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Commutative Tubes in Multiconductor-Transmission-Line Theory

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Abstract

In this paper, we study a certain class of solutions of the multiconductor transmission-line equations. This class is basically defined by two reasonable but otherwise arbitrary demands: (1) the commutativity between the propagation matrix and the characteristic impedance matrix and (2) the assumption that the matrices of interest are real matrices times complex valued functions (e.g. functions of the constitutive parameters of the medium). On the basis of the above requirements, it can be shown that all matrices which are relevant for the MTL equations and for their solutions can be simultaneously diagonalized with only one set of eigenvectors (e.g. the eigenvectors of the per-unit-length inductance matrix). Of course, the sets of eigenvalues corresponding to different matrices are different. We investigate the MTL equations and their solutions for different environments and with different properties, including lossy lines and lossy media. A special section is devoted to the investigation of those mechanisms which may cause splitting of natural frequencies.

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I. Introduction

Transmission-line theory has become one of the basic tools for describing EMP induced voltages and currents on multiconductor transmission lines (MTL). Although one has to observe some restrictions in the application of the MTL theory, there are innumerable cases (see e.g. [1,5,8,9]) where its use turned out to be very successful, among others in the analysis of EMP/EMC interaction with modern complex electronic systems.

Dealing with transmission-line networks one finally has to use computers in order to solve the high dimensional coupled differential equations for the voltages and currents. But, before starting a computer, one should try hard enough to find analytical answers, even as solutions for complex problems. Every analytical solution has the great advantage that it can be interpreted on the basis of physics.

In this paper we present a certain class of analytical solutions of the MTL-equations (in the frequency domain). These solutions are obtained under the restrictive assumptions of the commutativity between the propagation matrix and the characteristic impedance matrix and that the matrices which enter the MTL-equations can be written as a sum of products of real matrices times (scalar) complex valued functions. Nevertheless, these subsidiary conditions still permit one to describe a host of configurations of multiconductor lines. Especially, we cover multiconductors above a perfectly conducting plane and those inside a shielding tube.

The organization of our paper is as follows: In Section 2 we present the MTL equations and cast their solutions into an appropriate form exhibiting especially the property of forward and backward traveling waves. Moreover, the important property of the matrices under consideration to be symmetric is explicitly worked out.

In Section 3 we focus our attention on a set of (pairwise) commuting matrices. The commutativity of this whole set of matrices (which are the basic ingredients in the MTL equations) turns out as a result of our (working-) hypothesis. This hypothesis requires the propagation matrix and the characteristic impedance matrix to commute. Since (real) symmetric and pairwise commuting matrices can simultaneously be diagonalized by one set of (of course linearly independent) eigenvectors, the solutions simplify considerably.

Sections 4 through 7 deal with concrete examples of multiconductor-line properties and with the surrounding medium. These properties are expressed in terms of per-unit-length impedance and per-unit-length admittance matrices. In Section 4 these matrices are chosen for lossless conductors in a lossy, uniform, and isotropic medium. In Section 5 we add small changes to the per-unit-length admittance matrix whereas in Section 6 we assume lossy wires in a lossless medium. Section 7 deals with small changes added to the per-unit-length impedance matrix.

In Section 8 we permit general ansaetze for the per-unit-length impedance and per-unit-length admittance matrices (but still under the restriction of commutativity). A special relation between the eigenvalues of these matrices is chosen to make the expressions for the eigenvalues of the propagation matrix become perfect squares. This procedure is performed in close analogy to the "equalization" [7] of long telephone lines.

In Section 9 we are looking for resonances of an MTL with passive terminations. We choose the lines to be open ended or to be short circuited. Especially, mechanisms which lead to the splitting of natural modes are investigated.

We close our paper in Section 10 with a few concluding remarks. Appendixes cover some of the mathematical details.

II. Basic Equations for an N-Wire Transmission Line Tube

In this section we briefly list the basic transmission line equations for an N-wire transmission-line tube in order to provide the necessary quantities, denotations, and definitions which are used in the next section. Thereby, we rely on [1 (Section III)] and the formalism established therein. In Figure 1, we show a per-unit-length model of a multiconductor transmission line.

An N-wire transmission line consists of N conductors and a reference which can be chosen to be infinity or ground. We are most interested in those physical quantities which mainly describe such a system. These are the N modes of propagation and the (closely related) characteristic impedance matrix.

The generalized transmission line equations for a single section of an N-wire system are the well-known telegrapher equations governing the voltage and current propagation, respectively.

$$\frac{d}{dz}\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right) + \left(\tilde{V}_n^{(s)'}(z,s)\right) \tag{2.1}$$

$$\frac{d}{dz}\left(\tilde{I}_n(z,s)\right) = -\left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{V}_n(z,s)\right) + \left(\tilde{I}_n^{(s)'}(z,s)\right) \tag{2.2}$$

where

$s = \Omega + j\omega \equiv$ Laplace-transform variable or complex frequency

$\sim \equiv$ Laplace transform over time (two sided)

$z =$ position along the tube

$\left(\tilde{V}_n(z,s)\right) \equiv$ voltage vector at z (2.3)

$\left(\tilde{I}_n(z,s)\right) \equiv$ current vector at z

$\left(\tilde{Z}'_{n,m}(s)\right) \equiv$ per-unit-length series impedance matrix

$\left(\tilde{Y}'_{n,m}(s)\right) \equiv$ per-unit-length shunt admittance matrix

$\left(\tilde{V}_n^{(s)'}(z,s)\right) \equiv$ per-unit-length series voltage source vector

$\left(\tilde{I}_n^{(s)'}(z,s)\right) \equiv$ per-unit-length shunt current source vector

We note that all vectors have N components, and all matrices are $N \times N$.

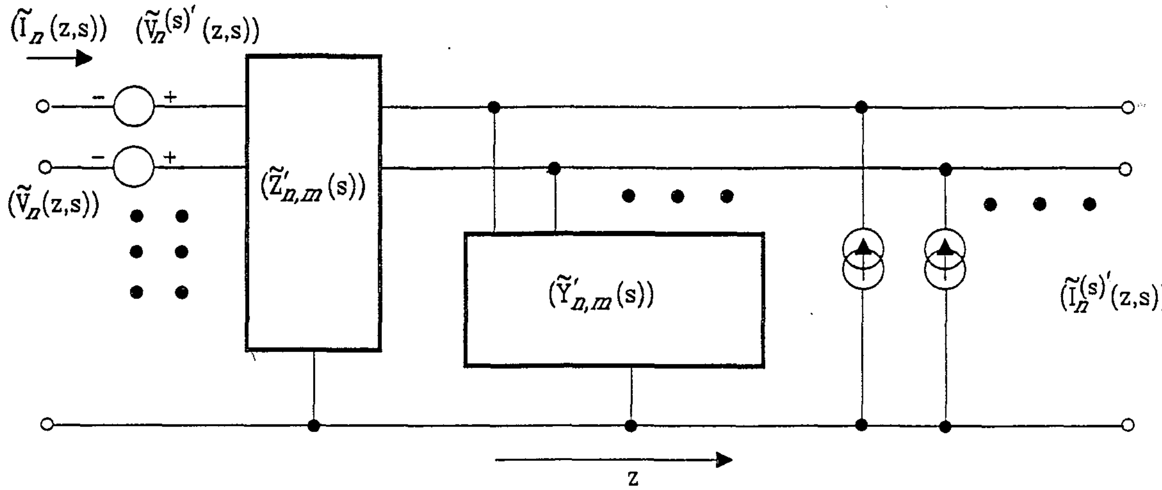

Figure 1. Per-Unit-Length Model of a Multiconductor Transmission Line

Equations (2.1) and (2.2) can be very elegantly combined [1] to result in the combined voltage equation.

$$\frac{d}{dz}\left(\tilde{V}_n(s)\right)_q + q\left(\tilde{\gamma}_{c_{n,m}}(s)\right)\cdot\left(\tilde{V}_n(z,s)\right)_q = \left(\tilde{V}_n^{(s)'}(z,s)\right)_q \tag{2.4}$$

the solution of which reads

$$\left(\tilde{V}_n(z,s)\right)_q = \exp\left\{-q\left(\tilde{\gamma}_{c_{n,m}}(s)\right)\left[z-z_o\right]\right\}\cdot\left(\tilde{V}_n(z_o,s)\right)_q + \int_{z_o}^{z}\exp\left\{-q\left(\tilde{\gamma}_{c_{n,m}}(s)\right)\left[z-z'\right]\right\}\cdot\left(\tilde{V}_n^{(s)'}(z',s)\right)_q dz' \tag{2.5}$$

Here the matrix $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ is called the propagation matrix, and it is defined by

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \text{principal value of } \left[\left(\tilde{Z}'_{n,m}(s)\right)\cdot\left(\tilde{Y}'_{n,m}(s)\right)\right]^{1/2} \tag{2.6}$$

We have the separation index

$q = \pm 1$ for forward and backward traveling combined N-vector waves, respectively. (2.7)

The combined voltage vectors are defined by

$$\left(\tilde{V}_n(z,s)\right)_q \equiv \left(\tilde{V}_n(z,s)\right) + q\left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right)$$

$$\left(\tilde{V}_n^{(s)'}(z,s)\right)_q \equiv \left(\tilde{V}_n^{(s)'}(z,s)\right) + q\left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z,s)\right) \tag{2.8}$$

where we have introduced the characteristic impedance matrix via

$$\left(\tilde{Z}_{c_{n,m}}(s)\right) \equiv \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}_{n,m}(s)\right)^{-1} = \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^{-1} \left(\tilde{Z}'_{n,m}(s)\right) \tag{2.9}$$

Thus, we obtain the important results

$$\left(\tilde{V}_n(z,s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right)$$

for forward traveling waves and (2.10)

$$\left(\tilde{V}_n(z,s)\right) = -\left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{I}_n(z,s)\right)$$

for backward traveling waves.

Knowing the forward and backward traveling waves (e.g. through appropriate experiments), we can reconstruct the voltage and current vectors. The combined current vectors are simply related to the combined voltage vectors through the characteristic admittance matrix

$$\left(\tilde{I}_n(z,s)\right)_q = q\left(\tilde{Y}_{c_{n,m}}(s)\right) \cdot \left(\tilde{V}_n(z,s)\right)_q \tag{2.11}$$

$$\left(\tilde{Y}_{c_{n,m}}(s)\right) \equiv \left(\tilde{Z}_{c_{n,m}}(s)\right)^{-1}$$

Once the combined voltages are evaluated, the total voltages and total currents are readily obtained. One finds

$$\left(\tilde{V}_n(z,s)\right) = \frac{1}{2}\left[\left(\tilde{V}_n(z,s)\right)_+ + \left(\tilde{V}_n(z,s)\right)_-\right] \tag{2.12}$$

$$\left(\tilde{I}_n(z,s)\right) = \frac{1}{2}\left(\tilde{Y}_{c_{n,m}}(s)\right) \cdot \left[\left(\tilde{V}_n(z,s)\right)_+ - \left(\tilde{V}_n(z,s)\right)_-\right] \tag{2.13}$$

With the aid of (2.5) we bring (2.12) into a very suggestive form expressing explicitly forward and backward traveling waves.

$$2\left(\tilde{V}_n(z,s)\right) = \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)z\right\} \cdot \left(\left(\tilde{V}_n(0,s)\right)_+ + \left(\tilde{V}_n^{(s)}(z,s)\right)_+\right)$$

$$+ \exp\left\{\left(\tilde{\gamma}_{c_{n,m}}(s)\right)(z-L)\right\} \cdot \left(\left(\tilde{V}_n(L,s)\right)_- + \left(\tilde{V}_n^{(s)}(z,s)\right)_-\right) \tag{2.14}$$

The constant (non z-dependent) vectors $\left(\tilde{V}_n(0,s)\right)_+$ and $\left(\tilde{V}_n(L,s)\right)_-$ are given by the boundary conditions of the total voltage and total current vector. We compute these vectors explicitly in Section 9.

The z-dependent vector fields $\left(\tilde{V}_n^{(s)}(z,s)\right)_+$ and $\left(\tilde{V}_n^{(s)}(z,s)\right)_-$ are integrals over the source vectors as

$$\left(\tilde{V}_n^{(s)}(z,s)\right)_+ \equiv \int_{z_o}^{z} dz' \exp\left\{\left(\tilde{\gamma}_{c_{n,m}}(s)\right)z'\right\} \cdot \left(\left(\tilde{V}_n^{(s)'}(z',s)\right)\right.$$

$$\left. + \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z',s)\right)\right) \tag{2.15}$$

$$\left(\tilde{V}_n^{(s)}(z,s)\right)_- \equiv \int_{L}^{z} dz' \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)(z'-L)\right\} \cdot \left(\left(\tilde{V}_n^{(s)'}(z',s)\right)\right.$$

$$\left. - \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{I}_n^{(s)'}(z',s)\right)\right) \tag{2.16}$$

Considering equation (2.6) we clearly recognize the necessity of an eigenmode expansion of the matrix product $(\tilde{Z}'_{n,m}(s)) \cdot (\tilde{Y}'_{n,m}(s))$. Once we start an eigenmode expansion we should do this as well for all matrices which occur in the above equations. We assume that those expansions are possible and that we are dealing with p.r. matrices (see [1]). The expansion of the propagation matrix yields.

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \sum_{\delta} \tilde{\gamma}_\delta(s) \left(\tilde{v}_{c_n}(s)\right)_\delta \left(\tilde{i}_{c_n}(s)\right)_\delta \tag{2.17}$$

where $\delta = 1,2,\ldots,N$ is the eigenindex and $\tilde{\gamma}_\delta(s)$ are the eigenvalues, and $\left(\tilde{v}_{c_n}(s)\right)_\delta$ and $\left(\tilde{i}_{c_n}(s)\right)_\delta$ are the normalized right and left eigenvectors, respectively, defined by the following equations:

$$\left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{v}_{c_n}(s)\right)_\delta = \tilde{\gamma}^2_\delta(s) \left(\tilde{v}_{c_n}(s)\right)_\delta$$

$$\left(\tilde{i}_{c_n}(s)\right)_{\delta'} \cdot \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) = \tilde{\gamma}^2_{\delta'}(s) \left(\tilde{i}_{c_n}(s)\right)_{\delta'} \tag{2.18}$$

In the case that $\gamma^2_\delta \neq \gamma^2_{\delta'}$ (sufficient, but often not necessary), there exists the biorthonormal relation

$$\left(\tilde{i}_{c_n}(s)\right)_{\delta'} \cdot \left(\tilde{v}_{c_n}(s)\right)_\delta = 1_{\delta,\delta'} \text{ (= Kronecker delta)} \tag{2.19}$$

Using (2.17) together with the fact that we equally well can expand a function F of $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ as

$$F\left[\left(\tilde{\gamma}_{c_{n,m}}(s)\right)\right] = \sum_\delta F\left[\tilde{\gamma}_\delta(s)\right] \left(\tilde{v}_{c_n}(s)\right)_\delta \left(\tilde{i}_{c_n}(s)\right)_\delta \tag{2.20}$$

equation (2.14) could be written in terms of eigenmodes.

In reference [1] it is shown that a reasonable normalization for the voltage and current modes in given by

$$\left(\tilde{i}_{c_n}(s)\right)_\delta = \left(\tilde{Y}_{c_{n,m}}(s)\right) \cdot \left(\tilde{v}_{c_n}(s)\right)_\delta$$

$$\left(\tilde{v}_{c_n}(s)\right)_\delta = \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left.\tilde{i}_{c_n}(s)\right)_\delta \tag{2.21}$$

In this case it becomes easily obvious (from (2.18), (2.19), and (2.21) that the characteristic matrices can be represented as

$$\left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{Y}_{c_{n,m}}(s)\right)^{-1} = \sum_\delta \left(\tilde{v}_{c_n}(s)\right)_\delta \left(\tilde{v}_{c_n}(s)\right)_\delta = \left(\tilde{Z}_{c_{n,m}}(s)\right)^T \tag{2.22}$$

and

$$\left(\tilde{Y}_{c_{n,m}}(s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right)^{-1} = \sum_\delta \left(\tilde{i}_{c_n}(s)\right)_\delta \left(\tilde{i}_{c_n}(s)\right)_\delta = \left(\tilde{Y}_{c_{n,m}}(s)\right)^T \tag{2.23}$$

These equations explicitly show that the characteristic matrices are symmetric. This property was not assumed in the beginning, but it turned out as a consequence of the assumed characteristics of the propagation matrix (p.r. matrix) and of the structure of the solution of the multiconductor-transmission-line equations. Usually the symmetric form of the characteristic impedance and admittance matrices is derived from the reciprocity principle. The present results then apply to reciprocal multiconductor transmission lines.

The expansion of $(\tilde{Z}'_{n,m}(s))$ and $(\tilde{Y}'_{n,m}(s))$ in terms of $\left(\tilde{v}_{c_n}(s)\right)_\delta$ and $\left(\tilde{i}_{c_n}\right)_\delta$ is also performed in [1]. We only quote the result here which we need for our investigations in the following sections.

$$\left(\tilde{Z}'_{n,m}(s)\right) = \sum_\delta \tilde{\gamma}_\delta(s) \left(\tilde{v}_{c_n}(s)\right)_\delta \left(\tilde{v}_{c_n}(s)\right)_\delta = \left(\tilde{Z}'_{n,m}(s)\right)^T \qquad (2.24)$$

$$\left(\tilde{Y}'_{n,m}(s)\right) = \sum_\delta \tilde{\gamma}_\delta(s) \left(\tilde{i}_{c_n}(s)\right)_\delta \left(\tilde{i}_{c_n}(s)\right)_\delta = \left(\tilde{Y}'_{n,m}(s)\right)^T \qquad (2.25)$$

Again, as in equations (2.22) and (2.23), the symmetry of the impedance and admittance per-unit-length matrices was not explicitly assumed at the start. It rather is a consequence of the development including, especially, the dynamic equations for $(\tilde{V}_n(z,s))$ and $(\tilde{I}_n(z,s))$. Moreover, it is a statement of reciprocity for these matrices.

For our further considerations in the next sections we assume reciprocity (symmetry) of the impedance and admittance per-unit-length matrices and of the characteristic matrices for granted.

III. Consequences of the Commutation Hypothesis for the Propagation Matrix and the Characteristic Impedance Matrix

Due to the structure of our solutions (2.12) and (2.14) of the multiconductor tube equations in the foregoing section, it becomes immediately obvious that two matrices describe the inherent physical properties of the system: The propagation matrix $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ and the characteristic impedance matrix $\left(\tilde{Z}_{c_{n,m}}(s)\right)$ (or equivalently the characteristic admittance matrix $\left(\tilde{Y}_{c_{n,m}}(s)\right)$. The first matrix describes the propagation of waves along the transmission lines, whereas the second one establishes the relation between the voltage and the current vector at every point along the lines. Both matrices can be measured by appropriately performed experiments. On the other hand, the properties of these matrices (e.g. symmetry, normality and/or commutativity) determine the mathematical degree of difficulty for the solution of the transmission line equations. It is well-known (from linear algebra) that real symmetric matrices and normal (in general complex) matrices can be diagonalized, and if there are real symmetric (or (complex) normal) matrices which commute with each other, they even can simultaneously (with the same set of eigenvectors) be diagonalized.

In some cases the propagation matrix takes the form of a complex number times the identity, e.g. the case of perfect transmission-line conductors embedded in a uniform medium (such as free space) [3]. In such a case, the eigenvalues are all the same and the diagonalization is not unique. Any set of N orthonormal N-component vectors will do for eigenvectors. Then $\left(\tilde{Z}_{c_{n,m}}(s)\right)$, as a measurable symmetric characteristic impedance matrix, can be used to generate an orthonormal set of eigenvectors, these also being applicable to $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$. More generally $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ need not be restricted to a complex number times the identity, but can be relaxed to the form of a matrix which commutes with $\left(\tilde{Z}_{c_{n,m}}(s)\right)$ as will be clear later.

Therefore, we are led to the following question: Assume we establish a commutation hypothesis for the propagation matrix and the characteristic impedance matrix, i.e.,

$$\underline{\text{Hypothesis}}:\quad \left(\tilde{\gamma}_{c_{n,m}}(s)\right)\cdot\left(\tilde{Z}_{c_{n,m}}(s)\right)=\left(\tilde{Z}_{c_{n,m}}(s)\right)\cdot\left(\tilde{\gamma}_{c_{n,m}}(s)\right) \tag{3.1}$$

what are then the consequences of this assumption, and which physical systems can still be described under this restriction? Note that $\left(\tilde{Z}_{c_{n,m}}(s)\right)$, $(\tilde{Z}'_{n,m}(s))$, and $(\tilde{Y}'_{n,m}(s))$ are already restricted to be symmetric (reciprocity).

It is the purpose of this section to find some consequences of the hypothesis (3.1). Examples which are an answer to the second part of the question are given in the following sections.

Our first observation is that

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(s)\right) \tag{3.2}$$

This follows from (2.9) and (3.1). Inserting (3.2) into the first relation of (2.9) gives

$$\left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right)^{-1} \tag{3.3}$$

and therefore

$$\left(\tilde{Z}_{c_{n,m}}(s)\right)^2 = \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right)^{-1} \tag{3.4}$$

Equation (3.4) represents the matrix generalization of the equivalent scalar relation for a two-conductor line.

Next we show that the per-unit-length impedance and per-unit-length admittance matrices commute. We have (see (2.6))

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right)^2 = \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) \tag{3.5}$$

On the other hand we derive with the aid of (2.9) and (3.1)

$$\begin{aligned}\left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\left(\tilde{Y}'_{n,m}(s) \cdot \tilde{Z}'_{n,m}(s)\right)\right) &= \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right) \\ &= \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^2 \end{aligned} \tag{3.6}$$

This implies the above statement (together with (3.5))

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right)^2 = \left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) = \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) \tag{3.7}$$

Using equations (3.7), (2.24), (2.25) and (3.5), it is easily seen that the propagation matrix itself is symmetric.

$$\left(\left(\tilde{\gamma}_{c_{n,m}}(s)\right)^2\right)^T = \left(\left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right)\right)^T = \left(\tilde{Y}'_{n,m}(s)\right)^T \cdot \left(\tilde{Z}'_{n,m}(s)\right)^T \tag{3.8}$$
$$= \left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) = \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) = \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^2$$

The symmetry-property of the $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ matrix will have an important impact on our further considerations.

We continue listing properties of symmetric matrices by formulating a theorem. The proof of this theorem is given in the second part of Appendix A.

<u>Theorem</u>: If $(A_{n,m})$ and $(B_{n,m})$ are two commuting $N \times N$ symmetric matrices, then the following statements hold:

(1) $(A_{n,m})^{-1}$ is symmetric. (3.9)

(2) $(A_{n,m})^{-1} \cdot (B_{n,m}) = (B_{n,m}) \cdot (A_{n,m})^{-1}$

and (3.10)

$(A_{n,m})^{-1} \cdot (B_{n,m})$ is symmetric.

(The assumptions imply that also the product $(A_{n,m}) \cdot (B_{n,m})$ is symmetric.)

(3) (This is an implication of (1) and (2))

$(A_{n,m})^{-1} \cdot (B_{n,m})^{-1} = (B_{n,m})^{-1} \cdot (A_{n,m})^{-1}$

and (3.11)

$(A_{n,m})^{-1} \cdot (B_{n,m})^{-1}$ is symmetric.

Before we apply the above theorem to our matrices of concern $\left(\text{i.e. } \left(\tilde{\gamma}_{c_{n,m}}(s)\right), \left(\tilde{Z}_{c_{n,m}}(s)\right), \left(\tilde{Z}'_{n,m}(s)\right), \left(\tilde{Y}'_{n,m}(s)\right)\right)$, we prove their mutual commutativity.

$$\text{(a)} \quad \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'_{n,m}\right) = \left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \tag{3.12}$$

$$\text{(b)} \quad \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) = \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \tag{3.13}$$

$$\text{(c)} \quad \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) \tag{3.14}$$

$$\text{(d)} \quad \left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) \tag{3.15}$$

<u>Proof</u>:

<u>To (a):</u> (Take into account equations (3.2) and (3.7)):

$$\begin{aligned} \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) &= \left(\tilde{Z}_{c_{n,m}}(s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) \\ &= \left(\tilde{Z}_{c_{n,m}}\right)^{-1} \cdot \left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) \end{aligned} \tag{3.16}$$

In (3.16) we replace $(\tilde{Z}'_{n,m}(s))$ by (applying (3.2), (2.9), (3.7))

$$\left(\tilde{Z}'_{n,m}(s)\right) = \left(\tilde{Y}'_{n,m}(s)\right)^{-1} \cdot \left(\tilde{Z}_{c_{n,m}}\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right)^{-1} \tag{3.17}$$

and observe (3.1). Then we obtain the result (3.12).

<u>To (b):</u> Statement (b) is a consequence of (3.1) and (3.2).

<u>To (c):</u> (Apply (3.2), the above theorem, and (b)):

$$\begin{aligned} \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right) &= \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(s)\right) \\ &= \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^{-1} \cdot \left(\tilde{Z}'_{n,m}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) \\ &= \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Z}'_{n,m}(s)\right) \end{aligned} \tag{3.18}$$

To (d): (Observe the above theorem):

$$\left(\tilde{Y}'_{n,m}(s)\right) \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{Y}'_{n,m}(s)\right)^T \cdot \left(\tilde{Z}_{c_{n,m}}(s)\right) = \left(\tilde{\gamma}_{c_{n,m}}(s)\right)^T$$
$$= \left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \left(\tilde{Z}_{c_{n,m}}(s)\right) \cdot \left(\tilde{Y}'_{n,m}(s)\right) \tag{3.19}$$

We summarize the important result: The set of matrices

$$\left\{\left(\tilde{\gamma}_{c_{n,m}}(s)\right), \left(\tilde{Z}_{c_{n,m}}(s)\right), \left(\tilde{Z}'_{n,m}(s)\right), \left(\tilde{Y}'_{n,m}(s)\right)\right\} \tag{3.20}$$

consists of mutual commuting symmetric matrices. Thus, the assumptions for the theorem are fulfilled, and therefore it applies to the above matrices.

We aim at a simultaneous diagonalization procedure for the set (3.20) of matrices. These matrices, however, are complex matrices. Nevertheless, we may use the theorems of matrix-theory for real symmetric matrices if we require the symmetric matrices in set (3.20) to be the product of a complex-valued function $f^{(c)}(s)$ with a real symmetric matrix $\left(A^{(r)}_{n,m}\right)$, i.e.

$$\left(A_{n,m}(s)\right) = f^{(c)}(s) \left(A^{(r)}_{n,m}\right) \tag{3.21}$$

(This is a special class of normal (complex) symmetric matrices.)

Now we use (3.20) and (3.21) - an alternate way is to choose (3.20) together with the requirement that these matrices are normal, i.e.

$$\left(A_{n,m}\right) \cdot \left(A_{n,m}\right)^{\dagger} = \left(A_{n,m}\right)^{\dagger} \cdot \left(A_{n,m}\right) \tag{3.22}$$

as our basis for further considerations. The symmetry of the real matrices implies their diagonalizability in terms of an orthogonal (real) matrix, whereas their commutativity (with each other) implies that these matrices can be diagonalized simultaneously with the same set of (real) orthonormal eigenvectors $(x_n)_\beta$. If $(A_{n,m}(s))$ is one of the matrices from set (3.20) we can represent it as a dyadic product

$$\left(A_{n,m}(s)\right) = \sum_{\beta=1}^{N} a_\beta(s) \left(x_n\right)_\beta \left(x_n\right)_\beta \tag{3.23}$$

where $a_\beta(s)$ are the (in general complex) eigenvalues of $(A_{n,m}(s))$, and $(x_n)_\beta$ are the right as well as left eigenvectors of $(A_{n,m}(s))$. Moreover, of g is some acceptable function of $(A_{n,m}(s))$, we have

$$g\left[\left(A_{n,m}(s)\right)\right] = \sum_{\beta=1}^{N} g\left(a_\beta(s)\right) \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.24)$$

In conclusion of this section we write the matrices of interest in their dyadic expansion:

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{\gamma}_\beta(s) \left(x_n\right)_\beta \left(x_n\right)_\beta = \sum_{\beta=1}^{N} \tilde{Z}_{c_\beta}^{-1}(s)\, \tilde{z}'_\beta(s) \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.25)$$

$$\left(\tilde{Z}_{c_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{z}_{c_\beta}(s) \left(x_n\right)_\beta \left(x_n\right)_\beta = \sum_{\beta=1}^{N} \left[\frac{\tilde{z}'_\beta(s)}{\tilde{y}'_\beta(s)}\right]^{1/2} \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.26)$$

$$\left(\tilde{Y}_{c_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{y}_{c_\beta}(s) \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.27)$$

$$\left(\tilde{Z}'_{n,m}(s)\right) = \sum_{\beta=1}^{N} \tilde{z}'_\beta(s) \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.28)$$

$$\left(\tilde{Y}'_{n,m}(s)\right) = \sum_{\beta=1}^{N} \tilde{y}'_\beta(s) \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.29)$$

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{\gamma}_\beta(s) \left(x_n\right)_\beta \left(x_n\right)_\beta = \sum_{\beta=1}^{N} \left[\tilde{z}'_\beta(s)\, \tilde{y}'_\beta(s)\right]^{1/2} \left(x_n\right)_\beta \left(x_n\right)_\beta \qquad (3.30)$$

As soon as we have found the similarity transformation matrix (which columns are the eigenvectors $(x_n)_\beta$ for one of our matrices), we only have to find the eigenvalues of the other matrices in order to represent them as a dyadic sum. It is sufficient to know only two sets of eigenvalues (e.g. $\tilde{\gamma}_\beta(s)$ and $\tilde{z}_{c_\beta}(s)$ or $\tilde{z}'_\beta(s)$ and $\tilde{y}'_\beta(s)$). The eigenvalues (due to (3.25) through (3.30)) of the other matrices are then simultaneously given. In special cases, however, (see Section 4) one may even express all the above eigenvalues in terms of the eigenvalues f_β of a real so-called geometrical factor matrix $(fg_{n,m})$. The $(x_n)_\beta$ are then the (real) eigenvectors of the (symmetric) matrix $(fg_{n,m})$.

In context with Section 2, we can establish the relation

$$\left(x_n\right)_\beta = \sqrt{z_{c_\beta}(s)}\,\left(\tilde{i}_n\right)_\beta = \left(\sqrt{z_{c_\beta}(s)}\right)^{-1}\left(\tilde{v}_n\right)_\beta \tag{3.31}$$

between the eigenvectors and thereby recover some of our above results.

In the remaining sections we describe systems which a priori fulfill our hypothesis (3.1) and equation (3.21).

IV. Perfectly Conducting Wires in a Lossy, Uniform and Isotropic Medium

We start our presentation of examples with a simple case. Let us assume that the per-unit-length impedance matrix and the per-unit-length admittance matrix are passive. Then they are p.r. matrices. Let us further assume that they can be written as

$$\left(\tilde{Z}'_{n,m}(s)\right) = s\left(\tilde{L}'_{n,m}(s)\right) = s\mu\left(fg_{n,m}\right) \tag{4.1}$$

$$\left(\tilde{Y}'_{n,m}(s)\right) = \left(\tilde{G}'_{n,m}(s)\right) + s\left(\tilde{C}'_{n,m}(s)\right) = \left(\sigma+s\epsilon\right)\left(fg_{n,m}\right)^{-1} \tag{4.2}$$

where

$\left(\tilde{L}'_{n,m}\right) \equiv$ per-unit-length inductance matrix

$\left(\tilde{G}'_{n,m}\right) \equiv$ per-unit-length conductivity matrix

$\left(\tilde{C}'_{n,m}\right) \equiv$ per-unit-length capacitance matrix

$\left(fg_{n,m}\right) \equiv$ dimensionless symmetric matrix of purely geometrical nature

The factors μ, σ, and ϵ characterize the surrounding medium. The elements of $(fg_{n,m})$ are positive - real and frequency independent (dispersionless). Since the matrices $(\tilde{L}'_{n,m})$ and $(\tilde{Y}'_{n,m}(s))$ commute, we now can apply the results of the foregoing section. This especially means that there exists an orthonormal real matrix (compare Appendix B) $(X_{n,m})$ which simultaneously diagonalizes $(fg_{n,m})$ and $(fg_{n,m})^{-1}$;

$$\left(X_{n,m}\right)^{-1} \cdot \left(fg_{n,m}\right) \cdot \left(X_{n,m}\right) = \begin{pmatrix} f_1 & & 0 \\ & f_2 & \\ 0 & & \ddots \; f_N \end{pmatrix} \tag{4.3}$$

$$\left(X_{n,m}\right)^{-1} \cdot \left(fg_{n,m}\right)^{-1} \cdot \left(X_{n,m}\right) = \begin{pmatrix} f_1^{-1} & & 0 \\ & f_2^{-2} & \\ 0 & & \ddots \; f_N^{-1} \end{pmatrix} \tag{4.4}$$

Here the f_β are the real, non-negative eigenvalues of $(fg_{n,m})$, i.e.

$$\left(fg_{n,m}\right) \cdot \left(x_n\right)_\beta = f_\beta \left(x_n\right)_\beta \qquad (\beta = 1,2,3,\ldots,N) \tag{4.5}$$

and the eigenvectors $(x_n)_\beta$ are the columns of the matrix $(X_{n,m})$. An eigenvalue with $f_{\beta_o} = 0$ cannot occur because otherwise $(fg_{n,m})$ would not be invertible (in contradiction to our assumption).

The eigenvalues of the propagation matrix and of the characteristic matrix are simply related to f_β:

$$\tilde{\gamma}_\beta(s) = \left[s\mu(\sigma+s\epsilon)\right]^{\frac{1}{2}} \text{ (p.r. square root)} \tag{4.6}$$

$$\tilde{z}_{c_\beta}(s) = \left[\frac{s\mu}{\sigma+s\epsilon}\right]^{\frac{1}{2}} f_\beta \quad \text{(p.r. square root)} \tag{4.7}$$

Observe that the eigenvalues of $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ are all equal. Thus, in the special case where $\sigma = 0$ we may introduce the phase velocities

$$\tilde{v}_\beta(s) \equiv s\tilde{\gamma}_\beta^{-1}(s) = \frac{1}{\sqrt{\epsilon\mu}} \equiv \tilde{v} \tag{4.8}$$

Then we find that all eigenmodes propagate at the same velocity $\tilde{v}$.

Going back to the original representation of the above matrices (with $\sigma \neq 0$) we have:

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \left[s\mu(\sigma+s\epsilon)\right]^{\frac{1}{2}} \left(1_{n,m}\right) \tag{4.9}$$

$$\left(\tilde{Z}_{c_{n,m}}(s)\right) = \left[\frac{s\mu}{\sigma+s\epsilon}\right]^{\frac{1}{2}} \left(fg_{n,m}\right) \tag{4.10}$$

Of course, due to the very simple relation of the matrices in (4.1) and (4.2), there was no urgent need for the diagonalization procedure (4.3). Equations (4.9) and (4.10) immediately follow from the definitions. Nevertheless, we invented the similarity transformation matrix $(X_{n,m})$ in order to stress the close (formal) analogy to the following sections.

V. Addition of Small Change to $(\tilde{Y}'_{n,m}(s))$

In the foregoing section we found for $\sigma = 0$ that all modes have the same speed of propagation (compare equation (4.8)). This is the case if the tube consists of N perfect conductors immersed in a uniform isotropic medium, and the $(\tilde{Z}'_{n,m}(s))$ and $(\tilde{Y}'_{n,m}(s))$ being frequency-independent symmetric real matrices times functions of the constitutive (space-independent) parameters of the medium.

In this section we describe the tube and the medium by the following per-unit-length impedance and per-unit-length admittance matrices, respectively:

$$\left(\tilde{Z}'_{n,m}(s)\right) = s\mu\left(fg_{n,m}\right)$$

$$\left(\tilde{Y}'_{n,m}(s)\right) = \left(\sigma+s\epsilon\right)\left(fg_{n,m}\right)^{-1} + \left(\tilde{Y}'_{e_{n,m}}(s)\right) \tag{5.2}$$

where the "extra" per unit-length admittance matrix $\left(\tilde{Y}'_{e_{n,m}}(s)\right)$ is constrained to be one that can be represented by

$$\left(\tilde{Y}'_{e_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{y}'_{e_\beta}(s)\ \left(x_n\right)_\beta\ \left(x_n\right)_\beta \tag{5.3}$$

(e.g. as a complicated function of $(fg_{n,m})$).

Again, the $(x_n)_\beta$ are the eigenvectors of $(fg_{n,m})$, and the functions $\tilde{y}'_{e_\beta}(s)$ are the eigenvalues of the matrix $\left(\tilde{Y}'_{e_{n,m}}(s)\right)$. Equation (5.3) guarantees that the per-unit-length impedance and admittance matrices are commutative.

Including the matrix $\left(\tilde{Y}'_{e_{n,m}}(s)\right)$ in equation (5.2) means that we allow a special class of transverse loading admittances (going beyond those being described by a $\sigma \neq 0$). This might be produced by extra permittivity and/or conductivity in an inhomogeneous and/or anisotropic form, or by the addition of lumped transverse admittance elements in sufficient number to approximate a uniform distribution along the length of the tube.

Our diagonalization procedure now leads to the following results for the eigenvalues of the propagation matrix and those of the characteristic impedance matrix, respectively:

$$\tilde{\gamma}_{\beta}(s) = \left[s\mu(\delta+s\epsilon)\right]^{\frac{1}{2}} \left(1 + (\sigma+s\epsilon)^{-1} f_{\beta}\, \tilde{y}'_{e_{\beta}}(s)\right)^{\frac{1}{2}} \tag{5.5}$$

$$\tilde{z}_{c_{\beta}}(s) = \left[\frac{s\mu}{\sigma+s\epsilon}\right]^{\frac{1}{2}} f_{\beta}\left(1 + (\sigma+s\epsilon)^{-1} f_{\beta}\, \tilde{y}'_{e_{\beta}}(s)\right)^{-\frac{1}{2}} \tag{5.6}$$

If we regard the extra loading as a small perturbation we may Taylor-expand the above eigenvalue functions. Doing this for the $\tilde{\gamma}_{\beta}(s)$-values we obtain

$$\tilde{\gamma}_{\beta}(s) = \left[s\mu(\sigma+s\epsilon)\right]^{\frac{1}{2}} \left\{1 + \frac{1}{2}(\sigma+s\epsilon)^{-1} f_{\beta}\, \tilde{y}_{e_{\beta}}(s) + O\left(\left((\sigma+s\epsilon)^{-1} f_{\beta}\, \tilde{y}'_{e_{\beta}}(s)\right)^{2}\right)\right\} \tag{5.7}$$

Now, different from (4.6), the eigenvalues $\tilde{y}'_{e_{\beta}}(s)$ prevent the degeneration of the eigenvalues of the propagation matrix.

With the aid of (5.7) we may derive a first order expression for the propagation matrix in its original representation.

$$\left(\tilde{\gamma}^{(1)}_{c_{n,m}}(s)\right) = \left[s\mu(\sigma+s\epsilon)\right]^{\frac{1}{2}} \left\{\left(1_{n,m}\right) + \frac{1}{2}(\sigma+s\epsilon)^{-1} \left(fg_{n,m}\right) \cdot \left(\tilde{Y}'_{e_{n,m}}(s)\right)\right\} \tag{5.8}$$

Of course, higher order approximations $\left(\tilde{\gamma}^{(k)}_{c_{n,m}}(s)\right)$ (the index k indicates the order of the approximation) can be obtained from higher order approximations for $\tilde{\gamma}_{\beta}(s)$.

The first order expression for the characteristic impedance matrix reads (derived from (5.6)):

$$\left(\tilde{Z}^{(1)}_{c_{n,m}}(s)\right) = \left[\frac{s\mu}{\sigma+s\epsilon}\right]^{\frac{1}{2}} \left(fg_{n,m}\right) \left\{\left(1_{n,m}\right) - \frac{1}{2}(\sigma+s\epsilon)^{-1} \left(fg_{n,m}\right) \cdot \left(\tilde{Y}'_{e_{n,m}}(s)\right)\right\} \tag{5.9}$$

Note that the approximate representations (5.8) and (5.9) are only possible after appropriate Taylor expansions for the corresponding eigenvalue functions.

A meaningful <u>direct</u> back-transformation (of the r.h.s.) of (5.5) or of (5.6) in terms of products of the per-unit-length impedance and per-unit-length admittance matrices does not exist.

VI. Lossy Wires in a Lossless Medium

In this example we describe lossy identical lines in a lossless medium. The commuting per-unit-length impedance matrix $(\tilde{Z}'_{n,m}(s))$ and per-unit-length admittance matrix $(\tilde{Y}'_{n,m}(s))$ may be given by

$$\left(\tilde{Z}'_{n,m}(s)\right) = s\mu\left(fg_{n,m}\right) + \tilde{z}'_w(s)\left(1_{n,m}\right) \qquad (6.1)$$

$$\left(\tilde{Y}'_{n,m}(s)\right) = s\epsilon\left(fg_{n,m}\right)^{-1} \qquad (6.2)$$

With the aid of the "wire function"

$$\tilde{z}'_w(s) = \tilde{r}'(s) + s\tilde{\ell}'_I(s) \text{ (resistive + inductive parts)} \qquad (6.3)$$

we have included per-unit-length "surface impedance" matrices of lossy conductors through

$$\tilde{z}'_w(s) \simeq \begin{cases} \tilde{r}' & \text{at low frequencies} \\ \sqrt{\dfrac{s\mu_w}{2\sigma_w}}\;\dfrac{1}{2\pi r_w} & \text{at high frequencies} \end{cases} \qquad (6.4)$$

where the N wires are all identical and have

$r_w \equiv$ wire radius (circular wires)

$\mu_w \equiv$ wire permeability

$\sigma_w \equiv$ wire conductivity (6.5)

If one wishes a more accurate expression for $\tilde{z}'_w(s)$ can be obtained in the form of Bessel functions which reduce to (6.4) in the limiting cases.

The matrix $\mu(fg_{n,m})$ is the per-unit-length external matrix computed as if the lines were lossless and the medium were uniform.

Of course, the ansaetze (6.1) and (6.2) together with (6.3) for the (special) losses only cover certain conductor configurations. Nevertheless, they cover two important configurations of N identical conductors with either (1) an infinite perfectly conducting ground pane or (2) a perfectly conducting shield surrounding the other N conductors.

After diagonalization of the matrices we obtain the following eigenvalues for the propagation matrix and the characteristic impedance matrix, respectively:

$$\tilde{\gamma}_\beta(s) = s\sqrt{\mu\epsilon}\left[1 + \left(s\mu f_\beta\right)^{-1} \tilde{z}_w'(s)\right]^{\frac{1}{2}} \tag{6.6}$$

$$\tilde{z}_{c_\beta}(s) = \sqrt{\frac{\mu}{\epsilon}}\, f_\beta\left[1 + \left(s\mu f_\beta\right)^{-1} \tilde{z}_w'(s)\right]^{\frac{1}{2}} = (s\epsilon)^{-1} f_\beta\, \tilde{\gamma}_\beta(s) \tag{6.7}$$

Assuming the perturbation (resistive/inductive) to be small, i.e.

$$\left|\left(s\mu f_\beta\right)^{-1} \tilde{z}_w'(s)\right| << 1 \tag{6.8}$$

for all relevant frequencies and $\beta = 1,2,\ldots,N$, we can derive a reasonable approximate expression for the propagation matrix. Observing that

$$\tilde{\gamma}_\beta(s) = s\sqrt{\mu\epsilon}\left\{1 + \frac{1}{2}\left(s\mu f_\beta\right)^{-1} \tilde{z}_w'(s) - \frac{1}{8}\left(s\mu f_\beta\right)^{-2} \tilde{z}_w'^{2}(s) + O\left[\left(\left(s\mu f_\beta\right)^{-1} \tilde{z}_w'(s)\right)^3\right]\right\} \tag{6.9}$$

we find in the original representation

$$\left(\tilde{\gamma}^{(2)}_{c_{n,m}}(s)\right) = s\sqrt{\mu\epsilon}\left\{\left(1_{n,m}\right) + \frac{1}{2}(s\mu)^{-1} \tilde{z}_w'(s) \left(fg_{n,m}\right)^{-1} - \frac{1}{8}(s\mu)^{-2} \tilde{z}_w'^{2}(s)\left[\left(fg_{n,m}\right)^{-1}\right]^2\right\} \tag{6.10}$$

From (6.9) we conclude that (in general) the phase velocities (to the extent that the concept of phase velocities makes physical sense) of the propagating eigenmodes are different from each other (due to the losses). Degeneration may occur if the configuration of the conductors fulfills certain symmetry conditions, e.g. if all conductors are equally spaced around a circle. (This will be the topic of a forthcoming paper.)

VII. Addition of Small Change to $(\tilde{Z}'_{n,m}(s))$

In this section we add a "small" contribution to the per-unit-length impedance matrix

$$\left(\tilde{Z}'_{n,m}(s)\right) = s\mu\left(fg_{n,m}\right) + \tilde{z}'_w(s)\left(1_{n,m}\right) + \left(\tilde{Z}'_{e_{n,m}}(s)\right) \tag{7.1}$$

with the "extra" per-unit-length impedance matrix is constrained as

$$\left(\tilde{Z}'_{e_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{z}'_{e_\beta}(s)\left(x_n\right)_\beta\left(x_n\right)_\beta \tag{7.2}$$

whereas the per-unit-length admittance matrix is the same as in (6.2).

This time we present the eigenvalues of the propagation matrix in two alternate forms

$$\begin{aligned}\tilde{\gamma}_\beta(s) &= s\sqrt{\epsilon\mu}\left(1 + \left(s\mu f_\beta\right)^{-1}\tilde{z}'_w(s)\right)^{\frac{1}{2}}\left(1 + \left(s\mu f_\beta + \tilde{z}'_w(s)\right)^{-1}\tilde{z}'_{e_\beta}(s)\right)^{\frac{1}{2}} \\ &= s\sqrt{\epsilon\mu}\left[1 + \left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right)\right]^{\frac{1}{2}}\end{aligned} \tag{7.3}$$

Assuming now "smallness" of the eigenvalues $\tilde{Z}'_{e_\beta}(s)$ only, we then may Taylor-expand the second factor of the first expression for $\tilde{\gamma}_\beta(s)$. However, the application of the inverse similarity transformation on the diagonalized propagation matrix

$$\left(X_{n,m}\right)\cdot\begin{pmatrix}\tilde{\gamma}_1(s) & & 0 \\ & \ddots & \\ 0 & & \gamma_n(s)\end{pmatrix}\cdot\left(X_{n,m}\right)^{-1} = \left(\tilde{\gamma}_{c_{n,m}}(s)\right) \tag{7.4}$$

does not lead to an expression of $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ in terms of matrices given in (7.1) and (6.2) due to the remaining square root in the first factor of the first-expression in (7.3). This changes if we consider the last two terms in (7.1) as small perturbations, i.e.

$$\left|\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right)\right| << 1 \tag{7.5}$$

and consequently can write down a series expansion for the second expression in (7.3) yielding (in first order, see the explanation after formula (5.8))

$$\tilde{\gamma}_{\beta}^{(1)}(s) = s\sqrt{\epsilon\mu}\left(1 + \frac{1}{2}\left(s\mu f_{\beta}\right)^{-1}\left(\tilde{z}_{w}'(s) + \tilde{z}_{e_{\beta}}'(s)\right)\right) \tag{7.6}$$

or after the inverse similarity transformation (in first order)

$$\left(\tilde{\gamma}_{c_{n,m}}^{(1)}(s)\right) = s\sqrt{\epsilon\mu}\left\{\left(1_{n,m}\right) + \frac{1}{2}\left(s\mu\right)^{-1}\left(fg_{n,m}\right)^{-1}\left(\tilde{z}_{w}'(s)\left(1_{n,m}\right) + \left(\tilde{Z}_{e_{n,m}}(s)\right)\right)\right\} \tag{7.7}$$

The advantage of (7.7) in comparison to (7.4) lies in the fact that in (7.7) we have not to know the eigenvectors constituting the similarity matrix but instead can use given matrices in their original representation.

In conclusion of this section we give an example. Choose N=2 and consider two identical wires at the same height above a lossy ground return described by the matrix

$$\left(\tilde{Z}_{e_{n,m}}(s)\right) = \tilde{z}_{g}'(s)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \tilde{z}_{g}'(s)\begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} \tag{7.8}$$

We immediately observe that only the common mode is present in (7.8) while the eigenvalue of the differential mode vanishes. The similarity transformation in this case reads

$$\left(X_{n,m}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \left(X_{n,m}\right)^{-1} \tag{7.9}$$

and via (7.4) we find

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \frac{1}{2}\begin{pmatrix}\tilde{\gamma}_1(s) + \tilde{\gamma}_2(s) & \tilde{\gamma}_1(s) - \tilde{\gamma}_2(s)\\ \tilde{\gamma}_1(s) - \tilde{\gamma}_2(s) & \tilde{\gamma}_1(s) + \tilde{\gamma}_2(s)\end{pmatrix} \tag{7.10}$$

Here we have to insert the eigenvalues $\tilde{\gamma}_1(s)$ and $\tilde{\gamma}_2(s)$ from (7.3). Note that the lossy ground only contributes to the common mode value. In case that the perturbation terms are small we also can use (7.7) with

$$\left(fg_{n,m}\right)^{-1} = \frac{1}{fg_{1,1}^{2} - fg_{1,2}^{2}}\begin{pmatrix}fg_{1,1} & -fg_{1,2}\\ -fg_{1,2} & fg_{1,1}\end{pmatrix} \tag{7.11}$$

to express the propagation matrix. Note that the two diagonal terms of $(fg_{n,m})$ are equal in this special case.

VIII. Equalization of Multiconductor Lines

The purpose of this section is twofold. On one had it generalizes and concisely summarizes the foregoing sections. Secondly it serves as a starting point for a more elaborate and deductive derivation of the so-far obtained results. Especially the equalization of the lines (i.e. the eigvenvalues of the propagation matrix are made to become a perfect square) is a major step.

The essential ingredients are the per-unit-length impedance and per-unit-length admittance matrices for which we make the following general ansaetze:

$$\left(\tilde{Z}'_{n,m}(s)\right) = s\mu\left(fg_{n,m}\right) + \tilde{z}'_{w}(s)\left(1_{n,m}\right) + \left(\tilde{Z}'_{e_{n,m}}(s)\right) \tag{8.1}$$

$$\left(\tilde{Y}'_{n,m}(s)\right) = \left(\sigma+s\epsilon\right)\left(fg_{n,m}\right)^{-1} + \left(\tilde{Y}'_{e_{n,m}}(s)\right) \tag{8.2}$$

with the usual assumption that the small losses are contained in

$$\left(\tilde{Z}'_{e_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{z}'_{e_\beta}(s)\left(x_n\right)_\beta\left(x_n\right)_\beta \tag{8.3}$$

and

$$\left(\tilde{Y}'_{e_{n,m}}(s)\right) = \sum_{\beta=1}^{N} \tilde{y}'_{e_\beta}(s)\left(x_n\right)_\beta\left(x_n\right)_\beta \tag{8.4}$$

In the first place we are again interested in the diagonal representation of the propagation matrix and of the characteristic impedance matrix. We easily find

$$\tilde{\gamma}^2_\beta(s) = (s\mu)(\sigma+s\epsilon)\left(1+\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s)+\tilde{z}'_{e_\beta}(s)\right)\right)\left(1+(\sigma+s\epsilon)^{-1} f_\beta\,\tilde{y}'_{e_\beta}(s)\right) \tag{8.5}$$

and

$$\begin{aligned}\tilde{z}^2_{c_\beta}(s) &= \left(\frac{s\mu}{\sigma+s\epsilon}\right) f^2_\beta\left(1+\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s)+\tilde{z}'_{e_\beta}(s)\right)\right)\left(1+(\sigma+s\epsilon)^{-1} f_\beta\,\tilde{y}'_{e_\beta}(s)\right)^{-1}\\ &= \left[s\mu f_\beta\left(1+\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s)+\tilde{z}'_{e_\beta}(s)\right)\right)\tilde{\gamma}^{-1}_\beta(s)\right]^2\end{aligned} \tag{8.6}$$

The last equation establishes an immediate relation between $\tilde{Z}_{c_\beta}(s)$ and $\tilde{\gamma}_\beta(s)$. Regard the distortion terms to be small compared to 1 (including the losses $\tilde{z}'_w(s)$ on the wires), i.e.

$$\left|\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right)\right| \ll 1 \;;\; \left|(\sigma+s\epsilon)^{-1} f_\beta \tilde{y}'_{e_\beta}(s)\right| \ll 1 \tag{8.7}$$

(for all relevant frequencies and $\beta = 1,2,\ldots,N$),

we perform a Taylor expansion (of lowest order) for $\tilde{\gamma}_\beta(s)$ and $\tilde{z}_{c_\beta}(s)$.

$$\tilde{\gamma}_\beta(s) = \sqrt{s\mu(\sigma+s\epsilon)}\left(1 + \frac{1}{2}\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right) + \frac{1}{2}(\sigma+s\epsilon)^{-1} f_\beta \tilde{y}'_{e_\beta}(s) + 0\left((\ldots)^2\right)\right) \tag{8.8}$$

$$\tilde{z}_{c_\beta}(s) = \sqrt{\frac{s\mu}{\sigma+s\epsilon}}\, f_\beta\left(1 + \frac{1}{2}\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right) - \frac{1}{2}(\sigma+s\epsilon)^{-1} f_\beta \tilde{y}'_{e_\beta}(s) + 0\left((\ldots)^2\right)\right) \tag{8.9}$$

The empty bracket in the "order-of-expansion-sign" has to be filled with either expression of equation (8.7).

Considering (8.6) and (8.9) it becomes evident that in the case of choosing

$$\left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right) = (\sigma+s\epsilon)^{-1} f_\beta \tilde{y}'_{e_\beta}(s) \tag{8.10}$$

we obtain for $\tilde{z}_{c_\beta}(s)$ the eigenvalues of a distortionless conductor system (compare (4.7)). At the same time $\tilde{\gamma}^2_\beta(s)$ represents a perfect square with now

$$\tilde{\gamma}_\beta(s) = \sqrt{s\mu(\sigma+s\epsilon)}\left(1 + \left(s\mu f_\beta\right)^{-1}\left(\tilde{z}'_w(s) + \tilde{z}'_{e_\beta}(s)\right)\right) \tag{8.11}$$

(p.r. square root)

The "equalization" of the two (main) factors in (8.5, 8.6), expressed by the relation (8.10), is completely analogous to the procedure performed in [7]. There the author investigates the problem of distortionless transmission lines.

By virtue of (8.10) we may write the "extra" admittance matrix as

$$\left(\tilde{Y}'_{e_{n,m}}(s)\right) = \left(\frac{\sigma+s\epsilon}{s\mu}\right) \left(fg_{n,m}\right)^{-2} \left[\tilde{z}'_w(s)\left(1_{n,m}\right) + \left(\tilde{Z}'_{e_{n,m}}(s)\right)\right] \tag{8.12}$$

In order to illustrate the physical meaning of this matrix more explicitly we reconsider our example of two identical conductors at the same height over a lossy ground return (compare Section 7). The two eigenvalues $\tilde{y}'_{e_\beta}(s)$, $\beta = 1,2$, are easily computed from (8.10) giving

$$\tilde{y}'_{e_1}(s) = \frac{(\sigma+s\epsilon)}{s\mu f_1^2} \left(\tilde{z}'_w(s) + \tilde{z}'_g(s)\right) \quad (\text{"common mode"}) \tag{8.13}$$

$$\tilde{y}'_{e_2}(s) = \frac{(\sigma+s\epsilon)}{s\mu f_2^2} \tilde{z}'_w(s) \quad (\text{"differential mode"}) \tag{8.14}$$

We may now even go one step further in our "equalization" procedure. Let us try to equalize the two eigenvalues of the propagation matrix on the basis of (8.11). Then we end up with the following requirement (cf. (8.11)):

$$\left(s\mu f_1\right)^{-1} \left(\tilde{z}'_w(s) + \tilde{z}'_g(s)\right) = \left(s\mu f_2\right)^{-1} \tilde{z}'_w(s)$$
$$\left(\text{i.e.} \quad f_1 \tilde{y}'_{e_1}(s) = f_2 \tilde{y}'_{e_2}(s)\right) \tag{8.15}$$

Since the eigenvalues of the matrix

$$\left(fg_{n,m}\right) = \begin{pmatrix} fg_{1,1} & fg_{1,2} \\ fg_{1,2} & fg_{1,1} \end{pmatrix} \tag{8.16}$$

are given by

$$f_1 = fg_{1,1} + fg_{1,2} \quad (\text{common mode}) \tag{8.17}$$

$$f_2 = fg_{1,1} - fg_{1,2} \quad (\text{differential mode}) \tag{8.18}$$

we can rewrite equation (8.15) (resolving w.r.t. $\tilde{z}_g(s)$) as

$$\tilde{z}'_g(s) = \left(\frac{f_1}{f_2} - 1\right) \tilde{z}'_w(s) = \frac{2\kappa}{1-\kappa} \tilde{z}'_w(s) \tag{8.19}$$

where we introduced the quantity κ as ratio

$$\kappa \equiv \frac{fg_{1,2}}{fg_{1,1}} \tag{8.20}$$

Note that f_1 (common mode) $\geq f_2$ (differential mode), and therefore $\tilde{z}_g'(s) \geq 0$ if $\tilde{z}_w'(s) > 0$. Note also that the common-mode impedance is of the form

$$\tilde{Z}_{\text{common}}(s) \equiv \frac{\tilde{V}_1\ (z,s)}{2\tilde{I}_1(z,s)} = \left[\frac{s\mu}{\sigma+s\epsilon}\right]^{\frac{1}{2}} \frac{f_1}{2} \tag{8.21}$$

Since this is normally defined with the sum of the wire currents. The differential-mode impedance is of the form

$$\tilde{Z}_{\text{differential}}(s) \equiv \frac{2\tilde{V}_1(z,s)}{\tilde{I}_1(z,s)} = \left[\frac{s\mu}{\sigma+s\epsilon}\right]^{\frac{1}{2}} 2f_2 \tag{8.22}$$

Since the differential voltage is the difference of the wire voltages. Thus, consistent with the above one may still often have

$$\underset{\text{(real)}}{\left(\tilde{Z}_{\text{common}}(s)\Big/\tilde{Z}_{\text{differential}}(s)\right)} = \frac{1}{4}\left(\frac{f_1}{f_2}\right) < 1 \tag{8.23}$$

and this is the case in typical shielded pairs.

Due to (6.4) $\tilde{z}_w'(s)$ is frequency dependent. Therefore, two cases are to be considered:

(1) In the high frequency domain we have besides (6.4)

$$\tilde{z}_g'(s) = \text{const.} \sqrt{\frac{s\mu_g}{2\sigma_g}} \tag{8.24}$$

and the constant turns out to be (cf. (8.19))

$$\text{const.} = \left(\frac{f_1}{f_2} - 1\right) \frac{1}{2\pi r_w} \sqrt{\frac{\mu_w \sigma_g}{\mu_g \sigma_w}} \tag{8.25}$$

(2) For low frequencies we set (by choice of "ground" materials)

$$\tilde{z}_g'(s) \equiv \tilde{r}_g' = \tilde{r}'\left(\frac{f_1}{f_2} - 1\right) \geq 0 \tag{8.26}$$

The 2×2 propagation matrix is obtained with the aid of the two eigenvectors

$$\left(x_n\right)_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \left(x_n\right)_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{8.27}$$

and is found to be

$$\left(\tilde{\gamma}_{c_{n,m}}(s)\right) = \tilde{\gamma}(s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.28}$$

where we used the definition

$$\tilde{\gamma}_1(s) = \tilde{\gamma}_c(s) \equiv \tilde{\gamma}(s) \tag{8.29}$$

Our result (8.28) is very important. Via our "equalization" steps (8.10) and (8.15) we finally arrived at the result that for both modes we have the same attenuation and the same velocity, and no dispersion occurring. However, we emphasize that the above "equalization" procedure only works in our example for N=2. In case of C_N symmetry and N=3 we also may equalize the eigenvalues of the propagation matrix due to the fact that the two "differential" modes are degenerated. This and more, however, will become the subject of a forthcoming paper.

Another simple configuration, covered by the above consideration for $f_1 = f_2$, is worth to be mentioned. If we deal with two identical coaxial cables for which the matrix $(fg_{n,m})$ "degenerates" to

$$\left(fg_{n,m}\right) = f_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8.30}$$

we also find equal eigenvalues for $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$, i.e. equal damping and equal speed for the propagating modes.

IX. Resonances on a Single Terminated Tube

In this section we want to show how the former results can be extended and used to calculate the natural frequencies of transmission lines with given (passive) terminations.

Inspecting our compact solution (2.14) we assume that its resonance behavior is inherent in the summands $(\tilde{V}_n(0,s))_+$ and $(\tilde{V}_n(L,s))_-$ since these vectors contain poles in the complex frequency plane. In the following we express these two vectors in such a way that the above statement immediately becomes obvious. This can be done in terms of the propagation matrix $\left(\tilde{\gamma}_{c_{n,m}}(s)\right)$ and the reflection matrices at $z = 0,L$ which are defined in terms of passive terminating impedance matrices $(\tilde{Z}_T(z_o,s))$ and the characteristic impedance matrix as

$$\left(\tilde{S}_{n,m}(z_o,s)\right) \equiv \left[\left(\tilde{Z}_{T_{n,m}}(z_o,s)\right) + \left(\tilde{Z}_{c_{n,m}}(s)\right)\right]^{-1} \cdot \left[\left(\tilde{Z}_{T_{n,m}}(z_o,s)\right) - \left(\tilde{Z}_{c_{n,m}}(s)\right)\right]$$

$$(\text{with } z_o = 0 \text{ and } z_o = L) \tag{9.1}$$

Then we find the scattering relations for both ends of the tube as

$$\left(\tilde{V}_n(0,s)\right)_+ = \left(\tilde{S}_{n,m}(0,s)\right) \cdot \left(\tilde{V}_n(0,s)\right)_- \tag{9.2}$$

$$\left(\tilde{V}_n(L,s)\right)_- = \left(\tilde{S}_{n,m}(L,s)\right) \cdot \left(\tilde{V}_n(L,s)\right)_+ \tag{9.3}$$

and thus have two equations for the four unknowns $\left(\tilde{V}_n(0,s)\right)_+$, $\left(\tilde{V}_n(0,s)\right)_-$, $\left(\tilde{V}_n(L,s)\right)_+$, and $\left(\tilde{V}_n(L,s)\right)_-$. By application of (2.12) through (2.14) we establish two other relations between the above quantities:

$$\left(\tilde{V}_n(L,s)\right)_+ = \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \cdot \left(\tilde{V}_n(0,s)\right)_+ + \int_0^L \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)(L-z')\right\} \cdot \left(\tilde{V}_n^{(s)'}(z',s)\right)_+ dz' \tag{9.4}$$

$$\left(\tilde{V}_n(0,s)\right)_- = \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \bullet \left(\tilde{V}_n(L,s)\right)_-$$

$$+ \int_L^0 \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)z'\right\} \bullet \left(\tilde{V}_n^{(s)'}(z',s)\right)_- dz' \qquad (9.5)$$

Equations (9.2) through (9.5) can easily be written in supermatrix form

$$\begin{pmatrix} \left(1_{n,m}\right) & \left(0_{n,m}\right) & -\left(\bar{S}_{n,m}(0,s)\right) & \left(0_{n,m}\right) \\ -\exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} & \left(1_{n,m}\right) & \left(0_{n,m}\right) & \left(0_{n,m}\right) \\ \left(0_{n,m}\right) & \left(0_{n,m}\right) & \left(1_{n,m}\right) & -\exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ \left(0_{n,m}\right) & -\left(\bar{S}_{n,m}(L,s)\right) & \left(0_{n,m}\right) & \left(1_{n,m}\right) \end{pmatrix} \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right)_+ \\ \left(\tilde{V}_n(L,s)\right)_+ \\ \left(\tilde{V}_n(0,s)\right)_- \\ \left(\tilde{V}_n(L,s)\right)_- \end{pmatrix}$$

$$= \begin{pmatrix} \left(0_n\right) \\ \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \bullet \int_0^L dz' \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)(L-z')\right\} \bullet \left(\tilde{V}_n^{(s)'}(z',s)\right)_+ \\ \int_L^0 dz' \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)z'\right\} \bullet \left(\tilde{V}_n^{(s)'}(z',s)\right)_- \\ \left(0_n\right) \end{pmatrix} \qquad (9.6)$$

The elements of the supermatrix are N × N matrices, and the elements of the supervectors are N-dimensional vectors. The description how to solve (9.6) w.r.t. the left-hand side supervector can be found in [2]. Before we, however, use these methods we reduce the 4 × 4 supermatrix to a 2 × 2 supermatrix by inserting (9.4) and (9.5) into (9.2) and (9.3). We then obtain (observe also (2.15) and (2.16))

$$\begin{pmatrix} \left(1_{n,m}\right) & -\left(\tilde{S}_{n,m}(0,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ -\left(\tilde{S}_{n,m}(L,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} & \left(1_{n,m}\right) \end{pmatrix} \odot \begin{pmatrix} \left(\tilde{V}_n(0,s)\right)_+ \\ \left(\tilde{V}_n(L,s)\right)_- \end{pmatrix}$$

$$= \begin{pmatrix} \left(\tilde{S}_{n,m}(0,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \cdot \left(\tilde{V}_n^{(s)}(0,s)\right)_- \\ \left(\tilde{S}_{n,m}(L,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \cdot \left(\tilde{V}_n^{(s)}(L,s)\right)_+ \end{pmatrix} \tag{9.7}$$

This equation system is easily resolved (compare e.g. [2]) with respect to the unknown vectors by applying the inverse of the 2 × 2 supermatrix. A brief calculation yields

$$\begin{aligned} \left(\tilde{V}_n(0,s)\right)_+ = \Big[&\left(1_{n,m}\right) - \left(\tilde{S}_{n,m}(0,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ &\cdot \left(\tilde{S}_{n,m}(L,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\}\Big]^{-1} \\ &\circ \left(\tilde{S}_{n,m}(0,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ &\circ \Big[\left(\tilde{S}_{n,m}(L,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ &\circ \left(\tilde{V}_n^{(s)}(L,s)\right)_+ + \left(\tilde{V}_n^{(s)}(0,s)\right)_-\Big] \end{aligned} \tag{9.8}$$

and

$$\begin{aligned} \left(\tilde{V}_n(L,s)\right)_- = \Big[&\left(1_{n,m}\right) - \left(\tilde{S}_{n,m}(L,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ &\circ \left(\tilde{S}_{n,m}(0,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\}\Big]^{-1} \\ &\circ \left(\tilde{S}_{n,m}(L,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ &\circ \Big[\left(\tilde{S}_{n,m}(0,s)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s)\right)L\right\} \\ &\circ \left(\tilde{V}_n^{(s)}(0,s)\right)_- + \left(\tilde{V}_n^{(s)}(L,s)\right)_+\Big] \end{aligned} \tag{9.9}$$

Insert the solutions (9.8, 9.9) into (9.4, 9.5) and obtain the last two vectors $(\tilde{V}_n(L,s))_+$ and $(\tilde{V}_n(0,s))_-$.

Equations (9.8) and (9.9) represent our desired results, and we make the following observations: (1) Both vectors contain an (inverse) matrix which might become singular for certain frequency values. (2) Since, however, singularity of one vector implies singularity of the other (compare (9.7)) (supposing the reflection matrices are non-singular for those cases) the possible poles for both vectors are the same. (3) The singularity condition reads (cf. [1]):

$$\det\left[\left(1_{n,m}\right) - \left(\tilde{S}_{n,m}(L,s_\alpha)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s_\alpha)\right)L\right\} \cdot \left(\tilde{S}_{n,m}(0,s_\alpha)\right) \cdot \exp\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s_\alpha)\right)L\right\}\right] = 0 \qquad (9.10)$$

One result becomes immediately obvious from (9.10): If one of the reflection matrices vanishes (i.e. if one end of the tube is terminated with its characteristic impedance), we have a perfectly matched transmission line which does not show any resonance behavior. In other words expression (9.10) has no zeros.

Let us now investigate the resonances of a tube which ends are either open ended, i.e.

$$\left(\tilde{S}_{n,m}(0,s_\alpha)\right)_o = \left(\tilde{S}_{n,m}(L,s_\alpha)\right)_o = \left(1_{n,m}\right) \qquad (9.11)$$

or short circuited, i.e.

$$\left(\tilde{S}_{n,m}(0,s_\alpha)\right)_{sc} = \left(\tilde{S}_{n,m}(L,s_\alpha)\right)_{sc} = -\left(1_{n,m}\right) \qquad (9.12)$$

In both cases (8.10) simplifies to

$$\det\left[\left(1_{n,m}\right) - \exp^2\left\{-\left(\tilde{\gamma}_{c_{n,m}}(s_\alpha)\right)L\right\}\right] = 0 \qquad (9.13)$$

We transform the matrix inside the square bracket of (9.13) into its diagonal form. Then we can write instead of (9.13)

$$\prod_{\beta=1}^{N}\left(1 - e^{-2\tilde{\gamma}_\beta(s_\alpha)L}\right) = 0 \qquad (9.14)$$

Thus only one factor in this product chain needs to vanish. We indicate this factor with a subscript zero and have

$$e^{2\tilde{\gamma}_{\beta_o}(s_\alpha)L} = 1 \tag{9.15}$$

With the aid of the natural logarithm equation (9.15) can be resolved with respect to $\tilde{\gamma}_{\beta_o}(s_\alpha)$ giving

$$\tilde{\gamma}_{\beta_o}(s_\alpha) = \frac{\pi m}{L} j \quad (m = \pm 1, \pm 2, \ldots) \tag{9.16}$$

as an implicit equation for $s_\alpha \equiv \Omega_\alpha + j\omega_\alpha$.

Now we will use equation (9.16) to compute the resonance frequencies (natural modes) in first order perturbation (see the foregoing sections) for three different situations.

A. Resonance Frequencies in a Special Lossy Medium

We refer to Section 5 (5.7) and obtain (with $\sigma = o$)

$$\left(\frac{m\pi}{L}\right)j = s^{(1)}_{\alpha_{\beta_o}} \sqrt{\epsilon\mu} \left\{1 + \frac{1}{2}\left(s^{(1)}_{\alpha_{\beta_o}} \epsilon\right)^{-1} f_{\beta_o} \tilde{y}'_{e_{\beta_o}}\right\} \tag{9.17}$$

Assuming that $\tilde{y}'_{e_{\beta_o}}$ does not depend on s, and setting

$$c \text{ (speed of light)} \equiv (\epsilon\mu)^{-1/2} \tag{9.18}$$

$$s^{(1)}_{\alpha_{\beta_o}} \equiv s^{(o)}_{\alpha_{\beta_o}} + \Delta s^{(1)}_{\alpha_{\beta_o}} = \Omega^{(o)}_{\alpha_{\beta_o}} + j\omega^{(o)}_{\alpha_{\beta_o}} + \Delta\Omega^{(1)}_{\alpha_{\beta_o}} + j\Delta\omega^{(1)}_{\alpha_{\beta_o}} \tag{9.19}$$

we get

$$\frac{\pi m}{L} j = \frac{1}{c}\left(\Omega^{(o)}_{\alpha_{\beta_o}} + \Delta\Omega^{(1)}_{\alpha_{\beta_o}} + j\left(\omega^{(o)}_{\alpha_{\beta_o}} + \Delta\omega^{(1)}_{\alpha_{\beta_o}}\right)\right) + \frac{1}{2} c\mu f_{\beta_o} \tilde{y}'_{e_{\beta_o}} \tag{9.20}$$

which in turn results in

$$\Omega^{(o)}_{\alpha_{\beta_o}} = 0$$

$$\Delta\Omega^{(1)}_{\alpha_{\beta_o}} = \frac{1}{2} c^2 \mu f_{\beta_o} \tilde{y}'_{e_{\beta_o}} \tag{9.21}$$

and

$$\omega^{(o)}_{\alpha_{\beta_o}} = \frac{m\pi c}{L} \quad (m = 1,2,3,\ldots)$$

$$\Delta\omega^{(1)}_{\alpha_{\beta_o}} = 0 \qquad (9.22)$$

$$(\beta_o \in \{1,2,\ldots,N\})$$

Observe that in this approximation we do not have frequency splitting. However, the damping is mode-dependent unless $f_{\beta_o} \tilde{y}'_{e_{\beta_o}}$ is independent of β.

B. Lossy Lines with Internal Inductance

A first order expansion for $\tilde{\gamma}_{\beta_o}(s_\alpha)$ following from (6.6) leads to the equation

$$\left(\frac{m\pi j}{L}\right) = s^{(1)}_{\alpha_{\beta_o}} \sqrt{\epsilon\mu} \left(1 + \frac{1}{2}\left(s^{(1)}_{\alpha_{\beta_o}} \mu f_{\beta_o}\right)^{-1} \tilde{z}'_{\omega}(s)\right) \qquad (9.23)$$

If we assume the losses to be caused by a frequency-independent internal per-unit-length inductance, i.e. $\tilde{z}'_{w}(s) = s^{(1)}_{\alpha_{\beta_o}} \tilde{\ell}'_{I}$, we obtain

$$\Omega^{(o)}_{\alpha_{\beta_o}} = 0 \quad , \quad \Delta\Omega^{(1)}_{\alpha_{\beta_o}} = 0 \qquad (9.24)$$

$$\omega^{(o)}_{\alpha_{\beta_o}} = \left(\frac{m\pi c}{L}\right) \ , \quad \Delta\omega^{(1)}_{\alpha_{\beta_o}} = \frac{1}{2}\,\frac{m\pi c}{L}\left(\mu f_{\beta_o}\right)^{-1} \tilde{\ell}'_{I} \qquad (9.25)$$

This time we found no attenuation in the first order approximation for all the modes, whereas the internal inductance gives rise to (mode-dependent) frequency splitting. With "frequency splitting" we refer to mode-dependent imaginary parts of the complex frequency s.

C. Lossy Lines with Resistance

In this example we assume $\tilde{z}'_{w}(s) \equiv \tilde{r}'$ and $\tilde{r}'$ to be independent of s. Again, because of the skin effect, this can only be an approximate value.

The following equation (derived from (6.6)) has to be resolved with respect to $s^{(1)}_{\alpha_{\beta_o}}$:

$$\frac{\pi m}{L} j = s^{(1)}_{\alpha_{\beta_o}} \sqrt{\epsilon\mu} \left(1 + \frac{1}{2} \frac{\tilde{r}'}{\mu f_{\beta_o} s^{(1)}_{\alpha_{\beta_o}}}\right)$$

$$= \frac{1}{c} \left(\Omega^{(o)}_{\alpha_{\beta_o}} + \Delta\Omega^{(1)}_{\alpha_{\beta_o}} + j\left(\omega^{(o)}_{\alpha_{\beta_o}} + \Delta\omega^{(1)}_{\alpha_{\beta_o}}\right)\right) + \frac{1}{2c} \frac{\tilde{r}'}{\mu f_{\beta_o}} \tag{9.26}$$

We find mode-dependent damping, but no frequency splitting:

$$\Omega^{(o)}_{\alpha_{\beta_o}} = 0 , \quad \Delta\Omega^{(1)}_{\alpha_{\beta_o}} = -\frac{1}{2} \frac{\tilde{r}'}{\mu f_{\beta_o}} \tag{9.27}$$

$$\omega^{(o)}_{\alpha_{\beta_o}} = \frac{m\pi c}{L} , \quad \Delta\omega^{(1)}_{\alpha_{\beta_o}} = 0 \tag{9.28}$$

The above considerations - even if they are somewhat artificial - show that frequency splitting is mainly caused by the internal inductance of the conductors (→ first order effect!). Other losses, like special losses in the medium or resistances of the lines, "only" contribute in higher order (second order and higher) to the frequency splitting. There is, however, (among others) another mechanism, space variation between the conductors, which causes a comparable frequency splitting [4] as the internal inductances.

X. Discussion and Concluding Remarks

In this paper we dealt with a certain class of solutions (in the frequency domain) of the MTL equations. These solutions were obtained on the basis of an eigenmode expansion for a commuting set of symmetric p.r. matrices which could be simultaneously diagonalized by only one set of orthonormal eigenvectors. In the basis of these eigenvectors the MTL equations completely decouple and scalarize, each scalar equation describing the propagation of one (voltage- or current-) eigenmode. The solution for every eigenmode (say for the voltage vector) $(\tilde{v}_\beta(z,s))$, $\beta = 1,2,\ldots,N$, can easily be represented as a sum of forward and backward running waves.

$$2\tilde{v}_\beta(z,s) = e^{-\tilde{\gamma}_\beta(s)z}\left(\tilde{v}_{+\beta}(L,s) + \tilde{v}^{(s)}_{+\beta}(z,s)\right) + e^{\tilde{\gamma}_\beta(s)(z-L)}\left(\tilde{v}_{-\beta}(L,s) + \tilde{v}^{(s)}_{-\beta}(z,s)\right) \tag{10.1}$$

Here, the mode vectors are indicated with a lower case letter, and they are defined via the equations

$$\tilde{v}_\beta(z,s) \equiv \left(x_n\right)_\beta \cdot \left(\tilde{V}_n(z,s)\right) \tag{10.2}$$

$$\tilde{v}_{q\beta}(z_o,s) \equiv \left(x_n\right)_\beta \cdot \left(\tilde{V}_n(z_o,s)\right)_q \tag{10.3}$$

$$\left(z_o = 0 \text{ and } q = 1 \ ; \ z_o = L \text{ and } q = -1\right)$$

$$\tilde{v}^{(s)}_{q\beta}(z,s) \equiv \left(x_n\right)_\beta \cdot \left(\tilde{V}^{(s)}_n(z,s)\right)_q \tag{10.4}$$

$$q = \pm 1$$

Equation (10.1) indicates the need of knowing the eigenvalues $\tilde{\gamma}_\beta(s)$ of the propagation matrix (of course, besides the knowledge of the boundary conditions and the source terms). We investigated various multiconductor lines which are embedded in different media. Our main emphasis laid on the discussion of the different properties of $\tilde{\gamma}_\beta(s)$

$$\tilde{\gamma}_\beta(s) = \left(\tilde{z}'_\beta(s)\ \tilde{y}'_\beta(s)\right)^{1/2} \tag{10.5}$$

for lossless as well as lossy lines and lossy media. Depending on the relation between the eigenvalues of the per-unit-length impedance matrix and those of the per-unit-length admittance matrix and on the degree of degeneration of their eigenvalues one may find entirely different values for the eigenvalues of the propagation matrix, ranging from a complete degeneracy of (10.5), i.e. all modes propagate without attenuation with the same phase velocity, over such modes which all have the same damping but distinct phase velocities (or vice versa) up to modes which all have different, real and imaginary parts. One may even study various multiconductor lines under the aspect to meet certain given sets of eigenvalues of the propagation matrix (see e.g. the discussion about "distortionless" transmission lines in [7]).

Another major advantage of knowing the $\tilde{\gamma}_\beta(s)$ values lies in their use in context with the search of poles of solution (2.14). These poles in the complex frequency plane represent the resonances of the MTL.

Of course, we are aware of the fact that our restrictions which we imposed on the MTLs are severe, and therefore many interesting cases for MTL-configurations and environments are excluded. But, nevertheless, the concept of dealing with eigenmode expansions for MTLs is very fundamental, and, in addition, even with our limiting assumptions we still cover the most common MTL configurations. More fundamentally, in the synthesis of MTLs to have certain desirable properties for a given application, the restrictions allow one to analytically associate some of the MTL properties with physically controllable parameters.

Appendix A. Commuting Real Symmetric Matrices

Let $(A_{n,m})$ be a given real symmetric matrix, i.e., $(A_{n,m})^T = (A_{n,m})$, and find all matrices $(B_{n,m})$ that commute with $(A_{n,m})$:

$$\left(A_{n,m}\right) \cdot \left(B_{n,m}\right) = \left(B_{n,m}\right) \cdot \left(A_{n,m}\right) \tag{A.1}$$

Due to well-known theorems of matrix-theory (e.g. [6]) we know that $(A_{n,m})$ (since it is a special case of a hermitian matrix) can be diagonalized with the aid of a unitary matrix $(X_{n,m})$ (even with a real orthogonal matrix), i.e.

$$\left(A^{(d)}_{n,m}\right) = \left(X_{n,m}\right)^{-1} \cdot \left(A_{n,m}\right) \left(X_{n,m}\right) \tag{A.2}$$

Here $(A^{(d)}_{n,m})$ denotes the diagonal matrix (similar to $(A_{n,m})$), and the matrix $(X_{n,m})$ has the property

$$\left(X_{n,m}\right)^{-1} = \left(X_{n,m}\right)^{\dagger} \tag{A.3}$$

$$\dagger \equiv T^{*} \equiv \text{adjoint}$$

The matrix $(X_{n,m})$ has columns as the eigenvectors $(x_n)_\beta$, i.e.

$$\left(X_{n,m}\right) = \left(\left(x_n\right)_1 , \left(x_n\right)_2 , \ldots, \left(x_n\right)_N\right) \tag{A.4}$$

the transposed matrix $(X_{n,m})^T$ then reads

$$\left(X_{n,m}\right)^T = \left(\left(x_n\right)_1 , \left(x_n\right)_2 , \ldots, \left(x_n\right)_N\right)^T \tag{A.5}$$

where the eigenvectors are now the rows. All matrices that commute with $(A_{n,m})$ are given in the following form (compare [6 (Chapter VIII, par. 2)]

$$\left(B_{n,m}\right) = \left(X_{n,m}\right)\left(B^{(d)}_{n,m}\right) \cdot \left(X_{n,m}\right)^{-1} \tag{A.6}$$

where $(B^{(d)}_{n,m})$ denotes an arbitrary matrix which commutes with $(A^{(d)}_{n,m})$. Since $(A^{(d)}_{n,m})$, however, is diagonal $(B^{(d)}_{n,m})$ is an arbitrary diagonal matrix, too.

This means that every matrix that commutes with $(A_{n,m})$ can be simultaneously diagonalized with $(A_{n,m})$. A more general version of this result reads as follows: Any finite or infinite set of pairwise commuting real symmetric matrices $\{(A_{n,m}), (B_{n,m}), (C_{n,m}), \ldots\}$ can be transformed into diagonal form by one and the same unitary transformation $(X_{n,m})$. Then the above matrices have a complete set of common orthonormal eigenvectors $(x_n)_\beta$. These eigenvectors constitute the matrix $(X_{n,m})$ (see equations A.4, A.5).

In the second part of this appendix we prove the theorem formulated in Section 3 (compare formulae (3.9) through (3.11)).

<u>Theorem</u>: If $(A_{n,m})$ and $(B_{n,m})$ are two commuting $N \times N$ symmetric matrices, then the following statements hold:

(1) $(A_{n,m})^{-1}$ is symmetric (A.7)

(2) $(A_{n,m})^{-1} \cdot (B_{n,m}) = (B_{n,m}) \cdot (A_{n,m})^{-1}$

and (A.8)

$(A_{n,m})^{-1} \cdot (B_{n,m})$ is symmetric

(3) $(A_{n,m})^{-1} (B_{n,m})^{-1} = (B_{n,m})^{-1} \cdot (A_{n,m})^{-1}$

and (A.9)

$(A_{n,m})^{-1} \cdot (B_{n,m})^{-1}$ is symmetric.

<u>Proof</u>:

<u>To (1)</u>: We have the trivial relation

$$(A_{n,m}) \cdot (A_{n,m})^{-1} = (A_{n,m})^{-1} \cdot (A_{n,m}) = (1_{n,m}) \qquad \text{(A.10)}$$

Transpose both sides of this equation and get

$$\left((A_{n,m}) \cdot (A_{n,m})^{-1}\right)^T = \left((A_{n,m})^{-1}\right)^T \cdot (A_{n,m}) \tag{A.11}$$

$$= (A_{n,m}) \cdot \left((A_{n,m})^{-1}\right)^T = (1_{n,m})$$

After multiplication with $(A_{n,m})^{-1}$ from left or right we obtain

$$\left((A_{n,m})^{-1}\right)^T = (A_{n,m})^{-1} \tag{A.12}$$

<u>To (2)</u>: We have

$$\left((A_{n,m}) \cdot (B_{n,m})\right)^{-1} = (B_{n,m})^{-1} \cdot (A_{n,m})^{-1} = (A_{n,m})^{-1} \cdot (B_{n,m})^{-1} \tag{A.13}$$

Multiply with $(B_{n,m})$ from the right

$$(B_{n,m})^{-1} \cdot (A_{n,m})^{-1} \cdot (B_{n,m}) = (A_{n,m})^{-1} \tag{A.14}$$

and this equation with $(B_{n,m})$ from the left

$$(A_{n,m})^{-1} \cdot (B_{n,m}) = (B_{n,m}) \cdot (A_{n,m})^{-1} \tag{A.15}$$

To show the symmetry of this product consider

$$\left((A_{n,m})^{-1} \cdot (B_{n,m})\right)^T = (B_{n,m}) \cdot (A_{n,m})^{-1} \tag{A.16}$$

$$= (A_{n,m})^{-1} \cdot (B_{n,m})$$

<u>To (3)</u>: The commutativity follows from

$$(A_{n,m})^{-1} \cdot (B_{n,m})^{-1} = \left((B_{n,m}) \cdot (A_{n,m})\right)^{-1} = \left((A_{n,m}) \cdot (B_{n,m})\right)^{-1}$$

$$= (B_{n,m})^{-1} \cdot (A_{n,m})^{-1} \tag{A.17}$$

whereas the symmetry of the above product is a trivial consequence of the subsequent relation

$$\left((A_{n,m})^{-1} \cdot (B_{n,m})^{-1}\right)^T = \left((B_{n,m})^{-1}\right)^T \cdot \left((A_{n,m})^{-1}\right)^T$$

$$= (B_{n,m})^{-1} \cdot (A_{n,m})^{-1} \qquad \text{(A.18)}$$

$$= (A_{n,m})^{-1} \cdot (B_{n,m})^{-1}$$

This completes our proof.

Appendix B. The Canonical Diagonalization of a Real Symmetric Matrix

Consider a non-singular positive-definite $N \times N$ real symmetric square matrix $(fg_{n,m})$ with eigenvalues f_β, and eigenvectors $(x_n)_\beta$ (right and left eigenvectors coincide) defined by

$$\left(fg_{n,m}\right)\left(x_n\right)_\beta = f_\beta\left(x_n\right)_\beta \tag{B.1}$$

with the orthonormalization condition

$$\left(x_n\right)_{\beta_1}\left(x_n\right)_{\beta_2} = 1_{\beta_1\,\beta_2} = \begin{cases} 1 & \text{for } \beta_1 = \beta_2 \\ 0 & \text{otherwise} \end{cases} \tag{B.2}$$

$$(\beta_1,\ \beta_2 = 1,2,\ldots,N)$$

Such matrices are very common in electromagnetics. They occur in the low-frequency limit of many field problems.

Since $(fg_{n,m})$ is real and symmetric it is orthogonally similar to a real diagonal matrix $(f^{(d)}_{n,m})$ (see, e.g., ref. [6 (Chapter IX, par. 13)], i.e., there exists a real orthogonal (even orthonormal) matrix $X_{n,m})$ such that

$$\left(f^{(d)}_{n,m}\right) = \left(X_{n,m}\right)^{-1}\cdot\left(fg_{n,m}\right)\cdot\left(X_{n,m}\right)$$

$$\left(fg^{(d)}_{n,m}\right) = \begin{pmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_N \end{pmatrix};\ \left(X_{n,m}\right)\cdot\left(X_{n,m}\right)^T = \left(1_{n,m}\right) \tag{B.3}$$

The columns of the matrix $(X_{n,m})$ are the eigenvectors $(x_n)_\beta$ of equation (B.1).

We need to briefly study the case where we have to deal with degenerate eigenvalues. We know that all eigenvalues of $(fg_{n,m})$ are real and positive. Moreover, the characteristic equation of a real matrix has real coefficients, so that with a root f_{β_o} of multiplicity p it also has the root $f^*_{\beta_o} = f_{\beta_o}$ of the same multiplicity. Thus, for possibly complex, linearly independent eigenvectors $(x^{(c)}_n)_1, \ldots, (x^{(c)}_n)_p$ there correspond the linearly independent eigenvectors $(x^{(c)}_n)^*_1, \ldots, (x^{(c)}_n)^*_p$ (corresponding to the eigenvalue $f^*_{\beta_o}(=f_{\beta_o})$).

Then from

$$\left(fg_{n,m}\right)\left(x_n^{(c)}\right)_{p_o} = f_{\beta_o}\left(x_n^{(c)}\right)_{p_o} \tag{B.4}$$

$$(p_o \in \{1,\ldots,p\})$$

it follows that

$$\left(fg_{n,m}\right)\left(x_n^{(c)}\right)^*_{p_o} = f_{\beta_o}\left(x_n^{(c)}\right)^*_{p_o} \tag{B.5}$$

Now we may go over from the complex basis $\left(\left(x_n^{(c)}\right)_{p_o}, \left(x_n^{(c)}\right)^*_{p_o}\right)$ to the corresponding real basis

$$\begin{aligned} \mathrm{Re}\left(\left(x_n^{(c)}\right)_{p_o}\right) &= \frac{1}{2}\left(\left(x_n^{(c)}\right)_{p_o} + \left(x_n^{(c)}\right)^*\right) \\ \mathrm{Im}\left(\left(x_n^{(c)}\right)_{p_o}\right) &= \frac{1}{2j}\left(\left(x_n^{(c)}\right)_{p_o} - \left(x_n^{(c)}\right)^*\right) \end{aligned} \tag{B.6}$$

It is easy to see that

$$\begin{aligned} \left(fg_{n,m}\right)\cdot \mathrm{Re}\left(\left(x_n^{(c)}\right)_{p_o}\right) &= f_{\beta_o}\ \mathrm{Re}\left(\left(x_n^{(c)}\right)_{p_o}\right) \\ \left(fg_{n,m}\right)\circ \mathrm{Im}\left(\left(x_n^{(c)}\right)_{p_o}\right) &= f_{\beta_o}\ \mathrm{Im}\left(\left(x_n^{(c)}\right)_{p_o}\right) \end{aligned} \tag{B.7}$$

In other words, in the case of degeneracy, we can construct an equivalent set of real eigenvectors from the set of complex eigenvectors taking the real- and imaginary part of the complex eigenvectors, respectively. A possibly necessary orthonormalization of this set of real (linearly independent) eigenvectors is performed by a subsequent Gram-Schmidt procedure.

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