Interaction Notes

Note 481

May 1990

The Treatment of Commuting Nonuniform Tubes in Multiconductor - Transmission-Line Theory

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ABSTRACT

In this paper we analytically study a certain class of nonuniform multiconductor transmission lines (NMTL). This class is described by circulant impedance and admittance matrices. Due to the properties of circulant matrices (which form a special set of normal matrices) it is shown that the NMTL equations can be simultaneously diagonalized with the aid of the position-independent so-called Fourier matrix. Thereby the NMTL equations decouple completely and the resulting equations for the modal components of the voltages and currents may be solved with techniques and methods already known from the analyses of single nonuniform transmission lines. Our discussion of application examples covers the high-frequency propagation on NMTL as well as the presentation of an exact solution (in the frequency domain) for a coupled transmission line model for a unit cell of a periodic array of wave launchers. By the same token we apply our methods to a diverging pair of conductors above a perfectly conducting plane. Our special interest, however, is focused on C_N and C_{N_c} symmetry of the NMTL configurations.

*On leave of absence from the NBC Defense Research and Development Institute, P. O. Box 1320, D-3042 Munster, F.R. of Germany.

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Table of Contents

I. Introduction

Multiconductor transmission-line theory is one of the basic tools to analyze the interaction of EMP/EMC with complex modern electronic systems. However, in the vast majority of all cases this theory is applied to uniform systems, i.e., those systems which can be described by means of non-space-dependent impedance and admittance matrices. On the other hand, there is an obvious lack of dealing with non-uniform systems, probably due to the increasing difficulty of the complex mathematical (differential) equations. Nevertheless, the more interesting real physical systems are described with the aid of space-dependent matrices. Therefore it seems worthwhile and necessary to us to investigate nonuniform multiconductor transmission lines (NMTL) in more detail.

In this paper we start our studies of NMTL restricting ourselves to those systems which permit a representation in terms of eigenmodes and eigenvalues. A large class of matrices which can be simultaneously diagonalized is that of pairwise commuting normal matrices. Normal matrices include real symmetric matrices [6] as well as circulant matrices, thereby covering a host of possible physical applications. The use of circulants (Appendix A) requires a certain symmetry of our NMTL configurations (e.g. C_N symmetry). However, dealing with them has the great advantage that the similarity-transformation matrix does not depend on the space coordinates and thus can be absorbed by the quantities behind the differential operators. This is the main reason that we in a first step begin our analysis of NMTL with circulants, trying to deliver a contribution for the closure of the above mentioned gap between the unequal treatment of MTL and NMTL.

The organization of our paper is as follows: In Section II we derive the sourceless NMTL equations and cast them into a form which makes a diagonalization procedure desirable. In Section III we perform this diagonalization and end up with decoupled, quasi-one-dimensional equations for the modal components of the voltages and currents. Furthermore we explicitly present the eigenvalues of circulants for even and odd dimensions. The eigenvectors of symmetric circulants are real and constitute an orthonormal matrix. By introduction of modal reflection coefficients it becomes possible to combine the decoupled second-order differential equations for the voltages and currents into first order nonlinear differential equations which in turn may be solved by methods of perturbation theory. Interesting applications are listed at the end of Section III.

In Section IV we derive the high-frequency limit of our diagonalized NMTL equations. This is done in close analogy to [4] and thereby serves as an interesting alternate way to obtain the results of [4].

Section V deals with a special unit cell of a wave-launcher array for which we find an exact solution in the frequency domain. This solution is expanded for both low and high frequencies, and useful transfer functions are calculated.

In Section VI we demonstrate with the description of two diverging coupled conductors above a conducting plane another example for the applicability of our general formalism.

We close our paper in Section VII with a few concluding remarks.

Finally, the appendices cover some more of the mathematical details.

II. Nonuniform Multiconductor Transmission Line

Consider a single section of a multiconductor-transmission-line tube as displayed in Figure 1. A multiconductor transmission line is one that consists of N conductors and a reference which may be, e.g., infinity or ground. Its physical construction and geometry is described by the per-unit-length impedance matrix $\left(\tilde{Z}_{n,m}\ (z,\ s)\right)$ and the per-unit-length admittance matrix $\left(\tilde{Y}'_{n,m}\ (z,\ s)\right)$. Due to the reciprocity principle, we assume these matrices to be symmetric throughout our paper [6].

The equations governing the voltage and current propagation on a single tube of N wires are the telegrapher equations

$$\frac{d}{dz}\left(\tilde{V}_n(z,\ s)\right) = -\left(\tilde{Y}'_{n,m}\ (z,\ s)\right) \cdot \left(\tilde{I}_n(z,\ s)\right) + \left(\tilde{V}^{(s)'}(z,\ s)\right) \tag{2.1}$$

$$\frac{d}{dz}\left(\tilde{I}_n(z,\ s)\right) = -\left(\tilde{Y}'_{n,m}\ (z,\ s)\right) \cdot \left(\tilde{V}_n(z,\ s)\right) + \left(\tilde{I}_n^{(s)'}(z,\ s)\right) \tag{2.2}$$

Here we have introduced the following quantities:

z ≡ position along the tube

s ≡ Laplace-transform variable (complex frequency) for transform over time (t)

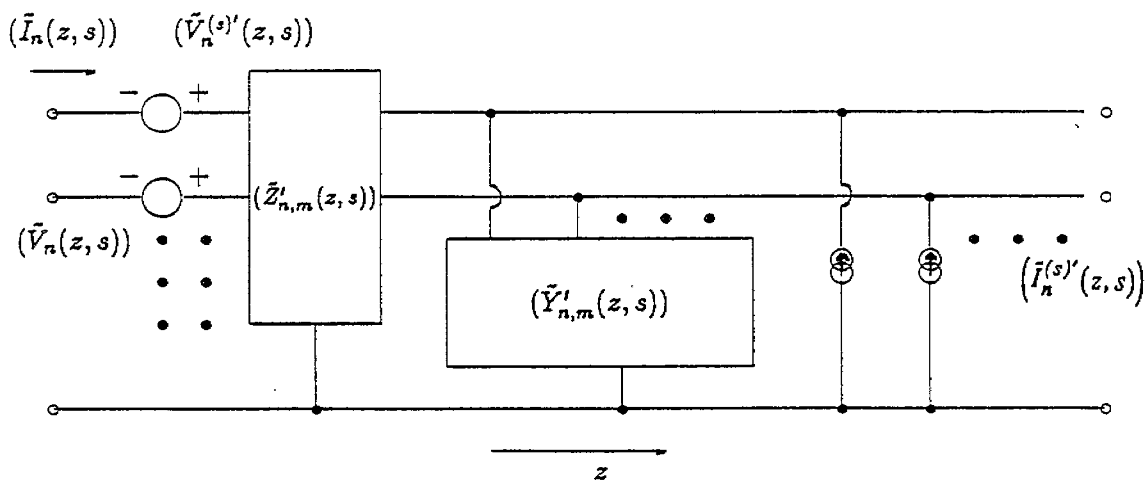

Figure 1. Per-unit-length model of a multiconductor transmission line.

$\left(\tilde{V}_n(z,s)\right)$ ≡ voltage vector at z

$\left(\tilde{I}'_{n,m}(z,s)\right)$ ≡ current vector at z

$\left(\tilde{Z}'_{n,m}(z,s)\right)$ ≡ per-unit-length series impedance matrix

$\left(\tilde{Y}'_{n,m}(z,s)\right)$ ≡ per-unit-length shunt admittance matrix (2.2')

$\left(\tilde{V}_n^{(s)'}(z,s)\right)$ ≡ per-unit-length series voltage source vector

$\left(\tilde{I}_n^{(s)'}(z,s)\right)$ ≡ per-unit-length shunt current source vector

Define the propagation matrix $\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)$ as a principal-value matrix

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right) \equiv \text{principal value of } \left[\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)\right]^{\frac{1}{2}} \tag{2.3}$$

and derive from this the characteristic impedance matrix and characteristic admittance matrix

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right) = \left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)\cdot\left(\tilde{Y}'_{c_{n,m}}(z,s)\right)^{-1} = \left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^{-1}\cdot\left(\tilde{Z}'_{c_{n,m}}(z,s)\right) \tag{2.4}$$

$$\left(\tilde{Y}_{c_{n,m}}(z,s)\right) = \left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^{-1} = \left(\tilde{Z}'_{n,m}(z,s)\right)^{-1}\cdot\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right) \tag{2.5}$$

$$= \left(\tilde{Z}_{c_{n,m}}(z,s)\right)$$

A reasonable interpretation of (2.3) assumes that the product $\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)$ is diagonalizable.

Then taking the p.r. square roots of the eigenvalues of $\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^2$ leads to the eigenmode expansion

$$\left(\tilde{\gamma}_{n,m}(z,s)\right) = \sum_{\delta=1}^{N}\tilde{\gamma}(z,s)\,\left(\tilde{\upsilon}_{c_n}(z,s)\right)_\delta\,\left(\tilde{i}_{c_n}(z,s)\right)_\delta \tag{2.6}$$

where the quantities $\left(\tilde{v}_{c_n}(z,s)\right)_\delta$ and $\left(\tilde{i}_{c_n}(z,s)\right)_\delta$ denote normalized right and left eigenvectors of the matrix $\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^2$, respectively. The eigenvalues $\tilde{\gamma}_\delta(z,s)$ are the principal values of $\left[\tilde{\gamma}_\delta^2(z,s)\right]^{\frac{1}{2}}$.

Under the assumption that $\left(\tilde{Z}'_{n,m}(z,s)\right)$ and $\left(\tilde{Y}'_{n,m}(z,s)\right)$ are not functions of z compact solutions of (2.1) and (2.2) are given in [3, 6]. In this paper we are interested in finding solutions of (2.1) and (2.2) letting the per-unit-length impedance and admittance matrices be functions of z, but assuming there to be no sources $\left(\tilde{V}_n^{(s)\prime}(z,s)\right)$ and $\left(\tilde{I}_n^{(s)\prime}(z,s)\right)$ along the tube. The only "excitation" comes from conditions at some coordinate z_0 of the tube (e.g. to be taken as $z_0 = 0$). In this case (2.1) and (2.2) reduce to

$$\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \tag{2.7}$$

$$\frac{d}{dz}\left(\tilde{I}_n(z,s)\right) = -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) \tag{2.8}$$

Resolve (2.7) with respect to $\left(\tilde{I}_n(z,s)\right)$ and replace this quantity in (2.8) giving

$$\frac{d}{dz}\left(\left(\tilde{Z}'_{n,m}(z,s)\right)^{-1}\frac{d}{dz}\left(\tilde{V}_n(z,s)\right)\right) = \left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) \tag{2.9}$$

Applying the chain rule of differentiation on (2.9) we obtain

$$\frac{d}{dz}\left(\left(\tilde{Z}'_{n,m}(z,s)\right)^{-1}\right)\cdot\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) + \left(\tilde{Z}'_{n,m}(z,s)\right)^{-1}\cdot\frac{d^2}{dz^2}\left(\tilde{V}_n(z,s)\right) = \left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) \tag{2.10}$$

Dot multiply this equation from the left with the matrix $\left(\tilde{Z}'_{n,m}(z,s)\right)$ and finally get

$$\begin{aligned}&\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\frac{d}{dz}\left(\left(\tilde{Z}'_{n,m}(z,s)\right)^{-1}\right)\cdot\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) + \\ &\qquad \frac{d^2}{dz^2}\left(\tilde{V}_n(z,s)\right) - \left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) = \left(0_n\right)\end{aligned} \tag{2.11}$$

Taking the derivative of the matrix itself instead of its inverse (2.11) becomes

$$\frac{d^2}{dz^2}\left(\tilde{V}_n(z,s)\right) - \left(\tilde{Z}'_{n,m}(z,s)\right)^{-1} \cdot \frac{d}{dz}\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot$$

$$\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) - \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{V}_n(z,s)\right) = (0_n) \tag{2.12}$$

For the current vector $\left(\tilde{I}_n(z,s)\right)$ we can derive an analogous equation. We obtain this equation from (2.12) replacing $\left(\tilde{Z}'_{n,m}(z,s)\right)$ by $\left(\tilde{Y}'_{n,m}(z,s)\right)$ and $\left(\tilde{V}_n(z,s)\right)$ by $\left(\tilde{I}_n(z,s)\right)$. This procedure results in

$$\frac{d^2}{dz^2}\left(\tilde{I}_n(z,s)\right) - \left(\tilde{Y}'_n(z,s)\right)^{-1} \cdot \frac{d}{dz}\left(\tilde{Y}'_{n,m}(z,s)\right) \cdot$$

$$\frac{d}{dz}\left(\tilde{I}_n(z,s)\right) - \left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{I}_n(z,s)\right) = (0_n) \tag{2.13}$$

Equations (2.12) and (2.13) are second-order differential equations for the voltage vector and the current vector, respectively. Since they are of the same mathematical structure it is sufficient to find an analytical solution of one of them. The solution of the other then results by analogy, of course, with different boundary conditions. In what follows we deal with equation (2.12).

Let us assume that we can diagonalize the matrix product $\left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right)$ with the aid of a so-far not-further-specialized similarity-transformation matrix $\left(U_{n,m}\right)$, i.e.,

$$\left(\tilde{\gamma}^{(d)}_{c_{n,m}}(z,s)\right)^2 = \left(U_{n,m}\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) \cdot \left(U_{n,m}\right) \tag{2.14}$$

Then we may left-multiply (2.12) by $\left(U_{n,m}\right)^{-1}$ and get

$$\left(U_{n,m}\right)^{-1} \cdot \frac{d^2}{dz^2}\left(\tilde{V}_n(z,s)\right) - \left(U_{n,m}\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\frac{d}{dz}\left(\tilde{Z}'_{n,m}(z,s)\right)\right) \cdot \left(U_{n,m}\right) \cdot \left(U_{n,m}\right)^{-1}$$

$$\cdot \left(\frac{d}{dz}\left(\tilde{V}_n(z,s)\right)\right) - \left(\tilde{\gamma}^{(d)}_{c_{n,m}}(z,s)\right)^2 \cdot \left(U_{n,m}\right)^{-1} \cdot \left(\tilde{V}_n(z,s)\right) = (0_n) \tag{2.15}$$

where we two times have inserted the identity matrix $\left(1_{n,m}\right) = \left(U_{n,m}\right) \cdot \left(U_{n,m}\right)^{-1}$. Looking at (2.15), it becomes desirable to diagonalize as well the matrix product $\left(\tilde{Z}'_{n,m}((z,s)\right)^{-1} \cdot \left(\frac{d}{dz}\left(\tilde{Z}'_{n,m}(z,s)\right)\right)$ with the same similarity transformation matrix $(U_{n,m})$. In order to realize this requirement we consult appropriate mathematical textbooks. There we find a very useful answer to our problem [16].

Theorem: Let there be given a set (finite or infinite) of pairwise commuting normal matrices

$$\left\{\left(A_{n,m}^{(1)}\right),\quad \left(A_{n,m}^{(2)}\right),\quad \left(A_{n,m}^{(3)}\right),\quad \ldots\right\} \tag{2.16}$$

where normal means that

$$\left(A_{n,m}^{(i)}\right)^{\dagger} \cdot \left(A_{n,m}^{(i)}\right) = \left(A_{n,m}^{(i)}\right) \cdot \left(A_{n,m}^{(i)}\right)^{\dagger} \tag{2.17}$$

(i = 1, 2, 3, . . .)

holds, then all these matrices can be transformed into diagonal form by one and the same unitary matrix $(U_{n,m})$.

Applying this theorem to our situation we then have to demand that our set of matrices

$$\left\{\left(\tilde{Z}'_{n,m}(z,s)\right)^{-1},\ \frac{d}{dz}\left(\tilde{Z}'_{n,m}(z,s)\right),\ \left(\tilde{Z}'_{n,m}(z,s)\right),\ \left(\tilde{Y}'_{n,m}(z,s)\right)\right\} \tag{2.18}$$

consists of normal, pairwise commuting matrices. Of course, such a requirement means a severe restriction of the class of physical permissible matrices. On the other hand; however, it is of special interest to find out which physical configurations can still be described by those matrices.

III. Eigenmode Expansion for Commuting Nonuniform Tubes

A special class of commuting normal matrices is given by circulant matrices (see Appendix A). The set of all non-singular circulant N X N matrices forms a commutative group (having even a far richer structure) with respect to matrix multiplication (Appendix A). Thus all elements of this group can be simultaneously diagonalized with the aid of one and the same unitary matrix $\left(U_{C_{n,m}}\right)$ (sometimes also called the Fourier matrix [19]) which columns

$$\left(u_{C_n}\right)_\beta = \frac{1}{\sqrt{N}} \begin{pmatrix} e^{j2\pi\frac{\beta}{N}} \\ e^{j2\pi\frac{2\beta}{N}} \\ e^{j2\pi\frac{3\beta}{N}} \\ \cdot \\ \cdot \\ e^{j2\pi\beta} \end{pmatrix} \tag{3.1}$$

$$\beta = 1,\ 2,\ 3,\ \ldots,\ N$$

are the orthonormal eigenvectors of the circulant matrices. The last components of these eigenvectors are all equal to one, i.e.

$$u_{C_{N;\beta}} = e^{j2\pi\beta} = 1 \tag{3.2}$$

The eigenvalues of a circulant matrix

$$\left(C_{n,m}\right) \equiv \text{circ}\ \left(C_1,\ C_2,\ C_3,\ \ldots,\ C_N\right) \tag{3.3}$$

are found to be (Appendix B2)

$$c_\beta = \sum_{k=1}^{N} C_{k+1}\ exp\left\{j2\pi\frac{k\beta}{N}\right\} \tag{3.4}$$

$$(\ C_{N+1} \equiv C_1\ )$$

A closer inspection of (3.1) reveals the fact that the eigenvectors $\left(u_{C_n}\right)_\beta$ are related to each other via the equations

$$\left(u_{C_n}\right)^*_\beta = \left(u_{C_n}\right)_{N-\beta} \tag{3.5}$$

$$\beta = \begin{Bmatrix} 1, 2, 3, \ldots, \frac{N}{2} - 1 & \text{if } N \text{ is even} \\ 1, 2, 3, \ldots, \frac{N-1}{2} & \text{if } N \text{ is odd} \end{Bmatrix} \tag{3.6}$$

and

$$\left(u_{C_n}\right)^*_{\frac{N}{2}} = \left(u_{C_n}\right)_{\frac{N}{2}} = \frac{1}{\sqrt{N}} \begin{pmatrix} -1 \\ +1 \\ -1 \\ \cdot \\ \cdot \\ +1 \end{pmatrix} \tag{3.7}$$

for an even N. These relations can easily be seen recalling that the complex components of the vectors in (3.1) all lie on the unit circle in the complex plane. They appear in complex conjugate pairs and constitute multiplication groups of complex numbers of magnitude 1.

Now let us assume that the set (2.18) consists of circulant matrices, i.e., we have for example

$$\left(\tilde{Z}'_{n,m}(z,s)\right) = \text{circ}\left(\tilde{Z}'_1(z,s), \tilde{Z}'_2(z,s), \ldots, \tilde{Z}'_N(z,s)\right) \tag{3.8}$$

$$\left(\tilde{Y}'_{n,m}(z,s)\right) = \text{circ}\left(\tilde{Y}'_1(z,s), \tilde{Y}'_2(z,s), \ldots, \tilde{Y}'_N(z,s)\right) \tag{3.9}$$

Assuming reciprocity these matrices have to be symmetric, too (cf. [3]), but we will discuss this additional symmetry property and its consequences later. The eigenvalue expansion for those matrices we have to deal with reads:

$$\left(\tilde{Z}'_{n,m}(z,s)\right) = \sum_{\beta=1}^{N} \tilde{z}'_\beta(z,s) \quad \left(u_{C_n}\right)_\beta \quad \left(u_{C_n}\right)^*_\beta \tag{3.10}$$

$$\left(\tilde{Y}'_{n,m}(z,s)\right) = \sum_{\beta=1}^{N} \tilde{y}'_\beta(z,s) \quad \left(u_{C_n}\right)_\beta \quad \left(u_{C_n}\right)^*_\beta \tag{3.11}$$

$$\left(\tilde{Z}'_{n,m}(z,s)\right)^{-1} = \sum_{\beta=1}^{N} \tilde{z}'_\beta(z,s)^{-1} \left(u_{C_n}\right)_\beta \left(u_{C_n}\right)^*_\beta \tag{3.12}$$

$$\frac{d}{dz}\left(\tilde{Z}'_{n,m}(z,s)\right) = \sum_{\beta=1}^{N} \frac{d}{dz}\tilde{z}'_\beta(z,s) \left(u_{C_n}\right)_\beta \left(u_{C_n}\right)^*_\beta \tag{3.13}$$

$$\begin{aligned}\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^2 &= \sum_{\beta=1}^{N} \tilde{\gamma}^2_\beta(z,s) \left(u_{C_n}\right)_\beta \left(u_{C_n}\right)^*_\beta \\ &= \sum_{\beta=1}^{N} \tilde{z}'_\beta(z,s)\ \tilde{y}'_\beta(z,s) \left(u_{C_n}\right)_\beta \left(u_{C_n}\right)^*_\beta\end{aligned} \tag{3.14}$$

$$\left(\tilde{Z}_{c_{n,m}}(z,s)\right)^2 = \sum_{\beta=1}^{N} \tilde{z}'_\beta(z,s)\left(\tilde{y}'_\beta(z,s)\right)^{-1} \left(u_{C_n}\right)_\beta \left(u_{C_n}\right)^*_\beta \tag{3.15}$$

In the basis of these eigenvectors the field equations (2.7) and (2.8), and (2.12) and (2.13) scalarize as follows (Appendix C):

$$\frac{d}{dz}\tilde{\upsilon}_\beta(z,s) = -\ \tilde{z}'_\beta(z,s)\ \tilde{i}_\beta(z,s) \tag{3.16}$$

$$\frac{d}{dz}\tilde{i}_\beta(z,s) = -\ \tilde{y}'_\beta(z,s)\ \tilde{\upsilon}_\beta(z,s) \tag{3.17}$$

$$\frac{d^2}{dz^2}\tilde{\upsilon}_\beta(z,s) - \frac{d}{dz}\ell n\ \tilde{z}'_\beta(z,s)\ \frac{d}{dz}\tilde{\upsilon}_\beta(z,s) - \tilde{\gamma}^2_\beta(z,s)\ \tilde{\upsilon}_\beta(z,s) = 0 \tag{3.18}$$

$$\frac{d^2}{dz^2}\tilde{i}_\beta(z,s) - \frac{d}{dz}\ell n\ \tilde{y}'_\beta(z,s)\ \frac{d}{dz}\tilde{i}_\beta(z,s) - \tilde{\gamma}^2_\beta(z,s)\ \tilde{i}_\beta(z,s) = 0 \tag{3.19}$$

Here we have used the definitions

$$\tilde{\upsilon}_\beta(z,s) \equiv \left(u_{C_n}\right)^*_\beta \cdot \left(\tilde{V}_n(z,s)\right) \tag{3.20}$$

$$\tilde{i}_\beta(z,s) \equiv \left(u_{C_n}\right)^*_\beta \cdot \left(\tilde{I}_n(z,s)\right) \tag{3.21}$$

$(\beta = 1, 2, \ldots, N)$

for the components of the modal voltage and current vectors, respectively. The structure of the above equations (3.16) through (3.19) remind us of the analogous equations in the usual one-dimensional (N = 1) transmission line theory. So we may borrow all the techniques and methods to find solutions of (3.16) - (3.19) from this theory, being aware, however, of the fact that the modal vector components have to fulfill quite different boundary conditions than the scalar voltage and current in the N = 1 transmission line theory.

Imposing symmetry (reciprocity) on our physical matrices $\left(\tilde{Z}'_{n,m}(z,s)\right)$, $\left(\tilde{Y}'_{n,m}(z,s)\right)$ and $\left(\tilde{Z}_{C_{n,m}}(z,s)\right)$, will not change the field equations (3.16) through (3.19). It has, however, an impact on the number of independent matrix elements of the above matrices and on their eigenvectors. Symmetry for $\left(\tilde{Z}'_{n,m}(z,s)\right)$, e.g., implies that

$$\tilde{Z}'_i(z,s) = \tilde{Z}'_{N+2-i}(z,s) \tag{3.22}$$

$$(i = 2,\dots,N)$$

and therefore a symmetric circulant matrix has at most $\left(\frac{N}{2}+1\right)$ different elements if N is even, and $\left(\frac{N+1}{2}\right)$ different elements if N is odd. In accord with these numbers, the number of different eigenvalues reduces as well. In the case that N is even we find from (3.4) (and Appendix B)

$$b_1 = b_{N-1}\ ,\ b_2 = b_{N-2}\ ,\ \dots\ ,\ b_{\frac{N}{2}-1} = b_{\frac{N}{2}+1} \tag{3.23}$$

$$b_{\frac{N}{2}} = B_1 - 2B_2 + 2B_3 - 2B_4 + \cdots + 2qB_{\frac{N}{2}} - qB_{\frac{N}{2}+1} \tag{3.24}$$

$$q = \begin{Bmatrix} +1 \text{ if } \left(\frac{N}{2}+1\right) \text{ is even} \\ -1 \text{ if } \left(\frac{N}{2}+1\right) \text{ is odd} \end{Bmatrix} \tag{3.25}$$

$$b_N = B_1 + 2\left(B_2 + B_3 + \cdots + B_{\frac{N}{2}}\right) + B_{\frac{N}{2}+1} \tag{3.26}$$

and thus have (at most) $\frac{N}{2}+1$ different eigenvalues. For an odd number N we obtain a similar relation:

$$b_1 = b_{N-1} \;,\; b_2 = b_{N-2} \;,\; \dots \;,\; b_{\frac{N+1}{2}} = b_{\frac{N+1}{2}} \tag{3.27}$$

$$b_N = B_1 + 2\left(B_2 + B_3 + \dots + B_{\frac{N+1}{2}}\right) \tag{3.28}$$

and count (at most) $\frac{N+1}{2}$ different eigenvalues.

The property (3.5) of the eigenvectors $\left(u_{C_n}\right)_\beta$ together with the properties (3.23) through (3.28) for the eigenvalues of symmetric circulant matrices now suggest that one introduce a real representation of the similarity transformation by virtue of new (real) eigenvectors $(w_n)_\beta$ (see Appendix B).

This new set $(w_n)_\beta$ of eigenvectors constitutes a new (real) orthonormal matrix$\left(W_{n,m}\right)$ which transforms symmetric circulant matrices into diagonal form, still having the (e.g. complex) eigenvalues (3.23) through (3.28) depending on whether N is even or odd, respectively. Equations (3.10) through (3.15), and (3.20) and (3.21) may now be rewritten in terms of the new eigenvectors $(w_n)_\beta$, but without any * indication since we deal with real eigenvectors.

A few remarks with respect to the fundamental differential equations (3.18) and (3.19) are in order. Due to the eigenvalue relations (3.23) through (3.28) we only have to solve (3.18) or (3.19) for the first $\left(\frac{N}{2}+1\right)$ voltage or current components $\tilde{\upsilon}_\beta(z,s)$ or $\tilde{i}_\beta(z,s)$ if N is even and for the first $\left(\frac{N+1}{2}\right)$ voltage or current components if N is odd. The remaining components are contained in the former solutions (up to possibly different boundary conditions). Once we have obtained the voltage vector $\left(\tilde{\upsilon}_n(z,s)\right)$ or the current vector $\left(\tilde{i}_n(z,s)\right)$ by the solutions of (3.18) or (3.19), respectively, we can easily compute the original voltage vector $\left(\tilde{V}_n(z,s)\right)$ or the original current vector$\left(\tilde{I}_n(z,s)\right)$ as

$$\left(\tilde{V}_n(z,s)\right) = \left(W_{n,m}\right) \cdot \left(\tilde{\upsilon}_n(z,s)\right) \tag{3.29}$$

$$\left(\tilde{I}_n(z,s)\right) = \left(W_{n,m}\right) \cdot \left(\tilde{i}_n(z,s)\right) \tag{3.30}$$

and finally impose on these vectors the appropriate boundary conditions (and thereby fixing the so-far undetermined integration constants, see, e.g., Section V).

Since the per-unit-length impedance and admittance matrices are depending on the position along the lines the linear second-order differential equations (3.18) and (3.19) for the modal voltage and current vectors become difficult to be solved. There are, however, some special cases -- as the exponential lines [8, 9] and the Bessel lines [7] - where one can find exact analytical solutions for (3.18) and (3.19).

A promising ansatz to find other solutions of (3.16) and (3.17) may lie in the application of Lie algebraic theory to nonuniform transmission lines [18].

Since our modal voltage and current field equations are very similar to those known from one-dimensional (N = 1) transmission-line theory one may expect to apply all the experiences from this theory to our situation. Going along these lines, it is possible to transform (3.16) and (3.17) into a first order non-linear differential equation involving modal reflection coefficients $\rho_\beta(z,s)$. These coefficients are defined by

$$\rho_\beta(z,s) \equiv \left(\frac{\tilde{\upsilon}_\beta(z,s)}{\tilde{i}_\beta(z,s)} - \tilde{z}_{c\beta}(z,s) \right) \Bigg| \left(\frac{\tilde{\upsilon}_\beta(z,s)}{\tilde{i}_\beta(z,s)} + \tilde{z}_{c\beta}(z,s) \right) \tag{3.31}$$

By proper substitution of these quantities into (3.16), (3.17) we obtain, after some manipulation [10]

$$\frac{d\tilde{\rho}_\beta(z,s)}{dz} - 2\tilde{\gamma}_\beta(z,s)\,\tilde{\rho}_\beta(z,s) + \frac{1}{2}\left(1 - \tilde{\rho}_\beta^2(z,s)\right) \frac{d}{dz}\,\ln\left(\tilde{z}_{c\beta}(z,s)\right) = 0 \tag{3.32}$$

Note that this expression is exact and no restrictions have been imposed on it. Equation (3.32) can be reduced to a first order linear differential equation assuming $\left|\tilde{\rho}_\beta^2\right| << 1$ for all modal wave reflections. Thinking of a matching section between two (different) uniform transmission-line tubes one should assume that the modal reflections due to any mismatch are small. Insofar as the above assumption for $\tilde{\rho}_\beta^2$ is acceptable, then equation (3.32) reduces to

$$\frac{d\tilde{\rho}_\beta(z,s)}{dz} - 2\tilde{\gamma}_\beta(z,s)\,\tilde{\rho}_\beta(z,s) + \frac{1}{2}\,\frac{d}{dz}\,\ln\left(\tilde{z}_{c\beta}(z,s)\right) = 0 \tag{3.33}$$

Now, this equation can be solved exactly, and solutions describing different non-uniform transmission lines are published in the literature [11, 12, 13, 14]. Even perturbation solutions of the equivalent to the full equation (3.32) can be found in the literature [15]. In this paper it is not our main intention to find exact solutions for (3.32) (see, however, Section V). This will be the subject of a forthcoming paper.

Here, we rather wanted to establish the general aspects and results of the theory for commuting nonuniform tubes, to derive the important field equations for the modal quantities $\tilde{v}_\beta$, $\tilde{i}_\beta$, and $\tilde{\rho}_\beta$ and to give some examples for their application.

An interesting application might be the investigation of twisted cables (even with space-dependent pitch angle) inside of braided or non-braided shields. In Figure 2 we display two twisted cables inside a shield. A rotation of 2π at one end of the twisted cable-pair removes them into their uniform arrangement.

Another field of application is the description of antennas with rotationally-symmetric cross sections with the aid of diverging identical conical multiconductor lines (see Figure 3). We will deal with those applications in another forthcoming paper.

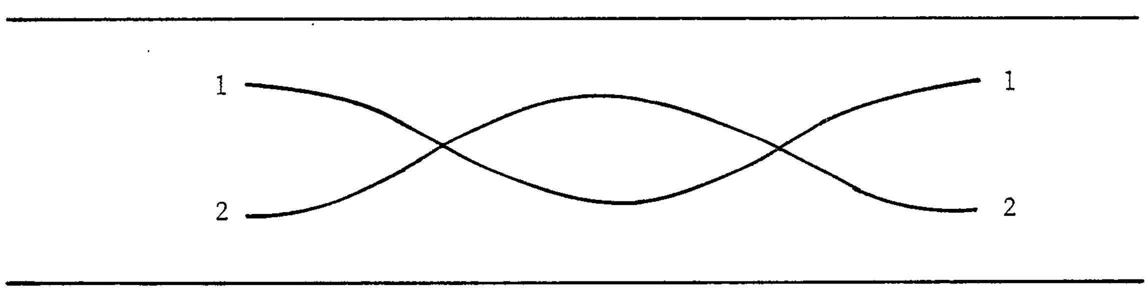

Figure 2. Two twisted cables inside a shield.

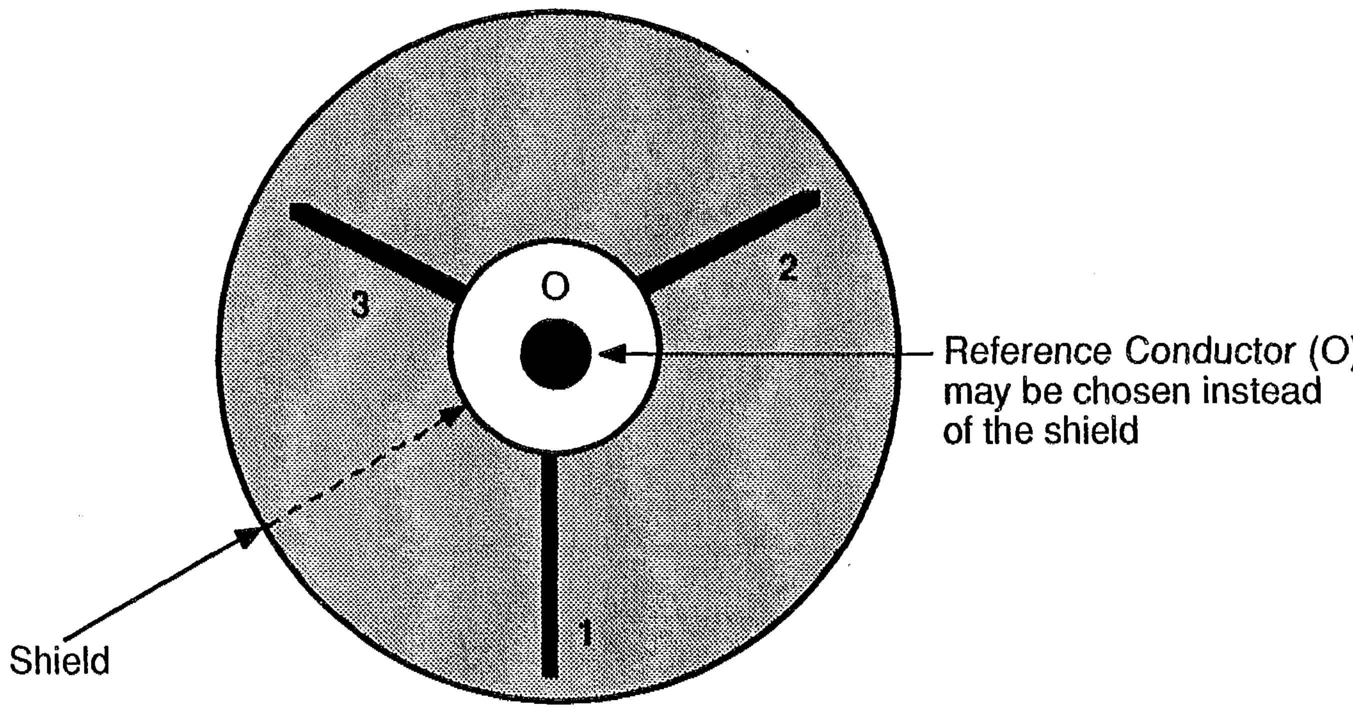

Figure 3. Projection of a Conically Diverging 3(+1)-Conductor-Line Inside a Shield Into a Plane.

IV. The High-Frequency Limit of the Nonuniform Tube

Let us consider a tube configuration where the symmetric circulant per-unit-length impedance-and admittance-matrices have the property that their product (which defines the square of the propagation matrix) is proportional to the identity matrix

$$\left(\tilde{\gamma}_{c_{n,m}}(z,s)\right)^2 = \left(\tilde{Z}'_{n,m}(z,s)\right) \cdot \left(\tilde{Y}'_{n,m}(z,s)\right) = \frac{s^2}{\upsilon^2}\left(1_{n,m}\right) \tag{4.1}$$

(i.e., all modes have the same speed of propagation).

Here we have assumed that the tube consists of N perfect conductors immersed in a uniform isotropic medium and that $\left(\tilde{Z}'_{n,m}\right)$ and $\left(\tilde{Y}'_{n,m}\right)$ are frequency-independent real matrices times functions of the constitutive parameters of the medium. These parameters in turn are taken as independent of z. In many practical cases we may approximate the medium by real constants ε and μ with $\sigma = 0$. With this form (3.18) becomes

$$\left(\frac{d^2}{dz^2} - \left(\frac{s}{\upsilon}\right)^2\right) \tilde{\upsilon}_\beta(z,s) - \frac{d}{dz} \ell n \left(\tilde{z}'_\beta(z,s)\right) \cdot \frac{d}{dz}\tilde{\upsilon}_\beta(z,s) = 0 \tag{4.2}$$

Now we try the ansatz (appropriate for forward travelling waves)

$$\tilde{\upsilon}_\beta(z,s) = exp(-\, s\, z/\upsilon)\ \phi_\beta(z,s)\ \tilde{\upsilon}_\beta(0,s) \tag{4.3}$$

$$\left(\phi_\beta(0,s) = 1\right)$$

giving

$$\frac{d^2}{dz^2}\phi_\beta - 2\,\frac{s}{\upsilon}\,\frac{d}{dz}\phi_\beta - \frac{d}{dz}\,\ell n\left(\tilde{z}'_\beta(z,s)\right) \cdot \left\{\frac{d}{dz}\phi_\beta - \frac{s}{\upsilon}\phi_\beta\right\} = 0 \tag{4.4}$$

Taking the limit $s \to \infty$ in (4.4) and neglecting $\frac{d}{dz}\phi_\beta$ with respect to $(s/\upsilon)\phi_\beta$ we get (see also [4])

$$\frac{d}{dz}\phi_\beta(z) = \frac{1}{2}\,\frac{d}{dz}\,\ell n\left(\tilde{z}'_\beta(z)\right)\,\phi_\beta(z) \tag{4.5}$$

i.e., a simple first order differential equation.

In (4.5) we dropped the s dependence from ϕ_β and $\bar{z}'_\beta$. Equation (4.5) can easily be solved giving

$$\phi_\beta(z) = c_\beta \sqrt{\bar{z}'_\beta(z)} \tag{4.6}$$

where the c_β denote the integration constants. They are fixed by the condition

$$\phi_\beta(0) = 1 = c_\beta \sqrt{\bar{z}'_\beta(0)} \tag{4.7}$$

resulting in

$$c_\beta = \frac{1}{\sqrt{\bar{z}'_\beta(0)}} \tag{4.8}$$

Thus for $s \to \infty$ (in the right half-plane) we have

$$\tilde{\upsilon}_\beta(z, s) = exp\ (-\ s\, z / \upsilon) \sqrt{\frac{\bar{z}'_\beta(z)}{\bar{z}'_\beta(0)}}\ \ \tilde{\upsilon}_\beta(0, s) \tag{4.9}$$

This result was obtained in close analogy to procedures known in quantum mechanics. There it is referred to as the W K B approximation.

V. A Special Unit Cell of Wave-Launcher Array

In recent papers [1, 2] one of us (C. E. Baum) studied (among others) a special case of a unit cell of a periodic array of wave launchers, based on a two-wire (plus reference) transmission-line model. See Figure 4 for the equivalent circuit of the unit cell of the wave-launcher array.) In this section we rely on special parts of these papers and derive (decoupled) non-uniform transmission line equations which are valid for the entire frequency domain, thus going beyond the cited papers which present solutions for the high-frequency (early time) domain. In what follows the reader is referred to [1, 2] as far as notation and more detailed explanation is concerned. Here, for convenience, we take I_1 , and I_2 as the currents and V_1 and V_2 as the voltages (instead of $2V$ and $2V_2$ as appropriate for differential systems).

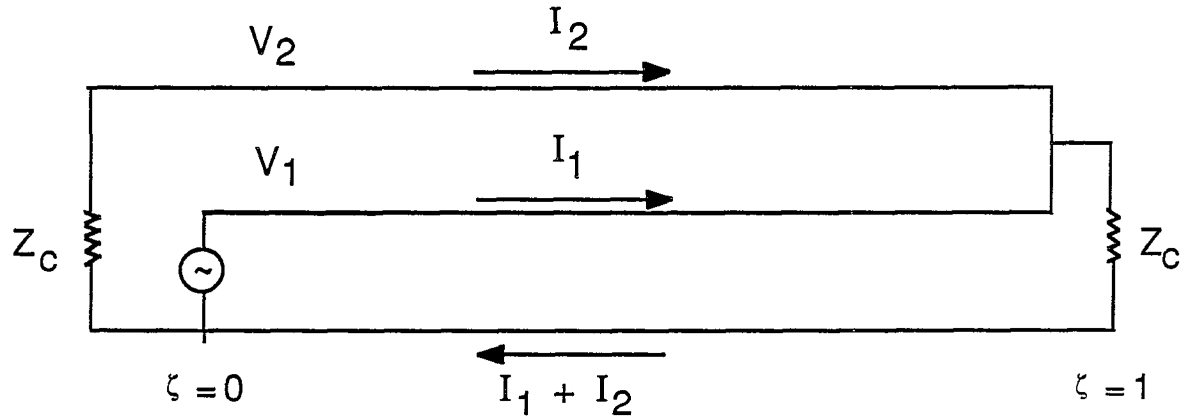

Figure 4. Equivalent circuit for the unit-cell of the wave-launcher array.

Our considerations in this section is restricted to simple per-unit-length impedance and admittance matrices of the form

$$\left(\tilde{Z}'_{n,m}(z,s)\right) = s\mu_o \begin{pmatrix} fg_{1,1}(z) & fg_{1,2}(z) \\ fg_{1,2}(z) & fg_{1,1}(z) \end{pmatrix} \tag{5.1}$$

$$\begin{aligned} \left(\tilde{Y}'_{n,m}(z,s)\right) &= s\varepsilon_o \left(fg_{n,m}\right)^{-1} \\ &= \frac{s\varepsilon_o}{det\left(\left(fg^{(z)}_{n,m}\right)\right)} \begin{pmatrix} fg_{1,1}(z) & -fg_{1,2}(z) \\ -fg_{1,2}(z) & fg_{1,1}(z) \end{pmatrix} \end{aligned} \tag{5.2}$$

The eigenvalues and eigenvectors of the matrix $(fg_{n,m})$ are easily computed giving

$$f_1(z) = fg_{1,1}(z) - fg_{1,2}(z) \qquad \text{(differential mode)} \tag{5.3}$$

$$f_2(z) = fg_{1,1}(z) + fg_{1,2}(z) \qquad \text{(common mode)} \tag{5.4}$$

$$(w_n)_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \;\; ; \;\; (w_n)_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.5}$$

The modal voltage components are from (3.20)

$$\tilde{\upsilon}_1(z,s) = \frac{1}{\sqrt{2}}\left(-\tilde{V}_1(z,s) + \tilde{V}_2(z,s)\right) \tag{5.6}$$

$$\tilde{\upsilon}_2(z,s) = \frac{1}{\sqrt{2}}\left(\tilde{V}_1(z,s) + \tilde{V}_2(z,s)\right) \tag{5.7}$$

where $\tilde{V}_1$ and $\tilde{V}_2$ denote the two components of the original voltage vector $\left(\tilde{V}_n(z,s)\right)$. Inversion of (5.6) and (5.7) leads to

$$\tilde{V}_1(z,s) = \frac{1}{\sqrt{2}}\left(-\tilde{\upsilon}_1(z,s) + \tilde{\upsilon}_2(z,s)\right) \tag{5.8}$$

$$\tilde{V}_2(z,s) = \frac{1}{\sqrt{2}}\left(\tilde{\upsilon}_1(z,s) + \tilde{\upsilon}_2(z,s)\right) \tag{5.9}$$

In what follows we specify the characteristic impedance matrix to be

$$\left(Z_{c_{n,m}}(\zeta)\right) = Z_o\left(fg_{n,m}(\zeta)\right) = Z_c\begin{pmatrix} 1 & f(\zeta) \\ f(\zeta) & 1 \end{pmatrix} \tag{5.10}$$

with

$Z_o \quad = \sqrt{\frac{\mu_o}{\varepsilon_o}} =$ characteristic impedance of free space

$2a \quad \doteq$ width of unit cell

$2b \quad \equiv$ height of unit cell

$Z_c \quad \equiv \frac{b}{a} Z_o$

$\zeta \quad \equiv \frac{z}{\ell} + 1$, $\ell \equiv$ length of wave launcher (5.10)

$(0 \leq \zeta \leq 1)$

$f(\zeta) \quad \equiv$ monotonic function of ζ with

$f(0) = 0$ and $f(1) = 1.$

Before we deal with the exact field equations we first derive the high frequency solutions. These read (from (4.9))

$$\upsilon_\beta(\zeta) = \left[\frac{f_\beta(\zeta)}{f_\beta(0)}\right]^{\frac{1}{2}} \upsilon_\beta(0) \tag{5.11}$$

with the initial condition at $\zeta = 0$

$$(V_n(0)) = V_o\begin{pmatrix}1\\0\end{pmatrix} \tag{5.12}$$

i.e.,

$$\upsilon_2(0) = \frac{1}{\sqrt{2}} V_0\begin{pmatrix}1\\0\end{pmatrix} \tag{5.13}$$

Equation (5.11) represents the solutions in time domain for lossless, dispensionless transmission lines where the factor $\exp(-s\,z\,/\,\upsilon)$ was removed since the above result is taken in retarded time.

The modal voltages are

$$\upsilon_1(\zeta) = \left[1 - f(\zeta)\right]^{\frac{1}{2}} \upsilon_1(0) \tag{5.14}$$

$$\upsilon_2(\zeta) = \left[1 + f(\zeta)\right]^{\frac{1}{2}} \upsilon_2(0) \tag{5.15}$$

and therefore (apply (5.8) and (5.9) the components of the original voltage vector are

$$V_1(\zeta) = \frac{V_o}{2}\left\{\left(1 + f(\zeta)\right)^{\frac{1}{2}} + \left(1 - f(\zeta)\right)^{\frac{1}{2}}\right\} \tag{5.16}$$

$$V_2(\zeta) = \frac{V_o}{2}\left\{\left(1 + f(\zeta)\right)^{\frac{1}{2}} - \left(1 - f(\zeta)\right)^{\frac{1}{2}}\right\} \tag{5.17}$$

At $\zeta = 1$ this gives

$$V_1(1) = \frac{V_o}{\sqrt{2}} = V_2(1) \tag{5.18}$$

consistent with the fact that the wave launcher plates meet the electric boundaries (see [1]) or $\pm$b at $\zeta = 1$. The results (5.16) and (5.17) agree with the corresponding ones of [1].

In the next step we have a closer look at the exact voltage field equations. They are obtained from (3.18) and (3.37). We derive from (3.18)

$$\left(\frac{d^2}{d\zeta^2} - \left(\frac{s\ell}{c}\right)^2\right) \tilde{v}_1(\zeta, s) + \frac{\frac{d}{d\zeta}f(\zeta)}{(1-f(\zeta))} \frac{d}{d\zeta} \tilde{v}_1(\zeta, s) = 0 \tag{5.19}$$

$$\left(\frac{d^2}{d\zeta^2} - \left(\frac{s\ell}{c}\right)^2\right) \tilde{v}_2(\zeta, s) - \frac{\frac{d}{d\zeta}f(\zeta)}{(1+f(\zeta))} \frac{d}{d\zeta} \tilde{v}_2(\zeta, s) = 0 \tag{5.20}$$

$$\left(\varepsilon_o\mu_o \equiv c^{-2} \;;\; c = \text{speed of light}\right)$$

introducing the dimensionless quantity $\Gamma = (s\ell / c)$ and the sign factor

$$\chi = \begin{cases} -1 & \text{for } \beta = 1 \\ +1 & \text{for } \beta = 1 \end{cases} \tag{5.21}$$

and choosing $f(\zeta) = \zeta$, the above field equations can be compactly written as

$$\left(\frac{d^2}{d\zeta^2} - \Gamma^2\right) \tilde{v}_\beta(\zeta, \Gamma) - \frac{\chi}{1+\chi\zeta} \frac{d}{d\zeta} \tilde{v}_\beta(\zeta, \Gamma) = 0 \tag{5.22}$$

In anology to the solution procedures performed in [7] we find the solutions of (5.22) in terms of modified Bessel functions I_ν and $K_\nu(\nu = 0, 1)$.

$$\tilde{v}_\beta(\zeta,\Gamma) = (1+\chi\zeta) \left[\tilde{A}_\beta(\Gamma)\ I_1(\Gamma(1+\chi\zeta)) + \tilde{B}_\beta(\Gamma)\ K_1(\Gamma(1+\chi\zeta))\right] \tag{5.23}$$

Here the functions $\tilde{A}_\beta(\Gamma)$ and $\tilde{B}_\beta(\Gamma)$ $(\beta = 1, 2)$ are the four "integration constants" emerging by the integration of (5.22). The modal current vector components are derived from (5.23) by using (3.16).

$$\frac{d}{d\zeta} \tilde{v}_\beta(\zeta, \Gamma) = - \ell\ \tilde{z}'_\beta(\zeta, \Gamma)\ \tilde{i}_\beta(\zeta, \Gamma) = - \Gamma\ Z_o\left(\frac{b}{a}\right) (1 + \chi\zeta)\ \tilde{i}_\beta(\zeta, \Gamma) \tag{5.24}$$

We obtain

$$\tilde{i}_\beta(\zeta, \Gamma) = - \frac{1}{Z_c} \left[\tilde{A}_\beta(\Gamma)\ I_o(\Gamma(H\chi\zeta)) - \tilde{B}_\beta(\Gamma)\ K_o(\Gamma(1 + \chi\zeta))\right] \tag{5.25}$$

$$Z_c \equiv (b / a)\ Z_o$$

Equations (5.23) and (5.25) are the (exact) solution of the problem in terms of the four (so-far) arbitrary integration-functions $\tilde{A}_\beta$ and $\tilde{B}_\beta$. Note that these functions occur (due to (5.24) in the modal voltages as well as in the corresponding currents. They will be fixed through appropriate boundary conditions which have to be imposed on the original vectors $\left(\tilde{V}_n(\zeta, \Gamma)\right)$ and $\left(\tilde{I}_n(\zeta, \Gamma)\right)$.

In matrix notation the solutions (5.23) and (5.25) read

$$\begin{pmatrix} -Z_c\tilde{i}_\beta(\zeta, \Gamma) \\ \\ \dfrac{\tilde{v}_\beta(\zeta, \Gamma)}{(1+\chi\zeta)} \end{pmatrix} = \begin{pmatrix} I_0(\Gamma(1+\chi\zeta)) & -K_0(\Gamma(1+\chi\zeta)) \\ I_1(\Gamma(1+\chi\zeta)) & K_1(\Gamma(1+\chi\zeta)) \end{pmatrix} \cdot \begin{pmatrix} \tilde{A}_\beta(\Gamma) \\ \tilde{B}_\beta(\Gamma) \end{pmatrix} \tag{5.26}$$

This representation suits best for a resolution with respect to $\tilde{A}_\beta(\Gamma)$ and $\tilde{B}_\beta(\Gamma)$ by matrix inversion.

A simple calculation which observes the Wronskian relation [17]

$$W\ \{I_o(z),\ -K_o(z)\} \equiv det\begin{pmatrix} I_o(z) & -K_o(z) \\ \frac{d}{dz}I_o(z) & \frac{d}{dz}-K_o(z)\end{pmatrix} = det\begin{pmatrix} I_o(z) & -K_o(z) \\ I_1(z) & K_{1o}(z)\end{pmatrix}$$

$$= I_o(z)\ K_1(z) + K_o(z)\ I_1(z) = 1/z \tag{5.27}$$

yields

$$\begin{pmatrix}\tilde{A}_\beta(\Gamma)\\ \tilde{B}_\beta(\Gamma)\end{pmatrix} = K_1(\Gamma(1+\chi\zeta))\begin{pmatrix} K_1(\Gamma(1+\chi\zeta)) & K_o(\Gamma(1+\chi\zeta)) \\ -I_1(\Gamma(1+\chi\zeta)) & I_o(\Gamma(1+\chi\zeta))\end{pmatrix}\cdot\begin{pmatrix} -Z_c\Gamma(1+\chi\zeta)\tilde{i}_\beta(\zeta,\Gamma) \\ \Gamma\quad \tilde{\upsilon}_\beta(\zeta,\Gamma)\end{pmatrix} \tag{5.28}$$

Since we finally have to deal with the original quantities we represent them as well in a very condensed form,

$$\left(\tilde{V}_n(\zeta,\Gamma)\right) = \frac{1}{\sqrt{2}}\sum_{\beta=1}^{2}(1+\chi\zeta)\left[\tilde{A}_\beta(\Gamma)\ I_1(\Gamma(1+\chi\zeta)) + \tilde{B}_\beta(\Gamma)\ K_1(\Gamma(1+\chi\zeta))\right]\begin{pmatrix}\chi\\1\end{pmatrix} \tag{5.29}$$

$$\left(\tilde{I}_n(\zeta,\Gamma)\right) = \frac{1}{\sqrt{2}}\frac{1}{Z_c}\sum_{\beta=1}^{2}\chi\left[\tilde{A}_\beta(\Gamma)\ I_o(\Gamma(1+\chi\zeta)) - \tilde{B}_\beta(\Gamma)\ K_o(\Gamma(1+\chi\zeta))\right]\begin{pmatrix}\chi\\1\end{pmatrix} \tag{5.30}$$

Now we are ready to determine the "constants" $\tilde{A}_\beta(\Gamma)$ and $\tilde{B}_\beta(\Gamma)$ by the boundary conditions. These conditions are:

For $\zeta = 1$ $\qquad \tilde{V}_1(1,\Gamma) = \tilde{V}_2(1,\Gamma)$ (5.31)

$$\tilde{V}_1(1,\Gamma) = Z_c\left(\tilde{I}_1(1,\Gamma) + \tilde{I}_2(1,\Gamma)\right) \tag{5.32}$$

For $\zeta = 0$ $\qquad \tilde{V}_1(0,\ \Gamma) = \tilde{Z}_{in}(\Gamma)\ \tilde{I}_1(0,\Gamma)$ (5.33)

$$\tilde{V}_2(0,\ \Gamma) = -\tilde{Z}_c(\Gamma)\ \tilde{I}_2(0,\Gamma) \tag{5.34}$$

Equation (5.31) implies that

$$\tilde{\upsilon}_1(1,\Gamma) = 0 \text{ and } \tilde{\upsilon}_2(1,\Gamma) = \sqrt{2}\quad \tilde{V}_1(1,\Gamma) \tag{5.35}$$

and from (5.28) we obtain for $\beta = 1$ and $\zeta \to 1$ the simple relations:

$$\tilde{A}_1(\Gamma) = -\ Z_c\ \tilde{i}_1(1,\Gamma) \tag{5.36}$$

$$\tilde{B}_1(\Gamma) \equiv 0 \tag{5.37}$$

For $\beta = 2$ and $\zeta \to 1$ we derive under observation of (5.32)

$$\tilde{A}_2(\Gamma) = \sqrt{2}\ \Gamma\ (K_o(2\Gamma) - K_1(2\Gamma))\ \tilde{V}_1(1,\Gamma) \tag{5.38}$$

$$\tilde{B}_2(\Gamma) = \sqrt{2}\ \Gamma\ (I_o(2\Gamma) + I_1(2\Gamma))\ \tilde{V}_1(1,\Gamma) \tag{5.39}$$

which in turn combines $\tilde{A}_2(\Gamma)$ and $\tilde{B}_2(\Gamma)$ as

$$\tilde{A}_2(\Gamma) = Q(2\Gamma)\ \tilde{B}_2(\Gamma) \tag{5.40}$$

where we have defined the function $Q(2\Gamma)$ as the following ratio

$$Q(2\Gamma) \equiv \frac{K_o(2\Gamma) - K_1(2\Gamma)}{I_o(2\Gamma) + I_1(2\Gamma)} \tag{5.41}$$

Application of the boundary condition (5.34) on the components $\tilde{V}_2$ and $\tilde{I}_2$ in (5.29) and (5.30), respectively, reveals another expression for

$$\tilde{A}_1(\Gamma) = \frac{I_o(\Gamma) - I_1(\Gamma)}{I_o(\Gamma) + I_1(\Gamma)}\ \tilde{A}_2(\Gamma) - \frac{K_o(\Gamma) + K_1(\Gamma)}{I_o(\Gamma) + I_1(\Gamma)}\ \tilde{B}_2(\Gamma) \tag{5.42}$$

Thus, together with (5.38) through (5.40), the integration functions can be expressed in terms of modified Bessel functions and the voltage components $\tilde{V}_1(1,\Gamma)$.

More desirable than expressing the functions $\tilde{A}_1$, $\tilde{A}_2$, and $\tilde{B}_2$ in proportionality to $\tilde{V}_1(1,\Gamma)$ is expressing these quantities in terms proportional to the initially impressed voltage $\tilde{V}_1(0,\Gamma)$. In other words we are interested in calculating the transfer function $\tilde{T}_+(\Gamma)$ defined by

$$\tilde{T}_+(\Gamma) \equiv \frac{\tilde{V}_1(1,\Gamma)}{\tilde{V}_1(0,\Gamma)} \tag{5.43}$$

In addition to this transfer function it is very informative to find expressions for the following ratios:

$$\tilde{T}_-(\Gamma) \equiv \frac{\tilde{V}_2(0,\Gamma)}{\tilde{V}_1(0,\Gamma)} \tag{5.44}$$

$$\tilde{Z}_{in}(\Gamma) \equiv \frac{\tilde{V}_1(0,\Gamma)}{\tilde{I}_1(0,\Gamma)} \tag{5.45}$$

The input impedance function $\tilde{Z}_{in}(\Gamma)$ occurred already in (5.33).

In order to compute explicit expressions for (5.43) through (5.45) we use equations (5.29), (5.30), (5,40), (5.41), and (5.42) and finally obtain:

$$\tilde{T}_+(\Gamma) \equiv \frac{I_o(\Gamma) + I_1(\Gamma)}{I_o(2\Gamma) + I_1(2\Gamma)} \; \frac{1}{\left[1+2\Gamma\; I_1(\Gamma) \;\; (I_1(\Gamma)Q(2\Gamma) + K_1(\Gamma))\right]} \tag{5.46}$$

$$\tilde{T}_-(\Gamma) = \frac{I_o(\Gamma)\; (2Q(2\Gamma)\; I_1(\Gamma) + K_1(\Gamma)) - K_o(\Gamma)\; I_1(\Gamma)}{2I_1(\Gamma)\; (Q(2\Gamma)\; I_1(\Gamma) + K_1(\Gamma)) + \dfrac{1}{\Gamma}} \tag{5.47}$$

$$\tilde{Z}_{in}(\Gamma) = Z_c \; \frac{\dfrac{1}{\Gamma} + 2I_1(\Gamma)\; (Q(2\Gamma)\; I_1(\Gamma) + K_1(\Gamma))}{\dfrac{1}{\Gamma} - 2I_0(\Gamma)\; (Q(2\Gamma)\; I_0(\Gamma) - K_0(\Gamma))} \tag{5.48}$$

In deriving the above formulae we again used the special Wronskian (5.27).

As expected, the two transfer functions $\tilde{T}_+(\Gamma)$ and $\tilde{T}_-(\Gamma)$, and the input impedance function $\tilde{Z}_{in}(\Gamma)$ can entirely be expressed in terms of modified Bessel functions. These expressions are, of course, valid for the whole frequency domain, $o \leq \omega \leq \infty$. Therefore we can especially investigate the high- and low-frequency limits of equations (5.46) through (5.48). For this purpose we need to know the corresponding limits of the modified Bessel functions. In the high-frequency limit we have

$$I_o(\Gamma) \sim \frac{e^{\Gamma}}{\sqrt{2\pi\Gamma}} \;\; \left\{1 + \frac{1}{8\Gamma}\right\} \qquad\qquad (\Gamma \to + \infty) \tag{5.49}$$

$$I_1(\Gamma) \sim \frac{e^{\Gamma}}{\sqrt{2\pi\Gamma}} \;\; \left\{1 - \frac{3}{8\Gamma}\right\} \tag{5.50}$$

$$K_o(\Gamma) \sim \sqrt{\frac{\pi}{2\Gamma}} \;\; e^{-\Gamma} \left\{1 - \frac{1}{8\Gamma}\right\} \tag{5.51}$$

$$K_1(\Gamma) \sim \sqrt{\frac{\pi}{2\Gamma}} \; e^{-\Gamma} \left\{1 + \frac{3}{8\Gamma}\right\} \tag{5.52}$$

For the dc limit (i.e. $(\Gamma \rightarrow 0)$) we approach the Bessel functions by

$$I_o(\Gamma) \sim 1 \tag{5.53}$$

$$I_1(\Gamma) \sim \frac{1}{2}\Gamma \tag{5.54}$$

$$K_o(\Gamma) \sim -\ \ell n(\Gamma) \tag{5.55}$$

$$K_1(\Gamma) \sim \frac{1}{\Gamma} \tag{5.56}$$

On the basis of the above approximate formulae we easily derive:

For the high-frequency limit $(\Gamma \rightarrow +\infty)$:

$$\tilde{T}_+(\Gamma) = \frac{1}{\sqrt{2}}\, e^{-\Gamma} + O\!\left(\frac{1}{\Gamma}\right) e^{-\Gamma} \tag{5.57}$$

$$\tilde{T}_-(\Gamma) = \frac{1}{4\Gamma} + O\!\left(\frac{1}{\Gamma^2}\right) \tag{5.58}$$

$$\tilde{Z}_{in}(\Gamma) = Z_c + O\!\left(\frac{1}{\Gamma}\right) \tag{5.59}$$

For the low frequency limit $(\Gamma \rightarrow 0)$:

$$\tilde{T}_+(\Gamma) = 1 + O(\Gamma) \tag{5.60}$$

$$\tilde{T}_-(\Gamma) = 1 + O(\Gamma) \tag{5.61}$$

$$\tilde{Z}_{in}(\Gamma) = \frac{1}{2}\, Z_c + O(\Gamma) \tag{5.62}$$

These results confirm our exact solutions. The high-frequency results agree with those obtained at the beginning of this section (cf. equations (5.12) and (5.18)). In the dc limit the results are obvious. The factor (1/2) in (5.62) is due to the parallel connection of the two loads Z_C (see Figure 4).

Finally we would like to transform our results into the time domain. Analytically this becomes extremely difficult. However, starting with the high-frequency solution, one might consider forward and backward (reflected) running waves step by step on the basis of a kind of perturbation analysis. This will become the subject of forthcoming investigations.

IV. Coupled Two-Conductor Lines Above a Conducting Plane

In this section we demonstrate with another example the applicability of our general formalism. In [5] we studied two straight diverging wires above a perfectly conducting plane. Here we would like to show that the equations which were used are those which result from (3.18) with the appropriate eigenvalues. For the problem in [5] we had to deal with the following per-unit-length matrices:

$$\left(\tilde{Z}'_{n,m}(z,s)\right) = s\left(L'_{n,m}(z)\right) = s\mu\, f_{g_{1,1}} \begin{pmatrix} 1 & \kappa(z,\theta) \\ \kappa(z,\theta) & 1 \end{pmatrix} - s\mu\, f_{g_{1,1}} \begin{pmatrix} 0 & \kappa(z,\theta)\sin^2(\theta) \\ \kappa(z,\theta)\sin^2(\theta) & 0 \end{pmatrix} \tag{6.1}$$

$$\left(\tilde{Y}'_{n,m}(z,s)\right) = s\left(C'_{n,m}(z)\right) = s\varepsilon \left(f_{g_{n,m}}\right)^{-1} \tag{6.2}$$

with

$$\left(f g_{n,m}(z)\right) = f g_{1,1} \begin{pmatrix} 1 & \kappa(z,\theta) \\ \kappa(z,\theta) & 1 \end{pmatrix} \tag{6.3}$$

and

$$\kappa(z,\theta) \equiv \left(f g_{1,2}(z) \,/\, f g_{1,1}\right) \tag{6.4}$$

The matrices $\left(\tilde{Z}'_{n,m}\right)$ and $\left(\tilde{Y}'_{n,m}\right)$ are symmetric circulant matrices. Therefore our theory applies, and we only need to calculate the eigenvalues of the matrices given by (6.1) and (6.2). This is not a difficult task. We find the eigenvalues for $\left(\tilde{Z}'_{n,m}(z,s)\right)$ as

$$\tilde{Z}'_1(z,s) = s\mu\; f g_{1,1} \left(1 - \kappa(z,\theta)\; \cos^2(\theta)\right) \tag{6.5}$$

$$\tilde{Z}'_2(z,s) = s\mu\; f g_{1,1} \left(1 + \kappa(z,\theta)\; \cos^2(\theta)\right) \tag{6.6}$$

The eigenvalues for $\left(\tilde{Y}'_{n,m}(z,s)\right)$ turn out to be

$$\tilde{y}_1(z,s) = \frac{s\varepsilon}{f g_{1,1}\; \left(1 - \kappa(z,\theta)\right)} \tag{6.7}$$

$$\tilde{y}_2(z,s) = \frac{s\varepsilon}{f g_{1,1}\,\left(1 + \kappa(z,\theta)\right)} \tag{6.8}$$

With the modal current components

$$\tilde{i}_1(z,s) = \frac{1}{\sqrt{2}}\left(-\tilde{I}_1(z,s) + \tilde{I}_2(z,s)\right) \tag{6.9}$$

$$\tilde{i}_2(z,s) = \frac{1}{\sqrt{2}}\left(\tilde{I}_1(z,s) + \tilde{I}_2(z,s)\right) \tag{6.10}$$

we derive from (3.19) the final current equations for the differential and the common mode, respectively:

$$\frac{d^2}{dz^2}\,\tilde{i}_\beta(z,s) + \chi\,\frac{\frac{d}{dz}\kappa(z,\theta)}{\left(1+\chi\kappa(z,\theta)\right)}\,\frac{d}{dz}\,\tilde{i}_\beta(z,s) - s^2\varepsilon\mu\frac{\left(1 + \chi\,\kappa(z,\theta)\,\cos^2(\theta)\right)}{\left(1 + \chi\,\kappa(z,\theta)\right)}\,\tilde{i}_\beta(z,s) = 0 \tag{6.11}$$

$$\chi = \begin{cases} -1 & \text{for the differential mode, i.e., } \beta = 1 \\ +1 & \text{for the common mode, i.e., } \beta = 2 \end{cases} \tag{6.12}$$

As expected, equation (6.11) coincides with the corresponding equation in [5]. There we solved (6.11) by application of perturbation theory.

VII. Discussion and Concluding Remarks

In this paper we dealt with a certain class of solutions of the homogeneous (sourceless) NMTL equations. These solutions were obtained under the restrictive assumption that the physical matrices which enter the NMTL equations are circulants. Of course, circulants require a relatively high symmetry of the NMTL configurations, like, e.g., C_N symmetry, but on the other hand they permit an eigenmode expansion which decouples the original NMTL equations and simplifies them considerably. Moreover, despite the above restriction, a host of applications are still possible. These applications include antennas, twisted conductor lines, and cables above ground, to name only a few.

It is worth mentioning that in the framework of an NMTL-eigenmode-expansion approach all appearing eigenvalues are in general functions of frequency and position. However, it is significant and useful that for our usual case of symmetric impedance and admittance matrices (reciprocity) the assumed circulant matrices (e.g. from C_N rotation symmetry) are bicirculant and therefore the eigenvalues are in general doubly degenerate. In particular we have

$$\begin{aligned}\text{largest number of distinct eigenvalues} &= \begin{cases} \frac{N}{2}+1 & \text{for N even} \\ \frac{N+1}{2} & \text{for N odd} \end{cases} \\ &= \text{largest number of distinct } \tilde{\gamma}_\beta(s) \\ &= \text{largest number of distinct modal speeds} \end{aligned} \tag{7.1}$$

This roughly cuts in half the number of propagating modes to be considered. Furthermore, if the impedance- and admittance-matrices are real and frequency independent (the typical lossless case), then all the eigenvalues of these matrices are real as well as satisfying (7.1), and the propagation matrix is s times a real matrix.

We did not investigate the resonance behavior of our solutions for the wave-launcher unit cell. For such an investigation it is advantagous to express the solutions in terms of Bessel functions of the first and second kind and then look for the zeros in the common denominator.

We understand the present paper as a first, more formal step, towards powerful analytical solutions for interesting real systems. This will be the subject for forthcoming papers.

Appendix A. Some Properties of Circulant Matrices

By a circulant matrix of order N, or circulant for short, we mean a square matrix of the form

$$\left(C_{n,m}\right) \equiv circ\ \left(C_1, C_2, C_3, \dots, C_N\right)$$

$$= \begin{pmatrix} C_1, C_2, C_3, \dots, C_N \\ C_N, C_1, C_2, \dots, C_{N-1} \\ \vdots \quad \vdots \quad \vdots \qquad \vdots \\ C_2, C_3, C_4, \dots, C_1 \end{pmatrix} \tag{A.1}$$

A circulant has at most N different elements occurring already in one of its rows (or columns). We observe that the circulant results if we cyclically permute the elements of the first row, beginning with C_N in the second row, C_{N-1} in the third row, etc.

Circulants can easily be identified with the aid of the generic permutation matrix $\left(\Pi_{n,m}\right)$, defined by

$$\left(\Pi_{n,m}\right) \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & 1 \\ 1 & 0 & 0 & 0 \cdots & 0 \end{pmatrix} = circ\ (0,\ 1,\ 0,\ \dots,\ 0) \tag{A.2}$$

Theorem A1: Let $\left(A_{n,m}\right)$ be an N X N matrix. Then $\left(A_{n,m}\right)$ is a circulant if and only if

$$\left(A_{n,m}\right) \cdot \left(\Pi_{n,m}\right) = \left(\Pi_{n,m}\right) \cdot \left(A_{n,m}\right) \tag{A.3}$$

In words: The circulants comprise all the (square) matrices that commute with $\left(\Pi_{n,m}\right)$, or are invariant under the simularity transformation

$$\left(A_{n,m}\right) = \left(\Pi_{n,m}\right) \cdot \left(A_{n,m}\right) \cdot \left(\Pi_{n,m}\right)^{-1} \tag{A.4}$$

Proof: See [19].

Corollary A1: $\left(A_{n,m}\right)$ is a circulant if and only if

$\left(A_{n,m}\right)^{\dagger}$ is a circulant.

Proof: Consider the conjugate transpose of the product

$$\left(\left(A_{n,m}\right) \cdot \left(\Pi_{n,m}\right)\right)^{\dagger} = \left(\left(\Pi_{n,m}\right) \cdot \left(A_{n,m}\right)\right)^{\dagger} \tag{A.5}$$

and evaluate both sides of (A.5)

$$\left(\Pi_{n,m}\right)^{\dagger} \cdot \left(A_{n,m}\right)^{\dagger} = \left(A_{n,m}\right)^{\dagger} \cdot \left(\Pi_{n,m}\right)^{\dagger} \tag{A.6}$$

Since we obviously have

$$\left(\Pi_{n,m}\right)^{\dagger} = \left(\Pi_{n,m}\right)^{-1} \tag{A.7}$$

we obtain from (A.6)

$$\left(\Pi_{n,m}\right)^{-1} \cdot \left(A_{n,m}\right)^{\dagger} = \left(\Pi_{n,m}\right) \cdot \left(A_{n,m}\right)^{\dagger} \tag{A.8}$$

Multiplying (A.8) with $\left(\Pi_{n,m}\right)$ from the left we get in conclusion of the proof

$$\left(A_{n,m}\right)^{\dagger} \cdot \left(\Pi_{n,m}\right) = \left(\Pi_{n,m}\right) \cdot \left(A_{n,m}\right)^{\dagger} \tag{A.9}$$

Corollary A2: $\left(A_{n,m}\right)$ is circulant if and only if

$\left(A_{n,m}\right)^{T}$ is a circulant.

Proof: Analogous to the proof of corollary A1.

In order to show that circulants form a special class of normal matrices we still have to prove some more properties.

Theorem A2:

(1) If $\left(A_{n,m}\right)$ and $\left(B_{n,m}\right)$ are circulants, then $\left(A_{n,m}\right) \cdot \left(B_{n,m}\right)$ is a circulant.

(2) If $\left(A_{n,m}\right)$ is a circulant and k is a non-negative integer, then $\left(A_{n,m}\right)^{k}$ is a circulant.

(3) If $\left(A_{n,m}\right)$ is a non-singular circulant, then $\left(A_{n,m}\right)^{-1}$ is circulant.

Proof For the proof one has mainly to use Theorem A1.

To (1): We have to show

$$(\Pi_{n,m}) \cdot (A_{n,m}) \cdot (B_{n,m}) = ((A_{n,m}) \cdot (B_{n,m})) \cdot (\Pi_{n,m}) \tag{A.10}$$

We know that

$$(\Pi_{n,m}) \cdot (A_{n,m}) = (A_{n,m}) \cdot (\Pi_{n,m}) \tag{A.11}$$

$$(\Pi_{n,m}) \cdot (B_{n,m}) = (B_{n,m}) \cdot (\Pi_{n,m}) \tag{A.12}$$

holds. This in turn implies

$$\begin{aligned} &((\Pi_{n,m}) \cdot (A_{n,m})) \cdot ((\Pi_{n,m}) \cdot (B_{n,m})) = \\ &((A_{n,m}) \cdot (\Pi_{n,m})) \cdot ((B_{n,m}) \cdot (\Pi_{n,m})) = \\ &((\Pi_{n,m}) \cdot (A_{n,m})) \cdot ((B_{n,m}) \cdot (\Pi_{n,m})) \end{aligned} \tag{A.13}$$

or

$$(\Pi_{n,m})^2 \cdot (A_{n,m}) \cdot (B_{n,m}) = (\Pi_{n,m}) \cdot (A_{n,m}) \cdot (B_{n,m}) \cdot (\Pi_{n,m}) \tag{A.14}$$

Left multiplication of (A.14) with $(\Pi_{n,m})^{-1}$ yields statement (1).

To (2): From (1) we know that $(A_{n,m})^2$ is a circulant. The rest follows by induction.

To (3): We have to show

$$(A_{n,m})^{-1} \cdot (\Pi_{n,m}) = (\Pi_{n,m}) \cdot (A_{n,m})^{-1} \tag{A.15}$$

Starting with

$$(A_{n,m}) \cdot (\Pi_{n,m}) = (\Pi_{n,m}) \cdot (A_{n,m}) \tag{A.16}$$

we arrive at

$$(\Pi_{n,m}) \cdot (A_{n,m})^{-1} = (A_{n,m})^{-1} \cdot (\Pi_{n,m}) \tag{A.17}$$

by a twofold multiplication with $(A_{n,m})^{-1}$, once from the left and once from the right. Combining the results (2) and (3) we conclude that also $\left((A_{n,m})^{-1}\right)^k$ is circulant. Theorem A2 among others states that non-singular circulants form a group with respect to matrix multiplication.

What we still need to show is the commutativity of circulants. This can most easily be done by virtue of a second representation of circulants. This can be obtained with the aid of the permutation matrices $(\Pi_{n,m})^k$, k = 0, 1, 2, . . ., N-1. We have

$$(C_{n,m}) = circ\ (C_1, C_2, \ldots, C_N) = C_1 (1_{n,m}) + C_2\ (\Pi_{n,m})$$
$$+C_3(\Pi_{n,m})^2 + \ldots + C_N(\Pi_{n,m})^{N-1} \tag{A.18}$$

Therefore $(C_{n,m})$ is a circulant if and only if

$(C_{n,m}) = p((\Pi_{n,m}))$ for some polynomial p(z). Associating with the N-tuple

$$\gamma \equiv (C_1, C_2, \ldots, C_N) \tag{A.19}$$

the polynomial

$$P_\gamma(z) = C_1 + C_2\ z + \ldots + C_N\ z^{N-1} \tag{A.20}$$

the so-called representer of the circulant, we may write

$$(C_{n,m}) = circ\ (\gamma) = P_\gamma((\Pi_{n,m})) \tag{A.21}$$

Now, having two circulants $(C^{(1)}_{n,m})$ and $(C^{(2)}_{n,m})$ of the same order, they can be represented with the help of their corresponding representers P_{γ_1} and P_{γ_2}. Then, inasmuch as polynomials in the same matrix commute, it follows that all circulants of the same order commute. Therefore, since with $(C_{n,m})$ also $(C_{n,m})^\dagger$ is a circulant (see (A.5), $(C_{n,m})$ and $(C_{n,m})^\dagger$ commute and hence <u>all circulants are normal matrices</u>.

Appendix B: Diagonalization of Circulants

B1: Introduction of the Fourier Matrix $\left(U_{c_{n,m}}\right)$.

We start with the definition of the Fourier matrix.

Define the quantity

$$u \equiv u_{C_{1;1}} = e^{j\frac{2\pi}{N}} \tag{B.1}$$

In a good deal of what follows, u might be taken as any primitive Nth root of unity, but we prefer to standardize the selection as above. First we list some properties of u.

(a) $u^N = 1$

(b) $u\, u^* = 1$ (* complex conjugate)

(c) $u^* = u^{-1}$

(d) $u^{*k} = u^{-k} = u^{N-k}$

(e) $u + u^2 + u^3 + \ldots + u^{N-1} + 1 = 0$

(Closed vector chain along the unit circle in the complex plane).

With the aid of the quantity u we establish the Fourier matrix as

$$\left(U_{c_{n,m}}\right) \equiv \frac{1}{\sqrt{N}} \begin{pmatrix} u^1 & u^2 & \ldots\ldots & u^{N-1} & 1 \\ u^2 & u^4 & \ldots\ldots & u^{2(N-1)} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ u^{N-1} & u^{2(N-1)} & \ldots\ldots & u^{(N-1)(N-1)} & 1 \\ 1 & 1 & \ldots\ldots & 1 & 1 \end{pmatrix} \tag{B.3}$$

The sequence u^k, $k = 1, 2, 3, \ldots, N, \ldots$ is periodic; hence there are only N distinct elements in $\left(U_{c_{n,m}}\right)$.

Therefore $\left(U_{c_{n,m}}\right)$ can alternatively be written as

$$\left(U_{c_{n,m}}\right) \equiv \frac{1}{\sqrt{N}} \begin{pmatrix} u^1 & u^2 & \ldots\ldots & u^{N-1} & 1 \\ u^2 & u^4 & \ldots\ldots & u^{N-2} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ u^{N-1} & u^{N-2} & \ldots\ldots & u & 1 \\ 1 & 1 & \ldots\ldots & 1 & 1 \end{pmatrix} \tag{B.4}$$

From (B.4) we easily derive that $\left(U_{c_{n,m}}\right)$ and $\left(U_{c_{n,m}}\right)^{\dagger}$ are symmetric, i.e.

$$\left(U_{c_{n,m}}\right) = \left(U_{c_{n,m}}\right)^T \tag{B.5}$$

$$\left(U_{c_{n,m}}\right)^{\dagger} = \left(\left(U_{c_{n,m}}\right)^{\dagger}\right)^T = \left(U_{c_{n,m}}\right)^* \tag{B.6}$$

Another important property of $\left(U_{c_{n,m}}\right)$ is its unitarity.

Theorem B1: The Fourier matrix $\left(U_{c_{n,m}}\right)$ is unitary, i.e.,

$$\left(U_{c_{n,m}}\right) \cdot \left(U_{c_{n,m}}\right)^{\dagger} = \left(U_{c_{n,m}}\right)^{\dagger} \cdot \left(U_{c_{n,m}}\right) = \left(1_{n,m}\right) \tag{B.7}$$

or $$\left(U_{c_{n,m}}\right)^{-1} = \left(U_{c_{n,m}}\right)^{\dagger} \tag{B.8}$$

or $$\left(U_{c_{n,m}}\right)^{-1} = \left(\left(U_{c_{n,m}}\right)^*\right)^T \tag{B.9}$$

Proof: The unitarity of $\left(U_{c_{n,m}}\right)$ is a result of the geometric series identity

$$\frac{1}{N} \sum_{r=0}^{N-1} u^{r(\ell-k)} = \frac{1}{N} \cdot \frac{1-u^{N(\ell-k)}}{1-u^{\ell-k}} = \begin{cases} 1 \text{ if } \ell = k \\ 0 \text{ if } \ell \neq k \end{cases} \tag{B.10}$$

Consider $\left(U_{c_{n,m}}\right)^{\dagger} \cdot \left(U_{c_{n,m}}\right)$ and take the product of the k^{th} row of $\left(U_{c_{n,m}}\right)^{\dagger}$ with the ℓ^{th} column of $\left(U_{c_{n,m}}\right)$ to calculate the $\left(k^{th}, \ell^{th}\right)$ element of the above product matrix. The result is (B.10), and thus concludes our proof.

B2: Eigenvalues of Circulants

In this subsection we show that the Fourier matrix diagonalizes circulants, and at the same time we calculate the eigenvalues of circulants. We again formulate a theorem.

Theorem B2:

Let N be a fixed integer ≥ 1. Let $u = exp\,(2\pi j/N)$ and define the matrix $\left(\Xi_{n,m}\right)$ by

$$\left(\Xi_{n,m}\right) \equiv \text{diag}\left(u,\ u^2,\ u^3,\ \ldots,\ u^{N-1},\ 1\right) \tag{B.11}$$

noting that

$$\left(\Xi_{n,m}\right)^k = \text{diag}\left(u^k,\ u^{2k},\ u^{3k},\ \ldots,\ u^{k(N-1)},\ 1\right) \tag{B.12}$$

Then

$$\left(\Pi_{n,m}\right) = \left(U_{c_{n,m}}\right) \cdot \left(\Xi_{n,m}\right) \cdot \left(U_{c_{n,m}}\right)^{-1} \tag{B.13}$$

Proof: From (B.4) we read off that the ℓ^{th} row of $\left(U_{c_{n,m}}\right)$ is

$$\frac{1}{\sqrt{N}}\left(u^{\ell},\ u^{2\ell},\ \ldots,\ u^{(N-1)\ell},\ u^{N\ell}\right) \tag{B.14}$$

Hence the ℓ^{th} row of $\left(U_{c_{n,m}}\right) \cdot \left(\Xi_{n,m}\right)$ is

$$\frac{1}{\sqrt{N}}\left(u^{\ell}\ u,\ u^{2\ell}\ u^2,\ \ldots, u^{(N-1)\ell}\ u^{(N-1)},\ u^{N\ell}\ u^N\right)$$

$$\equiv \frac{1}{\sqrt{N}}\left(\cdots,\ u^{\ell r}\ u^r\ \cdots\right)\ , r = 1,\ 2,\ 3,\ \cdots,\ N \tag{B.15}$$

The k^{th} column of $\left(U_{c_{n,m}}\right)^{-1}$ is

$$\frac{1}{\sqrt{N}}\left(u^{*k},\ u^{*2k},\ \ldots,\ u^{*(N-1)k},\ u^{N\ell}\ u^{*Nk}\right)$$

$$\equiv \frac{1}{\sqrt{N}}\left(\cdots,\ u^{*kr}\ \cdots\right),\ r = 1,\ 2,\ 3, \ldots,\ N \tag{B.16}$$

Thus the $(\ell\, k) - th$ element of $\left(U_{c_{n,m}}\right)\cdot\left(\Xi_{n,m}\right)\cdot\left(U_{c_{n,m}}\right)^{-1}$ is

$$\frac{1}{N}\left(u^{\ell+1-k} + u^{2(\ell-k)+2} + \ldots + u^{N\ell+N-Nk}\right)$$

$$= \frac{1}{N}\sum_{r=1}^{N} u^{r(\ell-k+1)} = \begin{cases} 1 \text{ if } \ell = k-1 \\ 0 \text{ if } \ell \neq k-1 \end{cases} \tag{B.17}$$

which finishes our proof.

Now, another appropriate expression for circulants can be derived from (A.21) and B(.13). We get

$$\left(C_{n,m}\right) = P_\gamma\left[\left(U_{c_{n,m}}\right)\cdot\left(\Xi_{n,m}\right)\cdot\left(U_{c_{n,m}}\right)^{-1}\right]$$

$$= \left(U_{c_{n,m}}\right)\cdot P_\gamma\ \left(\left(\Xi_{n,m}\right)\right)\cdot\left(U_{c_{n,m}}\right)^{-1} \tag{B.18}$$

$$\left[\underbrace{\left(\Pi_{n,m}\right)\cdot\left(\Pi_{n,m}\right)\cdot\ \ldots\ \cdot\left(\Pi_{n,m}\right)}_{\text{k-factors}} = \left(U_{c_{n,m}}\right)\cdot\left(\Xi_{n,m}\right)^k\cdot\left(U_{c_{n,m}}\right)^{-1}\right]$$

with

$$P_\gamma\left(\left(\Xi_{n,m}\right)\right) = C_1\left(1_{n,m}\right) + C_2\left(\Xi_{n,m}\right) + C_3\left(\Xi_{n,m}\right)^2$$

$$+ \ldots + C_N\ \left(\Xi_{n,m}\right)^{N-1} \tag{B.19}$$

$$= \text{diag}\ \left(P_\gamma(u),\ P_\gamma\left(u^2\right),\ \ldots,\ P_\gamma\left(u^{N-1}\right),\ P_\gamma(1)\right)$$

Recall that

$$P_\gamma(u) = C_1 + C_2\, u + C_3\, u^2 + \ldots + C_N\, u^{N-1}. \tag{B.20}$$

With the above intermediate results we arrive at the fundamental theorem:

Theorem B3:

If $\left(C_{n,m}\right)$ is a circulant, it may be diagonalized by the Fourier matrix $\left(U_{c_{n,m}}\right)$. More explicitly,

$$\left(C_{n,m}\right) = \left(U_{c_{n,m}}\right) \cdot \left(C^{(d)}_{n,m}\right) \cdot \left(U_{c_{n,m}}\right)^{-1} \tag{B.21}$$

where

$$\left(C^{(d)}_{n,m}\right) = \text{diag}\left(P_\gamma(u),\, P_\gamma\left(u^2\right),\, \ldots,\, P_\gamma\left(u^{N-1}\right),\, P_\gamma\left(u^N\right)\right) \tag{B.22}$$

The eigenvalues of $\left(C_{n,m}\right)$ are therefore

$$c_\beta = P_\gamma\left(u^\beta\right) = \sum_{k=1}^{N} C_{k+1}\, \exp\left(2\pi j\, \frac{k\beta}{N}\right) \qquad \left(C_{N+1} \equiv C_1\right) \tag{B.23}$$

Proof: Use (B.13) and (B.18).

Since the simularity transformation performed with the aid of $\left(U_{c_{n,m}}\right)$ diagonalizes circulants the column vectors of $\left(U_{c_{n,m}}\right)$ are the (right) eigenvectors which correspond to the eigenvalues (B.23), and the row vectors are the left eigenvectors.

If a circulant matrix $\left(C_{n,m}\right)$ is a symmetric circulant, or bicirculant matrix we have

$$C_i = C_{N+2-i} \qquad (i = 2, 3, \ldots, N) \tag{B.24}$$

We denote such a matrix with

$$\left(B_{n,m}\right) = \text{bicirc}\left(B_1, B_2, \ldots, B_N\right) = \text{bicirc}\left(B_1, B_N, \ldots, B_3, B_2\right) \tag{B.25}$$

In this case the eigenvalues reduce to

$$b_\beta = B_1 + B_2\left(u^\beta + u^{(N-1)\beta}\right) + B_3\left(u^{2\beta} + u^{(N-2)\beta}\right) + \cdots$$

$$= B_1 + B_2\left(u^\beta + \left(u^\beta\right)^*\right) + B_3\left(u^{2\beta} + \left(u^{2\beta}\right)^*\right) + \cdots \qquad \text{(B.26)}$$

$$= B_1 + 2B_2\ Re\left(u^\beta\right) + 2B_3\ Re\left(u^{2\beta}\right) + \cdots$$

i.e., for an even number N we find $\frac{N}{2} + 1$ eigenvalues

$$b_1 = b_{N-1},\ b_2 = b_{N-2},\ \ldots,\ b_{\frac{N}{2}-1} = b_{\frac{N}{2}+1} \qquad \text{(B.27)}$$

$$b_{\frac{N}{2}} = B_1 - 2B_2 + 2B_3 - 2B_4 + - \ldots + 2qB_{\frac{N}{2}} - qB_{\frac{N}{2}+1} \qquad \text{(B.28)}$$

$$q = \begin{cases} +1 \text{ if } \left(\frac{N}{2}+1\right) \text{ is even} \\ -1 \text{ if } \left(\frac{N}{2}+1\right) \text{ is odd} \end{cases} \qquad \text{(B.29)}$$

$$b_N = B_1 + 2\left(B_2 + B_3 + \ldots + B_{\frac{N}{2}}\right) + B_{\frac{N}{2}+1} \qquad \text{(B.30)}$$

For an off number N we derive $\frac{N+1}{2}$ eigenvalues

$$b_1 = b_{N-1},\ b_2 = b_{N-2},\ \ldots,\ b_{\frac{N-1}{2}} = b_{\frac{N+1}{2}} \qquad \text{(B.31)}$$

$$b_N = B_1 + 2\left(B_2 + B_3 + \ldots + B_{\frac{N+1}{2}}\right) \qquad \text{(B.32)}$$

Observe that symmetric circulants of even order N and those of order N + 1 have the same number of eigenvalues. This number as well corresponds to the number of different elements in a symmetric circulant. Thus the configurations of those physical systems which can be described by bicirculants exhibit a relatively high symmetry. Furthermore for real bicirculants (B_n real) the eigenvalues b_β are all real.

Now, formula (3.5) together with (B.26) through (B.32) suggest that one introduce a real representation of the similarity transformation matrix using the following real eigenvectors of bicirculants:

$$(w_n)_\beta \equiv \frac{1}{\sqrt{2}}\left(\left(u_{c_n}\right)_\beta + \left(u_{c_n}\right)_{N-\beta}\right) = \sqrt{2}\ Re\left(\left(u_{c_n}\right)_\beta\right)$$

$$= \sqrt{\frac{2}{N}}\begin{pmatrix} \cos\left(\frac{2\pi\beta}{N}\right) \\ \vdots \\ \cos\left(\frac{2\pi(N-1)\beta}{N}\right) \\ 1 \end{pmatrix} \tag{B.33}$$

$$\beta = \begin{cases} 1,\ 2,\ \ldots,\ \frac{N}{2} - 1 \text{ if } N \text{ is even} \\ 1,\ 2,\ \ldots,\ \frac{N-1}{2} \text{ if } N \text{ is odd} \end{cases} \tag{B.34}$$

$$(w_n)_\beta \equiv \frac{1}{\sqrt{2}\,j}\left(\left(u_{c_n}\right)_\beta - \left(u_{c_n}\right)_{\beta-N}\right) = \sqrt{2}\ I_m\left(\left(u_{c_n}\right)_\beta\right)$$

$$= \sqrt{\frac{2}{N}}\begin{pmatrix} \sin\left(\frac{2\pi\beta}{N}\right) \\ \vdots \\ \sin\left(\frac{2\pi(N-1)\beta}{N}\right) \\ 0 \end{pmatrix} \tag{B.35}$$

$$(w_n)_{\frac{N}{2}} \equiv \frac{1}{\sqrt{N}}\begin{pmatrix} -1 \\ +1 \\ -1 \\ \vdots \\ -1 \\ +1 \end{pmatrix} \quad \text{for } N \text{ even} \tag{B.36}$$

$$(\omega_n)_N \equiv \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{B.37}$$

The above eigenvectors $(w_n)_\beta$ constitute a real, orthonormal matrix $(W_{n,m})$ which transforms bicirculants into diagonal form

$$(W_{n,m})^{-1} \cdot (B_{n,m}) \cdot (W_{n,m}) = \begin{pmatrix} b_1 & & & & & & \\ & b_2 & & & & \bigcirc & \\ & & \ddots & & & & \\ & & & b_2 & & & \\ & \bigcirc & & & b_1 & & \\ & & & & & b_N \end{pmatrix} \tag{B.38}$$

Note that in this form we have cos for the eigenvectors for the lower β $(1, 2, \ldots)$ and sin for the upper β $(N, N\text{-}1, \ldots)$, but this could just as easily be the reverse. Since there are at most $\frac{N}{2} + 1$ eigenvalues for even N and $\frac{N+1}{2}$ for odd N, what we really have is modal degeneracy so that or a particular eigenvalue (except for $\beta = N$, and (for N even) $\beta = \frac{N}{2}$) one needs in general a linear combination of the two eigenvectors for each independent eigenvalue in solving general problems.

Appendix C: Diagonal Form of the Telegrapher Equations

Consider the telegrapher equations without source terms

$$\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(\tilde{I}_n(z,s)\right) \tag{C.1}$$

$$\frac{d}{dz}\left(\tilde{I}_n(z,s)\right) = -\left(\tilde{Y}'_{n,m}(z,s)\right)\cdot\left(\tilde{V}_n(z,s)\right) \tag{C.2}$$

and assume that the impedance and admittance matrices are circulants. In this case they can simultaneously be diagonalized by $\left(U_{C_{n,m}}\right)$, i.e.,

$$\left(\tilde{Z}'^{(d)}_{n,m}(z,s)\right) = \left(U_{C_{n,m}}\right)^{-1}\cdot\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(U_{C_{n,m}}\right) \tag{C.3}$$

$$\left(\tilde{Y}'^{(d)}_{n,m}(z,s)\right) = \left(U_{C_{n,m}}\right)^{-1}\cdot\left(\tilde{Z}'_{n,m}(z,s)\right)\cdot\left(U_{C_{n,m}}\right) \tag{C.4}$$

Multiply equations (C.1) and (C.2) from the left with $\left(U_{C_{n,m}}\right)^{-1}$ and observe (C.3) and (C.4), then one obtains the (coupled) equations for the modal voltage and current vectors $\left(\tilde{\upsilon}_n(z,s)\right)$ and $\left(\tilde{i}_n(z,s)\right)$, respectively.

$$\frac{d}{dz}\left(\tilde{\upsilon}_n(z,s)\right) = -\left(\tilde{Z}'^{(d)}(z,s)\right)\cdot\left(\tilde{i}_n(z,s)\right) \tag{C.5}$$

$$\frac{d}{dz}\left(\tilde{i}_n(z,s)\right) = -\left(\tilde{Y}'^{(d)}_{n,m}(z,s)\right)\cdot\left(\tilde{\upsilon}_n(z,s)\right) \tag{C.6}$$

In terms of the components $\tilde{\upsilon}_\beta(z,s)$ and $\tilde{i}_\beta(z,s)$ of the modal vectors the last two equations read:

$$\frac{d}{dz}\,\tilde{\upsilon}_\beta(z,s) = -\,\tilde{z}'_\beta(z,s)\;\tilde{i}_\beta(z,s) \tag{C.7}$$

$$\frac{d}{dz}\,\tilde{i}_\beta(z,s) = -\,\tilde{y}'_\beta(z,s)\;\tilde{\upsilon}_\beta(z,s) \tag{C.8}$$

$$(\beta = 1,\, 2,\, \ldots,\, N)$$

Here the functions $\tilde{z}'_{\beta}(z, s)$ and $\tilde{y}'_{\beta}(z, s)$ denote the eigenvalue functions of the corresponding matrices $\left(\tilde{Z}'_{n,m}(z, s)\right)$ and $\left(\tilde{Y}'_{n,m}(z, s)\right)$.

Appendix D: General Integral Solution for the Telegrapher Equations

The combined telegrapher equations (C.7) and (C.8) read in terms of the modal reflection coefficients $\tilde{\rho}_\beta(z,s)$:

$$\frac{d}{dz}\,\tilde{\rho}_\beta(z,s) - 2\,\tilde{\gamma}_\beta(z,s)\,\tilde{\rho}_\beta(z,s) + \frac{1}{2}\left(1 - \tilde{\rho}_\beta^2(z,s)\right)\frac{d}{dz}\,\ell n\!\left(\tilde{z}_{c\beta}(z,s)\right) = 0 \qquad (\beta = 1,\, 2,\, \ldots,\, N) \tag{D.1}$$

Here the reflection-coefficient functions are defined by

$$\tilde{\rho}_\beta(z,s) \equiv \left(\frac{\tilde{\upsilon}_\beta(z,s)}{\tilde{i}_\beta(z,s)} - \tilde{z}_{c\beta}(z,s)\right) \Bigg/ \left(\frac{\tilde{\upsilon}_\beta(z,s)}{\tilde{i}_\beta(z,s)} + \tilde{z}_{c\beta}(z,s)\right) \tag{D.2}$$

Defining the complex modal "phase" angles $\tilde{\theta}_\beta(z,s)$ by means of the relation (see [10])

$$e^{\tilde{\theta}_\beta(z,s)} \equiv \tilde{\rho}_\beta(z,s) \tag{D.3}$$

equation (D.1) "simplifies" to

$$\frac{d}{dz}\,\tilde{\theta}_\beta(z,s) = 2\,\tilde{\gamma}_\beta(z,s) + 2\,\tilde{\xi}_\beta(z,s)\,\sinh\left(\tilde{\theta}_\beta(z,s)\right) \tag{D.4}$$

where we introduced the quantity

$$\tilde{\xi}_\beta(z,s) \equiv \frac{1}{2}\,\frac{d}{dz}\,\ell n\left(\tilde{z}_{c\beta}(z,s)\right) \tag{D.5}$$

The solution of (D.4) can easily be obtained in integral form:

$$\tilde{\theta}_\beta^{(n)}(z,s) = \tilde{\theta}_\beta(0,s) + 2\int_0^z \tilde{\gamma}_\beta(z',s)\,dz' + 2\int_0^z \tilde{\xi}_\beta(z',s)\,\sinh\left(\tilde{\theta}_\beta(z',s)\right)dz' \tag{D.6}$$

Here $\tilde{\theta}_\beta(0,s)$ denotes some initial value function which we arbitrarily have chosen at $z = 0$. Of course, every other value z_0 could have been taken.

One way to solve the integral equations (D.6) may be the method of iteration. Applying this method we get for the n^{th} order approximation

$$\tilde{\theta}_{\beta}^{(n)}(z,s) = \tilde{\theta}_{\beta}(0,s) + 2\int_{0}^{z} \tilde{\gamma}_{\beta}(z',s)\, dz' + 2\int_{0}^{z} \tilde{\xi}_{\beta}(z',s)\, \sinh\left(\tilde{\theta}_{\beta}^{(n-1)}(z',s)\right) dz' \tag{D.7}$$

The lowest order approximation is given by

$$\tilde{\theta}_{\beta}^{(o)}(z,s) = \tilde{\theta}_{\beta}(0,s) + 2\int_{0}^{z} \tilde{\gamma}_{\beta}(z',s)\, dz' \tag{D.8}$$

and the first-order approximation by

$$\tilde{\theta}_{\beta}^{(1)}(z,s) = \tilde{\theta}_{\beta}^{(o)}(z,s) + 2\int_{0}^{z} \tilde{\xi}_{\beta}(z',s)\, \sinh\left(\tilde{\theta}_{\beta}^{(o)}(z,s)\right) dz' \tag{D.9}$$

As soon as we know an approximate solution $\tilde{\theta}_{\beta}^{(n)}(z,s)$, we then obtain the corresponding approximation for $\tilde{\rho}(z,s)$ via equation (D.3). In a further step we find approximate solutions for $\tilde{i}_{\beta}$ or $\tilde{\upsilon}_{\beta}$ with the aid of equations (C.7) and (C.8), observing that

$$\upsilon_{\beta} = \left(1 - \tilde{\zeta}_{\beta}\right)^{-1} \cdot \tilde{z}_{c\beta}\left(1 + \tilde{\zeta}_{\beta}\right) \tilde{i}_{\beta} \tag{D.10}$$

holds.

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