Interaction Notes

Note 496

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# Transforms and Symmetries in Target Identification

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Abstract

This paper considers the use of window and wavelet transforms for the analysis of transient/ broad-band electromagnetic scattering data. By appropriate choice of window position and width target substructures can be isolated under favorable conditions. The resulting target features and their associated scattering signatures can be used in target identification, both from the properties of the individual signatures and their relative time of arrival at the observer for a scattering-center analysis. An important concept is that of partial symmetries associated with individual features. This leads to a set of convenient decompositions of the target-feature signatures which aid in identification of a particular target in its various aspects (orientations) with respect to the observer.

# I. Introduction

A proposed approach to target identification is based on what are known as scattering centers [10, 18]. As illustrated in fig. 1.1 the target or scatterer is viewed in an approximate way as being describable by some finite number (say N) of scattering centers which one can label $P_n$ for $n = 1, 2, \cdots, N$. These can be considered as being located at spatial coordinates $\vec{r}_n$, but one should recognize that these centers are not in general points but have some non-zero spatial extent. On a highly conducting surface (such as the skin of an aircraft) these can be local structures such as apertures, small antennas, engine inlets, etc., as well as abrupt changes such as edges and corners. For certain angles of incidence and scattering, curved surfaces can provide specular points, which, however, move as the angles of incidence and scattering change.

As one rotates the target over all angles the relative distances between the $P_n$ (i.e. the $|\vec{r}_n - \vec{r}_m|$) remain invariant (assuming the $P_n$ are true points). By rotating each target in some set (library) of targets one can try to match a particular target to the observed relative delay times of the signals from the $P_n$ as an identification procedure. If one can successfully do this then one also has some estimate of the orientation of the target, i.e. the angles found in rotating the target to match the relative delay times. One partial limitation concerns line-of-sight for incident and scattered fields. As an example, in fig. 1.1 center $P_5$ is shadowed, so waves have to propagate around the target, changing both the time to reach the observer and perhaps significantly decreasing the magnitude of the scattered signal. Under such conditions, at a minimum such points can be removed from consideration for unfavorable angles.

The incident field is taken as a plane wave of the form

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o \vec{1}_p \tilde{f}^{(inc)}(s) e^{-\gamma \vec{1}_i \cdot \vec{r}}, \quad \vec{E}^{(inc)}(\vec{r},t) = E_o \vec{1}_p f^{(inc)}\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = \frac{E_o}{Z_o} \vec{1}_i \times \vec{1}_p \tilde{f}^{(inc)}(s) e^{-\gamma \vec{1}_i \cdot \vec{r}}, \quad \vec{H}^{(inc)}(\vec{r},t) = \frac{E_o}{Z_o} \vec{1}_i \times \vec{1}_p f^{(inc)}\left(t - \frac{\vec{1}_i \cdot \vec{r}}{c}\right)$$

$$\vec{1}_i \equiv \text{direction of incidence} \;, \quad \vec{1}_p \equiv \text{incident polarization}$$

$$\vec{1}_i \cdot \vec{1}_p = 0 \;, \quad f^{(inc)}(t) \equiv \text{incident waveform}$$

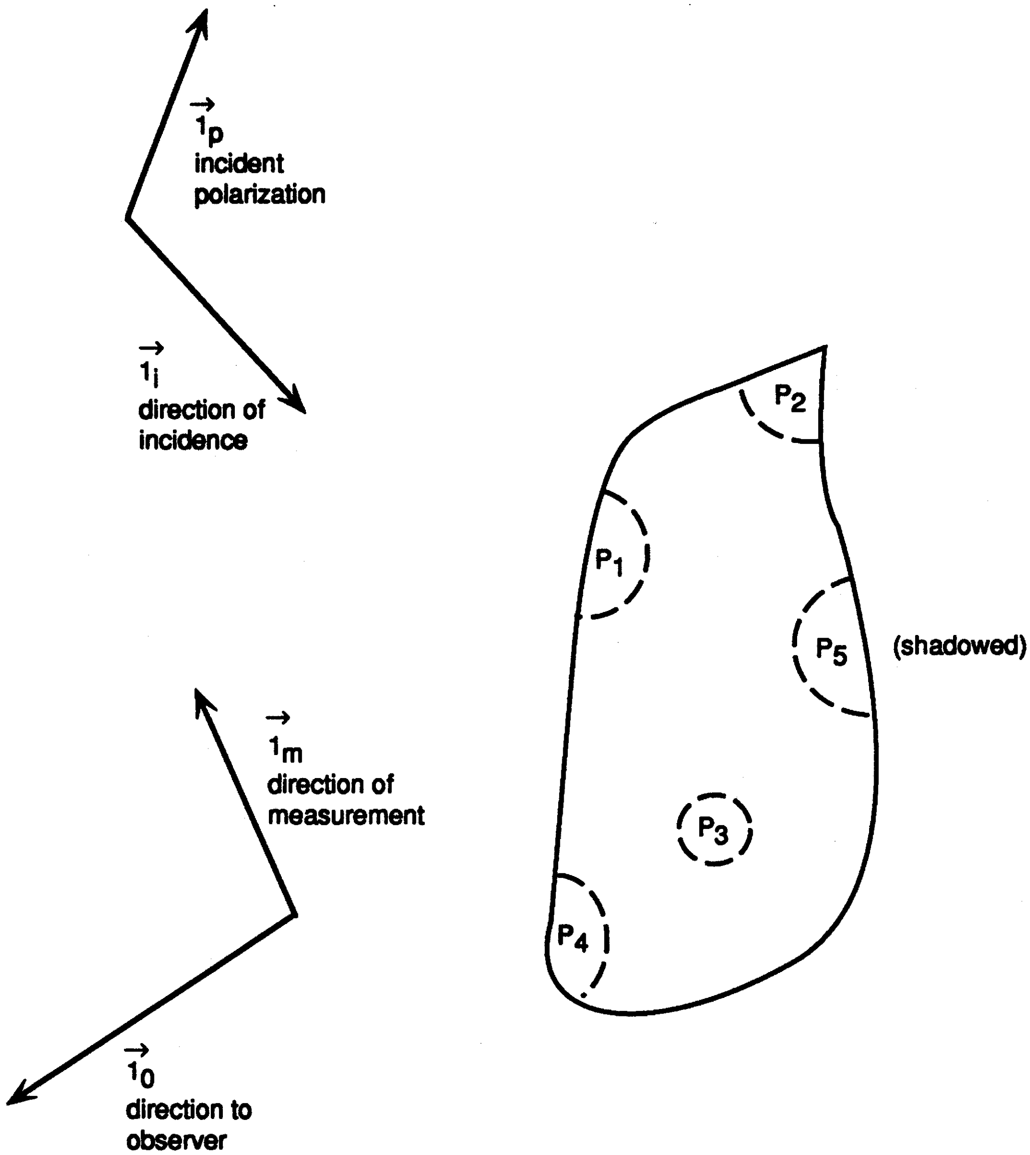


Fig. 1.1. Scattering Centers

$$\gamma \equiv \frac{s}{c} \equiv \text{propagation constant}$$

$$c = [\mu_0 \epsilon_0]^{-\frac{1}{2}} = \text{speed of light}$$

$$Z_o = \left[\frac{\mu_o}{\epsilon_o}\right]^{\frac{1}{2}} = \text{wave impedance of free space}$$

(1.1)

$E_0 \equiv$ amplitude factor (V/m)

$\sim \equiv$ Laplace-transform (two sided)

$s \equiv \Omega + j\omega \equiv$ Laplace-transform variable or complex frequency

where $\vec{r} = \vec{0}$ is some reference position on or near the target. The far-scattered electric field is given by

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i;s) \cdot \tilde{\vec{E}}^{(inc)}(\vec{0},s)$$

$$\vec{E}_f(\vec{r},t) = \frac{1}{4\pi r} \overleftrightarrow{\Lambda}(\vec{1}_o, \vec{1}_i;t) \circ \vec{E}^{(inc)}\left(\vec{0}, t - \frac{\vec{1}_o \cdot \vec{r}}{c}\right)$$

$\vec{1}_o \equiv$ direction to observer at large $r$

$\vec{1}_m \equiv$ direction of measurement of scattered electric field

$\vec{1}_o \cdot \vec{1}_m = 0$

(1.2)

$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i;s) \equiv$ dyadic $(2 \times 2)$ scattering length (convolution operator in time domain)

$\circ \equiv$ convolution with respect to time

Looking at the form of the scattered fields, note the presence of polarization (vector) information. In its scalar form (such as for sonar) such information is lacking. An implementation of scattering-center concepts without polarization information may be unnecessarily limited. Considering the special (but common) case of backscattering for which

$$\vec{1}_o = -\vec{1}_i \tag{1.3}$$

then the relative time of arrival from the various $P_n$ is invariant to a rotation of the target about an axis parallel to $\vec{1}_o$. This can be viewed as a benefit when trying to identify the target due to the fewer rotations required for matching to the target library. Then again, the inclusion of polarization information can separately identify this last rotation angle.

Polarization can halp find the target orientation with respect to the observer as discussed in [1, 2, 3, 20]. The pole terms in the singularity expansion method (SEM) have dyadic residues which for backscattering take the simple form $\vec{c}_\alpha\ \vec{c}_\alpha$ (for the typical case of non-degenerate natural modes) with $\vec{c}_\alpha$ (complex 2-component vector) containing the target orientation information. Similarly for the early-time scattering from structures with dilation symmetry (cones, wedges, half spaces) there is a real 2 x 2 dyadic coefficient [6, 7]. For backscattering this can be diagonalized to find principal axes related to target orientation. To obtain this information one needs both polarizations (usual h, v radar polarizations) from which one can obtain both in-line and cross-polarization information. Perhaps the above concepts (and others) can be incorporated into the modelling of the scattering centers to improve the target identification procedures.

## II. Waveform Transforms

### A. Laplace/Fourier Transform

A commonly used transform is the two-sided Laplace transform (LT)

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\; e^{-st}\, dt \;\equiv\; LT[f(t)]$$

$$f(t) = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\; e^{st}\, ds \tag{2.1}$$

$Br$ ≡ Bromwich contour along Re[s] = $\Omega_{Br}$
from $\Omega_{Br}$ - j∞ to $\Omega_{Br}$ + j∞ in strip of convergence

The kernel of this transform is $e^{-st}$ with the $t$ variable integrated out. This can also be thought of as a Fourier transform since

$$\tilde{f}(j\omega) = \int_{-\infty}^{\infty} f(t)\; e^{-j\omega t}\, dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega)\; e^{j\omega t}\, dt \tag{2.2}$$

which is, however, not quite as general unless one also allows the inverse transform to be moved away from the $\Omega = 0$ axis. Also note the conjugate symmetry

$$\tilde{f}(s^*) = \tilde{f}^*(s) \tag{2.3}$$

when dealing with real-valued time functions (as here). This transform also applies to vector and dyadic functions of time.

For later use we have the generalized Parseval theorem [8]

$$\int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s)\, \tilde{f}_2(-s)\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(j\omega)\, f_2(-j\omega)\, d\omega \tag{2.4}$$

Note the use of frequency reversal ($-s$ or $-j\omega$) which keeps the analytic properties of the functions in the complex frequency plane. Also we have

$$LT[f(-t)] = \tilde{f}(-s) \tag{2.5}$$

which with (2.4) gives an alternate form as

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(t)\, f_2(-t)\, dt &= \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s)\, \tilde{f}_2(s)\, ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(j\omega)\, \tilde{f}_2(j\omega)\, d\omega \end{aligned} \tag{2.6}$$

There are also the convolution formulas

$$\begin{aligned} LT[f_1(t)\, f_2(t)] &= \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s')\, \tilde{f}_2(s-s')\ ds' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}_1(j\omega')\, \tilde{f}_2(j\omega - j\omega')\ d\omega' \end{aligned} \tag{2.7}$$

$$LT\left[\int_{-\infty}^{t} f_1(t')\, f_2(t-t')\, dt'\right] = \tilde{f}_1(s)\ \tilde{f}_2(s)$$

B. Triwave Transform

Consider the kernel

$$\frac{1}{t_1}\, g\!\left(\frac{t-t_o}{t_1}\right) e^{-st} \quad , \quad t_1 > 0 \quad , \quad t_o \text{ real} \tag{2.8}$$

which has three parameters (hence triwave)

$$\begin{aligned} s &\equiv \text{complex frequency} \\ t_o &\equiv \text{time shift} \\ t_1 &\equiv \text{time dilation (width of window or wavelet)} \end{aligned} \tag{2.9}$$

The time $t$ is integrated out in the transforms. Here we have

$$\frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) \equiv \text{window or wavelet} \tag{2.10}$$

which can be normalized in various ways. The factor $t_1^{-1}$ preceeding $g$ is often replaced by $t_1^{-\frac{1}{2}}$, but we find the present also-used form more convenient. For present purposes $g$ is also taken as real. It also goes to zero as $t \to \pm\infty$ so that it weights $f(t)$ strongly near $t = t_0$.

The window or wavelet has a Laplace transform as

$$\begin{aligned} LT\left[\frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right)\right] &= \int_{-\infty}^{\infty} \frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) e^{-st}\, dt \\ &= e^{-st_o} \int_{-\infty}^{\infty} g(\tau)\; e^{\Sigma\tau}\, d\tau \\ &= e^{-st_o}\; \underset{\tau\to\Sigma}{LT}\left[g(\tau)\right] = e^{-st_o}\, \tilde{g}(\Sigma) \\ \tau \equiv \frac{t-t_o}{t_1} \quad &, \quad \Sigma \equiv st_1 \end{aligned} \tag{2.11}$$

Note that the frequency spectrum of $g$ is also dilated by $t_1$, while the time shift is evidenced by the usual shifting theorem of the Laplace transform.

Using all three parameters the triwave transform (TT) is given by

$$\tilde{f}(s,t_o,t_1) \equiv \int_{-\infty}^{\infty} f(t)\, \frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) e^{-st}\, dt \equiv TT\left[f(t)\right] \tag{2.12}$$

Utilizing the convolution theorem this has the alternate form

$$\begin{aligned} \tilde{f}(s,t_o,t_1) &= \frac{e^{-st_o}}{2\pi j} \int_{Br} \tilde{f}(s')\, \tilde{g}\left(st_1 - s't_1\right) e^{s't_o}\; ds' \\ \tilde{f}(j\omega,t_o,t_1) &= \frac{e^{-j\omega t_o}}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega')\, \tilde{g}\left(j\omega t_1 - j\omega' t_1\right)\; e^{j\omega' t_o}\, d\omega' \end{aligned} \tag{2.13}$$

So this transform can be regarded as an integral over time or frequency.

With these preliminaries let us consider two special cases of "biwave transforms", the window Laplace/Fourier transform and the wavelet transform, each of which uses two of the above three parameters. There is much literature concerning these, as for example [11-15, 17, 19].

C. Window Laplace/Fourier Transform

The window Laplace transform (WLT) is given by

$$\tilde{f}(s,t_o) \equiv \int_{-\infty}^{\infty} f(t)\; h(t-t_o)\, e^{-st}\, dt \equiv WLT[f(t)]$$

$$h(t-t_o) \equiv \frac{1}{t_1} g\left(\frac{t-t_o}{t_1}\right)\Bigg|_{t_1 \text{ constant}} \tag{2.14}$$

where a $t_0$, $\omega$ plane is often referred to as phase space. In this transform the $t_1$ parameter is not used, but without loss of generality one can retain the triwave form and note that

$$\tilde{f}(s,t_o) = \tilde{f}(s,t_o,t_1) \tag{2.15}$$

Choosing different values of $t_1$ gives different windows, and one can regard this as giving additional flexibility. For various applications some widths of the window may be more appropriate than others.

An important aspect of the WLT is that it can be inverted as an integral over $t_0$ and $\omega$ (or $s$) which is often referred to as phase space. For this purpose first define

$$f_n(t,t_o,t_1) \equiv f_n(t)\frac{1}{t_1} g\left(\frac{t-t_o}{t_1}\right)$$

$$LT[f_n(t,t_o,t_1)] = \tilde{f}_n(s,t_o,t_1) \tag{2.16}$$

Then we have

$$X \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(t',t_o,t_1)\, f_2(t',t_o,t_1)\, dt' \;\; dt_o$$

$$= \int_{-\infty}^{\infty} f_1(t')\, f_2(t') \left\{ \int_{-\infty}^{\infty} \frac{1}{t_1^2}\, g^2\!\left(\frac{t'-t_o}{t_1}\right) dt_o \right\} dt' \tag{2.17}$$

$$= \left\{ \frac{1}{t_1} \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \right\} \left\{ \int_{-\infty}^{\infty} f_1(t')\, f_2(t')\, dt' \right\}$$

which can also be written in frequency domain giving a Parseval-like formula as

$$X = \frac{1}{2\pi j} \int_{-\infty}^{\infty}\int_{Br} \tilde{f}_1(s,t_o,t_1)\;\; \tilde{f}_2(-s,t_o,t_1)\, ds \;\; dt_o$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}_1(j\omega,t_o,t_1)\;\; \tilde{f}_2(-j\omega,t_o,t_1)\, d\omega \;\; dt_o \tag{2.18}$$

As a special case choose

$$f_1(t') = f(t') \quad , \quad f_2(t') = \delta(t'-t)$$

$$\underset{t'\to s}{LT}\left[f_2(t',t_o,t_1)\right] = \int_{-\infty}^{\infty} \delta(t'-t)\, \frac{1}{t_1}\, g\left(\frac{t'-t_o}{t_1}\right) e^{-st'}\, dt' \tag{2.19}$$

$$= \frac{1}{t_1}\, g\left(\frac{t-t_o}{t_1}\right) e^{-st}$$

Then we have the inversion of the WLT as

$$f(t) = \int_{-\infty}^{\infty} f_1(t')\;\; f_2(t')\, dt = \left\{ \frac{1}{t_1} \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \right\} X$$

$$= \left\{ \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \right\}^{-1} \frac{1}{2\pi j} \int_{-\infty}^{\infty}\int_{Br} \tilde{f}(s,t_o,t_1)\;\; g\!\left(\frac{t-t_o}{t_1}\right) e^{st}\, ds\, dt_o \tag{2.20}$$

$$= \left\{ \int_{-\infty}^{\infty} g^2(\tau)\, d\tau \right\}^{-1} \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(j\omega,t_o,t_1)\;\; g\!\left(\frac{t-t_o}{t_1}\right) e^{j\omega t}\, d\omega\, dt_o$$

Note for $t_1 > 0$ there is an admissability condition for this inversion formula as

$$0 < \int_{-\infty}^{\infty} g^2(\tau)\, d\tau < \infty \tag{2.21}$$

This integral is often taken as unity by some authors, thereby giving a normalization condition for $g(\tau)$. There are various possible choices for $g(\tau)$, e.g. a gate or pulse function as

$$\begin{aligned} g(\tau) &= u(\tau) - u(\tau - 1) \\ \int_{-\infty}^{\infty} g^2(\tau)\, d\tau &= \int_{o}^{1} d\tau = 1 \quad , \quad \int_{-\infty}^{\infty} g(\tau)\, d\tau = 1 \end{aligned} \tag{2.22}$$

If we let $t_1 \to 0+$ we have

$$\begin{aligned} \frac{1}{t_1} g\left(\frac{t - t_o}{t_1}\right) &= \begin{cases} t_1^{-1} \text{ for } 0 < t - t_o < t_1 \\ 0 \text{ otherwise} \end{cases} \\ &= \delta(t - t_o) \text{ as } t_1 \to 0+ \end{aligned} \tag{2.23}$$

noting the unit time integral. Then (2.12) gives

$$\begin{aligned} \lim_{t_1 \to 0} \tilde{f}(s, t_o, t_1) = \tilde{f}(s, t_o, 0) &\int_{-\infty}^{\infty} f(t)\, \delta(t - t_o)\; e^{-st}\, dt \\ &= e^{-st_o}\; f(t_o) \\ f(t_o) = e^{st_o}\; \tilde{f}(s, t_o, 0) & \end{aligned} \tag{2.24}$$

which is a way of interpreting a narrow window as sampling the waveform at the window. Another delta-function like window is [11]

$$g(\tau) = e^{-\tau} u(\tau)$$

$$\int_{-\infty}^{\infty} g^2(\tau)\, d\tau = \frac{1}{2} \quad , \quad \int_{-\infty}^{\infty} g(\tau)\, d\tau = 1 \tag{2.25}$$

$$\frac{1}{t_1} g\left(\frac{t-t_o}{t_1}\right) = \delta(t-t_o) \text{ as } t_1 \to 0+$$

While this second window function has a unit integral (like the first one) and thereby (2.24) applies, it has a different normalization factor (integral of the square) for the inversion as in (2.20).

D. Wavelet Transform

The wavelet transform (WT) is given by

$$\begin{aligned} \hat{f}(t_o, t_1) &\equiv \int_{-\infty}^{\infty} f(t) \frac{1}{t_1} g\left(\frac{t-t_o}{t_1}\right) dt \equiv WT[f(t)] \\ &= \tilde{f}(0, t_o, t_1) \end{aligned} \tag{2.26}$$

This has a frequency-domain form by setting $s = 0$ in (2.13) as

$$\begin{aligned} \hat{f}(t_o, t_1) &= \frac{1}{2\pi j} \int_{Br} \tilde{f}(s')\, \tilde{g}(-s't_1)\, e^{s't_o}\, ds' \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(j\omega')\, \tilde{g}(-j\omega' t_1)\, e^{j\omega' t_o}\, d\omega' \end{aligned} \tag{2.27}$$

where the signs can also be reversed on $s', \omega'$. So, even though the transform in (2.26) has frequency nowhere exhibited, it can still be regarded as a transform over the frequency spectrum as in (2.27). One can also relate the WT to the WLT by observing

$$WT\left[f(t)\, e^{-st}\right] = WLT[f(t)] = \tilde{f}(s, t_o, t_1) \tag{2.28}$$

As in (2.16) form

$$\int_{-\infty}^{\infty} f_n(t, t_o, t_i)\, dt = \hat{f}_n(t_o, t_1) \tag{2.29}$$

Then for an inversion of the transform we have

$$\begin{aligned} \Xi &\equiv \int_o^{\infty}\int_{-\infty}^{\infty} \hat{f}_1(t_o,t_1)\ \hat{f}_2(t_o,t_1)\ dt_o\, \frac{dt_1}{t_1} \\ &= \frac{1}{(2\pi)^2}\int_o^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\tilde{f}_1(j\omega')\ \tilde{g}\,(-j\omega' t_1)\, e^{j\omega' t_o}\, d\omega'\right\} \\ &\qquad \left\{\int_{-\infty}^{\infty}\tilde{f}_2(-j\omega'')\ \tilde{g}\,(-j\omega'' t_1)\, e^{-j\omega'' t_o}\, d\omega''\right\} dt_o\, \frac{dt_1}{t_1} \end{aligned} \tag{2.30}$$

Integrate over $t_o$ noting that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega_o t_o}\, dt_o = \delta(\omega_o) \quad , \ \omega_o \equiv \omega' - \omega'' \tag{2.31}$$

is an inverse Fourier transform with frequency and time roles interchanged. Then we have

$$\begin{aligned} \Xi &= \frac{1}{2\pi}\int_o^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}_1(j\omega')\ \tilde{f}_2(-j\omega'')\,\tilde{g}\,(-j\omega' t_1)\,\tilde{g}\,(j\omega'' t_1)\ \delta(\omega'-\omega'')\ d\omega'\ d\omega''\,\frac{dt_1}{t_1} \\ &= \frac{1}{2\pi}\int_o^{\infty}\int_{-\infty}^{\infty}\tilde{f}_1(j\omega')\ \tilde{f}_2(-j\omega')\,\tilde{g}\,(-j\omega' t_1)\,\tilde{g}\,(j\omega' t_1)\ d\omega'\,\frac{dt_1}{t_1} \end{aligned} \tag{2.32}$$

Now integrate over $t_1$ giving

$$\int_o^{\infty}\tilde{g}\,(j\omega' t_1)\,\tilde{g}\,(-j\omega' t_1)\ \frac{dt_1}{t_1} = \int_o^{\infty}\tilde{g}\,(j\xi)\,\tilde{g}\,(-j\xi)\ \frac{d\xi}{\xi} \tag{2.33}$$

Then using the Parseval relation

$$\Xi = \left\{ \int_o^\infty \left| \tilde{g}(j\xi) \right|^2 \frac{d\xi}{\xi} \right\} \frac{1}{2\pi} \int_{-\infty}^\infty \tilde{f}_1(j\omega') \tilde{f}_2(-j\omega') \, d\omega'$$

$$= \left\{ \int_o^\infty \left| \tilde{g}(j\xi) \right|^2 \frac{d\xi}{\xi} \right\} \int_{-\infty}^\infty f_1(t') f_2(t') \, dt \qquad (2.34)$$

Then as a special case choose

$$f_1(t') = f(t') \quad , \quad f_2(t') = \delta(t' - t)$$

$$\underset{t'}{WT}\left[f_2(t')\right] = \int_{-\infty}^\infty \delta(t' - t) \frac{1}{t_1} g\left(\frac{t' - t_o}{t_1}\right) dt' \qquad (2.35)$$

$$= \frac{1}{t_1} g\left(\frac{t - t_o}{t_1}\right)$$

The inversion of the WT is then

$$f(t) = \int_{-\infty}^\infty f_1(t') f_2(t') \, dt' = \left\{ \int_o^\infty \left| \tilde{g}(j\xi) \right|^2 \frac{d\xi}{\xi} \right\}^{-1} \Xi$$

$$= \left\{ \int_o^\infty \left| \tilde{g}(j\xi) \right|^2 \frac{d\xi}{\xi} \right\}^{-1} \int_o^\infty \int_{-\infty}^\infty \hat{f}(t_o, t_1) \frac{1}{t_1} g\left(\frac{t - t_o}{t_1}\right) dt_o \, dt_1 \qquad (2.36)$$

Note there is an admissability condition for this inversion formula as

$$0 < \int_o^\infty \left| \tilde{g}(j\xi) \right|^2 \frac{d\xi}{\xi} < \infty \qquad (2.37)$$

For integrability near 0 we have

$$\left| \tilde{g}(j\xi) \right|^2 \xi^{-1} = O\left(\xi^{-1+\upsilon}\right) \quad , \quad \upsilon > 0 \text{ as } \xi \to 0$$

$$\tilde{g}(j\xi) = O\left(\xi^{\frac{\upsilon}{2}}\right) \text{ as } \xi \to 0 \qquad (2.38)$$

and near $\infty$ we have

$$\left|\tilde{g}(j\xi)\right|^2 \xi^{-1} = O\left(\xi^{-1-\upsilon}\right) \quad , \quad \upsilon > 0$$

$$\text{as } \xi \to 0 \tag{2.39}$$

$$\tilde{g}(j\xi) = O\left(\xi^{-\frac{\upsilon}{2}}\right) \text{ as } \xi \to 0$$

There are various possible choices of $g$ which meet this admissability condition, e.g.

$$g(\tau) = u(\tau) - 2u\left(\tau - \frac{1}{2}\right) + u(\tau - 1)$$

$$= \begin{cases} 1 \text{ for } 0 < \tau < \frac{1}{2} \\ -1 \text{ for } \frac{1}{2} < \tau < 1 \\ 0 \text{ otherwise} \end{cases} \tag{2.40}$$

This is a doublet or differentiating kind of wavelet. The zero time integral implied by (2.38) excludes the unipolar functions discussed in the previous subsection in connection with the WLT.

III. Symmetry Basis for Transforms

Now let us look at the waveform transforms from the point of view of symmetry [9]. As indicated in Section II B there are three parameters in the transforms of the waveform: $s$, $t_0$, and $t_1$.

The first parameter, $s$, comes from the assumed time-translation symmetry and linearity of the problem. Our scattering problem can be represented by (1.2). The target is then represented by the dyadic scattering length which is a convolution operator in time domain, but which simply (dot) multiplies in frequency domain. This effectively factors out the incident field, allowing one to use different waveforms and polarizations to emphasize various features of the target as present in $\overleftrightarrow{\Lambda}$. The properties of the scattering dyadic in $s$ plane are important sources of target features for target identification. There is the SEM representation [1, 2, 3, 5, 16, 20] (class 1 form) for backscattering

$$\begin{aligned}
\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_i,s) &= \sum_{\alpha} \frac{e^{-(s-s_\alpha)t_i}}{s-s_\alpha} \vec{c}_\alpha(\vec{1}_i)\,\vec{c}_\alpha(\vec{1}_i) \\
&\quad + \text{entire function} \\
s_\alpha &\equiv \text{aspect-independent natural frequencies} \\
\vec{c}_\alpha(\vec{1}_i) &\equiv \text{residue vector} \\
&\equiv \text{scatterer polarization vector} \\
\vec{c}_\alpha(\vec{1}_i)\cdot\vec{1}_i &= 0
\end{aligned} \tag{3.1}$$

One can also consider the low-frequency behavior (low-frequency method, LFM) [1] as

$$\begin{aligned}
\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o,\vec{1}_i;s) &= \gamma^2\left[-\overleftrightarrow{1}_o\cdot\overleftrightarrow{P}\cdot\overleftrightarrow{1}_i + \vec{1}_o\times\overleftrightarrow{M}\times\vec{1}_i\right] \text{ as } s\to 0 \\
\overleftrightarrow{1}_i &\equiv \overleftrightarrow{1}-\vec{1}_i\vec{1}_i \quad , \quad \overleftrightarrow{1}_o \equiv \overleftrightarrow{1}-\vec{1}_o\vec{1}_o \\
\overleftrightarrow{P} &\equiv \text{electric polarizability} \\
\overleftrightarrow{M} &\equiv \text{magnetic polarizability}
\end{aligned} \tag{3.2}$$

These polarizabilities (dimensions $m^3$) are characteristic of the overall size of the target and give some estimates of ratios of major dimensions (length to width, etc.). At high frequencies (high-frequency method, HFM) [1] one can consider various asymptotic representations valid as $s \to \infty$. These have the general form

$$\tilde{\overleftrightarrow{\Lambda}}(\vec{1}_o, \vec{1}_i; s) = \sum_p \overleftrightarrow{D}_p(\vec{1}_o, \vec{1}_i) \frac{e^{-st_p}}{s^p} \quad \text{as } s \to \infty \tag{3.3}$$

where the $p$ index includes not only integers. The $t_p$ represent the times the observer sees various discontinuities encountered by the incident wave. However, it is often found useful to include other kinds of important terms such as creeping waves with exponential decay as they travel around curved surfaces. Note that (3.3) is asymptotic, which can be distinguished from exact time-domain results valid for some non-zero clear time [6, 7]. The point to remember is that all these results come from the behavior in the complex frequency plane which in turn is a consequence of the time-translation symmetry and linearity.

The second parameter, $t_0$, shifts the window in time to times $t$ near $t_0$. The width of this window (or wavelet) is given roughly by $t_1$ with details depending on the exact form of the function $g$ that one uses. Suppose that there is something of particular interest in the scattering information at around time $t_0$, say corresponding to some scattering center. With $t_1$ appropriately chosen then the associated waveform (or signature) can be isolated from those waveforms coming earlier or later in time. Depending on target aspect there may be signatures from more than one target feature (e.g. scattering centers) arriving in the same time window. If one is fortunate enough to have one significant signature in the window, or can suppress other signatures by filtering or choice of polarization, then the problem simplifies considerably. In this case it is the local target feature as embedded in its local surroundings that is significant. Suppose that the local surroundings can be approximated as part of a simple infinite structure (an infinite perfectly conducting plane, an infinite wedge, etc.). The fact that it is actually finite is (to an approximation) ignored for purposes of analysis. The local target feature (window, small antenna, etc.) is then considered as a perturbation on an ideal infinite structure. Such infinite structures have translation symmetry, and translation in space is related to translation in time via the $t_0$ parameter. Thus the window sometimes allows one to impose symmetries on the target that are not strictly present. One might think of these as partial symmetries pertaining to portions of the target. These partial symmetries are brought out by optimal choice of the window function. As compared to the clear-time concept in [6, 7] based on causality, the window function accomplishes a similar effect by excluding certain times from consideration. Note that windowing can be accompanied by filtering, such as a filter with zeros at the substructure natural frequencies [16], both to help identify the substructure and to shorten the time duration of the resonant signature and thereby allow a smaller window width $t_1$ .

The third parameter, $t_1$ , adjusts the width of the window or wavelet. This parameter represents a dilation in time. Recall from [6, 7] that scattering from objects with dilation symmetry (generalized cones, wedges, and half spaces) has a simple form (or parameterization). Furthermore, if one restricts the observation time to exclude scattering from truncations of such structures, the special form involving time integral, identity, etc. operating on the incident waveform is preserved for all smaller times. So as one makes $t_1$ approximately less than the clear time, the behavior continues for all lesser $t_1$ down to 0+ where the beginning of the wavelet is adjusted to the time of first signal from the dilation-symmetry feature of interest. This is an invariance property (symmetry) with respect to $t_1$ .

## IV. Some Partial Geometric Symmetries

With the window function placed around some local target feature with appropriate time $t_0$ and width $t_1$, then under favorable circumstances one can think in terms of the scattering from the particular target substructure or feature. Suppose that this substructure has geometric symmetries such as [9]:

1. Symmetry planes: reflection symmetry $R$ (e.g. a rectangular window, certain blade antennas, etc.)

2. Continuous two-dimensional rotation symmetry $C_\infty$ (e.g. a circular window, certain engine intakes, etc.)

3. Discrete two-dimensional rotation symmetry $C_N$ (e.g. a propeller)

Then the scattering exhibits such symmetries. For the important case of backscattering [2, 3, 16, 20] certain results have been shown. Symmetry planes decompose the substructure natural frequencies into two sets: symmetric and antisymmetric. $C_\infty$ symmetry plus $R_a$ (symmetry plane containing the symmetry axis) makes the natural modes degenerate (two fold) and the scattering dyadic diagonalizable with real eigenvectors (parallel and perpendicular to plane of incidence containing symmetry axis). $C_N$ symmetry for $N \geq 3$ with direction of incidence along the symmetry axis makes the scattering dyadic a scalar times the 2 x 2 identity. Perhaps other partial symmetries such as translation symmetry, can be found for certain target substructures (e.g. certain array antennas).

V. Partial Dilation Symmetry

As discussed in [7] dilation symmetry of a scatterer is given by invariance under a transformation of the form

$$\vec{r}\,' = \chi \vec{r} \quad , \quad t' = \chi t \quad , \quad s' = \frac{s}{\chi} \tag{5.1}$$

which applies to a class of structures which might be termed generalized cones. These include perfectly conducting and uniform dielectric cones, finite length wedges, and finite facial dimensioned half spaces, and in general things with constitutive parameters scaling as

$$\tilde{\overleftrightarrow{\epsilon}}(\vec{r},s) = \tilde{\overleftrightarrow{\epsilon}}(\vec{r}\,',s') \quad , \quad \tilde{\overleftrightarrow{\mu}}(\vec{r},s) = \tilde{\overleftrightarrow{\mu}}(\vec{r}\,',s') \quad , \quad \tilde{\overleftrightarrow{\sigma}}(\vec{r},s) = \chi\, \tilde{\overleftrightarrow{\sigma}}(\vec{r}\,',s')$$

$$\tilde{\overleftrightarrow{Y}}_s(\vec{r}_s, s) = \tilde{\overleftrightarrow{Y}}_s(\vec{r}\,'_s, s') \text{ (sheet admittances on surfaces described by only } \theta_s,\ \phi_s) \tag{5.2}$$

While $\vec{r} = \vec{0}$ is the apex of a cone, specialization of (5.2) by imposition of translation symmetry leads to wedges and half spaces.

This kind of target feature is characterized in backscattering by a fairly simple signature which is valid for a clear time based on the time that a signal from a truncation or from multiple scattering (e.g. cone tip to cone tip on a finite wedge) can reach the observer. Letting $\vec{r} = \vec{0}$ be a point on the substructure from which the first signal backscatters (i.e. a cone tip, edge of a wedge, face of a half space) the scattering assumes a simple form. From a cone tip we have the factored form

$$\vec{E}_f^{(c)}(\vec{r},t) = \frac{1}{4\pi r} \overleftrightarrow{K}^{(c)}(\vec{1}_i) \cdot \int_{-\infty}^{t+\frac{\vec{1}_i \cdot \vec{r}}{c}} \vec{E}^{(inc)}(\vec{0},t'')\, c\, dt''$$

$$\vec{1}_i = -\vec{1}_o \quad \text{(backscattering)} \tag{5.3}$$

From a finite wedge with ends as cones we have two such cone terms, one delayed from the first by the additional round-trip travel time to the second. If both cone-tip signals arrive at the same time this is normal incidence on the edge giving two terms as

$$\vec{E}_f^{(e)}(\vec{r},t) = \frac{\ell}{4\pi r}\overleftrightarrow{K}^{(e)}(\vec{1}_i)\cdot\vec{E}^{(inc)}\left(\vec{0},t+\frac{\vec{1}_i\cdot\vec{r}}{c}\right)$$

$$+\frac{1}{4\pi r}\overleftrightarrow{K}^{(c)}(\vec{1}_i)\cdot\int_{-\infty}^{t+\frac{\vec{1}_i\cdot\vec{r}}{c}}\vec{E}^{(inc)}(\vec{0},t'')\,c\,dt'' \tag{5.4}$$

$\ell \equiv$ length of edge

From a finite-dimensioned half space truncated by planes which terminate the face in edges and cone tips, (5.3) and (5.4) apply depending on the arrival time at the observer from the various cone tips. If the direction of incidence is normal to the face there are now three terms as

$$\vec{E}_f^{(f)}(\vec{r},t) = \frac{A}{4\pi r}\overleftrightarrow{K}^{(f)}(\vec{1}_i)\cdot\frac{1}{c}\frac{\partial}{\partial t}\vec{E}^{(inc)}\left(\vec{0},t+\frac{\vec{1}_i\cdot\vec{r}}{c}\right)$$

$$+\frac{\ell_o}{4\pi r}\overleftrightarrow{K}^{(e)}(\vec{1}_i)\cdot\vec{E}^{(inc)}\left(\vec{0},t+\frac{\vec{1}_i\cdot\vec{r}}{c}\right)$$

$$+\frac{1}{4\pi r}\overleftrightarrow{K}^{(c)}(\vec{1}_i)\cdot\int_{-\infty}^{t+\frac{\vec{1}_i\cdot\vec{r}}{c}}\vec{E}^{(inc)}(\vec{0},t'')\,dt'' \tag{5.5}$$

$A \equiv$ area of face

$\ell_o \equiv$ perimeter of face

Note that in all cases the number of such terms is finite (in general a small number) and the result is valid up to some $t_V$ based on clear times discussed above.

Considering each of the scattering terms above note that each is the dot product of a real 2 x 2 dyadic with a simple temporal operator on the incident field. One can write a general term as

$$\vec{E}_f(\vec{r},t) = \frac{1}{4\pi r} \overleftrightarrow{K}^{(d)}(\vec{1}_i) \cdot I_t^{d-2}\left[\vec{E}^{(inc)}\left(\vec{0}, t + \frac{\vec{1}_i \cdot \vec{r}}{c}\right)\right]$$

$I_t^{d-2} \equiv$ integral operator (over time) of order d - 2

| | | | |
|---|---|---|---|
| cone tip : | d = 3 | (time integral) | (5.6) |
| edge : | d = 2 | (identity) | |
| face : | d = 1 | (time derivative) | |

Reciprocity implies

$$\overleftrightarrow{K}^{(d)^T}(\vec{1}_i) = \overleftrightarrow{K}^{(d)}(\vec{1}_i) \tag{5.7}$$

As discussed in [4] a real symmetric matrix can be diagonalized with real eigenvalues and real eigenvectors, i.e.

$$\overleftrightarrow{K}^{(d)}(\vec{1}_i) = \sum_{\beta=1}^{2} k_\beta^{(d)}(\vec{1}_i)\ \vec{1}_\beta(\vec{1}_i)\ \vec{1}_\beta(\vec{1}_i)$$

$$\vec{1}_{\beta_1}(\vec{1}_i) \cdot \vec{1}_{\beta_2}(\vec{1}_i) = 1_{\beta_1,\beta_2} \quad \text{(orthonormal)} \tag{5.8}$$

$$k_\beta^{(d)}(\vec{1}_i) \ , \ \vec{1}_\beta(\vec{1}_i) \ \text{real}$$

As real eigenvectors transverse to the direction of incidence, these represent principal axes or directions of the substructure in the usual h, v (horizontal vertical) radar coordinate system. Such dilation substructures can then help in obtaining target orientation.

Since the three different dilation types give different orders of temporal integration one would like to determine this order in the scattering data and thereby better identify the substructure. Note that for simple forms of the incident field (step, ramp, etc.) the scattering becomes a non-negative power of $t$ times a unit step (or derivatives of this). This behavior is invariant out to some time of validity as discussed previously. For window width $t_1$ less than this the above behavior applies for this and all smaller $t_1$. This is the dilation feature of the wavelet. In some sense the target signature is invariant to $t_1$ because of the dilation symmetry of the target substructure. In a wavelet plane described by $t_0$ , $t_1^{-1}$

(like phase space $t_0$, $\omega$) as $t_1^{-1}$ exceeds some value, for appropriate $t_0$ the pattern should illustrate this invariance and help in determining the dimension parameter $d$.

## VI. Concluding Remarks

By a target feature let us mean some geometrical shape including constitutive parameters of some substructure or piece of the target. A special case of this is the target as a whole. Associated with particular features there are target signatures by which we mean temporal waveforms, or more generally properties of the scattering dyadic, including polarization. Various symmetries of the features (and target as a whole) produce symmetries in the signatures which help parameterize (decompose) the signatures and help in target identification.

Particle physicists have often referred to the elementary-particle zoo to describe the menagerie of such "beasts". Furthermore symmetry plays an important role in elementary-particle theory. By analogy we can speak of a target-feature zoo or target-feature/signature zoo. As discussed in this paper symmetry also plays an important role in this zoo. In a manner similar to which elementary particles are grouped by quantum symmetries, let us group our feature/signature pairs according to the symmetries involved. So let us divide our target feature zoo into $N_h$ habitats as

$$zoo = \bigcup_{n_h=1}^{N_h} (\text{habitat})_{n_h} \tag{6.1}$$

Some of the habitats and their significances for target features and signatures are:

| habitat | consequences |
|---|---|
| time translation ($t$) and linearity | s plane representations and filters |
| dilation ($t_1$) | exhibition of dilation symmetric substructures |
| window location ($t_0$) | substructure viewed as though on infinite structure (space translation) |
| reciprocity | symmetric $\overleftrightarrow{\Lambda}$ in backscatter, principal axes for polarization |
| rotation/reflection (point symmetry groups) | symmetries in $\overleftrightarrow{\Lambda}$, polarization |

Specific features can involve more than one symmetry habitat (intersection of habitats).

An important concept is the idea of partial symmetries associated with target substructures. These are exhibited by the $t_0$ parameters of the triwave kernel, both in the form of the WLT and the WT. This windowing isolates one region of time, and under favorable conditions a corresponding region of space on the target for consideration. This windowing is a multiplication in time domain, and can be compared to a filter which multiplies in frequency domain (and is equivalently a convolution in time domain). One can combine the two. For example, one can isolate a substructure resonance which can also be modified by a pulse with zeros at the natural frequencies [16] (either before or after windowing) and thereby both identify the resonance and narrow the pulse for scattering-center analysis. Dispersion curves, such as for waveguides have also been observed in the phase-space diagrams coming out of these transforms [17]. If one has some information concerning these characteristics, then in principle filters can be formed to modify this behavior in a manner analogous to poles in the complex frequency plane, again helping with the identification.

Suppose that one has characterized some set of targets by the various entities in the target-feature zoo. Note that various of the corresponding signatures factor with certain of the factors being aspect independent (at least over some range of angles of incidence in backscattering). Then one can try to fit such signature functions to the target scattering and thereby identify the target. Of course, in a realistic radar application there is significant noise which complicates the problem. Various techniques such as filtering, windowing, and averaging can be used to reduce the noise. Finally, probability techniques [18] can be used to help bring the information out of the remaining noise.

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