Interaction Notes
Note 486

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Representation of Surface Current Density and Far Scattering in EEM and SEM With Entire Functions

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Abstract

In the SEM representation of various electromagnetic response parameters there is in general an entire function included for completeness. Conditions for presence and absence of this entire function are developed here. For the surface current density on a finite-size perfectly conducting object in free space entire-function-free representations are possible in both class-1 and class-2 forms of the coupling coefficients. For the far scattering from such an object, however, the class-1 form leads to the necessary inclusion of such an entire function which can be quantified. This class-1 form is still useful and appropriate for late-time representation and associated target identification. However, for scattering-length (or cross-section) calculations the class-2 form (with no additional entire function) is more suitable. In this case the coefficients are frequency dependent for each pole associated with a natural mode.

Contents

I. Introduction

Continuing on from recent papers [7,8,21,25] this paper considers some of the properties of the eigenmode expansion method (EEM) and singularity expansion method (SEM) with emphasis on the far scattered fields. The general scattering object is described in fig. 1.1 with surface S in a minimum circumscribing sphere of radius a. The incident field is a plane wave as

$$\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) = E_o\tilde{f}(s)\; e^{-\gamma\vec{1}_1\cdot\vec{r}}\;\vec{1}_p$$

$$\vec{E}^{(inc)}(\vec{r},t) = E_o\tilde{f}\left(t-\frac{\vec{1}_1\cdot\vec{r}}{c}\right)\vec{1}_p$$

$$\vec{1}_1 = \text{direction of incidence} \qquad (1.1)$$

$$\overset{\leftrightarrow}{1}_1 = \overset{\leftrightarrow}{1} - \vec{1}_1\vec{1}_1 \equiv \text{ dyadic transverse to incidence}$$

$$\vec{1}_p = \text{ direction of polarization}$$

$$\overset{\leftrightarrow}{1}_1\cdot\vec{1}_p = \vec{1}_p\;,\;\vec{1}_p\cdot\vec{1}_1 = 0$$

$$\gamma = \frac{s}{c}$$

s = Laplace-transform variable (2 sided) or complex frequency

$c = (\mu_o\varepsilon_o)^{-\frac{1}{2}} \equiv$ speed of light

$\sim\equiv$ Laplace transform (two sided)

The incident polarization is represented by a linear combination of the two usual (for radar) polarization unit vectors

$\vec{1}_h \equiv$ horizontal polarization

$\vec{1}_v \equiv$ "vertical" polarization

$$\vec{1}_v\times\vec{1}_h = \vec{1}_1\;,\;\vec{1}_h\times\vec{1}_1 = \vec{1}_v\;,\;\vec{1}_1\times\vec{1}_v = \vec{1}_h \qquad (1.2)$$

The surface current density is related to the incident electric field through the impedance or E-field integral equation as

$$\tilde{\vec{E}}_t^{(inc)}(\vec{r}_s,s) = \left\langle \tilde{\overset{\leftrightarrow}{Z}}_t(\vec{r}_s,\vec{r}_s\,';s);\tilde{\vec{J}}_s(\vec{r}_s\,',s)\right\rangle$$

$$= \overset{\leftrightarrow}{1}_s(\vec{r}_s)\cdot\tilde{\vec{E}}^{(inc)}(\vec{r}_s,s)\quad,\quad \vec{r}_s,\vec{r}_s\,'\in S$$

$$\overset{\leftrightarrow}{1}_S(\vec{r}_s) \equiv \overset{\leftrightarrow}{1} - \vec{1}_S(\vec{r}_s)\vec{1}_S(\vec{r}_s) \qquad (1.3)$$

$\vec{1}_S(\vec{r}_s) \equiv$ unit normal to S (outward) at $\vec{r}_s$

$$\tilde{\overset{\leftrightarrow}{Z}}_t(\vec{r}_s,\vec{r}_s\,';s) = \overset{\leftrightarrow}{1}_S(\vec{r}_s)\cdot\tilde{\overset{\leftrightarrow}{Z}}(\vec{r}_s,\vec{r}_s\,';s)\cdot\overset{\leftrightarrow}{1}_S(\vec{r}_s\,')$$

$$= s\mu_o\overset{\leftrightarrow}{1}_S(\vec{r}_s)\cdot\tilde{\overset{\leftrightarrow}{G}}_o(\vec{r}_s,\vec{r}_s\,';s)\cdot\overset{\leftrightarrow}{1}_S(\vec{r}_s\,')$$

$$= \frac{Z_o\gamma^2}{4\pi}\overset{\leftrightarrow}{1}_S(\vec{r}_s)\cdot\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\vec{1}_R\vec{1}_R + \left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\left[\overset{\leftrightarrow}{1}-\vec{1}_R\vec{1}_R\right]\right\}\cdot\overset{\leftrightarrow}{1}_S(\vec{r}_s\,')$$

$$R \equiv |\vec{r}_s-\vec{r}_s\,'|\quad,\quad \zeta \equiv \gamma R$$

$$\vec{1}_R = \frac{\vec{r}_s-\vec{r}_s\,'}{|\vec{r}_s-\vec{r}_s\,'|}\quad (\text{for } \vec{r}_s\neq\vec{r}_s\,')$$

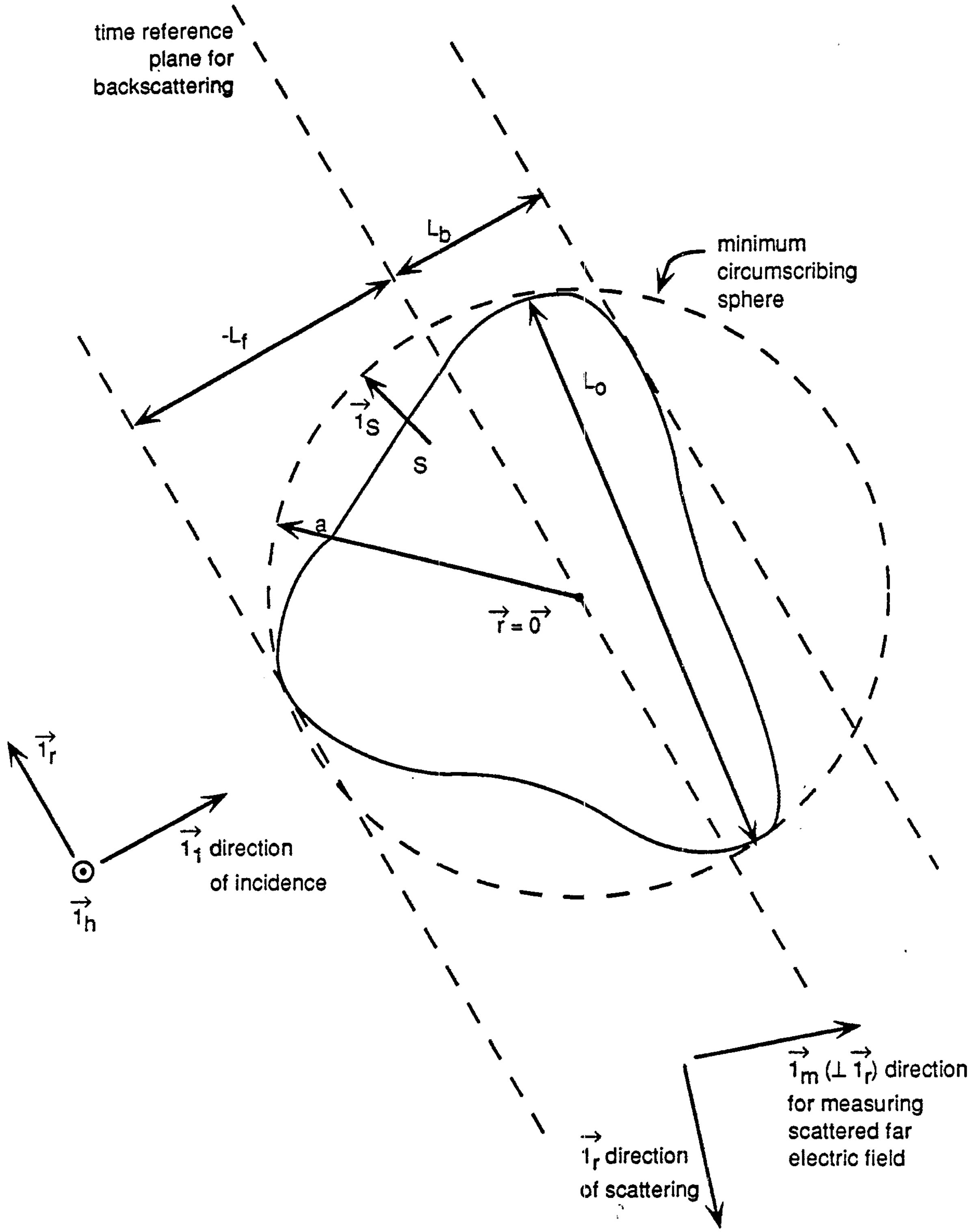

Fig. 1.1. Finite-Size Scatterer in Free Space Illuminated by Plane Wave

$$Z_o = \left(\frac{\mu_o}{\varepsilon_o}\right)^{\frac{1}{2}} \equiv \text{wave impedance of free space}$$

The formal solution to (1.3) is

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = \left\langle \tilde{\vec{\vec{Z}}}_t^{-1}(\vec{r}_s,\vec{r}_s\,';s);\tilde{\vec{E}}^{(inc)}(\vec{r}_s\,',s)\right\rangle \tag{1.4}$$

Besides the surface currents on the body we are interested in the scattered far fields. As indicated in Fig. 1.1 the far field is assumed to be observed propagating in the $\vec{1}_r$ and the component in the $\vec{1}_m$ direction measured with

$$\vec{1}_m \cdot \vec{\vec{1}}_r = 0 \tag{1.5}$$

Of course $\vec{1}_m$ can be decomposed into two orthogonal polarizations. The far field is expressed as

$$\tilde{\vec{E}}_f(\vec{r},s) = -\frac{s\mu_o e^{-\gamma r}}{4\pi r}\left\langle \vec{\vec{1}}_r e^{\gamma \vec{1}_r\cdot\vec{r}_s};\tilde{\vec{J}}_s(\vec{r},s)\right\rangle \tag{1.6}$$

$$\vec{\vec{1}}_r \equiv \vec{\vec{1}} - \vec{1}_r\vec{1}_r \equiv \text{dyadic transverse to scattering direction}$$

For plane-wave incidence as in (1.1) we have

$$\begin{aligned}\tilde{\vec{E}}_f(\vec{r},s) &= \frac{e^{-\gamma r}}{4\pi r}\,\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_r,\vec{1}_1;s\right)\cdot\tilde{\vec{E}}^{(inc)}\left(\vec{0},s\right)\\ &= E_o\tilde{f}(s)\frac{e^{-\gamma r}}{4\pi r}\,\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_r,\vec{1}_1;s\right)\cdot\vec{1}_p\end{aligned} \tag{1.7}$$

where the scattering dyadic (2x2 in terms of the transverse components of incident and far-scattered fields) is

$$\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_r,\vec{1}_1;s\right) = -s\mu_o\left\langle \vec{\vec{1}}_r e^{\gamma\vec{1}_r\cdot\vec{r}_s};\tilde{\vec{\vec{Z}}}_t^{-1}(\vec{r}_s,\vec{r}_s\,';s);\vec{\vec{1}}_1 e^{-\gamma\vec{1}_1\cdot\vec{r}_s\,'}\right\rangle = \tilde{\vec{\vec{\Lambda}}}^T\left(-\vec{1}_1,-\vec{1}_r;s\right) \text{ (reciprocity)} \tag{1.8}$$

For the important case of monostatic or back scattering [9] we have

$$\vec{1}_r = -\vec{1}_1$$

$$\begin{aligned}\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_1,s\right) &\equiv \tilde{\vec{\vec{\Lambda}}}\left(-\vec{1}_1,\vec{1}_1;s\right) = -s\mu_o\left\langle \vec{\vec{1}}_1 e^{-\gamma\vec{1}_1\cdot\vec{r}_s};\tilde{\vec{\vec{Z}}}_t(\vec{r}_s,\vec{r}_s\,';s);\vec{\vec{1}}_1 e^{-\gamma\vec{1}_1\cdot\vec{r}_s\,'}\right\rangle\\ &= \tilde{\vec{\vec{\Lambda}}}^T\left(\vec{1}_1,s\right) \text{ (symmetric)}\end{aligned} \tag{1.9}$$

For comparison to other standard forms [9] we have (in frequency domain) the scattering length

$$\ell \equiv \sqrt{4\pi r}\,\frac{\left|\tilde{\vec{E}}_f(\vec{r},j\omega)\right|}{\left|\tilde{\vec{E}}^{(inc)}(\vec{r},j\omega)\right|} = A^{\frac{1}{2}} \tag{1.10}$$

$A \equiv$ cross section

In more general form we have

$$\tilde{\vec{\vec{\ell}}}\left(\vec{1}_r,\vec{1}_1;s\right) \equiv \frac{1}{\sqrt{4\pi}}\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_r,\vec{1}_1;s\right) \quad , \quad \tilde{\vec{\ell}}\left(\vec{1}_r,\vec{1}_1;s\right) \equiv \tilde{\vec{\vec{\ell}}}\left(\vec{1}_r,\vec{1}_1;s\right)\cdot\vec{1}_p \tag{1.11}$$

$$\tilde{A}(\vec{1}_r, \vec{1}_1; s) = \tilde{\vec{\ell}}(\vec{1}_r, \vec{1}_1; s) \cdot \tilde{\vec{\ell}}(\vec{1}_r, \vec{1}_1; -s)$$

which for $s = j\omega$ reduces to

$$\tilde{A}(\vec{1}_r, \vec{1}_1; j\omega) = \tilde{\vec{\ell}}(\vec{1}_r, \vec{1}_1; j\omega) \cdot \tilde{\vec{\ell}}^*(\vec{1}_r, \vec{1}_1; -s) = \left| \tilde{\vec{\ell}}(\vec{1}_r, \vec{1}_1; j\omega) \right| \tag{1.12}$$

For backscattering (1.9) applies.

Figure 1.1 also gives some characteristic dimensions for the scatterer which define associated times as

$$t_f \equiv \frac{L_f}{c} \equiv \text{front time} \tag{1.13}$$

$\equiv$ time (negative) incident wave first reaches body

$$t_b \equiv \frac{L_b}{c} \equiv \text{back time}$$

$\equiv$ time (positive) incident wave passes body (reaches back)

(Note incident, not total field.)

$$t_o \equiv \frac{L_o}{c} \equiv \text{transit time for maximum linear dimension}$$

$$t_a \equiv \frac{a}{c}$$

$a \equiv$ radius of a minimum circumscribing sphere (establishing coordinate center)

Another time is the latest time that any place on S (exterior) is reached by the total field as

$$t_\ell \equiv \frac{L_\ell}{c} \equiv \text{latest (total) excitation time}$$

$\equiv$ first time after which surface current density has been excited on all of S (exterior) (1.14)

$$t_\ell \equiv t_b$$

For a body that is loaded on S so that waves can pass through S (with attenuation, but no delay) then t_m is the same as t_b. For perfectly conducting bodies t_m is larger due to the geodesic path on and/or outside S waves must propagate to reach points on the backside (side shadowed from the incident wave propagating in $\vec{1}_1$ direction). Similar comments apply to the volume V surrounded by S if filled with a medium with propagation speed less than c (in high-frequency limiting sense). Note some relationships among the various times and distances

$$
\begin{aligned}
&L_o = \sup_{\vec{1}_1} L_b - L_f = ct_o \\
&t_o = \sup_{\vec{1}_1} t_b - t_f \\
&-\frac{a}{c} \le L_f \le 0 \\
&0 \le L_b \le \frac{a}{c} \\
&L_o \le 2\frac{a}{c} \quad \text{(with equality defining a special class of scatterers [13])}
\end{aligned}
\tag{1.15}
$$

As discussed in [7,8,21,25] the eigenmode-expansion-method (EEM) and singularity-expansion-method (SEM) parameters can be written for both surface current density and scattering dyadic. Various properties of the scattering dyadic are developed in terms of the natural modes based on symmetries in the scatterer and presence or lack of modal degencracy. The present paper considers some additional results for the eigenmodes in the far field, as well as convergence properties of various forms that these series expansions can take. Appendices consider the perfectly conducting sphere as an illustration of these results.

II. Eigenmode Representation

The eigenmodes diagonalize the impedance integral equation as

$$\left\langle \tilde{\overset{=}{Z}}_t(\vec{r}_s,\vec{r}_s{}';s); j_{s\beta}(\vec{r}_s{}',s) \right\rangle = \tilde{Z}_\beta(s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)$$

$$= \left\langle \tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s); \tilde{\overset{=}{Z}}_t(\vec{r}_s,\vec{r}_s{}';s) \right\rangle \quad \text{symmetric} \tag{2.1}$$

$$\tilde{Z}_\beta(s) = \left\langle \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s); \tilde{\overset{=}{Z}}_t(\vec{r}_s,\vec{r}_s{}';s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s) \right\rangle \equiv \text{eigenimpedances}$$

$$\tilde{Y}_\beta(s) \equiv \tilde{Z}_\beta^{-1}(s) \equiv \text{eigenadmittances}$$

$$\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \equiv \text{eigenmodes}$$

$$\left\langle \tilde{\vec{j}}_{s\beta_1}(\vec{r}_s{}',s); \tilde{\vec{j}}_{s\beta_2}(\vec{r}_s{}',s) \right\rangle = 1_{\beta_1,\beta_2} \quad \text{(orthonormal)}$$

This gives

$$\tilde{\overset{=}{Z}}{}_t^{\nu}(\vec{r}_s,\vec{r}_s;s) = \sum_\beta \tilde{Z}_\beta^{\nu}(s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s) \tag{2.2}$$

which includes representations for the inverse and identity on S.

The surface current density is now written as

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = \sum_\beta \tilde{Z}_\beta^{-1}(s)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s{}',s); \tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s) \right\rangle \tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \tag{2.3}$$

which for an incident plane wave as in (1.1) is

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = E_o\tilde{f}(s)\sum_\beta \tilde{Z}_\beta^{-1}(s)\vec{1}_p \cdot \tilde{\vec{C}}_\beta(\vec{1}_1,s)\tilde{\vec{j}}_{s\beta}(\vec{r}_s,s) \tag{2.4}$$

$$\tilde{\vec{C}}_\beta(\vec{1}_1,s) = \left\langle \vec{1}_1 e^{-\gamma\vec{1}_1\cdot\vec{r}_s{}'}; \tilde{\vec{j}}_{s\beta}(\vec{r}_s{}',s) \right\rangle$$

As in [6] the eigenmodes are well behaved (bounded) provided the symmetric product for orthogonalization of the given mode as in (2.1) is non zero. Then note that natural frequencies are the zeros of the eigenimpedances as

$$\tilde{Z}_\beta(s_{\beta,\beta'}) = O \tag{2.5}$$

giving some set of natural frequencies associated with each eigenimpedance. These include interior natural frequencies which, for closed perfectly conducting objects, are all first order and located on the $j\omega$ axis of the s plane [1]. The exterior natural frequencies are all in the left half plane (LHP) away from the $j\omega$ axis and are usually first order, although it is noted that special cases of resistively loaded bodies can have second order poles [3,22].

Now the interior natural frequencies for a closed perfectly conducting object cannot appear as such in the external scattering. This requires that the first order zero of $\tilde{Z}_\beta(s)$ on the $j\omega$ axis be canceled by a zero (at least first order) as

$$\tilde{\vec{C}}_\beta\left(\vec{1}_1, s_{\beta,\beta'}\right) = \vec{0} \text{ for } \mathrm{Re}\left[s_{\beta,\beta'}\right] = 0 \tag{2.6}$$

Note that this result is for all $\vec{1}_p$ and $\vec{1}_1$ (or in general for more complicated (non-plane-wave) exterior incident fields). Normally we expect such zeros as in (2.6) to be first order (e.g. the sphere), except that in some cases for special $\vec{1}_1$ and $\vec{1}_p$ this vector function can be zero for all frequencies. So in general we have

$$\tilde{Z}_\beta^{-1}(s)\tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right) = \text{analytic vector function near interior } s_{\beta,\beta'} \text{ for all } \vec{1}_1 \tag{2.7}$$

and the surface current density is well behaved near interior natural frequencies for external incident fields. Stated another way the eigenterm in (2.7) exhibits only exterior poles.

The far-scattering dyadic in (1.8) has the eigenmode form

$$\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_r, \vec{1}_1; s\right) = -s\mu_o \sum_\beta \tilde{Z}_\beta^{-1}(s)\tilde{\vec{C}}_{r_\beta}\left(\vec{1}, s\right)\tilde{\vec{C}}_\beta\left(\vec{1}, s\right)$$

$$C_{r_\beta}\left(\vec{1}_r, s\right) = C_\beta\left(-\vec{1}_r, s\right) = \left\langle \vec{1}_r e^{\gamma \vec{1}_r \cdot \vec{r}_s'} ; \tilde{\vec{j}}_{s\beta}\left(\vec{r}_s', s\right)\right\rangle \tag{2.8}$$

As in (2.6) the $\tilde{\vec{C}}_{r_\beta}$ also has a zero (for all $\vec{1}_r$) at all interior $s_{\beta,\beta'}$. Thus for each eigenterm in (2.8) we have a zero (at least first order) as

$$\tilde{Z}_\beta^{-1}(s)\tilde{\vec{C}}_{r_\beta}\left(\vec{1}_r, s\right)\tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right) = \vec{O} \quad \text{for } \mathrm{Re}\left[s_{\beta,\beta'}\right] = O \tag{2.9}$$

This says that each far-scattering eigenterm has poles at the associated exterior natural frequencies and zeros at the associated internal natural frequencies (at lest first order). Of course, the sum over the eigenterms in (2.8) will not in general be zero at such interior $s_{\beta,\beta'}$, but this may help in sorting out eigenmodes in experimental data. Note that for backscattering (2.8) takes the simpler form

$$\tilde{\vec{\vec{\Lambda}}}\left(\vec{1}_1, s\right) = \tilde{\vec{\vec{\Lambda}}}\left(-\vec{1}_1, \vec{1}_1; s\right) - s\mu_o \sum_\beta \tilde{Z}_\beta^{-1}(s)\tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right)\tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right) \tag{2.10}$$

This zero for each eigenterm in the far scattered field at associated interior natural frequencies is expected to normally be of first order, since a second or higher order zero would require that both the integral over S (for first order) and its derivative with respect to s (in (2.4) giving $\tilde{\vec{C}}_\beta$) be zero at the interior $s_{\beta,\beta'}$. Considering the case of the perfectly conducting sphere (Appendix D) we find that these zeros are first order.

The asymptotic behavior of the eigenterms in the s plane has been considered in [6]. The eigenimpedances behave asymptotically as

$$\tilde{Z}_\beta(s) = \begin{cases} \frac{1}{2} Z_o \text{ in RHP} \\ O_e(-\gamma L_o) \text{ in LHP} \end{cases} \tag{2.11}$$

RHP (right half plane) => $\mathrm{Re}[s] \to +\infty$

LHP (left half plane) => $\mathrm{Re}[s] \to -\infty$

where (as in Appendix A) the O_e (exponential order) symbol gives the exponent of e in the asymptotic bound, neglecting a multiplying function bounded (above and below in this case) by some power of s. So we also have [6]

$$\tilde{Y}_\beta(s) = \tilde{Z}_\beta^{-1}(s) = \begin{cases} \frac{2}{Z_o} \text{ in RHP} \\ O_e(\gamma L_o) \text{ in LHP} \end{cases} \tag{2.12}$$

Inverting the eigenadmittance into time domain gives a convolution operator, the kernel $Y_\beta(t)$ of which can be found by closure of the Bromwich contour as

$$\begin{aligned} &\text{closure in RHP for } t < 0 \\ &\quad (\text{for which } \tilde{Y}_\beta(t) = 0) \\ &\text{closure in LHP for } t > -t_o = -\frac{L_o}{c} \end{aligned} \tag{2.13}$$

So there exists a time window $t_o > 0$ during which the contour can be closed in either half plane.

The inverse Kernel as in (2.2) is

$$\tilde{\bar{\bar{Z}}}_t^{-1}(\vec{r}_s, \vec{r}_s\,'; s) = \sum_\beta \tilde{Y}_\beta(s) \tilde{\bar{\bar{j}}}_{s_\beta}(\vec{r}_s, s) \tilde{\bar{\bar{j}}}_{s_\beta}(\vec{r}_s\,', s) \tag{2.14}$$

The βth term has asymptotic estimates as in (2.12) with the slowly varying (with s) eigenmodes [6] giving

$$\tilde{Y}_\beta(s) \tilde{\bar{\bar{j}}}_{s_\beta}(\vec{r}_s, s) \tilde{\bar{\bar{j}}}_{s_\beta}(\vec{r}_s\,', s) = \begin{cases} O_e(0) \text{ in RHP} \\ O_e(0) \text{ in LHP} \end{cases} \tag{2.15}$$

So at least considering the individual eigenterms the result in (2.13) applies. However, there are questions concerning the relative order of performing summation and integration. Appendix B goes further into these asymptotic estimates. Provided $\vec{r}_s\,' \neq \vec{r}_s$ the Bromwich contour can be closed in both RHP and LHP for $-t_o < t < 0$.

The numerator terms behave like

$$\tilde{\bar{\bar{C}}}_\beta(\vec{1}_1, s) = \begin{cases} O_e(-\gamma L_f) \text{ in RHP} \\ O_e(-\gamma L_b) \text{ in LHP} \end{cases} \tag{2.16}$$

For the surface current density the eigenterms as in (2.4) (neglecting $E_o f(s)$) behave like

$$\tilde{Y}_\beta(s) \tilde{\bar{\bar{C}}}_\beta(\vec{1}_1, s) \tilde{\bar{\bar{j}}}_{s_\beta}(\vec{r}_s, s) = \begin{cases} O_e(-\gamma L_f) \text{ in RHP} \\ O_e(\gamma(L_o - L_b)) \text{ in LHP} \end{cases} \tag{2.17}$$

In time domain this gives a time t_f before which the result is zero (closing inversion contour in RHP) and a time $t_b - t_o$ after which the contour can be closed in the LHP. Note that

$t_s \equiv$ surface time window

$$\equiv -\left[[t_b - t_o] - t_f\right] = -\frac{1}{c}\left[-L_f + L_b - L_o\right] \geq 0 \tag{2.18}$$

so that there exists a time window (t_s) between $t_b - t_o$ and t_f with closure possible in both half planes (and giving zero response) [12,13,14,16]. Varying this over all $\vec{1}_1$ gives special cases with equality (zero window in (2.18)) when $\vec{1}_1$ is aligned along the maximum linear dimension (or axis, one or more) of the body. (See fig. 1.1). The sphere is a special case of this. The Bromwich contour for inversion of the Laplace transform lies in $\mathrm{Re}[s] > 0$ (strip of convergence) and can even move to the left to the first singularity in the term of concern.

This asymptotic consideration can now be extended to the far scattering via the eigenterms in the far-scattering dyadic in (2.8). This can be applied in the bistatic case if we evaluate $\tilde{\bar{\bar{C}}}_{r\beta}\left(\vec{1}_r\right)$ as in (2.16) with new L_f and L_b depending on $\vec{1}_r$. For simplicity the monostatic (backscattering) terms as in (2.10) go like

$$-s\mu_o \tilde{Z}_\beta^{-1}(s)\tilde{\bar{\bar{C}}}_\beta\left(\vec{1}_1,s\right)\tilde{\bar{\bar{C}}}_\beta\left(\vec{1}_1,s\right) = \begin{cases} O_e\left(-2\gamma L_f\right) \text{ in RHP} \\ O_e\left(\gamma(L_o - 2L_b)\right) \text{ in LHP} \end{cases} \tag{2.19}$$

(again neglecting powers of s in the order symbols). In time domain this gives a time $2t_f$ before which the result is zero (closing inversion contour in RHP), and a time $2t_b - t_o$ after which the contour can be closed in the LHP. This gives

$t_{b_s} \equiv$ backscattering time window

$$\equiv -\left[[2t_b - t_o] - 2t_f\right] = -\frac{1}{c}\left[-2L_f + 2L_b - L_o\right] \tag{2.20}$$

Note that depending on shape and orientation of the scatterer this time can be positive or negative. For a positive time window with times between $2t_b - t_o$ and $2t_f$ the contour can be closed in either half plane. Objects which permit this include disk-like ("flat") scatterers with $\vec{1}_1$ normal (broadside) to the plane of the scatterer. Then as in (1.12) L_f and L_b can both be near zero giving a positive t_{b_s} of t_o (or L_o / c). However, rotating such an object until $\vec{1}_1$ is along the L_o dimension ("axis") of the object gives a negative t_{b_s} of $-t_o$ (or $-L_o / c$). In this latter case then t_{b_s} represents the time for which the contour can be closed in neither half plane. For such "flat" or "thin" objects then there exist critical orientations for which $t_{b_s} = 0$. For sufficiently "fat" objects there may be no orientations with positive t_{b_s}. For example a sphere has a t_{b_s} of $-t_o \equiv -t_a$ ($\equiv -a / c$) independent of orientation with respect to $\vec{1}_1$. See also [18,19,20,23,24].

So there seems to be an essential difference between the properties of the surface current density and the far fields. One can also look at the forms of the solution taken directly from the integrals in (1.4) and (1.8), and obtain the same kind of results by looking at the inverse kernel as in [15] and evaluating the integrals over the incident field as done here. In this case contour closure in the LHP is done by a succession of contours ($\rightarrow$ radius ∞) threading between the poles in the LHP. Note that the full inverse kernel includes all the eigenmodes and has an infinite number of poles, including those from all of the infinity of eigenvalues (the pole locations in the LHP in general now extending to $-\infty$). The closure of the Bromwich contour in the RHP and LHP can be applied to the inverse kernel and surface current density (as in (1.4)) and the back-scattering dyadic (in (1.9)) with the various times the same as in this section.

III. SEM Representation: Class 1

The SEM representation begins with the natural frequencies and modes from the impedance integral equation as

$$\left\langle \tilde{\vec{\vec{Z}}}_t(\vec{r}_s,\vec{r}_s\,';s_\alpha);\vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle = \vec{0}$$
$$= \left\langle \vec{j}_s(\vec{r}_s);\tilde{\vec{\vec{Z}}}_t(\vec{r}_s,\vec{r}_s\,';s_\alpha)\right\rangle \tag{3.1}$$

$\vec{j}_{s_\alpha}(\vec{r}_s\,') \equiv$ natural modes (not functions of s)

$s_\alpha \equiv$ natural frequencies

Connecting this to the EEM form in (2.1) and (2.5) we have

$$\alpha = (\beta,\beta') \text{ (index set)}$$
$$s_\alpha = s_{\beta,\beta'}$$
$$\vec{j}_{s_\alpha}(\vec{r}_s) \equiv \vec{j}_{s_{\beta,\beta'}}(\vec{r}_s) = u_\alpha \tilde{\vec{j}}_{s_\beta}(s_{\beta,\beta'}) \tag{3.2}$$
$$u_\alpha = u_{\beta,\beta'} \equiv \text{normalization constant}$$

where the normalization constant is chosen for convenience (e.g. to make the natural mode have a particular peak magnitude, such as unity).

The inverse kernel can be expressed (for $\vec{r}_s\,' \neq \vec{r}_s$) as

$$\tilde{\vec{\vec{Z}}}_t^{-1}(\vec{r}_s,\vec{r}_s\,';s) = \sum_\alpha \frac{e^{-(s-s_\alpha)t_i}}{s-s_\alpha} \mathrm{U}_\alpha \vec{j}_{s_\alpha}(\vec{r}_s)\vec{j}_{s_\alpha}(\vec{r}_s\,')$$
$$+ \text{ possible entire function}$$
$$= e^{-st_i}\sum_\alpha \frac{e^{-s_\alpha t_i}}{s-s_\alpha} \mathrm{U}_\alpha \vec{j}_{s_\alpha}(\vec{r}_s)\vec{j}_{s_\alpha}(\vec{r}_s\,')$$
$$+ \text{ possible entire function} \tag{3.3}$$
$$\mathrm{U}_\alpha = \left\langle \vec{j}_{s_\alpha}(\vec{r}_s);\frac{\partial}{\partial s}\tilde{\vec{\vec{Z}}}_t(\vec{r}_s,\vec{r}_s\,';s)\bigg|_{s=s_\alpha};\vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle^{-1}$$
$$= u_\alpha^{-2}\left[\frac{\alpha}{\partial s}\tilde{Z}_\beta(s)\bigg|_{s=s_{\beta,\beta'}}\right]^{-1}$$

where we have assumed the usual case of first-order poles. (Second and higher order poles are discussed in [22].) The parameter

$$t_i \equiv \frac{L_i}{c} \equiv \text{initial time or turn-on-time} \tag{3.4}$$

is included for later use. This corresponds to a time shift of the scatterer response, recognizing that the initial time that the wave reaches the scatterer is an arbitrary definition. Note that the choice of t_i does not affect the residues at any of the poles. (Let $s \rightarrow$ each s_α). Note that in time domain

$$\frac{e^{-(s-s_\alpha)t_i}}{s-s_\alpha} \to e^{s_\alpha t} u(t-t_i) \tag{3.5}$$

so that the late-time behavior is not affected. However, in time domain the inverse kernel as in (1.4) is a convolution operator.

Now from (2.15) the contour can be closed as in (2.13) in both half planes with a time window t_o. Hence a particular eigenterm as in (2.12) or (2.15) can be written with no entire function. So in (3.3) if we choose $-t_o < t_i < 0$ an entire function is not needed for each (βth) eigenterm (summing over β' for given β). Then summing over β one can construct the inverse kernel. Similarly an individual (βth) eigenadmittance can be constructed as

$$\tilde{Y}_\beta = \sum_{\beta'} \frac{e^{-(s-s_{\beta,\beta'})t_i}}{s-s_{\beta,\beta'}} \mathrm{U}'_\alpha$$

+ possible entire function

+ other potential singularities (3.6)

$$\mathrm{U}_{\alpha'} = \left[\frac{\partial}{\partial s}\tilde{Z}_\beta(s)\bigg|_{s=s_{\beta,\beta'}}\right]^{-1} = \left\langle \tilde{\vec{j}}_{s\beta}\left(\vec{r}_s, s_{\beta,\beta'}\right); \frac{\partial}{\partial s}\tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s)\bigg|_{s=s_{\beta,\beta'}}; \tilde{\vec{j}}_{s\beta}\left(\vec{r}_s\,', s_{\beta,\beta'}\right)\right\rangle^{-1}$$

Again the entire function can be neglected by the choice of t_i in the above time window. Of course one can simplify matters by choosing t_o as zero (or 0_-, allowing for $\vec{r}_s\,' = \vec{r}_s$). The "other potential singularities" allow for potential branch points in the $\tilde{Z}_\beta(s)$ which is still somewhat open [17]. However, such do not exist in the complete sum (3.3), and are not important for present purposes.

Considering the convergence properties of the pole series as in [13] we have

$$\mathrm{U}_\alpha = O_e(\gamma_\alpha L_o) = O_e(s_\alpha t_o) \quad \text{as } \mathrm{Re}[s_\alpha] \to -\infty$$

$$e^{s_\alpha t_i} = O_e(s_\alpha t_i) \quad \text{as } \mathrm{Re}[s_\alpha] \to -\infty \tag{3.7}$$

Then we can think of the series as

$$\tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s) = e^{-st_i}\sum_\alpha O_e\left(s_\alpha(t_i+t_o)\right) \tag{3.8}$$

+ possible entire function

Consider for the present purposes that the sum over α is taken in the sense of sweeping to the left in the s plane and summing the poles as they are successively passed. Then provided the density of the poles, including the functions multiplying the exponentials (all $e^{s_\alpha(t_i+t_o)}$) is bounded by a function growing to the left slower than any exponential (e.g. a power of s_α) then this series converges provided

$$t_i + t_o > 0 \tag{3.9}$$

This is in general the case provided

$t_i > -t_o$ (3.10)

Where $t_o > 0$ is given by the scatterer geometry.

Convergence of the pole series is not in general the same as the criteria for lack of an entire function. As in (2.15) one can close the Bromwich contour in both RHP and LHP in this time window. So summing over β, β' up to some finite β is allowed, but one needs to be concerned with $\beta \to \infty$ and $|s| \to \infty$ along with interchange of the order of summation and integration. As discussed in Appendix B there are other ways to obtain an asymptotic estimate of the inverse kernel involving $\vec{r}_s$ and $\vec{r}_s'$ which avoid this problem. Fortunately for $\vec{r}_s \neq \vec{r}_s'$ one can choose $t_i = 0$ and be consistent with the results in Appendix B.

Similarly write the surface current density in (1.4) an d(1.1) for an incident plane wave as

$$\tilde{\vec{J}}_s(\vec{r}_s, s) = E_o \tilde{f}(s) \sum_{\alpha} \frac{e^{-(s-s_\alpha)t_i}}{s - s_\alpha} \eta_\alpha(\vec{1}_1, \vec{1}_p) \vec{j}_{s_\alpha}(\vec{r}_s)$$

$$+ \text{ possible entire function}$$

$$= E_o \sum_{\alpha} \frac{e^{-(s-s_\alpha)t_i}}{s - s_\alpha} \tilde{f}(s_\alpha) \eta_\alpha(\vec{1}_1, \vec{1}_p) \vec{j}_{s_\alpha}(\vec{r}_s)$$

$$+ \text{ singularities from } \tilde{f}(s)$$

$$+ \text{ possible entire function} \tag{3.11}$$

$$\eta_\alpha(\vec{1}_1, \vec{1}_p) = U_\alpha \vec{1}_p \cdot \left\langle e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'}, \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle$$

$$\equiv \text{ coupling coefficient}$$

This is the class-1 form where in time domain the Bromwich contour is closed on $\tilde{\vec{J}}_s$ in the LHP to give simple complex vectors (no frequency dependence) as residues except for the possible time shift t_i allowing for arrival time of the incident wave. Let us then consider the convergence properties of this series as in [13].

In the numerators in this pole series we have

$$\left\langle e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'}, \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle = O_e(-\gamma_\alpha L_b) = O_e(-s_\alpha t_b) \text{ as } \operatorname{Re}[s_\alpha] \to -\infty \tag{3.12}$$

The series then behaves as

$$\tilde{\vec{J}}_s(\vec{r}_{s,s}) = E_o \tilde{f}(s) e^{-st_i} \sum_{\alpha} O_e(s_\alpha(t_i - t_b + t_o)) \text{ as } \operatorname{Re}[s_\alpha] \to -\infty \tag{3.13}$$

with convergence provided

$$t_i - t_b + t_o > 0 \tag{3.14}$$

Here let us consider the sum in (3.13) as over β' for the eigenterms, and then over β to some large but finite β and finally letting $\beta \to \infty$. Comparing t_i to first arrival of the incident wave on the scatterer we have the condition for convergence

$$t_i - t_f > -t_f + t_b - t_o = -t_s \le 0 \tag{3.15}$$

In a worst-case sense when $\vec{1}_1$ is aligned along L_o then we require

$$t_i > t_f \tag{3.16}$$

or starting the series just after the wave touches the body, with perhaps t_i taken as t_f as a limiting acceptable case. For cases that $\vec{1}_1$ is oriented with $L_b - L_f < L_o$ then t_i can be taken as t_f or even a little earlier with convergence of the series. For times earlier than t_i the pole series is identically zero as is the surface current density. For positions $\vec{r}_s$ away from the point the incident wave first contacts the scatterer one can choose t_i later than t_f but on or before the surface current density begins at $\vec{r}_s$ with series convergence for all times t_i and later.

Compare (3.15) for $t_i - t_f$ to (2.18) for the time window t_s for closure of the Bromwich contour in both half planes. A positive t_s corresponds to $t_i - t_f$ which (as discussed above) occurs as long as we do not have both $\vec{r}_s$ at first contact point of the incident wave and $\vec{1}_1$ aligned along L_o. So pole-series convergence and times for which the contour can be closed in the LHP are tied together. Closure in the LHP means the pole series completely represents the surface current density (no entire function for such times). If the contour is closed in the LHP for times while the surface current density is zero and the pole series gives zero then there is no entire function at all.

For the far-scattering dyadic we can apply this procedure for convergence of the pole series to (1.8) for bistatic scattering or to (1.9) for monostatic scattering. Taking the latter case we have [9]

$$\begin{aligned}
\tilde{\overline{\overline{\Lambda}}}(\vec{1}_1, s) &= \sum_\alpha \frac{e^{-(s-s_\alpha)t_i}}{s-s_\alpha} \vec{c}_\alpha(\vec{1}_1)\vec{c}_\alpha(\vec{1}_1) \\
&\quad + \text{possible entire function}
\end{aligned} \tag{3.17}$$

$$\vec{c}_\alpha(\vec{1}_1) = w_\alpha \vec{C}_\alpha(\vec{1}_1)$$

$$\vec{C}_\alpha(\vec{1}_1) = \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle$$

$$W_\alpha = w_\alpha^2 = -s_\alpha \mu_o U_\alpha$$

$$= -s_\alpha \mu_o \left\langle \vec{j}_{s_\alpha}(\vec{r}_s) ; \frac{\partial}{\partial s} \tilde{\overline{\overline{Z}}}_t(\vec{r}_s, \vec{r}_s'; s)\Big|_{s=s_\alpha} ; j_{s_\alpha}(\vec{r}_s') \right\rangle$$

Considering the behavior of individual poles in the LHP we have

$$\begin{aligned}
W_\alpha &= O_e(\gamma_\alpha L_o) = O_e(s_\alpha t_o) \text{ as } \mathrm{Re}[s_\alpha] \to -\infty \\
\vec{C}_\alpha(\vec{1}_1) &= O_e(-\gamma_\alpha L_b) = O_e(-s_\alpha t_b) \text{ as } \mathrm{Re}[s_\alpha] \to -\infty
\end{aligned} \tag{3.18}$$

The series then behaves as

$$\tilde{\overline{\overline{\Lambda}}}(\vec{1}_1, s) = \sum_\alpha O_e(s_\alpha(t_i - 2t_b + t_o)) \text{as } \mathrm{Re}[s_\alpha] \to -\infty \tag{3.19}$$

with convergence provided

$t_i - 2t_b + t_o > O$ (3.20)

Comparing t_i to first arrival $2t_f$ of the backscatter signal we have the convergence condition

$$t_i - 2t_f > 2\left[-t_f + t_b\right] - t_o = -t_{bs} \qquad (3.21)$$

Note that the right-hand side is the negative of the backscattering time window t_{bs} in (2.20).

If one constrains t_i to be less than or equal to the first signal return time $2t_f$, then this requires that t_{bs} be positive. As we found in Section II this occurs under conditions in which one has a "flat" or "thin" object with near broadside incidence. As the scatterer is rotated t_{bs} becomes negative and t_i must be later than $2t_f$ (up to an amount t_o later) for the series to converge. Constraining t_i to assure convergence means that the backscattering response between t_i and $2t_f$ is not represented by a pole series, and as such can be regarded as an <u>entire function</u> of width up to t_o (in time domain) depending on scatterer orientation.

IV. SEM Representation: Class 2

In the class-2 form of the SEM representation one begins with the inverse kernel as in (3.3) with the choice of zero for t_i as

$$\tilde{\tilde{Z}}_t^{-1}(\vec{r}_s,\vec{r}_s\,';s) = \sum_\alpha [s-s_\alpha]^{-1} U_\alpha \, \vec{j}_{s_\alpha}(\vec{r}_s)\vec{j}_{s_\alpha}(\vec{r}_s\,') \tag{4.1}$$

where as discussed in Appendix B for $\vec{r}_s \neq \vec{r}_s\,'$ we can close the contour in both RHP and LHP and thereby have no remaining entire function. The convergence properties of this sum are discussed in Section III. Now substitute (4.1) into (1.4) to give

$$\begin{aligned}\tilde{\vec{J}}_s(\vec{r}_s,s) &= \sum_\alpha [s-s_\alpha]^{-1} U_\alpha \left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,',s);\vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle \vec{j}_{s_\alpha}(\vec{r}_s) \\ &= E_o \tilde{f}(s) \sum_\alpha U_\alpha [s-s_\alpha]^{-1} \vec{1}_p \cdot \tilde{\vec{C}}_\alpha(\vec{1}_1,s)\vec{j}_{s_\alpha}(\vec{r}_s) \end{aligned} \tag{4.2}$$

$$\tilde{\vec{C}}_\alpha(\vec{1}_1,s) \equiv \left\langle e^{-\gamma \vec{1}_1\cdot\vec{r}_s}, \vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle$$

for an incident plane wave as in (1.1). As with (4.1) there is no additional entire function aside from those in the symmetric products which are frequency-dependent coefficients of the poles. In time domain this is a convolution form where the symmetric products represent successive illumination and contribution to the integral as the incident wave passes over the scatterer. As often observed this class-2 form involves a smoothing which increases the convergence rate of the series, at the cost of a more complicated form of the series (frequency- or time-dependent residues).

Similarly the scattering dyadic can be found by operating on the term $e^{-\gamma \vec{1}_1\cdot\vec{r}_s}$ for the scattered field. For bistatic scattering (1.8) gives

$$\tilde{\tilde{\Lambda}}(\vec{1}_r,\vec{1}_1;s) = \sum_\alpha [s-s_\alpha]^{-1} W_\alpha \tilde{\vec{C}}_\alpha(-\vec{1}_r,s)\tilde{\vec{C}}_\alpha(\vec{1}_r,s) \tag{4.3}$$

which for monostatic (back) scattering reduces to

$$\tilde{\tilde{\Lambda}}(\vec{1}_1,s) = \tilde{\tilde{\Lambda}}(-\vec{1}_1,\vec{1}_1;s) = \sum_\alpha [s-s_\alpha]^{-1} W_\alpha \tilde{\vec{C}}_\alpha(\vec{1}_1,s)\tilde{\vec{C}}_\alpha(\vec{1}_1,s) \tag{4.4}$$

These forms involve no additional entire functions beyond those in the $\tilde{\vec{C}}_\alpha(\vec{1}_1,s)$. In time domain, of course, this involves a lot of convolution. However, in frequency domain (more generally in s plane) we gain the distinct advantage of not having the additional entire function required in (3.17) (class-1 form) which must be present for at least some scatterer orientations. Except in special cases then the class-2 form has much to offer for calculation of scattering length and cross section (as in (1.11) and (1.12).

One could consider various hybrids of class-1 and class-2 forms depending on how the various terms in (4.3) and (4.4) are separately treated, but it is not clear whether any improvement in calculational convenience would be achieved. If one goes to the trouble of integrating over S to find $\tilde{\tilde{C}}_\alpha(\vec{1}_1,s)$ at each s for each natural mode of interest, then applying it to both parts of the dyadic is not much more difficult, at least in frequency domain. For backscattering, of course, the symmetry (reciprocity) means that both vectors in each dyad are the same and in the integral need be done but once.

V. Some Characteristics of Entire Functions

By an entire function is meant a function $\tilde{f}(s)$ which has no singularities in the entire s plane [28,30]. Let us use the same applation for $f(t)$, to which the entire function corresponds in time domain, these two being related by the two-sided Laplace transform. While the present discussion uses a scalar notation, it applies to vector-valued and dyadic-valued functions as well, since the Laplace-transform pair relating s and t can be thought of as applying to each component.

A special, but important, type of entire function might be termed a "time-limited" or "gate" entire function, defined by

$$f(t) = 0 \text{ for } t < t_1 \text{ and } t > t_2 \qquad t_1 > t_2 \tag{5.1}$$

For the "gate" between t_1 and t_2 one might postulate a boundedness, but all one really needs is integrability in the sense that

$$f(s) = \int_{-\infty}^{\infty} g(t)e^{-st}dt = \int_{t_{1-}}^{t_{2+}} g(t)e^{-st}dt \tag{5.2}$$

exists for all finite s. This allows for δ functions and higher order distributions. Note then that such an $\tilde{f}(s)$ is an entire function.

That this is an entire function is also seen from the asymptotic evaluation for high frequencies as

$$\tilde{f}(s) = \begin{cases} O_e(-st_1) & \text{in RHP} \\ O_e(-st_2) & \text{in LHP} \end{cases} \tag{5.3}$$

This implies that one can close the Bromwich contour in the RHP for $t < t_1$ and obtain zero, and in the LHP similarly for $t > t_2$, as required. Refining this somewhat, let $f(t)$ have step-like behavior at the "end points" t_1 and t_2 giving

$$\tilde{f}(s) = \begin{cases} f(t_{1+})\dfrac{e^{-st_1}}{s} & \text{in RHP} \\ -f(t_{2-})\dfrac{e^{-st_2}}{s} & \text{in LHP} \end{cases} \tag{5.4}$$

as leading terms in the asymptotic expansions. Another kind of behavior at the end points, such as δ function behavior, changes the power of s which multiplies the exponentials in (5.4).

Note that this type of "gate" entire function corresponds to the entire-function behavior discussed in Sections II and III. In particular for the backscattered field we have observed that for each eigenmode there is a backscattering time window t_{bs} in (2.20) which, if negative, implies that for the time duration $-t_{bs}$ an entire-function representation is required. Letting the response be divided into three time regimes we have first zero response, second an entire function, and third a pole series (class 1). Let

t_1 and t_2 as above divide these three time regions, indicating that such a "gate" entire function can represent the middle time region.

This is not the only kind of entire function. A sufficient condition is that $f(t)$ decay to zero for both positive and negative t faster than any exponential. An example of this is the Gaussian function

$$f(t) = e^{-\left(\frac{t}{\tau}\right)^2} \quad , \tau > 0$$

$$\tilde{f}(s) = \sqrt{\pi}\, \tau\, e^{\left(\frac{s\tau}{2}\right)^2} \tag{5.5}$$

This can be closed in neither RHP nor LHP for any time. However, the inverse-transform integral can be readily performed along the $j\omega$ axis of the s plane.

VI. Time for Target Identification in Backscattering

One application of the foregoing results concerns the problem of target identification via the determination of the external natural frequencies [21]. As discussed in Section III there can be a time window (after first backscatter return) during which the pole series does not converge (at least not independent of grouping terms in some special way). As such an attempt to fit the data with the correct pole series for the target in this time regime will likely be troublesome. Without knowledge of the target aspect (orientation with respect to the direction of incidence and polarization) one will have difficulty computing the class-2 form of the pole series for convergence in this region (except possibly trying all possible aspects) which involves integration over the target. Fortunately the class-1 form does converge after a time t_o which is worst case for all aspects. Note that t_o can be known in advance for selected types of targets, merely based on the maximum linear dimension L_o.

Another consideration is the time for the incident wave to reach all the scatterer and return information to the observer. Even after a time t_o (or less depending on orientation) for which the class-1 pole series will converge there is still the question of which target, and hence which pole series, to use. One can hypothesize two targets with the same shape on the front (i.e. facing the radar) but different back shapes. If the target consisted of a non-perfectly-conducting surface S (i.e. transparent) then there is a time of $2\left[t_b - t_f\right]$ from first signal return to have information from all of the target. This represents a lower bound on what might be termed an identification time. Varying over all directions of incidence this gives $2t_o$ as the maximum of this time. Compare this to t_o for worst case class-1 series convergence.

For perfectly conducting scatterers the time to discriminate the target is further increased by increased propagation distances for the <u>total</u> fields (and hence surface current density) to reach the back side of the target. As defined in (1.14) and illustrated in fig. 6.1, t_ℓ represents the latest time for such fields to arrive at any point on S. This point is designated $\vec{r}_S$, but there could be more than one such point on S. Note that the propagation path to $\vec{r}_S$ is not in general a straight line, but includes diffracted and creeping rays. So the minimum time for target identification (after first signal return) is increased to

$$t_d \equiv 2\left[t_\ell - t_f\right] = 2t_o + 2\left[t_\ell - t_b\right] > 2t_o \qquad (6.1)$$

As illustrated in fig. 6.1 one can perform a *gedanken* experiment by altering the shape of S near $\vec{r}_S$ so as to increase t_ℓ. This can be accomplished by a protrustion (moving local S farther back), or by a reentrant depression which still increases t_ℓ. For the perfectly conducting sphere t_ℓ is $\pi t_a / 2$ and one may expect t_ℓ to be generally less than this. However, for some shapes (e.g. reentrant) t_ℓ can exceed this. The relationship of these various signal return times is also exhibited in fig. 6.1. Note that fig. 6.1 is a simplified description due to the three-dimensional shape of S. Ray paths can also go around S in the senses of into and out of the page.

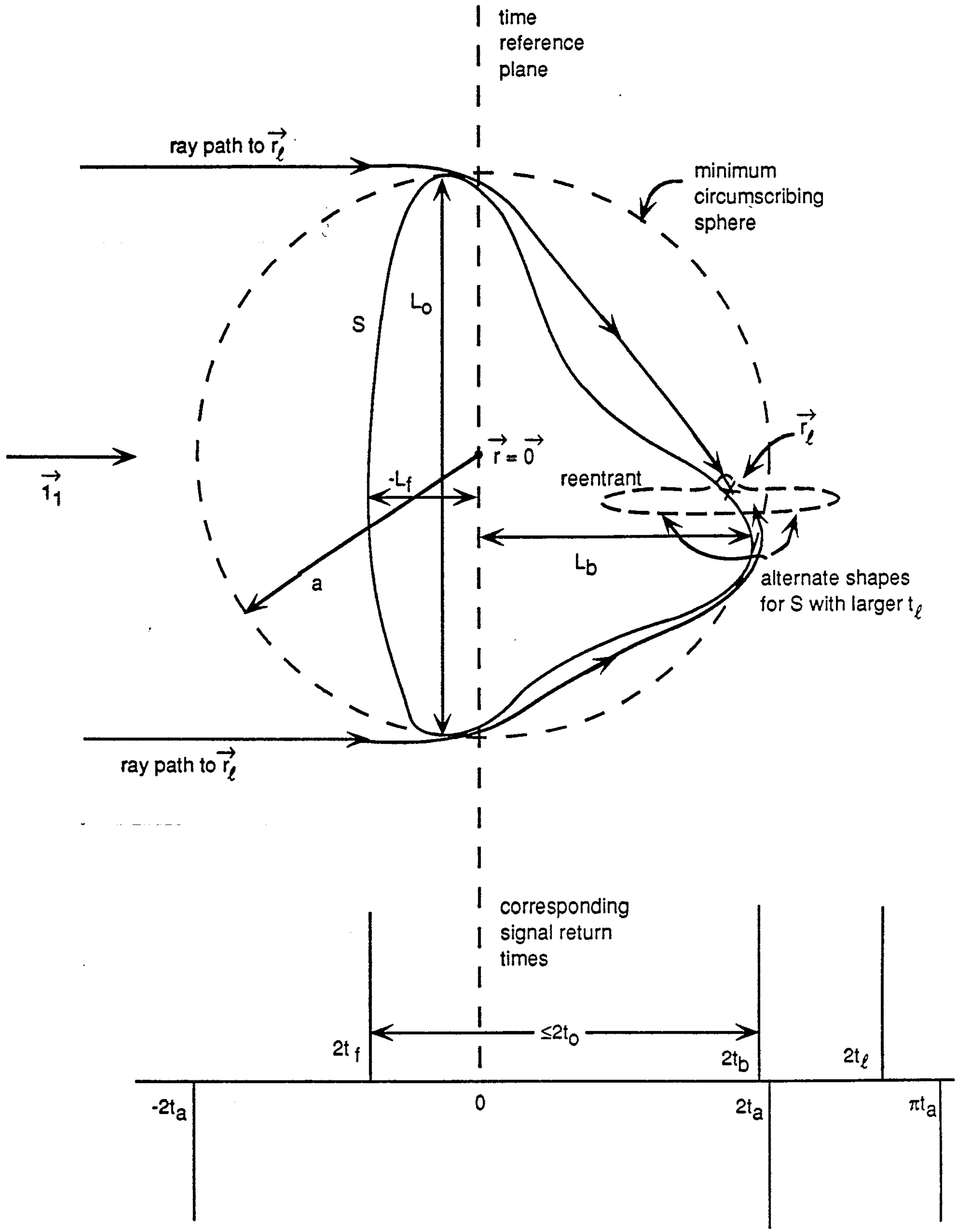

Fig. 6.1. Characteristic Times for Backscattering

VII. Concluding Remarks

Here we have alternative forms for representing surface currents and far scattering. The class-1 form corresponds to a pole expansion in the strict sense, i.e. no frequency dependence in the residues (which can be scalars, vectors, or dyadics for representing the physical problem at hand). For the surface current density class 1 is adequate and convenient. However, for early times or high frequencies other kinds of representations (not discussed here) are more efficient

For target discrimination class 1 is also appropriate. Even though the representation is valid only for late time it still gives a set of complex natural frequencies (and residues if desired) which can be used as an identifying characteristic, particularly since the natural frequencies are aspect independent. Here we have some estimates of what constitutes late time for this purpose including time for the pole series to converge and time to return signals from all of the target. This will also be influenced by the waveform of the incident-field pulse, this being convolved with the target impulse response. Special discriminating waveforms (filters) such as the K / E pulse can also be convolved with the signal returning from the target [21]. These also impact what one should think of as late time.

For the computation of scattering length (or cross section) the use of class-1 residues introduces an entire function which complicates matters. Using the class-2 form in the pole series this particular problem goes away. However, there is a price for this. Each $\tilde{\tilde{C}}_\alpha(\vec{1}_1, s)$ needs to be computed for all frequencies of interest, instead of only at the associated natural frequency s_α. Fortunately, the natural mode $\vec{j}_{s_\alpha}(\vec{r}_s)$ need be computed only once and can be used for all selections of s (=$j\omega$ typically for scattering length) and directions of incidence $\vec{1}_1$. Note that variation of $\vec{1}_1$ is equivalent to rotation of the scatterer and hence rotation of the natural modes in space.

Appendix A. Order Symbols

For asymptotic evaluation one often uses the order symbol O defined by

$$\tilde{F}(s) = O\left(\tilde{G}(s)\right) \text{ as } s \to s_o \tag{A.1}$$

as shorthand for

$$\left|\tilde{F}(s)\right| \le a\left|\tilde{G}(s)\right| \text{ for all } \left|s - s_o\right| < b$$
$$a > 0$$
$$b > 0 \text{ but sufficiently small} \tag{A.2}$$

This is extended to ∞ by requiring

$$\text{for all } |s| > b \text{ (sufficiently large)} \tag{A.3}$$

with possible restriction on the direction that $s \to \infty$, e.g.

$$\text{Re}|s| \to +\infty \text{ or RHP (right half plane)}$$
$$\text{Re}|s| \to -\infty \text{ or LHP (left half plane)} \tag{A.4}$$

So this order symbol is an asymptotic upper bound.

We can similarly define an asymptotic lower bound via

$$\tilde{F}(s) = O_-\left(\tilde{G}(s)\right)$$
$$\left|\tilde{F}(s)\right| \ge a\left|\tilde{G}(s)\right| \text{ for all } \left|s - s_o\right| < b$$
$$a > 0$$
$$b > 0 \text{ but sufficiently small} \tag{A.5}$$

which also applies for $s \to \infty$. If one wishes, for symmetry O can be thought of as O_+. Note that (A.5) implies

$$\tilde{F}^{-1}(s) = O\left(\tilde{G}^{-1}(s)\right) \tag{A.6}$$

and conversely.

For some purposes it is convenient to introduce the concept of exponential order O_e as

$$\tilde{F}(s) = O_e\left(\tilde{G}(s)\right) \text{ as } s \to s_o \tag{A.7}$$

with s_o as some form of $s \to \infty$ as above. This means that

$$\left|\tilde{F}(s)\right| = O\left(e^{\tilde{G}(s) + \chi|s|}\right) = e^{\chi|s|} O\left(e^{\tilde{G}(s)}\right) \text{ as } s \to s_o$$
$$\text{for all } \chi > O \tag{A.8}$$

Essentially, besides the exponential we allow any function which grows less rapidly toward ∞ than any exponential function. This allows for powers of s (including non-integer powers) and emphasizes the important exponential part. Often one deals with a function of order 1 [28,30] such as

$$e^{st'} = O_e(st') = O_e(\text{Re}[s]t') \text{ for } t' \text{ real} \tag{A.9}$$

indicating in effect the power of s in the exponent.

Similarly a lower bound can be defined via

$$\tilde{F}(s) = O_{e-}\left(\tilde{G}(s)\right) \text{ as } s \to s_o$$

$$\left|\tilde{F}(s)\right| = O_{-}\left(e^{\tilde{G}(s)-\chi|s|}\right) \text{ for all } \chi > O \tag{A.10}$$

Again this allows for functions which multiply $e^{\tilde{G}(s)}$ and which decay more slowly than any exponential. Note that

$$\tilde{F}(s) = O_{e-}\left(\tilde{G}(s)\right) \Rightarrow \tilde{F}^{-1}(s) = O_e\left(-\tilde{G}^{-1}(s)\right) \tag{A.11}$$

and conversely. Again O_e can be regarded as O_{e+}.

These order symbols can also be applied to vectors and matrices. The upper bounds can be applied to all the vector or matrix components (all functions of s here). They can also be applied to the norms of such vectors and matrices [5,10], these norms being the generalization of complex magnitude. For lower bounds such norms can also be used. Inverses as in (A.6) and (A.11) are not defined for vectors, but are defined for matrices, in which case the use of norms is appropriate.

Appendix B. Asymptotic Behavior of Kernel and Inverse Kernel for Perfectly Conducting Object

An important term in the present considerations is the kernel of the impedance or E-field integral equation in (1.3). This is closely related ot the dyadic Green's function of free space and gives the fields at $\vec{r}_S$ from a point current source (tangential to S) at $\vec{r}_S'$. As such we have the asymptotic estimate

$$\tilde{\overset{\leftrightarrow}{Z}}_t(\vec{r}_S,\vec{r}_S';s) = O_e\left(-\gamma\left|\vec{r}_S-\vec{r}_S'\right|\right) \text{ in RHP and LHP} \tag{B.1}$$

While this strictly requires $\vec{r}_S \neq \vec{r}_S'$, the singularity here is integrable [11]. Hence, when operating on a bounded surface current density the above estimate applies.

For the inverse kernel [15] has

$$\tilde{\overset{\leftrightarrow}{Z}}_t^{-1}(\vec{r}_S,\vec{r}_S';s) = \begin{cases} O_e\left(-\gamma\left|\vec{r}_S-\vec{r}_S'\right|\right) \text{ in RHP} \\ O_e\left(\gamma\left|\vec{r}_S-\vec{r}_S'\right|\right) \text{ in LHP} \end{cases} \tag{B.2}$$

with the assumption of a convex body. Note that this is defined so that the solution for the surface current is

$$\tilde{\vec{J}}_S(\vec{r}_S,s) = \left\langle \tilde{\overset{\leftrightarrow}{Z}}_t^{-1}(\vec{r}_S,\vec{r}_S';s);\tilde{\vec{E}}^{(inc)}(\vec{r}_S,s)\right\rangle \tag{B.3}$$

This suggests that one look at special forms of the incident field and see what surface currently density is produced.

As in fig. B.1 let there be some source close to but not exactly on S at $\vec{r}_S'$. This can be some current (small antenna) or equivalent magnetic current (magnetic frill). In time domain let this be turned on at $t = 0$. Fields then propagate away from this source with causality (speed of light) limitation for their arrival somewhere else (i.e. $\vec{r}_S$) on the body. Since S is assumed perfectly conducting no energy can penetrate through it. Letting the source be just <u>outside</u> S at $\vec{r}_S'$ the first surface current density (exterior) at $\vec{r}_S$ begins at $L^{(ex)}/c$ where as in fig. B.1 $L^{(ex)}(\vec{r}_S,\vec{r}_S')$ is the minimum distance for a signal to propagate between $\vec{r}_S$ and $\vec{r}_S'$ by an appropriate geodesic path outside S. Then we have

$$\begin{aligned} &\tilde{\vec{J}}_S^{(ex)}(\vec{r}_S,s) = O_e\left(-\gamma L^{(ex)}(\vec{r}_S,\vec{r}_S')\right) \text{ on RHP} \\ &\qquad L^{(ex)}(\vec{r}_S,\vec{r}_S') \geq \left|\vec{r}_S-\vec{r}_S'\right| \text{ (path outside S)} \end{aligned} \tag{B.4}$$

Similarly a source turned on at $\vec{r}_S'$ just inside S gives for the surface current density (interior)

$$\begin{aligned} &\tilde{\vec{J}}_S^{(in)}(\vec{r}_S,s) = O_e\left(-\gamma L^{(in)}(\vec{r}_S,\vec{r}_S')\right) \text{ on RHP} \\ &\qquad L^{(in)}(\vec{r}_S,\vec{r}_S') \geq \left|\vec{r}_S-\vec{r}_S'\right| \text{ (path inside S)} \end{aligned} \tag{B.5}$$

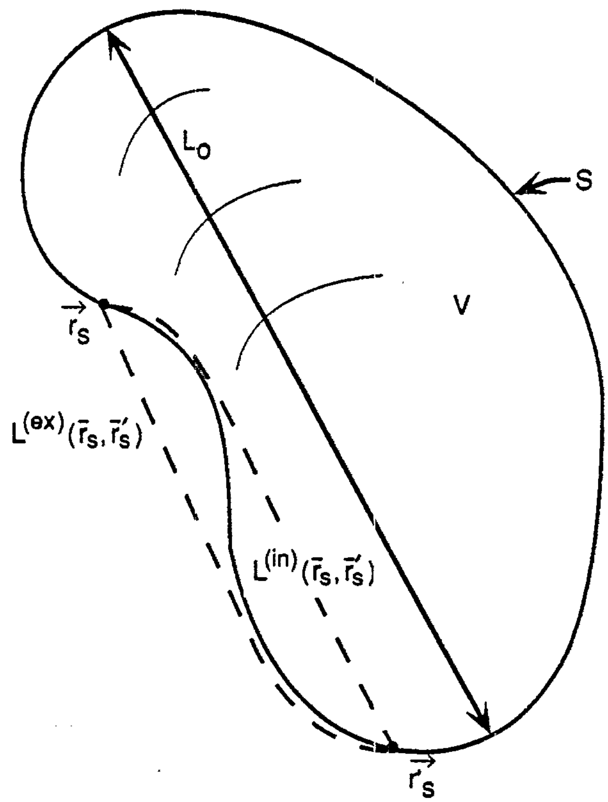

Fig. B.1. Object in Free Space with Perfectly Conducting Closed Surface S Surrounding Volume V

This suggests that one might define some effective inverse kernel as having the property

$$\tilde{Z}_t^{(ex)^{-1}}\left(\vec{r}_S,\vec{r}_S';s\right) = O_e\left(-\gamma L^{(ex)}\left(\vec{r}_S,\vec{r}_S';s\right)\right) \text{ in RHP}$$

$$\tilde{Z}_t^{(in)^{-1}}\left(\vec{r}_S,\vec{r}_S';s\right) = O_e\left(-\gamma L^{(in)}\left(\vec{r}_S,\vec{r}_S';s\right)\right) \text{ in RHP} \tag{B.6}$$

When used with some general $\tilde{\vec{E}}^{(inc)}$ (external or internal but not both), these asympotic estimates give the correct estimate for the surface current density. These are tighter RHP bounds then implied by (B.2). The difference accounts for cancellation of fields on the straight-line path by contributions from surface current density on intervening parts of S. If, however, S is not perfectly conducting but impedance loaded (non zero), then as in [22] $\tilde{\vec{\vec{Z}}}_t$ is modified by the addition of the impedance loading (possibly a function of $\vec{r}_S$). In this case signals can propagate from $\vec{r}_S$ to $\vec{r}_S'$ along the straight-line path and (B.2) applies in the RHP.

In the LHP (B.2) indicates that closure of the Bromwich contour can occur after a time $-\left|\vec{r}_S - \vec{r}_S'\right| / c$ which varies over the body from zero to $-L_o / c$. Comparing this to the EEM form in Section II we have the form in (2.14) where the eigenadmittances $\tilde{Y}_\beta(s)$ are $O_e(\gamma L_o)$ which is the smallest of the bounds in (B.2) corresponding to maximum $\left|\vec{r}_S - \vec{r}_S'\right|$. Of course, this applies to finite β for each eigenterm, and not necessarily to the infinite sum. First summing the poles over β' for each β and thereby constructing eigenterms seems to improve the convergence properties.

Summarizing [1,4,6,26,27,29,31] we have the Legendre functions

$$P_n(\xi) = P_n^{(o)}(\xi) \equiv \frac{1}{2^n n!} \frac{d^n}{d\xi^n}\left[\xi^2 - 1\right]^n$$

$$-1 \le \xi \le 1, n = 0,1,2\cdots$$

$$P_n^{(m)}(\xi) \equiv (-1)^m \left[1-\xi^2\right]^{\frac{m}{2}} \frac{d^m}{d\xi^m} P_n(\xi) \tag{C.1}$$

with special values we will use as

$$P_n^{(m)}(1) = \begin{cases} 1 \text{ for } m=0 \\ 0 \text{ for } 1 \le m \le n \end{cases}$$

$$\frac{d}{d\theta} P_n^{(1)}(\cos(\theta))\bigg|_{\theta=0} = \lim_{\theta \to 0} \frac{P_n^{(1)}(\cos(\theta))}{\sin(\theta)} = -\frac{n(n+1)}{2}$$

$$\frac{d}{d\theta} P_n^{(1)}(\cos(\theta))\bigg|_{\theta=\pi} = \lim_{\theta \to \pi} -\frac{P_n^{(1)}(\cos(\theta))}{\sin(\theta)} = (-1)^{n+1} \frac{n(n+1)}{2} \tag{C.2}$$

The spherical Bessel functions are

$$k_n(\zeta) = \frac{e^{-\zeta}}{\zeta} \sum_{p=O}^{n} \frac{(n+p)!}{p!(n-p)!} (2\zeta)^{-p}$$

$$i_n(\zeta) = \frac{1}{2}\left\{-k_n(-\zeta) + (-1)^{n+1} k_n(\zeta)\right\} \tag{C.3}$$

The $k_n(\gamma r)$ give outgoing waves while $k_n(-\gamma r)$ give incoming waves. These have asymptotic behavior for large arguments

$$k_n(\zeta) = \frac{e^{-\zeta}}{\zeta}\left[1 + O\left(\zeta^{-1}\right)\right] \text{ in RHP and LHP}$$

$$i_n(\zeta) = \begin{cases} \frac{e^{\zeta}}{2\zeta}\left[1+O\left(\zeta^{-1}\right)\right] \text{ in RHP} \\ (-1)^n \frac{e^{-\zeta}}{2\zeta}\left[1+O\left(\zeta^{-1}\right)\right] \text{ in LHP} \end{cases}$$

$$\left[\zeta k_n(\zeta)\right]' = -e^{-\zeta}\left[1 + O\left(\zeta^{-1}\right)\right] \text{ in RHP and LHP}$$

$$\left[\zeta i_n(\zeta)\right]' = \begin{cases} \frac{e^{\zeta}}{2}\left[1+O\left(\zeta^{-1}\right)\right] \text{ in RHP} \\ (-1)^{n+1} \frac{e^{-\zeta}}{2\zeta}\left[1+O\left(\zeta^{-1}\right)\right] \text{ in LHP} \end{cases} \tag{C.4}$$

where the prime indicates differentiation with respect to the argument of the Bessel functions. For small arguments we have

$$k_n(\zeta) = (2n-1)!!\,\zeta^{-n-1}[1+O(\zeta)] \text{ as } \zeta \to 0$$

$$i_n(\zeta) = \frac{\zeta^n}{(2n+1)!!}\left[1+O\left(\zeta^2\right)\right] \text{ as } \zeta \to 0$$

$$[\zeta k_n(\zeta)]' = -n(2n-1)!!\,\zeta^{-n-1}[1+O(\zeta)] \text{ as } \zeta \to 0$$

$$[\zeta i_n(\zeta)]' \frac{n+1}{(2n+1)!!}\zeta^n\left[1+O\left(\zeta^2\right)\right] \text{ as } \zeta \to 0 \tag{C.5}$$

There are the Wronskian relations

$$W\{i_n(\zeta), k_n(\zeta)\} \equiv i_n(\zeta)k_n'(\zeta) - i_n'(\zeta)k_n(\zeta) = -\zeta^2$$

$$W\{\zeta i_n(\zeta), \zeta k_n(\zeta)\} = -1 \tag{C.6}$$

First one constructs the spherical harmonics

$$Y_{n,m,\sigma} = Y_{n,m,{e \atop o}}(\theta,\phi) = P_n^{(m)}(\cos(m\theta))\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$$\sigma = \begin{cases} e \Rightarrow \cos(m\phi) \text{ (even, upper)} \\ 0 \Rightarrow \sin(m\phi) \text{ (odd, lower)}\end{cases}$$

$$\vec{P}_{n,m,\sigma}(\theta,\phi) = Y_{n,m,\sigma}(\theta,\phi)\vec{1}_r$$

$$\vec{Q}_{n,m,\sigma}(\theta,\phi) = \nabla_{\theta,\phi}Y_{n,m,\sigma}(\theta,\phi) = \nabla_{\theta,\phi}\left[\vec{1}_r \cdot \vec{P}_{n,m,\sigma}(\theta,\phi)\right]$$

$$= \vec{1}_\theta \frac{\partial}{\partial\theta}Y_{n,m,\sigma}(\theta,\phi) + \vec{1}_\phi \frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}Y_{n,m,\sigma}(\theta,\phi) \tag{C.7}$$

$$= \vec{1}_\theta\left[\frac{n(n-m+1)}{2n+1}\frac{P_{n+1}^{(m)}\cos(\theta)}{\sin(\theta)} - \frac{(n+1)(n+m)}{2n+1}\frac{P_{n-1}^{(m)}\cos(\theta)}{\sin(\theta)}\right]\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$$+\vec{1}_\phi\, m\frac{P_n^{(m)}\cos(\theta)}{\sin(\theta)}\begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix}$$

$$\vec{R}_{n,m,\sigma}(\theta,\phi) = \nabla_{\theta,\phi}\times\left[\vec{1}_r Y_{n,m,\sigma}(\theta,\phi)\right] = \nabla_{\theta,\phi}\times\vec{P}_{n,m,\sigma}(\theta,\phi)$$

$$= \vec{1}_\theta\, m\frac{P_n^{(m)}\cos(\theta)}{\sin(\theta)}\begin{Bmatrix}-\sin(m\phi)\\ \cos(m\phi)\end{Bmatrix}$$

$$+\vec{1}_\phi\left[-\frac{n(n-m+1)}{2n+1}\frac{P_{n+1}^{(m)}\cos(\theta)}{\sin(\theta)} - \frac{(n+1)(n+m)}{2n+1}\frac{P_{n-1}^{(m)}\cos(\theta)}{\sin(\theta)}\right]\begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix}$$

$\nabla_{\theta,\phi} \equiv$ surface gradient (∇_s) on unit sphere

$$\vec{Q}_{n,m,\sigma}(\theta,\phi) = 1_r \times \vec{R}_{n,m,\sigma}(\theta,\phi)$$

$$\vec{R}_{n,m,\sigma}(\theta,\phi) = -\vec{1}_r \times \vec{Q}_{n,m,\sigma}(\theta,\phi)$$

The three types of vector functions are mutually orthogonal on the unit sphere. They are each orthogonal to another of the same type if any indices (subscripts) are different as

$$\int_o^\pi \int_o^{2\pi} \vec{P}_{n,m,\sigma}(\theta,\phi)\cdot\vec{P}_{n',m',\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta$$
$$=\left[1+\left[1_{e,\sigma}-1_{o,\sigma}\right]1_{0,m}\right]\frac{2\pi}{2n+1}\frac{(n+m)!}{(n-m)!}1_{n,n'}1_{m,m'}1_{\sigma,\sigma'}$$
$$\int_o^\pi \int_o^{2\pi} \vec{Q}_{n,m,\sigma}(\theta,\phi)\cdot\vec{Q}_{n',m'\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta$$
$$=\int_o^\pi \int_o^{2\pi} \vec{R}_{n,m,\sigma}(\theta,\phi)\cdot\vec{R}_{n',m'\sigma'}(\theta,\phi)\sin(\theta)d\phi d\theta$$
$$=\left[1+\left[1_{e,\sigma}-1_{o,\sigma}\right]1_{0,m}\right]2\pi\frac{n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}1_{n,n'}1_{m,m'}1_{\sigma,\sigma'} \tag{C.8}$$

Note that the $\vec{Q}$ and $\vec{R}$ functions are zero for $n=0$. Then one constructs the wave functions

$$\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \Xi^{(\ell)}_{n,m,\sigma}(\gamma r,\theta,\phi) = f_n^{(\ell)}(\gamma r)Y_{n,m,\sigma}(\theta,\phi)$$
$$\vec{L}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \frac{1}{\gamma}\nabla\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$
$$= f_n^{(\ell)'}(\gamma\vec{r})P_{n,m,\sigma}(\theta,\phi)+\frac{f_n^{(\ell)}(\gamma r)}{\gamma r}\vec{Q}_{n,m,\sigma}(\theta,\phi)$$

longitudinal modes (non-zero divergence)

$$M^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \nabla\times\left[\vec{r}\,\Xi^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) = -\frac{1}{\gamma}\nabla\times\vec{N}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})\right. \tag{C.9}$$
$$= f_n^{(\ell)}(\gamma r)\vec{R}_{n,m,\sigma}(\theta,\phi)$$

H modes (when representing $\vec{E}$)

$$\vec{N}^{(\ell)}_{n,m,\sigma}(\gamma\vec{r}) \equiv \frac{1}{\gamma}\nabla\times M^{(\ell)}_{n,m,\sigma}(\gamma\vec{r})$$
$$= n(n+1)\frac{f_n^{(\ell)}(\gamma r)}{\gamma r}\vec{P}_{n,m,\sigma}(\theta,\phi)+\frac{\left[\gamma r f_n^{(\ell)}(\gamma r)\right]'}{\gamma r}\vec{Q}_{n,m,\sigma}(\theta,\phi)$$

E modes (when representing E)

The superscript ℓ corresponds to different choices of the spherical Bessel functions, where we let

$f_n^{(1)}(s) = i_n(\zeta)$ analytic at $\zeta = O$ (for incident wave)

$f_n^{(2)}(s) = k_n(\zeta)$ for outgoing wave

$f_n^{(3)}(\zeta) = k_n(-\zeta)$ for incoming wave (C.10)

with various linear combinations of the above also possible.

With these we can construct a dyadic plane wave with angles θ_1, ϕ_1 for propagation direction as [1,27]

$$\vec{1}_1 = \sin(\theta_1)\cos(\phi_1)\vec{1}_x + \sin(\theta_1)\sin(\phi_1)\vec{1}_y + \cos(\theta_1)\vec{1}_z$$

$$\overleftrightarrow{1}e^{-\gamma\vec{1}_1\cdot\vec{r}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{-\gamma\vec{1}_1\cdot\vec{r}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,o}\left[2-1_{0,m}\right](-1)^n(2n+1)\frac{(n-m)!}{(n+m)!}\left\{-\vec{P}_{n,m,\sigma}(\theta_1,\phi_1)\vec{L}^{(1)}_{n,m,\sigma}(\gamma\vec{r})\right.$$

$$\left.+\frac{1}{n(n+1)}\left[\vec{R}_{n,m,\sigma}(\theta_1,\phi_1)\vec{M}^{(1)}_{n,m,\sigma}(\gamma\vec{r}) - \vec{Q}_{n,m,\sigma}(\theta_1.\phi_1)\vec{N}^{(1)}_{n,m,\sigma}(\gamma\vec{r})\right]\right\} \tag{C.11}$$

Note the rotation symmetry in that one can always rotate the coordinates to make $\vec{1}_1$ coincide with $\vec{1}_z$. Then we have

$$\overleftrightarrow{1}e^{-\gamma z} = \sum_{n=o}^{\infty}(-1)^n(2n+1)\left\{-\vec{1}_z\vec{L}^{(1)}_{n,m,e}(\gamma\vec{r}) + \right. \tag{C.12}$$

$$\left.\frac{1}{n(n+1)}\left[\vec{1}_x\left[-\vec{M}^{(1)}_{n,1,o}(\gamma\vec{r}) + \vec{N}^{(1)}_{n,1,e}(\gamma\vec{r})\right] + \vec{1}_y\left[\vec{M}^{(1)}_{n,1,e}(\gamma\vec{r}) + \vec{N}^{(1)}_{n,1,o}(\gamma\vec{r})\right]\right]\right\}$$

Here only particular choices of m (0 for $\vec{L}$ and 1 for $\vec{M}$ and $\vec{N}$) remain and for $n=0$ the $\vec{M}$ and $\vec{N}$ are identically zero. Note that for a transverse electromagnetic plane wave we can use

$$\overleftrightarrow{1}_z = \overleftrightarrow{1} - \vec{1}_z\vec{1}_z = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y \tag{C.13}$$

$$\overleftrightarrow{1}_z e^{-\gamma z} = \sum_{n=1}^{\infty}(-1)^n \frac{(2n+1)}{n(n+1)}\left\{\vec{1}_x\left[-\vec{M}^{(1)}_{n,1,o}(\gamma\vec{r}) + \vec{N}^{(1)}_{n,1,e}(\gamma\vec{r})\right]\right.$$

$$\left.+\vec{1}_y\left[\vec{M}^{(1)}_{n,1,e}(\gamma\vec{r}) + \vec{N}^{(1)}_{n,1,o}(\gamma\vec{r})\right]\right\}$$

from which any polarization transverse to $\vec{1}_1$ can be constructed. For example, let the incident wave in (1.1) be specified as

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o\tilde{f}(s)\vec{1}_x e^{-\gamma z}$$

$$= E_o\tilde{f}(s)\sum_{n=1}^{\infty}(-1)^n \frac{(2n+1)}{n(n+1)}\left[-\vec{M}^{(1)}_{n,1,o}(\gamma\vec{r}) + \vec{N}^{(1)}_{n,1,e}(\gamma\vec{r})\right]$$

$$\tilde{\vec{H}}^{(inc)}(\vec{r},s) = Z_o E_o\tilde{f}(s)\vec{1}_y e^{-\gamma z} \tag{C.14}$$

$$= Z_o E_o\tilde{f}(s)\sum_{n=1}^{\infty}(-1)^n \frac{(2n+1)}{n(n+1)}\left[\vec{M}^{(1)}_{n,1,e}(\gamma\vec{r}) + \vec{N}^{(1)}_{n,1,o}(\gamma\vec{r})\right]$$

Figure D.1 gives the sphere of radius a with apposite coordinates. It has characteristic dimensions/times (as in (1.13) and (1.14))

$$L_f = -a \quad , \quad L_b = a \quad L_o = 2a$$
$$L_\ell = \frac{\pi}{2} a \tag{D.1}$$

with this L_ℓ applying to the perfectly conducting sphere. On the sphere surface we have

$$\vec{r}_s = a\vec{1}_r \quad , \quad \vec{r}_s' = a\vec{1}_r' \quad , \quad \vec{1}_S(\vec{r}_s) = \vec{1}_r$$
$$|\vec{r}_s - \vec{r}_s'| = a|\vec{1}_r - \vec{1}_r'| \tag{D.2}$$

The eigenimpedances are [6]

$$\tilde{Z}_{e,n}(s) = \tilde{Y}_{e,n}^{-1}(s) = -Z_o[\gamma a i_n(\gamma a)]'[\gamma a k_n(\gamma a)]' \quad E \text{ modes}$$
$$\tilde{Z}_{h,n}(s) = \tilde{Y}_{h,n}^{-1}(s) = -Z_o[\gamma a i_n(\gamma a)][\gamma a k_n(\gamma a)] \quad H \text{ modes}$$
$$\beta = \left(\begin{matrix}\text{e}\\ \text{h}\end{matrix}, n\right) \equiv \text{eigenmode index set}$$
$$\alpha = (\beta, \beta') = \left(\begin{matrix}\text{e}\\ \text{h}\end{matrix}, \beta, \beta'\right) \equiv \text{natural frequency index set} \tag{D.3}$$

For large γa we have

$$\tilde{Z}_{e,n}(s) = \begin{cases} \frac{Z_o}{2}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ in RHP} \\ \frac{Z_o}{2}(-1)^{n+1} e^{-2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ in LHP} \end{cases}$$
$$\tilde{Z}_{h,n}(s) = \begin{cases} \frac{Z_o}{2}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ in RHP} \\ \frac{Z_o}{2}(-1)^{n} e^{-2\gamma a}\left[1 + O\left((\gamma a)^{-1}\right)\right] \text{ in LHP} \end{cases} \tag{D.4}$$

For low frequencies we have

$$\tilde{Z}_{e,n}(s) = \frac{1}{sC_n}\left[1 + O\left((\gamma a)^2\right)\right] \text{ capacitive}$$
$$C_n = \frac{2n+1}{n(n+1)} \varepsilon_o a$$
$$\tilde{Z}_{h,n}(s) = sL_n\left[1 + O\left((\gamma a)^2\right)\right] \text{ inductive}$$
$$L_n = \frac{\mu_o a}{2n+1} \tag{D.5}$$

showing a distinctly different behavior for the two types of modes. Note that both types contain internal (i_n and $[\gamma a i_n]'$ related) and external (k_n and $[\gamma a k_n]'$ related) natural frequencies.

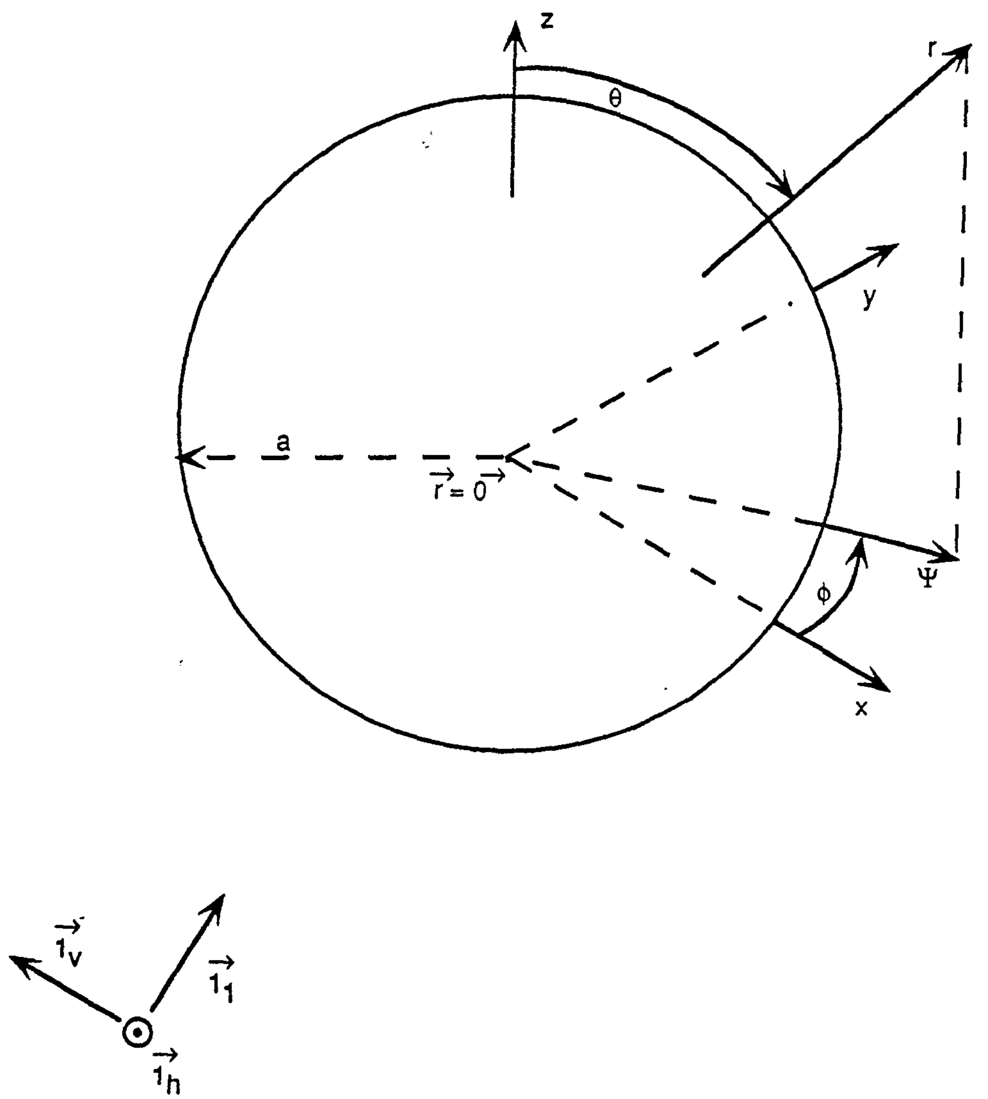

Fig. D.1. The Sphere

Noting that $\tilde{\vec{J}}_s(\vec{r}_s,s)$ has contributions from both incident an scattered fields with the scattered field inside the sphere the negative of the incident field, then using [1] we can define external and internal parts of the eigenimpedances by the ratio of electric field to surface current density on respective sides of S (or by considering each surface current density mode as a source and using (C.9) directly). For E modes we have

$$\tilde{Z}_{e,n}^{(ex)}(s) = \tilde{Y}_{e,n}^{(ex)-1}(s) = -Z_o \frac{[\gamma a k_n(\gamma a)]'}{\gamma a k_n(\gamma a)}$$

$$\tilde{Z}_{e,n}^{(in)}(s) = \tilde{Y}_{e,n}^{(in)-1}(s) = Z_o \frac{[\gamma a i_n(\gamma a)]'}{\gamma a i_n(\gamma a)}$$

$$\tilde{Y}_{e,n}(s) = \tilde{Y}_{e,n}^{(ex)}(s) + \tilde{Y}_{e,n}^{(in)}(s) \quad \text{(parallel combination)} \tag{D.6}$$

and for H modes we have

$$\tilde{Z}_{h,n}^{(ex)}(s) = \tilde{Y}_{h,n}^{(ex)-1}(s) = -Z_o \frac{\gamma a k_n(\gamma a)}{[\gamma a k_n(\gamma a)]'}$$

$$\tilde{Z}_{h,n}^{(in)}(s) = \tilde{Y}_{h,n}^{(in)-1}(s) = Z_o \frac{\gamma a i_n(\gamma a)}{[\gamma a i_n(\gamma a)]'}$$

$$\tilde{Y}_{h,n}(s) = \tilde{Y}_{h,n}^{(ex)}(s) + \tilde{Y}_{h,n}^{(in)}(s) \quad \text{(parallel combination)} \tag{D.7}$$

where the parallel combinations can be verified by a Wronskian in (C.6). These functions are all p.r. (positive real) as required of passive impedances and admittances. The external parts have a finite number ($\approx n$) first-order zeros and first-order poles, a way of ordering the eigenmodes. The internal parts have an infinite number of first-order zeros and first-order poles alternating on the $j\omega$ axis of the s plane as this is a reactance function (Foster's theorem). The external and internal parts are related with

$$\frac{\tilde{Z}_{e,n}^{(ex)}(s)}{Z_o} = Z_o \tilde{Y}_{h,n}^{(ex)}(s)$$

$$\frac{\tilde{Z}_{e,n}^{(in)}(s)}{Z_o} = Z_o \tilde{Y}_{h,n}^{(in)}(s) \tag{D.8}$$

thereby establishing some relationship between the E and H modes. For large γa these are

$$\frac{\tilde{Z}_{e,n}^{(ex)}(s)}{Z_o} = \begin{cases} 1+O\left((\gamma a)^{-1}\right) \text{ in RHP} \\ 1+O\left((\gamma a)^{-1}\right) \text{ in LHP} \end{cases}$$

$$\frac{\tilde{Z}_{h,n}^{(in)}(s)}{Z_o} = \begin{cases} 1+O\left((\gamma a)^{-1}\right) \text{ in RHP} \\ -1+O\left((\gamma a)^{-1}\right) \text{ in LHP} \end{cases} \tag{D.9}$$

Note how much better these are behaved in the LHP as compared to (D.4). For low frequencies we have

$$\frac{\tilde{Z}_{e,n}^{(ex)}(s)}{Z_o} = \frac{n}{\gamma a}\left[1+O(\gamma a)\right]$$

$$\frac{\tilde{Z}_{e,n}^{(in)}(s)}{Z_o} = \frac{n+1}{\gamma a}\left[1+O\left((\gamma a)^2\right)\right] \qquad \text{(D.10)}$$

This can be used to decompose the capacitances and inductances in (D.5) as

$$C_n^{(ex)} = \frac{\varepsilon_o a}{n} \quad , \quad C_n^{(in)} = \frac{\varepsilon_o a}{n+1}$$

$$L_n^{(ex)} = \frac{\mu_o a}{n} \quad , \quad L_n^{(in)} = \frac{\mu_o a}{n+1} \qquad \text{(D.11)}$$

which is a quite symmetrical relationship between the E and H modes.

The eigenmodes are the vector spherical harmonics weighted to be orthonormal as in (2.1) giving from (C.8)

$$\vec{j}_{s_{e,n,m,\sigma}}(\vec{r}_s) = a^{-1}\, \mathrm{v}_{n,m,\sigma}^{-\frac{1}{2}} \vec{Q}_{n,m,\sigma}(\theta,\phi) \quad E \text{ modes}$$

$$\vec{j}_{s_{h,n,m,\sigma}}(\vec{r}_s) = -a^{-1}\, \mathrm{v}_{n,m,\sigma}^{-\frac{1}{2}} \vec{R}_{n,m,\sigma}(\theta,\phi) \quad H \text{ modes}$$

$$\text{(D.12)}$$

$$\beta = \left\{ \begin{matrix} e \\ h \end{matrix}, n, m, \sigma \right\} \text{ index set}$$

$$n = 1, 2, \cdots$$

$$0 \le m \le n \quad , \quad \sigma = e, o$$

$$\mathrm{v}_{n,m,\sigma} = \left[1 + \left[1_{e,\sigma} - 1_{o,\sigma}\right] 1_{0,m}\right] 2\pi \frac{n(n+1)}{2n+1} \frac{(n+m)!}{(n-m)!}$$

These E and H modes are related via [4]

$$\vec{j}_{s_{h,n,m,\sigma}}(\vec{r}_s) = \vec{1}_r \times \vec{j}_{s_{e,n,m,\sigma}}(\vec{r}_s)$$

$$\vec{j}_{s_{e,n,m,\sigma}}(\vec{r}_s) = \vec{1}_r \times \vec{j}_{s_{h,n,m,\sigma}}(\vec{r}_s) \qquad \text{(D.13)}$$

Note that the modes in this case are not a function of frequency. However, they are degenerate with $n+1$ of these associated with each eigenimpedance.

With the above results one can construct the surface current density on the sphere via (2.4) with the additional term

$$\tilde{\vec{C}}_\beta(\vec{1}_1, s) = \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s{}'} ; \vec{j}_{s_\beta}(\vec{r}_s{}', s) \right\rangle \qquad \text{(D.14)}$$

This integral can be expressed explicitly via (C.11). Noting the symmetry a simpler form is found from (C.14) by specializing the direction of incidence to the z axis, with the incident electric field parallel to the x axis as

$$\tilde{\vec{C}}_{e,n,1,e}\left(\vec{1}_z,s\right)=(-1)^n\frac{2n+1}{n(n+1)}a\nu_{n,1,e}^{\frac{1}{2}}\frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a}\vec{1}_x$$

$$\tilde{\vec{C}}_{h,n,1,o}\left(\vec{1}_z,s\right)=(-1)^n\frac{2n+1}{n(n+1)}a\nu_{n,1,o}^{\frac{1}{2}}i_n(\gamma a)\vec{1}_x$$

$$\nu_{n,1,e}=\nu_{n,1,o}=2\pi\left(\frac{n^2(n+1)^2}{2n+1}\right) \tag{D.15}$$

In terms of the vector surface harmonics as in (D.12) we have the simpler form

$$\tilde{\vec{J}}_s(\vec{r}_s,s)=\frac{E_o}{Z_o}\tilde{f}(s)\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}\left[\left(-\frac{1}{\gamma a}\frac{1}{\left[\gamma a k_n(\gamma a)\right]'}\right)\vec{Q}_{n,1,e}(\theta,\phi)\right.$$
$$\left.-\frac{1}{(\gamma a)^2}\frac{1}{k_n(\gamma a)}\vec{R}_{n,1,e}(\theta,\phi)\right] \tag{D.16}$$

This is treated for its SEM form in [1,2].

Note the asymptotic forms for low frequency (contributed to from $n=1$) as

$$\tilde{\vec{J}}_s(\vec{r}_s,s)=\frac{E_o}{Z_o}\tilde{f}(s)\left[\frac{3}{2}\left[\gamma a\vec{Q}_{n,1,e}(\theta,\phi)+O\left((\gamma a)^2\right)\text{ for } E \text{ modes}\right]\right.$$
$$\left.-\frac{3}{2}\left[\vec{R}_{n,1,o}(\theta,\phi)+O(\gamma a)\text{ for } H \text{ modes}\right]\right]\text{ as } \gamma a\to O \tag{D.17}$$

where the two terms correspond to the induced electric an magnetic dipoles. For large s we have

$$\tilde{\vec{J}}_s(\vec{r}_s,s)=\frac{E_o}{Z_o}\tilde{f}(s)\sum_{n=1}^{\infty}(-1)^n\frac{2n+1}{n(n+1)}\left[-\frac{e^{-\gamma a}}{\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right]\vec{Q}_{n,1,e}(\theta,\phi)\right.$$
$$\left.-\frac{e^{-\gamma a}}{\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right]\vec{R}_{n,1,e}(\theta,\phi)\right]\text{ in RHP and LHP} \tag{D.18}$$

Considering this sum one eigenmode at a time note that (neglecting $\tilde{f}(s)$) the Bromwich contour can be closed in the RHP (giving zero) for $t<-a/c$. For $t>-a/c$ the contour can be closed in the LHP giving a finite number of poles (with class-1 coupling coefficients).

Shifting the time reference to $t+a/c$ note that the frequency dependence goes as $(\gamma a)^{-1}$ so there is no pole at ∞ (i.e. no entire function). There is even some leeway in including $\tilde{f}(s)$ as say s^{ℓ} (for integer ℓ) in the pole expansion (class 1). Step-function incidence ($\ell=-1$) does not produce a pole at $s=0$, but $\ell=-2$ (ramp function) does produce such a pole. Positive ℓ such as $\ell=1$ (a doublet) gives a constant behavior as $s \to \infty$ which does require an extra constant term (due to the finite number of poles for each β). While the above considerations apply for each eigenterm, there is still the sum over β which as $s \to \infty$ and in time domain can diverge such as when attempting to produce a distribution such as a delta function.

The far-field scattering is giving by a scattering dyadic as in (1.8) or the far field as in (1.7). For the special case of $+z$ propagation and $+x$ polarization of the incident wave in (C.14) we have

$$
\begin{aligned}
\tilde{\vec{E}}_f(\vec{r},s) &= -E_o \frac{e^{-\gamma r}\gamma Z_o}{4\pi r} \sum_{n=1}^{\infty} \Big[\tilde{Z}_{e,n}^{-1}(s)\tilde{\vec{C}}_{r_{e,n,1,e}}\left(\vec{1}_r,s\right)\tilde{\vec{C}}_{e,n,1,e}\left(\vec{1}_z,s\right)\cdot\vec{1}_x \\
&\quad +\tilde{Z}_{h,n}^{-1}\tilde{\vec{C}}_{r_{h,n,1,o}}\left(\vec{1}_r,s\right)\tilde{\vec{C}}_{h,n,1,o}\left(\vec{1}_z,s\right)\cdot\vec{1}_x\Big] \\
\tilde{\vec{C}}_{r_{e,n,1,e}}\left(\vec{1}_r,s\right) &= \tilde{\vec{C}}_{e,n,1,e}\left(-\vec{1}_r,s\right) = \left\langle \vec{1}_r e^{\gamma\vec{1}_r\cdot\vec{r}_s'};\tilde{\vec{J}}_{s_{e,n,1,e}}(\vec{r}_s',s)\right\rangle \\
&= (-1)^{n+1} 2\frac{2n+1}{n^2(n+1)^2} a\, v_{n,1,e}^{\frac{1}{2}} \frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a}\vec{Q}_{n,1,e}(\pi-\theta,\phi\pm\pi) \\
\tilde{\vec{C}}_{r_{h,n,1,o}}\left(\vec{1}_r,s\right) &= \tilde{\vec{C}}_{h,n,1,o}\left(-\vec{1}_r,s\right) = \left\langle \vec{1}_r e^{\gamma\vec{1}_r\cdot\vec{r}_s'};\tilde{\vec{J}}_{s_{h,n,1,e}}(\vec{r}_s',s)\right\rangle \\
&= (-1)^{n+1} 2\frac{2n+1}{n^2(n+1)^2} a\, v_{n,1,o}^{\frac{1}{2}} i_n(\gamma a)\vec{R}_{n,1,o}(\pi-\theta,\phi\pm\pi)
\end{aligned}
\tag{D.19}
$$

where the angles in the vector harmonics are associated with $-\vec{1}_r$ where $\vec{1}_r$ is the direction to the observer.

For backscattering the symmetry makes the far-scattering dyadic proportional to the transverse dyadic. It can be evaluated from the far field on the $-z$ axis for which the vector coefficients reduce to those in (D.15) giving

$$
\begin{aligned}
\tilde{\vec{\Lambda}}\left(\vec{1}_z,s\right)\cdot\vec{1}_x &= -\gamma Z_o \sum_{n=1}^{\infty}\Big[\tilde{Z}_{e,n}^{-1}\tilde{\vec{C}}_{e,n,1,e}\left(\vec{1}_z,s\right)\tilde{\vec{C}}_{e,n,1,e}\left(\vec{1}_z,s\right) \\
&\quad +\tilde{Z}_{h,n}^{-1}\tilde{\vec{C}}_{h,n,1,o}\left(\vec{1}_z,s\right)\tilde{\vec{C}}_{h,n,1,o}\left(\vec{1}_z,s\right)\Big]\cdot\vec{1}_x \\
\tilde{\vec{\Lambda}}\left(\vec{1}_z,s\right) &= \vec{1}_z 2\pi a Z_o \sum_{n=1}^{\infty}(2n+1)\Bigg[-\tilde{Z}_{e,n}^{-1}(s)\frac{\left[\left[\gamma a i_n(\gamma a)\right]'\right]^2}{\gamma a} \\
&\quad -\tilde{Z}_{h,n}^{-1}(s)\gamma a i_n^2(\gamma a)\Bigg] \\
&= \vec{1}_z 2\pi a \sum_{n=1}^{\infty}(2n+1)\left[\frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a\left[\gamma a i_n(\gamma a)\right]'} - \frac{i_n(\gamma a)}{\gamma a k_n(\gamma a)}\right] \\
&= \vec{1}_z 2\pi a \sum_{n=1}^{\infty}(2n+1)\left[(\gamma a)^2 k_n(\gamma a)\left[\gamma a k_n(\gamma a)\right]'\right]^{-1}
\end{aligned}
\tag{D.20}
$$

again using a WronsKian. Note that, with the result separated into E-mode and H-mode terms, the interior natural frequencies are first-order zeros of the scattering dyadic. The exterior natural frequencies are first order poles. However, at least for backscattering, combining the two kinds of modes for each n gives no zeros except t $s=0$

For low frequency only the $n=1$ term (electric and magnetic dipoles) contributes giving

$$\tilde{\tilde{\Lambda}}\left(\vec{1}_z,s\right)=-6\pi a(\gamma a)^2\vec{1}_z \text{ as } \gamma a\to 0 \tag{D.21}$$

For high frequencies the individual eigenterms behave like

$$\frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a\left[\gamma a i_n(\gamma a)\right]'}=\begin{cases}-\frac{e^{2\gamma a}}{2\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right] \text{ in RHP}\\ \frac{(-1)^n}{2\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right] \text{ in LHP}\end{cases}$$

$$-\frac{i_n(\gamma a)}{\gamma a k_n(\gamma a)}=\begin{cases}-\frac{e^{2\gamma a}}{2\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right] \text{ in RHP}\\ \frac{(-1)^{n+1}}{2\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right] \text{ in LHP}\end{cases} \tag{D.22}$$

This exhibits the closure of the Bromwich contour in the RHP for $t<-2a/c$, and in the LHP for $t>0$. However, if we consider the combination of these terms to form the nth term we have

$$\left[(\gamma a)^2 k_n(\gamma a)\left[\gamma a k_n(\gamma a)\right]'\right]^{-1}=-\frac{e^{2\gamma a}}{\gamma a}\left[1+O\left((\gamma a)^{-1}\right)\right] \text{ in RHP and LHP} \tag{D.23}$$

for which the contour can be closed in the LHP for $t>-2a/c$. In fact this nth term can be expressed as a finite number of poles times $e^{2\gamma a}$, requiring no additional entire function. It would be interesting to understand whether this result applies to other shapes or is peculiar to the sphere.

Considering the high-frequency behavior of $\tilde{\tilde{\Lambda}}$ one can obtain the well-known result for a sphere [31]. Considering the curvature of the surface where the wave first strikes the sphere, the scattered field falls off as $a/(2r)$ where $a/2$ is the distance of the effective apex of the expanding wave behind the surface of the sphere. In our present notation this gives

$$\tilde{\tilde{\Lambda}}\left(\vec{1}_z,s\right)\to 2\pi a e^{2\gamma a}\vec{1}_z \qquad \text{in RHP} \tag{D.24}$$

which also applies on the $j\omega$ axis (for large ω). This can be interpreted as a scattering length in (1.10) as

$$\ell\to -\sqrt{\pi}ae^{2\gamma a} \text{ in RHP and on } j\omega \text{ axis as } \omega\to\infty \tag{D.25}$$

or backscattering cross section

$$A=|\ell|^2\to\pi a^2 \text{ on } j\omega \text{ axis as } \omega\to\infty \tag{D.26}$$

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