p. ? Ch. 3 in B&H book

Interaction Notes

Note 493

8 May 1993

Scattering from Cones, Wedges, and Half Spaces

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Abstract

This paper considers the far-scattering from general cones based on the dilation symmetry of such objects. These include not only perfectly conducting cones, but also apply to various forms of the constitutive parameters (dielectric cones, etc.). The far-scattered field exactly factors into a dyadic angular function times the time integral of the incident field. Similar results apply to wedges and half spaces with the same dilation symmetry except that the frequency-dependent (or s) part of the scattering is proportional to $s^{-\frac{d-1}{2}}$ where d is the number of dimensions describing the expansion of the far fields.

I. Introduction

Symmetry is a powerful tool in electromagnetics [4]. Some of the symmetries are inherent in the Maxwell equations for suitable media constitutive parameters, including duality, reciprocity, and relativistic invariance. Other symmetries are associated with the geometrical symmetries of some object (antenna, scatterer) including rotations, reflections, translations, and dilations. These two kinds of symmetries can be combined as in the self-complementary planar structures arising from the Babinet principle.

This paper considers some consequences of dilation symmetry in a scatterer. For present purposes rotations (which in the antenna case lead to log-spirals, including on cones) and translations (which lead to log-periodic antennas) are excluded. This is associated with an incident plane wave of fixed direction of incidence and polarization. The scatterer with this dilation symmetry is a cone of arbitrary cross section, admitting perfectly conducting surfaces and certain forms of finite constitutive parameters.

II. Electrodynamic Similitude and Self-Similar Scatterers

There is a well-known scaling of the Maxwell equations [5, 7, 10] in which the cartesian coordinates are multiplied by a common real number as a scale factor; the time, the electric field, and the magnetic field are also each multiplied by other real numbers. This leads to a new set of equivalent equations depending on these four scale factors. The constitutive parameters of the new equivalent medium are also related to those of the original medium by these scale factors. One can also consider the more general case where the scale factors for the spatial coordinates are matrices (tensors) which can even be functions of position leading to special kinds of inhomogeneous and isotropic media (special lenses) [12].

For present purposes we restrict the scaling to the form

$$\vec{r}\,' = \chi\vec{r} \quad , \quad t' = \chi t \quad , \quad s' = \frac{s}{\chi} \tag{2.1}$$

$$\chi > 0 \quad \text{(scaling parameter)}$$

where distance and time are scaled the same. The medium in which the incident wave propagates is free space in both cases. While it will be convenient to deal in terms of time one can also think in terms of frequency as

$$\begin{aligned} \sim \;&\equiv\; \text{Laplace transform (two-sided)} \\ s \;&\equiv\; \Omega + j\omega \;\equiv\; \text{Laplace-transform variable} \\ &\qquad\qquad\;\;\; \equiv\; \text{complex frequency} \end{aligned} \tag{2.2}$$

which we note scales reciprocally to the time. As in fig. 2.1 we have some general cone of arbitrary cross section with apex at $\vec{r} = \vec{0}$. We have the usual cylindrical and spherical coordinate systems related to cartesian coordinates as

$$\begin{aligned} x &= \Psi\cos(\phi) \quad , \quad y = \Psi\sin(\phi) \\ z &= r\cos(\theta) \quad , \quad \Psi = r\sin(\theta) \end{aligned} \tag{2.3}$$

Consider the Maxwell equations

$$\nabla\times\tilde{\vec{E}}\left(\vec{r},s\right) = -s\tilde{\vec{B}}\left(\vec{r},s\right) \;,\; \nabla\times\tilde{\vec{H}}\left(\vec{r},s\right) = \tilde{\vec{J}}\left(\vec{r},s\right) + s\tilde{\vec{D}}\left(\vec{r},s\right) \tag{2.4}$$

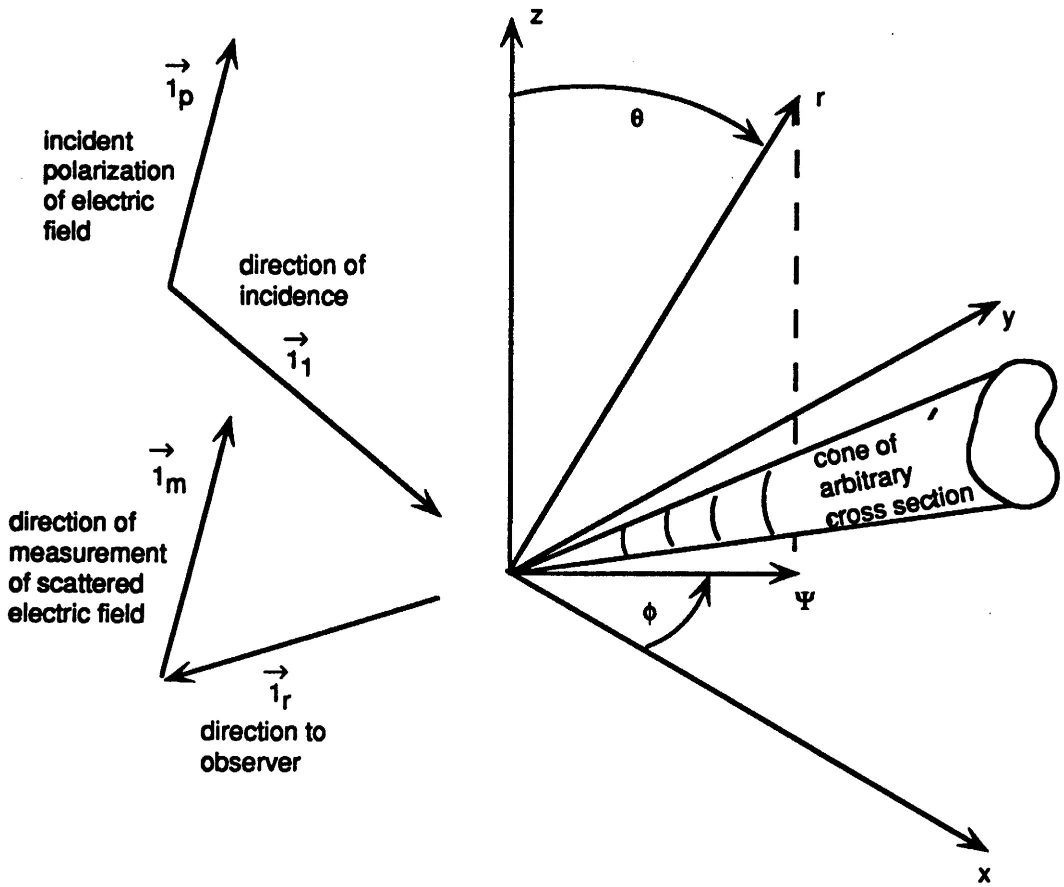

Fig. 2.1. General Cone Scatterer

These are invariant under the transformation (2.1) provided

$$
\begin{aligned}
&\tilde{\vec{E}}\left(\vec{r},s\right) = \tilde{\vec{E}}\left(\vec{r}',s'\right) \quad , \quad \tilde{\vec{H}}\left(\vec{r},s\right) = \tilde{\vec{H}}\left(\vec{r}',s'\right) \\
&\tilde{\vec{D}}\left(\vec{r},s\right) = \tilde{\vec{D}}\left(\vec{r}',s'\right) \quad , \quad \tilde{\vec{B}}\left(\vec{r},s\right) = \tilde{\vec{B}}\left(\vec{r}',s'\right) \\
&\tilde{\vec{J}}\left(\vec{r},s\right) = \chi \tilde{\vec{J}}\left(\vec{r}',s'\right) \text{ (current density, like } s\text{)} \\
&\tilde{\vec{J}}_s\left(\vec{r},s\right) = \tilde{\vec{J}}_s\left(\vec{r}',s'\right) \text{ (sheet or surface current density, like } \vec{H}\text{)}
\end{aligned}
\tag{2.5}
$$

The constitutive relations are written

$$
\begin{aligned}
&\tilde{\vec{D}}\left(\vec{r},s\right) = \overleftrightarrow{\epsilon} \cdot \tilde{\vec{E}}\left(\vec{r},s\right) \quad , \quad \tilde{\vec{B}}\left(\vec{r},s\right) = \overleftrightarrow{\mu} \cdot \tilde{\vec{H}}\left(\vec{r},s\right) \\
&\tilde{\vec{J}}\left(\vec{r},s\right) = \overleftrightarrow{\sigma} \cdot \tilde{\vec{E}}\left(\vec{r},s\right)
\end{aligned}
\tag{2.6}
$$

Here the constitutive parameters are taken as matrices (tensors) which can even be taken as frequency dependent, in which case the constitutive relations are invariant under (2.1) provided

$$
\begin{aligned}
&\tilde{\overleftrightarrow{\epsilon}}\left(\vec{r},s\right) \equiv \tilde{\overleftrightarrow{\epsilon}}(r,\theta,\phi;s) = \tilde{\overleftrightarrow{\epsilon}}(r',\theta,\phi;s') \\
&\tilde{\overleftrightarrow{\mu}}\left(\vec{r},s\right) \equiv \tilde{\overleftrightarrow{\mu}}(r,\theta,\phi;s) = \tilde{\overleftrightarrow{\mu}}(r',\theta,\phi;s') \\
&\tilde{\overleftrightarrow{\sigma}}\left(\vec{r},s\right) \equiv \tilde{\overleftrightarrow{\sigma}}(r,\theta,\phi;s) = \chi\, \tilde{\overleftrightarrow{\sigma}}(r',\theta,\phi;s')
\end{aligned}
\tag{2.7}
$$

with 0 and ∞ as acceptable values for conductivity components. Even more elaborate constitutive relations (such as for chiral media) can also be allowed. Consider the simple case that the constitutive parameters in (2.7) are frequency independent. Then we have

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}(\theta,\phi) \ , \ \overleftrightarrow{\mu} = \overleftrightarrow{\mu}(\theta,\phi) \ , \ \overleftrightarrow{\sigma} = \frac{1}{r}\overleftrightarrow{\Sigma}(\theta,\phi) \tag{2.8}$$

Note that relation times have the form ϵ/σ which scale as r, consistent with (2.1).

A special case of interest concerns conical sheets. These are described in terms of θ and ϕ alone (extending over $0 \le r \le \infty$), being cones in the usual sense. Perfectly conducting conical sheets evidently fit the foregoing scaling relationships. These include various cross-section shapes, including degenerate cases such as flat sheets. More generally one can have a sheet admittance which can be a dyadic in general [4] as

$$\begin{aligned}
\overleftrightarrow{Y}_s\left(\vec{r}_s, s\right) &= \overleftrightarrow{Y}_s(r_s,\theta_s,\phi_s;s) = \overleftrightarrow{Y}_s(r_s',\theta_s,\phi_s;s') \\
&\equiv \text{sheet admittance} \\
&= \overleftrightarrow{Z}_s^{-1}\left(\vec{r}_s, s\right) \\
\overleftrightarrow{Z}_s\left(\vec{r}_s, s\right) &\equiv \text{sheet impedance} \\
(r_s,\theta_s,\phi_s) &\equiv \text{spherical coordinates on surface}
\end{aligned} \tag{2.9}$$

Note that the components relate the sheet current density to the electric field tangential to the sheet as

$$\tilde{\vec{J}}_s\left(\vec{r}_s, s\right) = \overleftrightarrow{Y}_s\left(\vec{r}, s\right) \cdot \tilde{\vec{E}}\left(\vec{r}_s, s\right) \tag{2.10}$$

As such these are in effect 2 x 2 dyadics giving 4 components, or 3 independent components in the case of reciprocity. The scaling relationship in (2.9) is derived by substitution from (2.5) into (2.10).

A simple example of a sheet admittance is the frequency-independent kind as

$$\overleftrightarrow{Y}_s = \overleftrightarrow{Y}_s(\theta_s,\phi_s) \tag{2.11}$$

which is just a sheet conductance, anisotropic (dyadic) if one wishes, or just a scalar (perhaps even independent of θ_s and ϕ_s). Comparing this sheet conductance to the volume conductivity in (2.8), one can consider the conical sheet to have a finite thickness. Consistent with the scaling this thickness should be proportional to r_s, which when combined with the $1/r$ scaling of the conductivity in (2.8) gives the r-independent form in (2.11).

This set of scaling relationships can be used to define a generalized cone. There are numerous simple examples of such cones, including perfectly conducting cones, uniform dielectric cones, etc. These all have the properties discussed here.

III. Scattering of Plane Wave by Cone

Again referring to fig. 2.1 note the incident wave which we take of the form

$$\tilde{\vec{E}}^{(inc)}\left(\vec{r},s\right) = E_o\,\vec{1}_p\,\tilde{f}(s)\,e^{-\gamma\,\vec{1}_1\cdot\vec{r}}\,,\quad \vec{E}^{(inc)}\left(\vec{r},t\right) = E_o\,\vec{1}_p\,f\left(t-\frac{\vec{1}_1\cdot\vec{r}}{c}\right)$$

$$\tilde{\vec{H}}^{(inc)}\left(\vec{r},s\right) = \frac{E_o}{Z_o}\,\vec{1}_1\times\vec{1}_p\,\tilde{f}(s)\,e^{-\gamma\,\vec{1}_1\cdot\vec{r}}\,,\quad \vec{H}^{(inc)}\left(\vec{r},t\right) = \frac{E_o}{Z_o}\,\vec{1}_1\times\vec{1}_p\,f\left(t-\frac{\vec{1}_1\cdot\vec{r}}{c}\right)$$

$$\vec{1}_1 \equiv \text{direction of incidence}\;,\quad \vec{1}_p \equiv \text{polarization}$$

$$\vec{1}_1\cdot\vec{1}_p = 0\;,\quad f(t) \equiv \text{incident waveform}$$

$$Z_o = \left[\frac{\mu_o}{\epsilon_o}\right]^{\frac{1}{2}} \equiv \text{wave impedance of free space}$$

$$c = \left[\mu_o\,\epsilon_o\right]^{-\frac{1}{2}} \equiv \text{speed of light}$$

$$E_o \equiv \text{amplitude factor (V/m)} \tag{3.1}$$

$$\gamma \equiv \frac{s}{c} \equiv \text{propagation constant}$$

The waveform can be chosen for convenience. In time domain one can have it chosen as a step function, while in frequency domain one can have $\tilde{f}(s) = 1$ (delta function in time). In either case the wave arrives at $\vec{r} = \vec{0}$ when $t = 0$. Note that the linear time-invariance of the scatterer allows one to factor out the waveform and reinsert it as a common factor (in frequency domain) in all the fields, current density, etc.

With the factor $\vec{f}(s)$ removed, then the incident field can be scaled in $\vec{r}$ and s as in (2.5). The scattered field then scales similarly as does related parameters. The dilation symmetry of the problem effectively factors out radius. Note that we restrict the direction of incidence $\vec{1}_1$, or associated angles θ_1 and ϕ_1, such that the wave is incident from outside the cone. This even allows cases for which time-domain fields scatter from portions of the cone away from $\vec{r} = \vec{0}$ before reaching the origin. In this case negative times also appear in the scaling. For present purposes, however, we shall restrict the direction of incidence $\vec{1}_1$ and direction to the observer of the scattered field $\vec{1}_r$ such that the first scattered signal arrives at the observer from the origin at zero retarded time. In this case the scattered field is a spherically expanding wave allowing the usual radiation condition in three dimensions. For any combinations of $\vec{1}_1$ and $\vec{1}_r$ to exist satisfying this requirement the cone should be restricted to lie all on one side of a plane passing through $\vec{r} = \vec{0}$, except for touching the plane at the origin.

Our present interest is centered on the far scattered field for which we write [3]

$$\tilde{\vec{E}}_f\left(\vec{r},s\right) = \frac{e^{-\gamma r}}{4\pi r} \tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r, \vec{1}_1; s\right) \cdot \tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right) \tag{3.2}$$

where the superscript 3 is to indicate the three-dimensional expansion of the far fields. This scattering dyadic is defined such that the far field is the exact field in the limit as $r \to \infty$. Note that for reciprocal media in the cone we have already

$$\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r, \vec{1}_1; s\right) = \tilde{\overleftrightarrow{\Lambda}}^{(3)^T}\left(-\vec{1}_1, -\vec{1}_r; s\right) \qquad \text{(reciprocity)} \tag{3.3}$$

For the case of backscattering we have

$$\vec{1}_1 = -\vec{1}_r \qquad \text{(backscattering)} \tag{3.4}$$

and the scattering dyadic is symmetric. Note that $\vec{1}_r$ is the direction to the observer in the general case and we have

$$\tilde{\vec{E}}_f\left(\vec{r},s\right) \cdot \vec{1}_r = 0 \qquad \text{(transverse wave)} \tag{3.5}$$

In fig. 2.1 one can think of $\vec{1}_m$ as some direction with

$$\vec{1}_m \cdot \vec{1}_r = 0 \tag{3.6}$$

which one can use to define a component of the far field (say for measurement).

Note that since the incident and far-scattered fields each have only two components (perpendicular to directions of incidence and scattering, respectively), the scattering dyadic has the form of a 2 x 2 matrix with only four components. For backscattering the off-diagonal components are equal due to reciprocity, giving only three components to consider.

Now apply the scaling in (2.5) to the fields in (3.2). This requires that

$$\tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right) = \tilde{\vec{E}}^{(inc)}\left(\vec{0},s'\right)$$

$$e^{\gamma r}\,\tilde{\vec{E}}_f\left(\vec{r},s\right) = e^{\gamma' r'}\,\tilde{\vec{E}}_f\left(\vec{r}',s'\right) \qquad (3.7)$$

$$\gamma' \equiv \frac{s'}{c}$$

and implies that

$$\frac{1}{r}\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r,\vec{1}_1;s\right) = \frac{1}{r'}\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r,\vec{1}_1;s'\right) \qquad (3.8)$$

Using (2.1) we have

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r,\vec{1}_1;s\right) &= \frac{1}{\chi}\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r,\vec{1}_1;s'\right) \\ &= \frac{s'}{s}\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r,\vec{1}_1;s'\right)\end{aligned} \qquad (3.9)$$

For this to hold for all s and s' we have

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}^{(3)}\left(\vec{1}_r,\vec{1}_1;s\right) &= \frac{c}{s}\overleftrightarrow{K}^{(3)}\left(\vec{1}_r,\vec{1}_1\right) \\ \overleftrightarrow{K}^{(3)}\left(\vec{1}_r,\vec{1}_1\right) &= \overleftrightarrow{K}^{(3)^T}\left(-\vec{1}_1,-\vec{1}_r\right) \quad \text{(reciprocity)} \\ &= \text{real valued dyadic (dimensionless)}\end{aligned} \qquad (3.10)$$

where c is included for dimensional convenience. This is a very general result. It represents an exact factorization of the scattering from the cone into an angular part (dyadic) times a frequency part (scalar, $1/s$). It is consistent with the known eigenfunction solution for the perfectly conducting circular cone [11].

In time domain (3.2) is written as

$$\vec{E}_f\left(\vec{r}, t+\frac{r}{c}\right) = \frac{1}{4\pi r} \overleftrightarrow{\Lambda}^{(3)}\left(\vec{1}_r, \vec{1}_1; t\right) \circ \vec{E}^{(inc)}\left(\vec{0}, t\right)$$

$$\overleftrightarrow{\Lambda}^{(3)}\left(\vec{1}_r, \vec{1}_1; t\right) = c\,u(t)\, \overleftrightarrow{K}^{(3)}\left(\vec{1}_1, \vec{1}_r\right) \tag{3.11}$$

$\circ \equiv$ convolution with respect to time

where now $\overleftrightarrow{K}^{(3)}$ is evidently real. This gives the result

$$\vec{E}_f\left(\vec{r}, t+\frac{r}{c}\right) = \frac{c}{4\pi r} \overleftrightarrow{K}^{(3)}\left(\vec{1}_1, \vec{1}_r\right) \cdot \int_{-\infty}^{t} \vec{E}^{(inc)}\left(\vec{0}, t'\right) dt' \tag{3.12}$$

allowing the incident wave to first arrive at the origin at any convenient time. Thus our generalized cone acts as a time integration in the scattering process. Again we see in time domain the factorization into a temporal part times an angular part. While in frequency domain the cone is assumed to extend to ∞ (consistent with (2.7)), this is not required in time domain. The cone can be truncated and (3.12) will apply up until such time that a signal scattered from somewhere on the truncation can first reach the observer. For an incident wave first reaching the cone tip, one can calculate the time at which the truncation is first observed. This is referred to as the clear time [1], and appears in various contexts. Based only on causality (speed-of-light limitation) it gives the time for which a finite-size structure behaves just as though it were infinite or some other shape (with perhaps other constitutive parameters) different from the ideal case used for the computation or measurement.

Another simple way to view this far-field result is to look in time domain for the case of a step function wave $(f(t) = u(t))$. Then with the amplitude of the far field falling off as $1/r$ consider two different radii, r and r' at times $t + r/c$ and $t' + r'/c$, which by the scaling must have the same results. Going back from r' to say the smaller r, but at time $t' + r/c$, the field must be larger by the same factor. So a step-function incident field gives a ramp (proportional to t) in the far field in retarded time.

IV. Scattering of Plane Wave by Wedge

Now consider the wedge as a special case of the cone. As indicated in fig. 4.1 let the z axis lie on the edge of this infinite wedge. Furthermore let the wedge parameters be independent of z. Thus the wedge parameters are constrained to only be functions of Ψ and ϕ (cylindrical coordinates). Considering the scaling relationship in (2.1) we have

$$\Psi' = \chi\Psi \quad , \quad z' = \chi z \quad , \quad \phi' = \phi \tag{4.1}$$

as our coordinate scaling. Imposing the scaling of the fields as in Section II, as well as the z invariance now, gives

$$\begin{aligned} \tilde{\overleftrightarrow{\epsilon}}\left(\vec{r},s\right) &\equiv \tilde{\overleftrightarrow{\epsilon}}(\Psi,\phi;s) = \tilde{\overleftrightarrow{\epsilon}}(\Psi',\phi;s') \\ \tilde{\overleftrightarrow{\mu}}\left(\vec{r},s\right) &\equiv \tilde{\overleftrightarrow{\mu}}(\Psi,\phi;s) = \tilde{\overleftrightarrow{\mu}}(\Psi',\phi;s') \\ \tilde{\overleftrightarrow{\sigma}}\left(\vec{r},s\right) &\equiv \tilde{\overleftrightarrow{\sigma}}(\Psi,\phi;s) = \chi\,\tilde{\overleftrightarrow{\sigma}}(\Psi',\phi;s') \end{aligned} \tag{4.2}$$

for the scaling of the constitutive parameters. If these are chosen to be frequency independent we have

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}(\phi) \quad , \quad \overleftrightarrow{\mu} = \overleftrightarrow{\mu}(\phi) \quad , \quad \overleftrightarrow{\sigma} = \frac{1}{\Psi}\overleftrightarrow{\Sigma}(\phi) \tag{4.3}$$

which are the wedge version of (2.8).

Similarly, the admittance sheet is now on a plane of constant ϕ and takes the form

$$\begin{aligned} \tilde{\overleftrightarrow{Y}}_s\left(\vec{r}_s,s\right) &= \tilde{\overleftrightarrow{Y}}_s(\Psi_s,s) = \tilde{\overleftrightarrow{Y}}_s(\Psi_s';s') \\ (\Psi_s\,,\ \phi_s &= \text{constant},\ z_s) \equiv \text{cylindrical coordinates on surface} \end{aligned} \tag{4.4}$$

This 2 x 2 dyadic relates the Ψ_s and z_s components of the electric field and sheet current density. The frequency-independent version of this is

$$\overleftrightarrow{Y}_s \neq \text{function of coordinates on sheet of constant } \phi_s \tag{4.5}$$

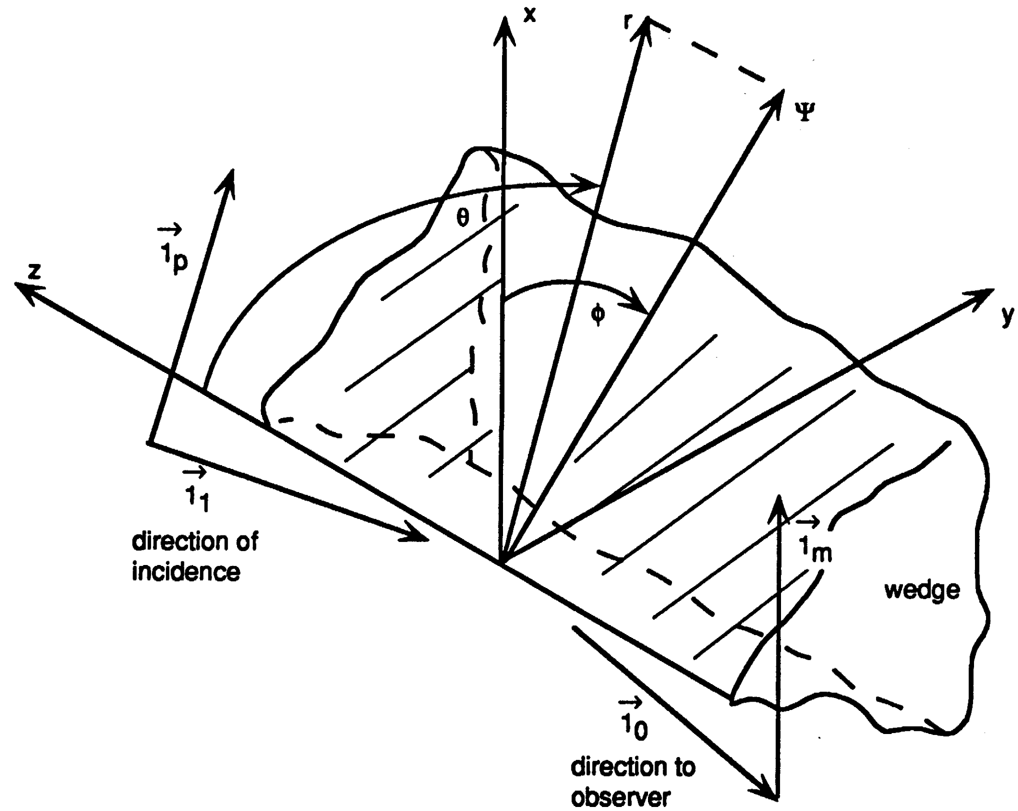

Fig. 4.1. General Wedge Scatterer

Special cases of this sheet conductance are a simple scalar and a perfectly conducting sheet. Note that multiple sheets are in general allowed at various choices of ϕ_S.

The incident field is taken the same as in (3.1). Now the incident polarization $\vec{1}_p$ is naturally referred to the plane of incidence (the plane containing the z axis and being parallel to $\vec{1}_1$). This gives the usual E (or TM) wave with $\vec{1}_p$ parallel to this plane and H (or TE) wave with $\vec{1}_p$ perpendicular to this plane. One can also specify the direction by two angles, θ_1 and ϕ_1 with

$$\cos(\theta_1) = \vec{1}_1 \cdot \vec{1}_z \tag{4.6}$$

and ϕ_1 giving the plane of incidence, this being perpendicular to $\vec{1}_1 \times \vec{1}_z$, and given by

$$\begin{aligned} \sin(\theta_1)\ \cos(\phi_1) &= \vec{1}_1 \cdot \vec{1}_x \\ \sin(\theta_1)\ \cos(\phi_1) &= \vec{1}_1 \cdot \vec{1}_y \\ \tan(\phi_1) &= \frac{\vec{1}_1 \cdot \vec{1}_y}{\vec{1}_1 \cdot \vec{1}_x} \end{aligned} \tag{4.7}$$

The scattered field is characterized by a direction of $\vec{1}_o$ to a distance observer. This is not necessarily the same as $\vec{1}_r$ since the origin can be anywhere along the edge of the wedge. Due to the z-independence of the scatterer both incident and scattered field have the same phase speed $c/\cos(\theta_1)$ along the z axis, giving

$$\theta_o = \theta_1 \quad , \quad \vec{1}_1 \cdot \vec{1}_z = \vec{1}_o \cdot \vec{1}_z \tag{4.8}$$

The observer can, however, take on any value of ϕ_0 outside the wedge. One can define a scattering plane by this ϕ_0 and refer the polarization for the scattered field by parallel and perpendicular to this plane. Let us also restrict both $\phi_1 \pm \pi$ and ϕ_o not only to not lie in the wedge, but also such that the first scattered signal to reach the observer comes from the edge of the wedge. Let us also restrict then the wedge to lie all on one side of a plane (containing the z axis) except for the edge. This allows there to exist a range of ϕ_0 (for a given ϕ_1) for which this assumption of first signal from the edge applys.

For the wedge problem we write the far scattered field as

$$\tilde{\vec{E}}_f\left(\vec{r};s\right) = \left[\frac{2}{\pi\Psi\cos(\theta_1)}\right]^{\frac{1}{2}} e^{-\gamma\Psi\sin(\theta_o)}\,\tilde{\overleftrightarrow{\Lambda}}^{(2)}\left(\vec{1}_o,\vec{1}_1;s\right)\cdot\tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right) \tag{4.9}$$

where we have let the observer be on the z = 0 plane without loss of generality. This form is based on the well-known radiation condition in two dimensions [11] and takes the form of the asymptotic expansion of an appropriate Bessel function for outgoing cylindrical waves [9]. Note that the phase factor of the incident wave along the z axis is included, but the significant point for our scaling is the $\Psi^{-\frac{1}{2}}$ dependence. The superscript 2 is used to distringuish the scattering dyadic for this two-dimensional case. Unlike the previous three-dimensional case where the scattering dyadic has dimensions of length, in the two-dimensional case it has dimensions of $(\text{length})^{\frac{1}{2}}$.

Applying the scaling in (2.5) to the fields in (4.9) requires that

$$\begin{gathered}\tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right) = \tilde{\vec{E}}^{(inc)}\left(\vec{0},s'\right)\\ e^{\gamma\Psi\sin(\theta_o)}\,\tilde{\vec{E}}_f\left(\vec{r},s\right) = e^{\gamma'\Psi'\sin(\theta_o)}\,\tilde{\vec{E}}_f\left(\vec{r}\,',s'\right)\end{gathered} \tag{4.10}$$

and implies that

$$\Psi^{-\frac{1}{2}}\,\tilde{\overleftrightarrow{\Lambda}}^{(2)}\left(\vec{1}_o,\vec{1}_1;s\right) = \Psi'^{-\frac{1}{2}}\,\tilde{\overleftrightarrow{\Lambda}}^{(2)}\left(\vec{1}_o,\vec{1}_1;s'\right) \tag{4.11}$$

Using (2.1) we have

$$\begin{aligned}\tilde{\overleftrightarrow{\Lambda}}^{(2)}\left(\vec{1}_o,\vec{1}_1;s\right) &= \chi^{-\frac{1}{2}}\,\tilde{\overleftrightarrow{\Lambda}}^{(2)}\left(\vec{1}_o,\vec{1}_1;s'\right)\\ &= \left[\frac{s'}{s}\right]^{\frac{1}{2}}\tilde{\overleftrightarrow{\Lambda}}^{(2)}\left(\vec{1}_o,\vec{1}_1;s'\right)\end{aligned} \tag{4.12}$$

For this to hold for all s and s' we have

$$\overset{\leftrightarrow}{\Lambda}^{(2)}\left(\vec{1}_o,\vec{1}_1;s\right) = \left[\frac{c}{s}\right]^{\frac{1}{2}} \overset{\leftrightarrow}{K}^{(2)}\left(\vec{1}_o,\vec{1}_1\right)$$

$$\overset{\leftrightarrow}{K}^{(2)}\left(\vec{1}_o,\vec{1}_1\right) = \text{real valued dyadic (dimensionless)} \tag{4.13}$$

$$= \overset{\leftrightarrow}{K}^{(2)^T}\left(-\vec{1}_1,-\vec{1}_o\right) \quad \text{(reciprocity)}$$

Again this is an exact factorization of the scattering from the wedge into an angular part (dyadic) times a frequency part (scalar, $s^{-\frac{1}{2}}$). It is consistent with the known eigenfunction solution for the perfectly conducting wedge [11].

In time domain (4.9) is written

$$\vec{E}_f\left(\vec{r},t+\frac{\Psi\cos(\theta_1)}{c}\right) = \left[\frac{2}{\pi\Psi\cos(\theta_1)}\right]^{\frac{1}{2}} \overset{\leftrightarrow}{\Lambda}^{(2)}\left(\vec{1}_o,\vec{1}_1;t\right) \circ \vec{E}^{(inc)}\left(\vec{0},t\right)$$

$$\overset{\leftrightarrow}{\Lambda}^{(2)}\left(\vec{1}_o,\vec{1}_1;t\right) = \left[\frac{c}{\pi t}\right]^{\frac{1}{2}} u(t)\overset{\leftrightarrow}{K}^{(2)}\left(\vec{1}_o,\vec{1}_1\right) \tag{4.14}$$

using well-known Laplace transform pairs. For a given incident waveform $f(t)$ one can perform the above convolution to obtain the far-scattered waveform. This kind of convolution is also referred to as fractional integration (the half integral in this case) [8]. For a delta-function incident field $\delta(t)$ the response is as above, $[\pi t]^{-\frac{1}{2}}\, u(t)$. For a step-function incident field $u(t)$ the response is $2[t/\pi]^{\frac{1}{2}}\, u(t)$.

As with the cone, the wedge can also be truncated beyond some Ψ which might also depend on ϕ. One can calculate a clear time based on causality for which the finite structure (with respect to Ψ) behaves as though it were infinite. Here, however, we still require an infinite extent in the $\pm z$ directions.

V. Scattering of Plane Wave by Half Space

Now reduce the number of dimensions to one by considering a half space as indicated in fig. 5.1. Now let the x, y plane (infinite in both directions) be the boundary of the scattering half space, $z \le 0$. Let the half-space parameters be constrained to be only functions of z. Considering the scaling relationship in (2.1) the cartesian coordinates scale as

$$x' = \chi x \quad , y' = \chi y \quad , z' = \chi z \tag{5.1}$$

Imposing the scaling of the fields as in Section II, as well as the x and y invariance, gives

$$\begin{aligned} \tilde{\overleftrightarrow{\epsilon}}\left(\vec{r},s\right) &\equiv \tilde{\overleftrightarrow{\epsilon}}(z,s) = \tilde{\overleftrightarrow{\epsilon}}(z' s') \\ \tilde{\overleftrightarrow{\mu}}\left(\vec{r},s\right) &\equiv \tilde{\overleftrightarrow{\mu}}(z,s) = \tilde{\overleftrightarrow{\mu}}(z' s') \\ \tilde{\overleftrightarrow{\sigma}}\left(\vec{r},s\right) &\equiv \tilde{\overleftrightarrow{\sigma}}(z,s) = \chi\, \tilde{\overleftrightarrow{\sigma}}(z',s') \end{aligned} \tag{5.2}$$

for the scaling of the constitutive parameters. If these are chosen to be frequency independent we have

$$\begin{aligned} &\overleftrightarrow{\epsilon} = \text{constant dyadic} \; , \; \overleftrightarrow{\mu} = \text{constant dyadic} \\ &\overleftrightarrow{\sigma} = -\frac{1}{z}\overleftrightarrow{\Sigma} \quad (z \text{ negative}) \end{aligned} \tag{5.3}$$

This is a curious form for the conductivity since near $z = 0$ a layer of finite thickness gives an infinite sheet conductance. However, zero and ∞ are acceptable constant values for the components of the conductivity dyadic. Note that admittance sheets as discussed in Sections III and IV are not appropriate here due to the requirement of both x and y invariance, except that a frequency-independent $\overleftrightarrow{Y}_s$ is allowed on the $z = 0$ plane.

The incident field is again as in (3.1). The plane of incidence is now taken as containing the z axis and parallel to $\vec{1}_1$ in fig. 5.1. Then an E(TM) wave has $\vec{1}_1$ in this plane and an H(TE) wave has $\vec{1}_1$ perpendicular to this plane. Now the scattered field is also a plane wave with $\vec{1}_o$ parallel to this plane of

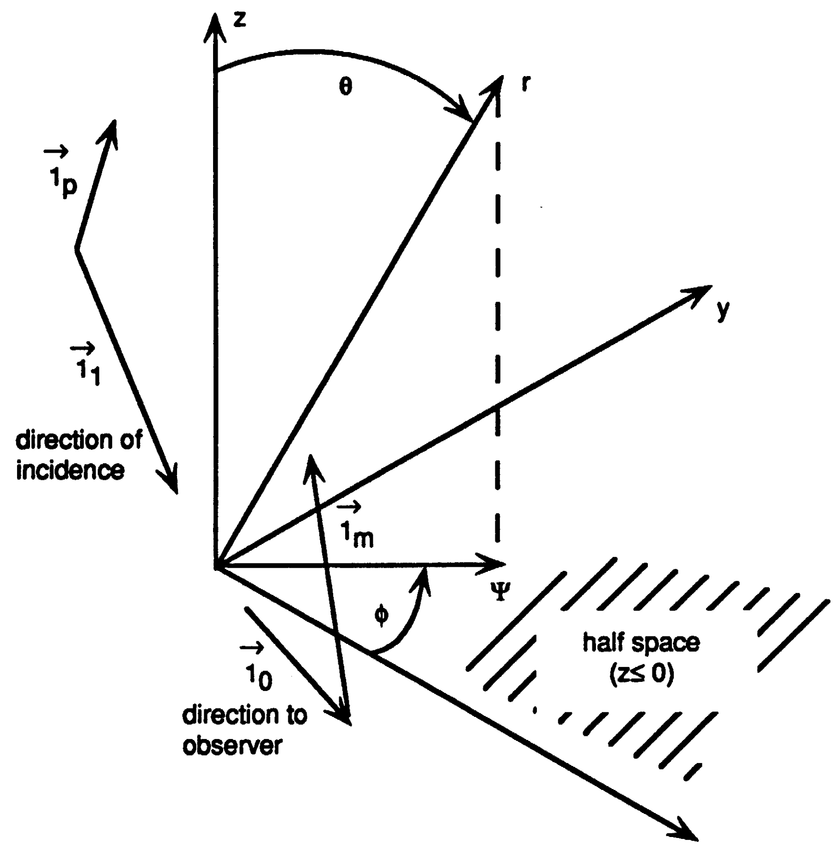

Fig. 5.1. General Half-Space Scatterer

incidence, to which the scattered polarization can also be referred. The incident and scattering directions are related as

$$\vec{1}_o \cdot \vec{1}_z = \cos(\theta_o) = -\vec{1}_1 \cdot \vec{1}_z = -\cos(\theta_1)$$

$$\sin(\theta_o) = \sin(\theta_1) \tag{5.4}$$

$$\theta_o + \theta_1 = \pi$$

For this half-space problem we write the scattered field as

$$\tilde{\vec{E}}^{(sc)}\left(\vec{r},s\right) = e^{-\gamma z \cos(\theta_o)} \tilde{\overleftrightarrow{\Lambda}}^{(1)}\left(\vec{1}_o,s\right) \cdot \tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right) \tag{5.5}$$

where we have let the observer be on the z axis without loss of generality. Note that this is not just a far field, but is the exact scattered field for $z > 0$. While the phase factor along the z axis is included, the significant point for our scaling is the lack of distance scaling (such as r^{-1} or $\Psi^{-\frac{1}{2}}$ previously). The superscript 1 is used to distinguish the scattering dyadic for this one-dimensional case. In the present case the scattering dyadic is dimensionless.

Applying the scaling in (2.5) to these fields requires that

$$\tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right) = \tilde{\vec{E}}^{(inc)}\left(\vec{0},s'\right)$$

$$e^{\gamma z \cos(\theta_o)} \tilde{\vec{E}}^{(sc)}\left(\vec{r},s\right) = e^{\gamma' z' \cos(\theta_o)} \tilde{\vec{E}}^{(sc)}\left(\vec{r}\,',s'\right) \tag{5.6}$$

and implies that

$$\tilde{\overleftrightarrow{\Lambda}}^{(1)}\left(\vec{1}_o,s\right) = \tilde{\overleftrightarrow{\Lambda}}^{(1)}\left(\vec{1}_o,s'\right) \tag{5.7}$$

This makes the scattering independent of s as

$$\overset{\leftrightarrow}{\Lambda}^{(1)}\left(\overrightarrow{1}_o,s\right) = \overset{\leftrightarrow}{K}^{(1)}\left(\overrightarrow{1}_o\right) = \text{real valued dyadic (dimensionless)}$$

$$= \overset{\leftrightarrow}{K}^{(1)^T}\left(-\overrightarrow{1}_1\right) \quad \text{(reciprocity)} \tag{5.8}$$

This result is consistent with the well-known solution for the scattering from a perfectly conducting infinite sheet, and from a uniform isotropic constant (frequency-independent) ϵ and μ half space [2, 7].

In time domain (5.5) is written

$$\overrightarrow{E}^{(sc)}\left(\overrightarrow{r},t + \frac{z\cos(\theta_o)}{c}\right) = \overset{\leftrightarrow}{\Lambda}^{(1)}\left(\overrightarrow{1}_o,t\right) \circ \overrightarrow{E}^{(inc)}\left(\overrightarrow{0},t\right)$$

$$\overset{\leftrightarrow}{\Lambda}^{(1)}\left(\overrightarrow{1}_o,t\right) = \delta(t)\ \overset{\leftrightarrow}{K}^{(1)}\left(\overrightarrow{1}_o\right) \tag{5.9}$$

This gives the result

$$\overrightarrow{E}^{(sc)}\left(\overrightarrow{r},t + \frac{z\cos(\theta_o)}{c}\right) = \overset{\leftrightarrow}{K}^{(1)}\left(\overrightarrow{1}_o\right) \cdot \overrightarrow{E}^{(inc)}\left(\overrightarrow{0},t\right) \tag{5.10}$$

so that the scattered waveform is the same as the incident waveform. In the language of fractional integration this is the zero integral. Note that the polarization can be rotated by the dyadic coefficient, this effect being present in the case of an anisotropic half space allowed by (5.2) and (5.3). In this one-dimensional case one can also truncate the half space at some z (negative), and have a clear time for which the results for an infinite (in z) half space apply.

VI. Concluding Remarks

The use of scaling invariance, applying to a generalized cone, with special cases of wedge and half space, has led to a remarkably simple result. While it does not give an explicit solution for the far-scattered field, it does give an exact factorization (or parameterization) of this field into a temporal (or frequency) part times an angular part. Since this factorization applies to a large class of structures of interest (perfectly conducting cones of arbitrary cross section, dielectric cones, etc.) it should be useful in target identification. There is also the interesting result that the frequency dependence of the scattering is given by $s^{-\frac{d-1}{2}}$ where d is the number of spatial dimensions describing the expansion of the far-scattered field (3 for cone, 2 for wedge, and 1 for half space). In time domain this can be thought of in terms of fractional integration as the $(d-1)/2$ integral. Furthermore, in time domain there can be certain truncations of the scatterer which still allow the time-domain result to be utilized up to some clear time based on speed-of-light limitations.

An interesting comparison can be made to the area-function result in [6]. Based on a physical optics approximation on a perfectly conducting scatterer ($\vec{J}_s = 2\ \vec{1}_S \times \vec{H}^{(inc)}$ on the illuminated side of the scatterer with $\vec{1}_S$ as the outward-pointing surface normal), the reference shows that the scattered field in time domain is proportional to the second derivative of the cross-section area as the incident wavefront sweeps over the scatterer. Assuming that the scatterer is a perfectly conducting cone the area increases as $t^2 u(t)$ with second derivative $2\ u(t)$. Since this is for a delta function incident wave this gives the scattered field as proportional to the time integral of the incident field, in agreement with our present results. However, the area-function approximation is polarization independent, whereas our present exact results retain the polarization dependence which must be there for scattering from perfectly conducting cones of arbitrary cross section. Perhaps comparisons of exact results to the area-function approximation for perfectly conducting cones can give some further insight into this kind of scattering.

While the present results do not give explicit calculation of the angular part of the scattering dyadic, the factoring out of the frequency part greatly simplifies such calculations. One does not need to recalculate for different times or frequencies since one calculation applies to all times and frequencies. Effectively one spatial dimension of the scattering problem has been removed by the imposition of dilation symmetry. This could be utilized to simplify numerical solutions by imposing the associated symmetry in the surface currents etc.

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