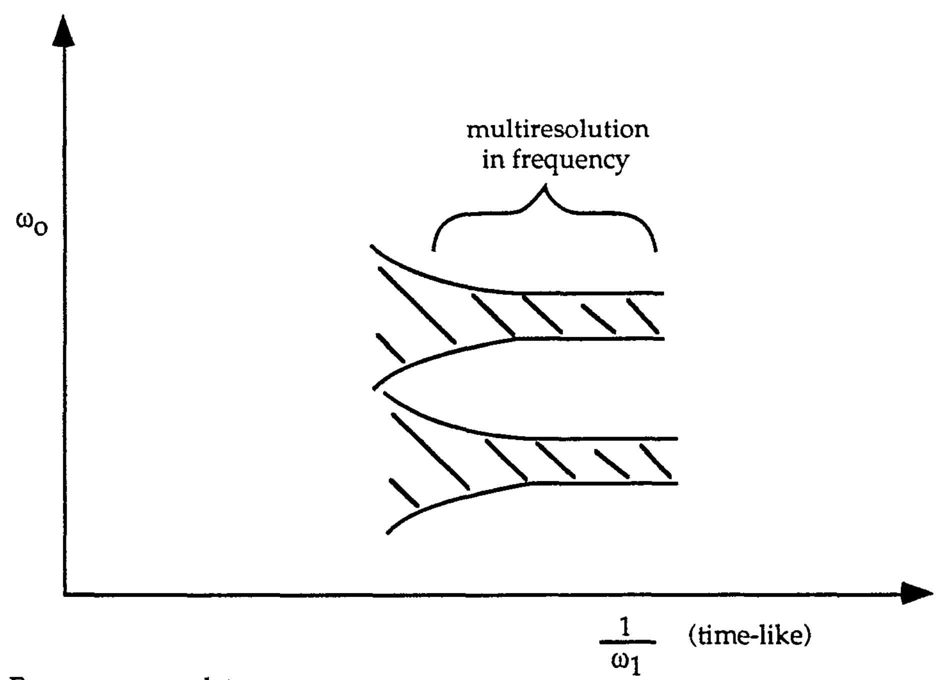
with an admissibility condition on g concerning its square integrability. The wavelet transform has

$$f(t) = \left\{ \int_0^{\infty} \left| \tilde{g}(j\xi) \right|^2 \frac{d\xi}{\xi} \right\}^{-1} \int_0^{\infty} \int_{-\infty}^{\infty} \hat{f}(t_o, t_1) \frac{1}{t_1}\, g\left(\frac{t - t_o}{t_1} \right) dt_o\, dt_1 \tag{1.10}$$

with an admissibility condition on g concerning the integrability of $|\tilde{g}|^2 / \xi$.



A. Temporal wavelet



B. Frequency wavelet

Fig. 1.1. Phase Space for Wavelets

II. Waveform Spectra Transforms

A. Frequency Triwave transform

Consider the kernel

$$\frac{1}{2\pi j}\,\frac{1}{\omega_1}\,\tilde{G}\left(\frac{s-s_o}{\omega_1}\right)e^{st} \quad , \quad \omega_1 > 0 \tag{2.1}$$

where s_0 is often taken as $j\omega_0$ (ω_0 real). This has three parameters

$$\begin{aligned} t &\equiv \text{time (real)} \\ s_0 &\equiv \text{frequency shift} \\ \omega_1 &\equiv \text{frequency dilation (width of filter or wavelet)} \end{aligned} \tag{2.2}$$

with the complex frequency s integrated out in the transforms. The filter or wavelet can also be expressed in time domain via

$$\begin{aligned} \frac{1}{\omega_1}\tilde{G}\left(\frac{s-s_o}{\omega_1}\right) &\equiv \text{filter or wavelet} \\ &= LT\left[e^{s_o t}G(\omega_1 t)\right] \end{aligned} \tag{2.3}$$

Note that if the filter is chosen as real for $s\text{-}s_0 = j\omega - j\omega_0$, then in time domain it is not necessarily real. One may wish to take this into account when selecting $\tilde{G}$ to perhaps correspond to some realizable filter (in a circuit-theory sense).

Using all three parameters the frequency triwave transform FTT is

$$\begin{aligned} FTT\left[\tilde{f}(s)\right] &= F(t,s_o,\omega_1) = \frac{1}{2\pi j}\int_{Br}\tilde{F}(s,s_o,\omega_1)\,e^{st}ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(j\omega,j\omega_o,\omega_1)\,e^{j\omega t}d\omega \\ \tilde{F}(s,s_o,\omega_1) &= \tilde{f}(s)\,\frac{1}{\omega_1}\,\tilde{G}\left(\frac{s-s_o}{\omega_1}\right) \end{aligned} \tag{2.4}$$

Utilizing the convolution theorem we have the alternate form

$$F(t,s_o\omega_1) = e^{s_ot}\int_{Br} f(t')\;G(\omega_1(t-t'))\,e^{-s_ot'}dt'$$

$$= e^{j\omega_ot}\int_{-\infty}^{\infty} f(t')\;G(\omega_1(t-t'))\,e^{-j\omega_ot'}dt' \tag{2.5}$$

as an integral over time.

Comparing to the previous paper [2] there is a mathematically dual form between the triwave transform (TT) and the frequency triwave transform (FTT) due to the formal similarity of the Laplace/Fourier transform and the inverse transform. In the time-domain form (TT) one multiplies *f(t)* by a window function $\frac{1}{t_1}\,g\left(\frac{t-t_o}{t_1}\right)$. This window multiplication can also be expressed as a convolution in frequency. In the frequency-domain form (FTT) one multiplies $\tilde{f}(s)$ by a filter function $\frac{1}{\omega_1}\tilde{G}\left(\frac{s-s_o}{\omega_1}\right)$. This filter multiplication can also be expressed as a convolution in time. However, linear time-invariant systems (such as electromagnetic scatterers) are not expressed the same way in both domains. There is convolution in time, but multiplication in frequency. Due to time translation invariance t = 0 is an arbitrary definition (whereas s = 0 is not arbitrary). In the present FTT we do not have the flexibility of moving t = 0 as we do by use of t_0 in the TT. However, the two approaches are somewhat complementary, the first for resolving local features via temporal events, and the second for resolving global features via frequency identification. Multiresolution in time concerns various scales of resolution t_1 around selected times t_0. Multiresolution in frequency concerns various scales of resolution ω_1 around selected frequencies ω_o. This comparison is illustrated in fig. 1.1.

B. Filter Inverse-Laplace/Fourier Transform

The filter inverse-Laplace/Fourier transform (FILT) is given by

$$F(t,s_o) \equiv FILT\left[\tilde{f}(s)\right] = FTT\left[\tilde{f}(s)\right] = F(t,s_o,\omega_1) \tag{2.6}$$

with dependence on t_1 suppressed in the triwave form (giving a "biwave transform"). Defining

$$\tilde{F}_n(s,s_o,\omega_1) \equiv \tilde{f}_n(s)\frac{1}{\omega_1}\tilde{G}\left(\frac{s-s_o}{\omega_1}\right) \tag{2.7}$$

and following the procedure in [2] with time and frequency interchanged gives a Parseval-like relation as

$$- \frac{1}{4\pi^2} \int_{Br} \int_{Br} \tilde{F}_1(s', s_o, \omega_1) \ \tilde{F}_2(-s', -s_o, \omega_1) \, ds_o \, ds'$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{F}_1(j\omega', j\omega_o, \omega_1) \ \tilde{F}_2(-j\omega', -j\omega_o, \omega_1) \, d\omega_o \, d\omega'$$

$$= \left\{ \frac{1}{2\pi\omega_1} \int_{-\infty}^{\infty} \tilde{G}(j\chi) \, \tilde{G}(-j\chi) \, d\chi \right\} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(j\omega') \, \tilde{f}_2(-j\omega') \, d\omega' \right\}$$

$$= \left\{ \frac{1}{2\pi j \omega_1} \int_{Br} \tilde{G}(\Sigma) \, \tilde{G}(-\Sigma) \, d\Sigma \right\} \left\{ \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s') \, \tilde{f}_2(-s') \, ds' \right\}$$

$$= \frac{1}{2\pi j} \int_{Br} \int_{-\infty}^{\infty} F_1(t', s_o, \omega_1) \ F_2(t', -s_o, \omega_1) \, dt' \, ds_o \tag{2.8}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F_1(t', j\omega_o, \omega_1) \ F_2(t', -j\omega_o, \omega_1) \, dt' \, d\omega_o$$

As a special case choose

$$\tilde{f}_1(s') = \tilde{f}(s') \quad , \quad \tilde{f}_2(-s') = \tilde{f}_2(-j\omega') = \delta(\omega' - \omega) \tag{2.9}$$

giving

$$F_2(t', -s_o, \omega_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_2(j\omega') \frac{1}{\omega_1} \tilde{G}\left(\frac{j\omega' + j\omega_o}{\omega_1} \right) e^{j\omega' t'} d\omega'$$

$$= \frac{1}{2\pi} \frac{1}{\omega_1} \tilde{G}\left(\frac{j\omega + j\omega_o}{\omega_1} \right) e^{j\omega t'} \tag{2.10}$$

$$\int_{-\infty}^{\infty} \tilde{f}_1(j\omega') \ \tilde{f}_2(-j\omega') \, d\omega' = \tilde{f}(j\omega)$$

and the inversion formula

$$\tilde{f}(s) = \tilde{f}(j\omega) = \left\{ \int_{-\infty}^{\infty} \tilde{G}(j\chi) \, \tilde{G}(-j\chi) \, d\chi \right\}^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(t', j\omega_o, \omega_1) \, \tilde{G}\left(\frac{j\omega + j\omega_o}{\omega_1} \right) e^{j\omega t'} dt' \, d\omega_o$$

$$= \left\{ \int_{Br} \tilde{G}(\Sigma) \, \tilde{G}(-\Sigma) \, d\Sigma \right\}^{-1} \int_{Br} \int_{-\infty}^{\infty} F(t', s_o, \omega_1) \, \tilde{G}\left(\frac{s + s_o}{\omega_1} \right) e^{st'} dt' \, ds_o \tag{2.11}$$

There is an admissibility condition

$$0 < \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{G}(j\chi)\,\tilde{G}(-j\chi)\,d\chi = \int_{-\infty}^{\infty} G^2(\tau)\,d\tau < \infty \tag{2.12}$$

where we have assumed that $\tilde{G}(j\chi)$ has been chosen such that

$$\tilde{G}(-j\chi) = \tilde{G}^*(j\chi) \tag{2.13}$$

giving real $G(\tau)$. If one chooses $\tilde{G}$ such that

$$\lim_{\omega_1 \to 0} \frac{1}{\omega_1}\tilde{G}\left(\frac{j\omega - j\omega_o}{\omega_1}\right) = \delta(\omega - \omega_o) \tag{2.14}$$

then (2.4) gives

$$\lim_{\omega_1 \to 0} F(t, j\omega_o, \omega_1) = F(t, j\omega_o, 0) = e^{j\omega_o t}\tilde{f}(j\omega_o) \tag{2.15}$$

thereby recovering the original $\tilde{f}(s)$.

A commonly used form of filter is a Gaussian function as

$$\tilde{G}(j\chi) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\chi^2}{2}}$$

$$\tilde{G}(\Sigma) = \frac{1}{\sqrt{2\pi}}\, e^{\frac{\Sigma^2}{2}} \tag{2.16}$$

$$G(\tau) = \frac{1}{2\pi}\, e^{-\frac{\tau^2}{2}}$$

In this case the filter function is readily extended into the complex frequency plane, and is real for $s = j\omega$. As one can verify for this case

$$\int_{-\infty}^{\infty} \frac{1}{\omega_1}\tilde{G}\left(\frac{j\omega - j\omega_o}{\omega_1}\right) d\omega = \int_{-\infty}^{\infty} G(j\chi)\,d\chi = 1 \tag{2.17}$$

as a good delta function should. The admissibility condition has

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{G}(j\chi)\,\tilde{G}(-j\chi)\,d\chi = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{-\chi^2}\,d\chi$$

$$= \int_{-\infty}^{\infty}G^2(\tau)\,d\tau = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{-\tau^2}\,d\tau = \frac{1}{4\pi^{3/2}} \tag{2.18}$$

Note that such a filter (or window in the case of the WLT) is often used because of its same form in both time and frequency. Except for shifts $(\omega_o, t',$ etc.$)$ we have

$$\chi = \frac{\omega}{\omega_1} \quad , \quad \tau = \omega_1\, t \tag{2.19}$$

showing the inverse relationship between localization in time and localization in frequency. This is analogous to the uncertainty principle in quantum mechanics.

C. Frequency Wavelet Transform

The frequency wavelet transform (FWT) is given by

$$FWT\left[\tilde{f}(s)\right] \equiv \hat{F}(s_o,\omega_1) = \frac{1}{2\pi j}\int_{Br}\tilde{F}(s,s_o,\omega_1)\,ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(j\omega,j\omega_o,\omega_1)\,d\omega \tag{2.20}$$

$$= F(0,s_o,\omega_1)$$

This has the time-domain form from (2.5) as

$$\hat{F}(s_o,\omega_1) = \int_{-\infty}^{\infty}f(t')\,G(-\omega_1\,t')\,e^{-s_o t'}\,dt' \tag{2.21}$$

Noting that

$$\hat{F}_n(s_o,\omega_1) = \frac{1}{2\pi j}\int_{Br}\tilde{F}_n(s,s_o,\omega_1)\,ds = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}_n(j\omega,j\omega_o,\omega_1)\,d\omega \tag{2.22}$$

and following the procedure in [2] with time and frequency interchanged gives a Parseval-like relation as

$$\frac{1}{2\pi j}\int_{o}^{\infty}\int_{Br}\hat{F}_1(s_o,\omega_1)\ \hat{F}_2(-s_o,\omega_1)\,ds_o\,\frac{d\omega_1}{\omega_1}$$

$$=\frac{1}{2\pi}\int_{o}^{\infty}\int_{-\infty}^{\infty}\hat{F}_1(j\omega_o,\omega_1)\ \hat{F}_2(-j\omega_o,\omega_1)\,d\omega_o\,\frac{d\omega_1}{\omega_1}$$

$$=\frac{1}{2\pi j}\int_{o}^{\infty}\int_{Br}\left\{\int_{-\infty}^{\infty}f_1(t')\,G(-\omega_1 t')e^{-s_o t'}dt'\right\}\left\{\int_{-\infty}^{\infty}f_2(t'')\,G(-\omega_1 t'')e^{s_o t''}dt''\right\}\,ds_o\,\frac{d\omega_1}{\omega_1}$$

$$=\int_{o}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_1(t')\,f_2(t'')\,G(-\omega_1 t')\,G(-\omega_1 t'')\,\delta(t''-t')\,dt'\,dt''\,\frac{d\omega_1}{\omega_1}$$

$$=\int_{o}^{\infty}\int_{-\infty}^{\infty}f_1(t')\,f_2(t')\,G^2(-\omega_1 t')\,dt'\,\frac{d\omega_1}{\omega_1}$$

$$=\left\{\int_{o}^{\infty}G^2(\tau)\,\frac{d\tau}{\tau}\right\}\int_{-\infty}^{\infty}f_1(t')\,f_2(t')\,dt'$$

$$=\left\{\int_{o}^{\infty}G^2(\tau)\,\frac{d\tau}{\tau}\right\}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}_1(j\omega')\,\tilde{f}_2(-j\omega')\,d\omega' \qquad (2.23)$$

$$=\left\{\int_{o}^{\infty}G^2(\tau)\,\frac{d\tau}{\tau}\right\}\frac{1}{2\pi j}\int_{Br}\tilde{f}_1(s')\,\tilde{f}_2(-s')\,ds'$$

As a special case choose

$$\tilde{f}_1(s') = \tilde{f}(s)\quad,\quad \tilde{f}_2(-s') = \tilde{f}_2(-j\omega') = \delta(\omega'-\omega)$$

$$\underset{\omega'}{FWT}\left[\tilde{f}_2(j\omega')\right] = \hat{F}_2(j\omega_o,\omega_1) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega'-\omega)\frac{1}{\omega_1}\tilde{G}\left(\frac{j\omega'-j\omega_o}{\omega_1}\right)d\omega \qquad (2.24)$$

$$= \frac{1}{2\pi\omega_1}\tilde{G}\left(\frac{j\omega-j\omega_o}{\omega_1}\right)$$

The inversion of the FWT is then

$$\tilde{f}(s) = \tilde{f}(j\omega) = \int_{-\infty}^{\infty} \tilde{f}_1(j\omega')\,\tilde{f}_2(-j\omega')\,d\omega'$$

$$= \left\{\int_o^{\infty} G^2(\tau)\frac{d\tau}{\tau}\right\}^{-1} \frac{1}{2\pi}\int_o^{\infty}\int_{-\infty}^{\infty} \hat{F}(j\omega_o,\omega_1)\,\frac{1}{\omega_1}\tilde{G}\left(\frac{j\omega - j\omega_o}{\omega_1}\right) d\omega_o\, d\omega_1 \tag{2.25}$$

$$= \left\{\int_o^{\infty} G^2(\tau)\frac{d\tau}{\tau}\right\}^{-1} \frac{1}{2\pi j}\int_o^{\infty}\int_{Br} \hat{F}(s_o,\omega_1)\,\frac{1}{\omega_1}\tilde{G}\left(\frac{s - s_o}{\omega_1}\right) d\omega_o\, d\omega_1$$

The admissibility condition is now

$$0 < \int_o^{\infty} G^2(\tau)\frac{d\tau}{\tau} < \infty \tag{2.26}$$

which restricts $G(\tau)$ for $\tau \to 0$ and $\tau \to \infty$.

III. Linear Array of Scatterers

With frequency wavelets one might ask what kind of target signatures are appropriate for viewing this way. What target signatures are characterized by frequencies? Of course we have the natural frequencies which are the poles in the singularity expansion method (SEM) [3]. Note, however, that these are complex, lying in the left-half s-plane. If they are only lightly damped then on the $j\omega$ axis these appear as narrow peaks which might be resolved by an FWT. This is illustrated in [4, 5] where waveguide dispersion curves (frequency versus time) are also brought out.

Let us consider another basically frequency phenomenon. As indicated in fig. 3.1 let there be a linear array of N elements, periodically spaced a uniform distance ℓ apart. Without loss of generality let the array be aligned along the z axis. The direction of incidence $\vec{1}_i$ then makes an angle θ_i with respect to this axis so that

$$0 \le \pi - \theta_i < \frac{\pi}{2}$$
$$\vec{1}_i \cdot \vec{1}_z = \cos(\theta_i) \qquad \text{(negative)} \tag{3.1}$$

In backscattering we have [2]

$$\vec{1}_o = -\vec{1}_i \tag{3.2}$$

and the additional delay for the successive signals to reach the observer is just

$$T_o = 2\,T_\ell \cos(\pi - \theta_i) = -2T_\ell \cos(\theta_i)$$
$$T_\ell \equiv \frac{\ell}{c} \tag{3.3}$$

Note that this is readily generalizable to bistatic scattering via

$$\vec{1}_o \cdot \vec{1}_z = \cos(\theta_o) \qquad \text{(positive)}$$
$$T_o = T_o\left[\cos(\theta_o) - \cos(\theta_i)\right] > 0 \tag{3.4}$$

with $\vec{1}_o$ as the direction to the far-field observer. Here T_0 is still constrained to be positive.

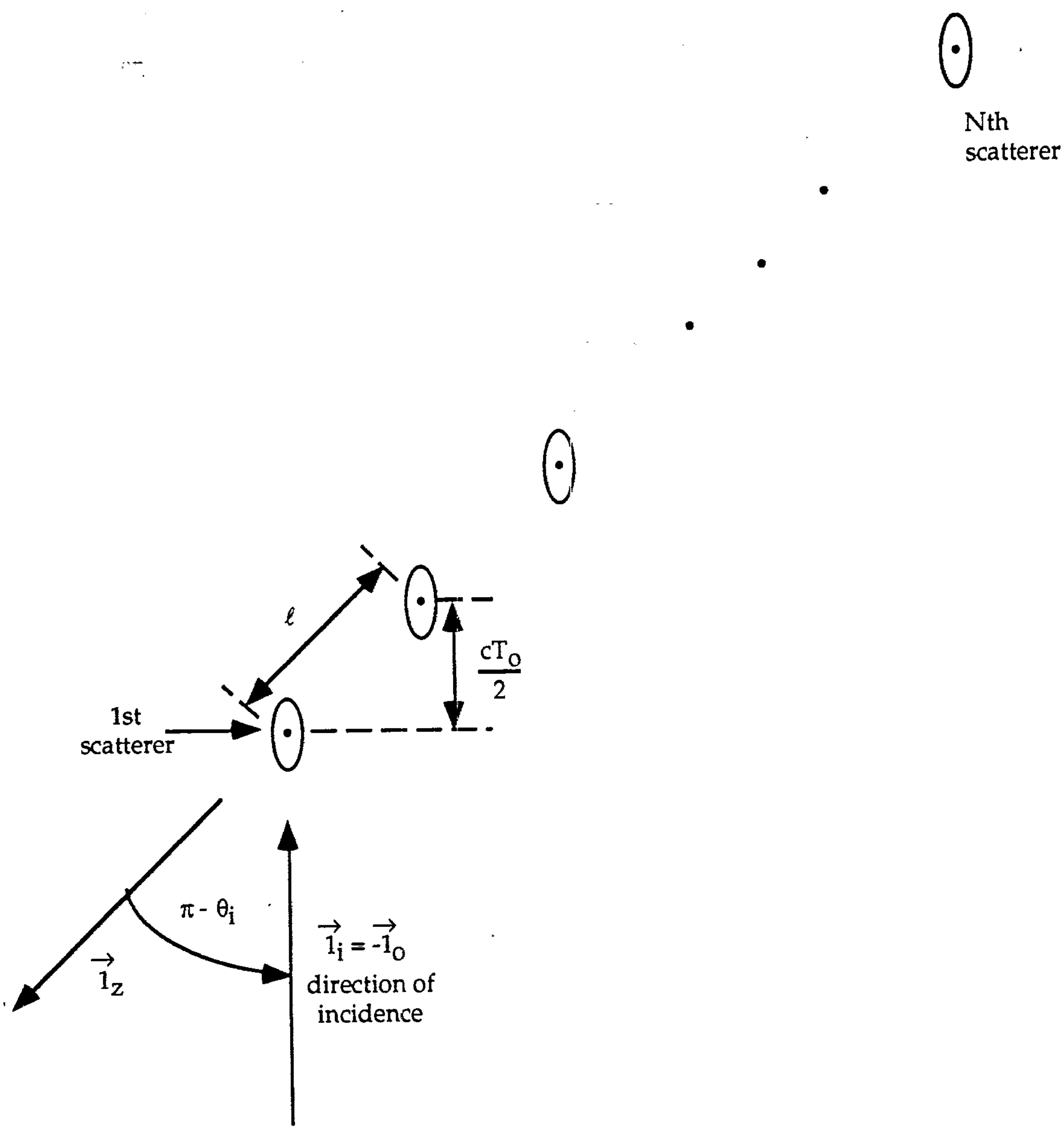


Fig. 3.1. Backscattering from Linear Array

With incident field and scattered far field as

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o \vec{1}_p \tilde{f}^{(inc)}(s)\, e^{-\gamma \vec{1}_i \cdot \vec{r}} \quad , \quad \vec{E}^{(inc)}(\vec{r},t) = E_o \vec{1}_p f^{(inc)}\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}\left(\vec{1}_o, \vec{1}_i ; s\right) \cdot \tilde{\vec{E}}^{(inc)}(\vec{0},s)$$

$$\vec{E}_f(\vec{r},t) = \frac{1}{4\pi r} \overleftrightarrow{\Lambda}\left(\vec{1}_o, \vec{1}_i ; t\right) \circ \vec{E}^{(inc)}\left(\vec{0}, t - \frac{\vec{1}_o \cdot \vec{r}}{c}\right) \tag{3.5}$$

$$\circ \equiv \text{convolution with respect to time}$$

then for backscattering we have

$$\tilde{\overleftrightarrow{\Lambda}}_b(\vec{1}_i ; s) = \tilde{\overleftrightarrow{\Lambda}}(-\vec{1}_i, \vec{1}_i ; s) \tag{3.6}$$

It is the properties of $\overleftrightarrow{\Lambda}$, the 2 x 2 dyadic scattering operator, that are of interest. Here lie the target signatures.

For convenience consider $\vec{r} = \vec{0}$ as somewhere on the first scatterer to send a signal back to the observer. Then neglecting multiple scattering among the N scatterers we can write

$$\overleftrightarrow{\Lambda}(\vec{1}_i, \vec{1}_i ; t) = \sum_{n=1}^{N} \overleftrightarrow{\Lambda}^{(0)}(\vec{1}_o, \vec{1}_i ; t-(n-1)T_o)$$

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i ; s) = \tilde{\overleftrightarrow{\Lambda}}^{(0)}(\vec{1}_o, \vec{1}_i ; s)\, \tilde{W}(s)$$

$$\tilde{W}(s) = \sum_{n=1}^{N} e^{-(n-1)sT_o} = \sum_{n=0}^{N-1} e^{-nsT_o}$$

$$= \frac{1-e^{-NsT_o}}{1-e^{-sT_o}} \tag{3.7}$$

$$W(t) = \sum_{n=0}^{N-1} \delta(t - nT_o)$$

Here W is like an antenna array factor [6], except for the factor of two (in backscattering) accounting for both incoming and outgoing waves. One can call this a scattering array factor.

Viewed in time domain, as the returns come from the successive elements of the scattering array the observer has not yet "seen" the Nth (last) scatterer, and for such times might regard N as ∞. In this case we have

$$\lim_{N\to\infty} \tilde{W}(s) = \left[1-e^{-sT_o}\right]^{-1} \text{ for } \operatorname{Re}[s] > 0 \tag{3.8}$$

where the limit is taken in the right half plane, and then extended to the left half plane by analytic continuation. This has poles at $s = s_m = j\omega_m$ where

$$\left.\begin{aligned} s_m T_o &= j\ 2\pi m \\ \omega_m T_o &= \ \ 2\pi m \end{aligned}\right\} m = 0, \pm 1, \pm 2, \cdots \tag{3.9}$$

Note that these poles are on the $j\omega$ axis, but are only true poles in the limit of $N \to \infty$ as defined above.

For finite N these quasi poles are bounded. Consider s near s_m as

$$s = s_m + \Delta s \quad , \quad e^{-Ns_m T_o} = 1$$

$$\tilde{W}(s) = \frac{1-e^{-NT_o\Delta s}}{1-e^{-T_o\Delta s}} \tag{3.10}$$

$$\tilde{W}(s_m) = N$$

which shows how, for large N, the function is approximating the poles in (3.8). An alternate form for $\tilde{W}$ is

$$\begin{aligned} \tilde{W}(s) &= e^{-\frac{N-1}{2}sT_o} \frac{\sinh\left(\frac{N}{2}sT_o\right)}{\sinh\left(\frac{1}{2}sT_o\right)} \\ &= e^{-\frac{N-1}{2}T_o\Delta s} \frac{\sinh\left(\frac{N}{2}T_o\Delta s\right)}{\sinh\left(\frac{1}{2}T_o\Delta s\right)} \end{aligned} \tag{3.11}$$

For frequencies on the $j\omega$ axis this is

$$\tilde{W}(j\omega) = e^{-j\frac{N-1}{2}T_o\Delta\omega} \frac{\sin\left(\frac{N}{2}T_o\Delta\omega\right)}{\sin\left(\frac{1}{2}T_o\Delta\omega\right)}$$

$$\left|\tilde{W}(j\omega)\right| = \frac{\left|\sin\left(\frac{N}{2}T_o\Delta\omega\right)\right|}{\left|\sin\left(\frac{1}{2}T_o\Delta\omega\right)\right|} \tag{3.12}$$

The first zero on the $j\omega$ axis is at

$$T_o\Delta\omega = \frac{2}{N} \quad , \quad \frac{\Delta\omega}{\omega_1} = \frac{1}{\pi N} \tag{3.13}$$

This can be thought of as defining a spectral line width which $\to 0$ as $N \to \infty$.

Since this set of quasi poles lies on the $j\omega$ axis one might use the transforms in Section II to analyze them with a filter which can emphasize a narrow band of frequencies on the $j\omega$ axis (e.g. (2.15). By varying ω_1 then multiresolution is attained as in fig. 1.1B. The width of the spectral line does not go to zero, but to some width determined by N as $\omega_1^{-1} \to \infty$.

Looking back at (3.7) note that $\tilde{W}(s)$ multiplies the scattering dyadic for a single scatterer. So one might like to have one or more s_m at frequencies for which $\tilde{\overleftrightarrow{\Lambda}}^{(0)}(s)$ is large. Then $\tilde{W}(s)$ raises the signal amplitude near the s_m by a factor of N (or spectral energy density by a factor of N^2). So this is potentially a large enhancement of the scattering at these special frequencies. Note that these are <u>not</u> natural frequencies since they are <u>aspect dependent</u> as indicated by (3.3) and (3.4). Furthermore, natural frequencies (except at $s = 0$) for the exterior scattering problem must lie to the left of the $j\omega$ axis [1]. This fact of aspect dependence can perhaps be exploited in target identification.

IV. Concluding Remarks

While formally similar, the time and frequency forms of these transforms seem to be suited for analysis of different kinds of target signatures. The temporal version allows one to concentrate on some localized portion of a target, such as a single one of the scatterers in fig. 3.1, and thereby analyze the properties of the associated scattering dyadic (e.g. $\tilde{\overleftrightarrow{\Lambda}}^{(0)}$). The frequency version allows one to concentrate on various characteristic frequencies such as natural frequencies including global features of the target. The natural frequencies are aspect independent, but those from arrays of scattering elements are on the $j\omega$ axis and are aspect dependent. So it would seem that in identifying some target out of a set of potential targets one may want a variety of techniques for analyzing the different kinds of target signatures. As discussed in [2] various symmetries and partial symmetries in the target features (including substructures) can be used to define a target-feature/signature zoo. Discrete spatial translation symmetry (periodicity) can be added as one of the habitats, with the linear array in Section III as one of the beasts therein.

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