Interaction Notes

Note 475

31 May 1989

Scattering, Reciprocity, Symmetry, EEM, and SEM

Carl E. Baum

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Abstract

This paper takes the reciprocity relation between incident and scattered fields and applies it to the EEM and SEM terms for the scattered fields. Besides the symmetry relations due to reciprocity there is found a simple relation relating the backscattering coupling coefficients. Including a symmetry plane in the scatterer and having directions of incidence and scattering near the symmetry plane, and having suitable polarizations allows one to separate the scatterer natural frequencies into two sets.

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I. Introduction

In the theory of electromagnetic scattering, reciprocity plays an important role [11, 12]. As long as the scatterer is perfectly conducting or is comprised of reciprocal media (media characterized by symmetrical constitutive parameter matrices) there is scattering reciprocity. In this context, reciprocity means the equivalence between incident and scattered waves, i.e., a wave incident at some direction and polarization on a scatterer produces a far scattered field at some scattering direction and polarization. The result is the same upon interchange of the directions of incidence and scattering with the polarizations remaining unchanged [17 (chap. 2)]. This basic scattering reciprocity is considered via integral equations in sections 2 and 3.

The scattered fields are expanded in terms of eigenmodes of the integral equation (eigenmode expansion method or EEM) in section 4. Here the reciprocity is evident in terms of the various symmetric products involving the eigenmodes which decompose the solution into a sum over the eigenmode index β. In section 5 this is extended to the pole terms in the singularity expansion method (SEM) which further decompose the solution according to the natural-frequency index α. Considered on a pole-by-pole basis the residues factor according to terms dependent separately on the various directions (incidence, scattering, and two polarizations). An important result is that the normalized backscattering coupling coefficient is the square of the normalized coupling coefficient for the natural current modes.

Section 6 discusses the application of scattering reciprocity to the forward scattering theorem. Specifically the total scattering (or extinction) of the incident wave is invariant to reversal of the direction of incidence.

If the scatterer has a symmetry plane P, then the geometrical symmetry combines with the reciprocity symmetry to give additional symmetries in the results. Section 7 discusses this and introduces the case of mirror scattering (a special case of bistatic scattering). Further specializing the direction of incidence as parallel to P, then defining polarizations parallel and perpendicular to P, the backscattered fields have no cross-polarized components. Furthermore, parallel (or "vertical") polarization only excites one set of natural modes (the symmetric modes), and the scattered fields only contain the corresponding natural frequencies $s_{sy,\alpha'}$. Likewise perpendicular (or horizontal) polarization only excites the remaining set of natural modes (the antisymmetric modes) and the scattered fields only contain the $s_{as,\alpha'}$.

II. Surface Current Density on Perfectly Conducting Scatterer

As indicated in Fig. 2.1 let us assume there is some perfectly conducting scatterer with volume V and boundary surface S. The surface current density $\vec{J}_S$ on the scatterer is related to the incident electric field $\vec{E}^{(inc)}$ through the impedance or E-field integral equation [4, 6] as

$$\vec{E}^{(inc)}(\vec{r}_s,s) = \left\langle \overset{\leftrightarrow}{Z}_t(\vec{r}_s,\vec{r}_s\,';s);\vec{J}_s(\vec{r}_s\,';s)\right\rangle$$

$$\vec{r}_s,\vec{r}_s\,' \in S$$

$$\overset{\leftrightarrow}{Z}_t(\vec{r}_s,\vec{r}_s\,';s) = \overset{\leftrightarrow}{1}_t(\vec{r}_s)\cdot\overset{\leftrightarrow}{Z}(\vec{r}_s,\vec{r}_s\,';s)\cdot\overset{\leftrightarrow}{1}_t(\vec{r}_s\,')$$

$$= -s\mu_o\overset{\leftrightarrow}{1}_t(\vec{r}_s)\cdot\overset{\leftrightarrow}{G}_o(\vec{r}_s,\vec{r}_s\,';s)\cdot\overset{\leftrightarrow}{1}_t(\vec{r}_s\,')$$

$$= +\frac{Z_o\gamma^2}{4\pi}\overset{\leftrightarrow}{1}_t(\vec{r}_s)\cdot\left\{\left[-2\zeta^{-3}-2\zeta^{-2}\right]e^{-\zeta}\vec{1}_R\vec{1}_R\right.$$

$$\left.+\left[\zeta^{-3}+\zeta^{-2}+\zeta^{-1}\right]e^{-\zeta}\left[\overset{\leftrightarrow}{1}-\vec{1}_R\vec{1}_R\right]\right\}\cdot\overset{\leftrightarrow}{1}_t(\vec{r}_s\,')$$

$$R \equiv \left|\vec{r}_s-\vec{r}_s\,'\right|$$

$$\vec{1}_R = \frac{\vec{r}_s-\vec{r}_s\,'}{\left|\vec{r}_s-\vec{r}_s\,'\right|} \quad (\text{for } \vec{r}_s \neq \vec{r}_s\,')$$

$$\zeta \equiv \gamma R \tag{2.1}$$

$$\overset{\leftrightarrow}{1} \equiv \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z \quad (\text{identify dyad})$$

$$\overset{\leftrightarrow}{1}_t(\vec{r}_S) \equiv \overset{\leftrightarrow}{1} - \vec{1}_S(\vec{r}_S)\vec{1}_S(\vec{r}_S) \quad (\text{transverse dyad to S at } \vec{r}_S)$$

$$\vec{1}_S(\vec{r}_S) \equiv \text{ outward unit normal to S at } \vec{r}_S$$

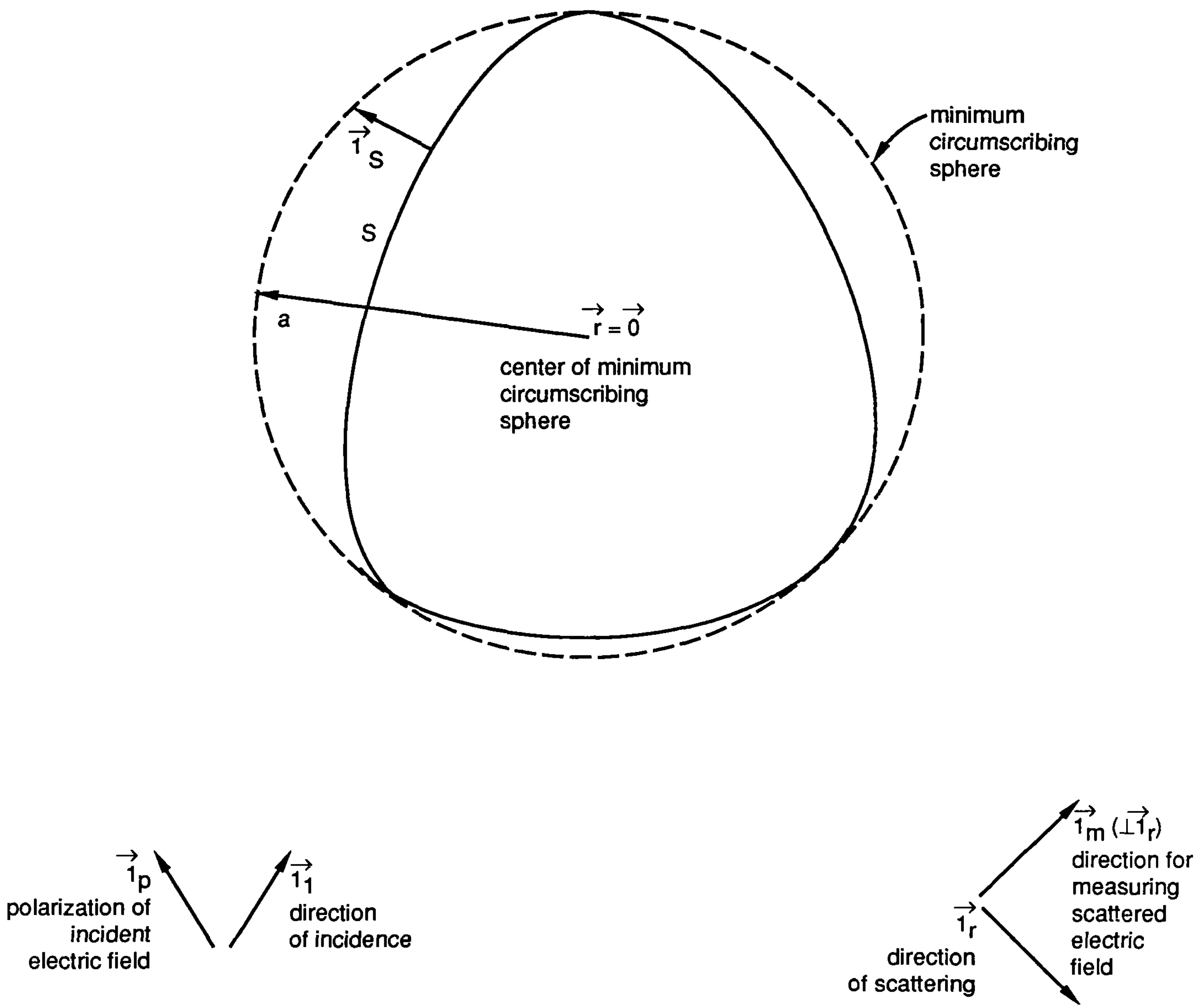

Fig. 2.1. Incident and Scattered Waves

$\gamma = \frac{s}{c}$ (propagation constant)

$s = \Omega + jw$ (complex frequency)

$c = \left(\mu_o \epsilon_o\right)^{-\frac{1}{2}}$ (speed of light)

$Z_o = \left(\frac{\mu_o}{\epsilon_o}\right)^{\frac{1}{2}}$

$\sim \equiv$ Laplace transform (two-sided)

Note that the delta function at $\vec{r}_S = \vec{r}_S'$ has been removed by the transverse dyads [13]. The formal solution to the integral equation is, of course,

$$\tilde{\vec{J}}_s\left(\vec{r}_s, s\right) = \left\langle \tilde{\overset{\leftrightarrow}{Z}}_t^{-1}\left(\vec{r}_s, \vec{r}_s'; s\right); \tilde{\vec{E}}^{(inc)} \vec{r}_s'; s \right\rangle \tag{2.2}$$

An important property of this impedance kernel is its symmetry which takes the form of a generalized transpose as

$$\tilde{\overset{\leftrightarrow}{Z}}_t\left(\vec{r}_S, \vec{r}_S'; s\right) = \tilde{\overset{\leftrightarrow}{Z}}_t^{T}\left(\vec{r}_S', \vec{r}_S; s\right) \tag{2.3}$$

i.e, besides taking the matrix (or dyadic) transpose $\vec{r}_S$ and $\vec{r}_S'$ are interchanged. The inverse is defined via

$$\left\langle \tilde{\overset{\leftrightarrow}{Z}}_t\left(\vec{r}_s, \vec{r}_s''; s\right); \tilde{\overset{\leftrightarrow}{Z}}_t^{-1}\left(\vec{r}_s'', \vec{r}_s'; s\right) \right\rangle = \overset{\leftrightarrow}{1}_t\left(\vec{r}_s\right) \delta_s\left(\vec{r}_s - \vec{r}_s'\right) = \left\langle \tilde{\overset{\leftrightarrow}{Z}}_t^{-1}\left(\vec{r}_s, \vec{r}_s''; s\right); \tilde{\overset{\leftrightarrow}{Z}}_t\left(\vec{r}_s'', \vec{r}_s'; s\right) \right\rangle$$

$\delta_s \equiv$ surface delta function (2.4)

By taking the generalized transpose of (2.4) one can show that (2.3) also applies to the inverse kernel, i.e.,

$$\tilde{\overset{\leftrightarrow}{Z}}_t^{-1}\left(\vec{r}_s, \vec{r}_s'; s\right) = \tilde{\overset{\leftrightarrow}{Z}}_t^{-1^T}\left(\vec{r}_s', \vec{r}_s; s\right) \tag{2.5}$$

This strictly applies only where the inverse exists, i.e., for $(\vec{r}_s \neq \vec{r}_s')$ and away from natural frequencies s_α.

This concept of symmetry in the kernel is very important, leading to the idea of reciprocity [10, 11, 12] which will be used later. While the formulation here is in terms of a perfectly conducting body it also applies to any body comprised of reciprocal media, even anisotropic media. One can have sheet impedances on s or volume distributions of $\overset{\leftrightarrow}{\epsilon}$, $\overset{\leftrightarrow}{\mu}$ and $\overset{\leftrightarrow}{\sigma}$ (as long as these dyadic functions of space and frequency are symmetric, i.e., reciprocal). In the volume case the surface integrals are replaced by volume integrals.

Now diagonalize $\tilde{\tilde{Z}}_t$ (i.e., expand in eigenmodes) via

$$\left\langle \tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s\,',s)\right\rangle = \tilde{Z}_\beta(s)\,\tilde{\vec{j}}_{s_\beta}(\vec{r}_s;s)$$

$$= \left\langle \tilde{\vec{j}}_{s_\beta}(\vec{r}_s\,',s);\tilde{\tilde{Z}}_t(\vec{r}_s\,',\vec{r}_s;s)\right\rangle$$

$$\tilde{Z}_\beta(s) \equiv \text{eigenimpedances}$$

$$\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) \equiv \text{eigenmodes}$$

$$\left\langle \tilde{\vec{j}}_{s_{\beta_1}}(\vec{r}_s\,',s);\tilde{\vec{j}}_{s_{\beta_2}}(\vec{r}_s\,',s)\right\rangle = 1_{\beta_1,\beta_2} \text{ (biorthonormal)} \tag{2.6}$$

This gives

$$\tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s) = \sum_\beta \tilde{Z}_\beta(s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s\,',s)$$

$$\tilde{\tilde{Z}}_t^{-1}(\vec{r}_s,\vec{r}_s\,';s) = \sum_\beta \tilde{Z}_\beta^{-1}(s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s\,',s)$$

$$\vec{1}_t(\vec{r}_s)\delta_s(\vec{r}_s-\vec{r}_s\,') = \sum_\beta \tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s)\tilde{\vec{j}}_{s_\beta}(\vec{r}_s\,',s) \tag{2.7}$$

The symmetry (generalized transpose) here is apparent. The surface current density is now

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = \sum_\beta \tilde{Z}_\beta^{-1}(s)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,',s);\tilde{\vec{j}}_{s_\beta}(\vec{r}_s\,',s)\right\rangle \tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) \tag{2.8}$$

The SEM form of the solution is [4, 16]

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = E_o \sum_\alpha \tilde{f}(s_\alpha)\eta_\alpha \vec{j}_{s_\alpha}(\vec{r}_s)[s-s_\alpha]^{-1}$$

$$+ \text{ singularities of } \tilde{f}(s)$$

$$+ \text{ possible entire function}$$

$$\tilde{f}(s) = \text{incident waveform (Laplace transformed)}$$

$$\left\langle \tilde{\tilde{Z}}(\vec{r}_s,\vec{r}_s\,';s_\alpha);\vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle = \vec{0}$$

$$= \left\langle \vec{j}_{s_\alpha}(\vec{r}_s\,');\tilde{\tilde{Z}}(\vec{r}_s\,',\vec{r}_s;s_\alpha)\right\rangle$$

$$s_\alpha \equiv \text{natural frequency}$$

$$\vec{j}_{s_\alpha}(\vec{r}_s) \equiv \text{natural mode}$$

$$\eta_\alpha = \frac{E_o^{-1}\tilde{f}^{-1}(s_\alpha)\left\langle \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,',s_\alpha);\vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle}{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s);\frac{\partial}{\partial s}\tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s\,';s)\Big|_{s=s_\alpha};\vec{j}_{s_\alpha}(\vec{r}_s\,')\right\rangle}$$

$$\equiv \text{coupling coefficient}$$

$$E_o \equiv \text{scaling constant (volts/meter) for incident field} \tag{2.9}$$

Note that coupling coefficients here are taken with no s dependence, i.e., class 1.

Now let the incident field be a plane wave with electric field as

$$\tilde{\vec{E}}^{(inc)}(\vec{r},s) = E_o\,\tilde{f}(s)\,\vec{1}_p\,e^{-\gamma\vec{1}_1\cdot\vec{r}}$$

$$\vec{1}_1 \equiv \text{direction of incidence}$$

$$\vec{1}_p \equiv \text{polarization}$$

$$\vec{1}_1\cdot\vec{1}_p = 0 \tag{2.10}$$

As in fig. 2.1, phase or arrival time is chosen by the location of coordinate center $\vec{r} = \vec{0}$, here taken for general finite sized scatterers as the center of the minimum circumscribing sphere of radius a [6, 14]. The surface current density is then

$$\tilde{\vec{J}}_s(\vec{r}_s, s) = E_o \tilde{f}(s)\left\langle \tilde{\vec{\vec{Z}}}_t^{-1}(\vec{r}_s, \vec{r}_s'; s); \vec{1}_p \, e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle$$

$$= E_o \tilde{f}(s) \left\langle \tilde{\vec{\vec{Z}}}_t^{-1}(\vec{r}_s, \vec{r}_s'; s), e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_p$$

$$= E_o \tilde{f}(s) \vec{1}_p \cdot \left\langle e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}, \tilde{\vec{\vec{Z}}}_t^{-1}(\vec{r}_s', \vec{r}_s; s) \right\rangle \tag{2.11}$$

noting the symmetry of the kernel. In the EEM term this is

$$\tilde{\vec{J}}_s(\vec{r}_s, s) = E_o \tilde{f}(s) \sum_{\beta} \tilde{Z}_\beta^{-1}(s)\left[\vec{1}_p \cdot \left\langle e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}, \tilde{\vec{j}}_{s_\beta}(\vec{r}_s', s) \right\rangle\right] \tilde{\vec{j}}_{s_\beta}(\vec{r}_s, s) \tag{2.12}$$

The SEM form is

$$\tilde{\vec{J}}_s(\vec{r}_s, s) = E_o \sum_{\alpha} \tilde{f}(s_\alpha)\, \eta_\alpha(\vec{1}_1, \vec{1}_p) \vec{j}_{s_\alpha}(\vec{r}_s)[s - s_\alpha]^{-1}$$

$$+ \text{ singularities of } \tilde{f}(s)$$

$$+ \text{ possible entire function}$$

$$\eta_\alpha(\vec{1}_1, \vec{1}_p) = \frac{\vec{1}_p \cdot \left\langle e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'}, \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle}{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\vec{\vec{Z}}}_t(\vec{r}_s, \vec{r}_s'; s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle}$$

$$\gamma_\alpha \equiv \frac{s_\alpha}{c} \tag{2.13}$$

Noting that $\vec{1}_p$ is always orthogonal to $\vec{1}_1$ we have a transverse dyad $\vec{\vec{1}}_1$ with the property

$$\vec{\vec{1}}_1 \equiv \vec{\vec{1}} - \vec{1}_1 \vec{1}_1 \equiv \text{incident transverse dyad}$$

$$\vec{\vec{1}}_1 \cdot \vec{1}_p = \vec{1}_p, \vec{\vec{1}}_1 \cdot \vec{1}_1 = \vec{0} \tag{2.14}$$

This transverse dyad can be included wherever $\vec{1}_p$ is used. Since it is only a function of $\vec{1}_1$ then including it in integrals where $\vec{1}_1$ is already present it does not complicate matters. It can be thought to simplify things by removing the components "parallel" to $\vec{1}_1$ which don't enter the answer anyway. Including this we have alternate forms for (2.11) in regular form

$$\tilde{\vec{J}}_s(\vec{r}_s, s) = E_o \tilde{f}(s) \left\langle \tilde{\overleftrightarrow{Z}}_t^{-1}(\vec{r}_s, \vec{r}_s'; s); \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_p$$

$$= E_o \tilde{f}(s) \vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}; \tilde{\overleftrightarrow{Z}}_t^{-1}(\vec{r}_s', \vec{r}_s; s) \right\rangle \tag{2.15}$$

for (2.12) in EEM form

$$\tilde{\vec{J}}_s(\vec{r}_s, s) = E_o \tilde{f}(s) \sum_\beta \tilde{Z}_\beta^{-1}(s) \left[\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'}; \vec{j}_{s_\beta}(\vec{r}_s', s) \right\rangle \right] \vec{j}_{s_\beta}(\vec{r}_s, s) \tag{2.16}$$

for (2.13) in SEM form

$$\eta_\alpha(\vec{1}_1, \vec{1}_p) = \frac{\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle}{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\overleftrightarrow{Z}}_t(\vec{r}_s, \vec{r}_s'; s) \Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle} \tag{2.17}$$

Note that $\vec{1}_1$ just weights the integral over the surface-current-density modes to remove any $\vec{1}_1$ component.

III. Scattered Fields

The far scattered electric field is calculated from the surface current density as [1]

$$\tilde{\vec{E}}_f(\vec{r},s) = \frac{-s\mu_0 e^{-\gamma r}}{4\pi r}\left\langle \vec{1}_r e^{+\gamma \vec{1}_r \cdot \vec{r}_s} ; \tilde{\vec{J}}_s(\vec{r}_s,s)\right\rangle$$

$$\vec{1}_r \equiv \vec{1} - \vec{1}_r\,\vec{1}_r \equiv \text{scattered transverse dyad}$$

$$\vec{1}_r \cdot \vec{1}_m = \vec{1}_m, \vec{1}_r \cdot \vec{1}_r = \vec{0} \tag{3.1}$$

Note again as in fig. 2.1 that the coordinate center has to be specified, here as the center of the minimum circumscribing sphere. Note that (3.1) is for the far field, i.e., the leading r^{-1} term as $r \to \infty$ with the property that

$$\vec{1}_r \cdot \tilde{\vec{E}}_f(\vec{r}, s) = 0 \tag{3.2}$$

As indicated in fig. 2.1 we have

$\vec{1}_r \equiv$ scattering direction

$\vec{1}_m \equiv$ direction for measuring (sampling) scattered electric field

$$\vec{1}_r \cdot \vec{1}_m = 0 \quad (\text{constraint on } \vec{1}_m) \tag{3.3}$$

Note that we also then have

$$\vec{1}_m \cdot \tilde{\vec{E}}_f(\vec{r}, s) = \frac{-s\mu_0 e^{-\gamma r}}{4\pi r}\vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma \vec{1}_r \cdot \vec{r}_s} ; \tilde{\vec{J}}_s(\vec{r}_s,s)\right\rangle$$

$$= \frac{-s\mu_0 e^{-\gamma r}}{4\pi r}\vec{1}_m \cdot \left\langle e^{\gamma \vec{1}_r \cdot \vec{r}_s} , \tilde{\vec{J}}_s(\vec{r}_s,s)\right\rangle \tag{3.4}$$

Combining the far field results with (2.15) for the surface current density gives

$$\vec{1}_m \cdot \tilde{\vec{E}}_f(\vec{r}, s) = \frac{-E_o \tilde{f}(s) s\mu_o e^{-\gamma r}}{4\pi r} \vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma \vec{1}_r \cdot \vec{r}_s} ; \tilde{\tilde{Z}}_t^{-1}(\vec{r}_s, \vec{r}_s'; s) ; \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_p$$

$$= \frac{-E_o \tilde{f}(s) s\mu_o e^{-\gamma r}}{4\pi r} \vec{1}_p \cdot \left\langle \vec{1}_r e^{-\gamma \vec{1}_r \cdot \vec{r}_s} ; \tilde{\tilde{Z}}_t^{-1}(\vec{r}_s, \vec{r}_s'; s) ; \vec{1}_r e^{\gamma \vec{1}_r \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_m \tag{3.5}$$

Note the symmetry (reciprocity) in this formula where we have used the generalized transpose result in (2.5) for the inverse kernel. Here we have the result that the interchange

$$\left(\vec{1}_1, \vec{1}_p\right) \leftrightarrow \left(-\vec{1}_r, \vec{1}_m\right) \tag{3.6}$$

leaves the result unchanged. Referring to fig. 2.1, the scattering reciprocity result is that an incident plane wave characterized by $\left(\vec{1}_1, \vec{1}_p\right)$ giving a scattered by $\left(\vec{1}_r, \vec{1}_m\right)$ at a distance r implies that a second incident wave characterized by $\left(-\vec{1}_r, \vec{1}_m\right)$ gives the same scattered field with directions $\left(-\vec{1}_1, \vec{1}_p\right)$ at the same distance r.

Note that this symmetry is merely a result of reciprocity, the generalized transpose symmetry of the operator, including any reciprocal loading of the scatterer. While the foregoing consider scattering of a plane wave, the result is even more general. One could have any two (reciprocal) antennas at distances r_1 and r_2 from the scattering coordinate center. The ratio of received (open-circuit) voltage at one antenna to source current into the other antenna is independent of interchange of roles of transmitting and receiving antennas. This is one form of the reciprocity theorem and is often termed reaction [10].

A special case of (3.5) is that of backscattering with backscattered electric field measured in the same direction as the incident field so that

$$\vec{1}_r \equiv -\vec{1}_1, \quad \vec{1}_m \equiv \vec{1}_p \tag{3.7}$$

In this case (3.5) becomes

$$\vec{1}_p \cdot \tilde{\vec{E}}_f(\vec{r}, s) = \frac{-E_o \tilde{f}(s) s\mu_o}{4\pi r} \vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s} ; \tilde{\tilde{Z}}_t^{-1}(\vec{r}_s, \vec{r}_s'; s) ; \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_p \tag{3.8}$$

which might be termed self-reciprocity, i.e., the incident-to-scattered-field transfer function is not only the same on interchange of incident and scattered (reciprocity), but also in the case of (3.7) in effect these are in some sense the same thing, (i.e., which is called incident and which is called scattered). Note that since this reciprocity applies to all frequencies, then by inverse Laplace (2-sided) or Fourier transformation it applies to arbitrary transient waveforms as well, the scattering taking the form of a convolution operator.

Another special case of (3.5) is cross-polarized backscattering so that

$$\vec{1}_r \equiv -\vec{1}_1, \quad \vec{1}_m \cdot \vec{1}_p = 0 \tag{3.9}$$

In this case we have from (3.5) that incident with polarization $\vec{1}_p$ into backscattering with polarization $\vec{1}_m$ is the same as with $\vec{1}_p$ and $\vec{1}_m$ interchanged. This can be interpreted as a symmetric backscattering matrix.

While the reciprocity here is stated in terms of electric fields and currents, one could consider source currents at one antenna producing magnetic fields at the other. With a sign interchange there is a similar reciprocity relation (or "magnetic reaction") as discussed in [8]. Furthermore, while these results have been stated in terms of (complex) frequency domain, since they apply for all frequencies they apply for time-domain waveforms as well (i.e., consider the inverse Laplace transform).

IV. EEM Scattered Fields

Applying (3.5) in the context of EEM as in (2.8) and (2.16) gives

$$\vec{1}_m \cdot \tilde{\vec{E}}_p(\vec{r},s) = \frac{-E_o \tilde{f}(s) s \mu_o e^{-\gamma r}}{4\pi r} \sum_\beta \tilde{Z}_\beta^{-1}(s) \vec{1}_m \cdot \tilde{\overset{\leftrightarrow}{C}}_{f_\beta}\left(\vec{1}_r, \vec{1}_1; s\right) \cdot \vec{1}_p$$

$$\tilde{\overset{\leftrightarrow}{C}}_{f_\beta}\left(\vec{1}_r, \vec{1}_1; s\right) = \tilde{\vec{C}}_{r_\beta}\left(\vec{1}_r, s\right) \tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right)$$

$$\tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right) = \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}\,'_s} ; \tilde{\vec{J}}_{s_\beta}\left(\vec{r}\,'_s, s\right) \right\rangle$$

$$\tilde{\vec{C}}_{r_\beta}\left(\vec{1}_r, s\right) = \left\langle \vec{1}_r e^{\gamma \vec{1}_r \cdot \vec{r}\,'_s} ; \tilde{\vec{J}}_{s_\beta}\left(\vec{r}_s, s\right) \right\rangle$$

$$\vec{1}_m \cdot \tilde{\overset{\leftrightarrow}{C}}_{f_\beta}\left(\vec{1}_r, \vec{1}_1; s\right) \cdot \vec{1}_p = \left[\vec{1}_m \cdot \tilde{\vec{C}}_{r_\beta}\left(\vec{1}_r, s\right)\right] \left[\vec{1}_p \cdot \tilde{\vec{C}}_\beta\left(\vec{1}_1, s\right)\right]$$

$$= \left[\vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma \vec{1}_r \cdot \vec{r}_s} ; \tilde{\vec{J}}_{s_\beta}\left(\vec{r}_s, s\right) \right\rangle\right] \left[\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}\,'_s} ; \tilde{\vec{J}}_{s_\beta}\left(\vec{r}\,'_s, s\right) \right\rangle\right]$$

$$= \left\langle \vec{1}_m e^{\gamma \vec{1}_r \cdot \vec{r}_s} ; \tilde{\vec{J}}_{s_\beta}\left(\vec{r}_s, s\right) \right\rangle \left\langle \vec{1}_p e^{-\gamma \vec{1}_1 \cdot \vec{r}\,'_s} ; \tilde{\vec{J}}_{s_\beta}\left(\vec{r}\,'_s, s\right) \right\rangle \tag{4.1}$$

Note in this form how $\vec{1}_m$ and $\vec{1}_p$ factor out (in dot-product sense) with factors depending separately on $\vec{1}_r$ and $\vec{1}_1$, so that the four unit vectors all appear in separate factors. In this form reciprocity is apparent as

$$\vec{1}_m \cdot \tilde{\overset{\leftrightarrow}{C}}_{f_\beta}\left(\vec{1}_r, \vec{1}_1; s\right) \cdot \vec{1}_p = \vec{1}_p \cdot \tilde{\overset{\leftrightarrow}{C}}_{f_\beta}\left(-\vec{1}_1, -\vec{1}_r; s\right) \cdot \vec{1}_m \tag{4.2}$$

Note that $\tilde{\vec{C}}_{r_\beta}$ and $\tilde{\vec{C}}_\beta$ are simply related via

$$\tilde{\vec{C}}_{r_\beta}\left(\vec{1}_r, s\right) = \tilde{\vec{C}}_\beta\left(-\vec{1}_r, s\right) \tag{4.3}$$

so there is really only one vector function to consider. In the dyadic then we have

$$\begin{aligned}\tilde{\tilde{C}}_{f_\beta}\left(\vec{1}_r,\vec{1}_1;s\right)&=\tilde{\tilde{C}}_{r_\beta}\left(\vec{1}_r,s\right)\tilde{\tilde{C}}_\beta\left(\vec{1}_1,s\right)=\tilde{\tilde{C}}_\beta\left(-\vec{1}_r,s\right)\tilde{\tilde{C}}_\beta\left(\vec{1}_1,s\right)\\&=\tilde{\tilde{C}}_\beta\left(-\vec{1}_r,s\right)\tilde{\tilde{C}}_{r_\beta}\left(-\vec{1}_1,s\right)=\tilde{\tilde{C}}_{r_\beta}\left(\vec{1}_r,s\right)\tilde{\tilde{C}}_{r_\beta}\left(-\vec{1}_1,s\right)\\&=\tilde{\tilde{C}}^T_{f_\beta}\left(-\vec{1}_1,-\vec{1}_r;s\right)\end{aligned}\tag{4.4}$$

which is scattering reciprocity in terms of only the directions of incidence and scattering.

For backscattering we have in dyadic form

$$\vec{1}_r=-\vec{1}_1$$

$$\tilde{\tilde{C}}_{r_\beta}\left(\vec{1}_r,s\right)=\tilde{\tilde{C}}_\beta\left(\vec{1}_1,s\right)$$

$$\begin{aligned}\tilde{\tilde{C}}_{f_\beta}\left(-\vec{1}_1,\vec{1}_1;s\right)&=\tilde{\tilde{C}}_\beta\left(\vec{1}_1,s\right)\tilde{\tilde{C}}_\beta\left(\vec{1}_1,s\right)\\&=\tilde{\tilde{C}}^T_{f_\beta}\left(-\vec{1}_1,\vec{1}_1;s\right)\end{aligned}\tag{4.5}$$

In scalar form we have for in-line or parallel polarization

$$\vec{1}_m=\vec{1}_p,\quad \vec{1}_r=-\vec{1}_1$$

$$\begin{aligned}\vec{1}_m\cdot\tilde{\tilde{C}}_{f_\beta}\left(\vec{1}_r,\vec{1}_1;s\right)\cdot\vec{1}_p&=\vec{1}_p\cdot\tilde{\tilde{C}}_{f_\beta}\left(-\vec{1}_1,\vec{1}_1;s\right)\cdot\vec{1}_p\\&=\left[\vec{1}_p\cdot\tilde{\tilde{C}}_\beta\left(\vec{1}_1,s\right)\right]^2\end{aligned}\tag{4.6}$$

which is a way of saying that for each eigenmode the coupling to the target is the same as the coupling of the target currents to the backscattered fields, at least in terms of angular dependence, i.e., $\vec{1}_1$ and $\vec{1}_p$. Note that in (4.1) there is a sum over β for arbitrary frequencies, so this factorization by $\vec{1}_p$ and $\vec{1}_1$ does not apply to arbitrary frequencies for the scattered or backscattered fields.

V. SEM Pole Scattered Fields

Applying (3.5) in the context of SEM poles as in (2.9) and (2.17) gives

$$\vec{1}_m \cdot \tilde{\vec{E}}_f(\vec{r}, s) = \frac{E_o}{4\pi r} e^{-\gamma r} \sum_{\alpha} \tilde{f}(s_\alpha)\, \eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_p; \vec{1}_r, \vec{1}_m\right)[s - s_\alpha]^{-1}$$

$$+ \text{ singularities of } \tilde{f}(s)$$

$$+ \text{ possible entire function}$$

$$\eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_p; \vec{1}_r, \vec{1}_m\right) = \eta_\alpha\left(\vec{1}_1, \vec{1}_p\right)\eta_{r_\alpha}\left(\vec{1}_r, \vec{1}_m\right)$$

$$\equiv \text{ far coupling coefficient (class 1)}$$

$$\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) = \frac{\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle}{\left\langle \vec{j}_{s_\alpha}\left(\vec{r}_s\right); \frac{\partial}{\partial s} \tilde{\vec{Z}}_t\left(\vec{r}_s, \vec{r}_s'; s\right)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle}$$

$$\equiv \text{ coupling coefficient (class 1)}$$

$$\eta_{r_\alpha}\left(\vec{1}_r, \vec{1}_m\right) = -s_\alpha \mu_o \vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle$$

$$\equiv \text{ recoupling coefficient (class 1)} \qquad (5.1)$$

The coupling coefficient gives the coupling of the incident field to the natural mode. The recoupling coefficient couples the natural mode to the far scattered field.

It is convenient to normalize these coupling coefficients so that they have peak magnitude of 1 over all angles concerned and assume the value +1 (real) at the angles concerned (or one set of these angle combinations). For η_α let this angle combination be $\vec{1}_{1_0}$ and $\vec{1}_{p_0}$ giving

$$\sup_{\vec{1}_1, \vec{1}_p} \left|\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right)\right| = \left|\eta_\alpha\left(\vec{1}_{1_o}, \vec{1}_{p_o}\right)\right|$$

$$\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right) \equiv \frac{\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right)}{\eta_\alpha\left(\vec{1}_{1_o}, \vec{1}_{p_o}\right)}$$

$$\sup_{\vec{1}_1, \vec{1}_p} \left|\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right)\right| = 1 = \eta_\alpha^{(n)}\left(\vec{1}_{1_o}, \vec{1}_{p_o}\right) \tag{5.2}$$

In another form, noting that the 3-term symmetric product that is the denominator of η_α is not a function of $\vec{1}_1$ or $\vec{1}_p$, we have

$$\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right) = \frac{\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle}{\vec{1}_{p_o} \cdot \left\langle \vec{1}_{1_o} e^{-\gamma_\alpha \vec{1}_{1_o} \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle}$$

$$= \frac{\left\langle \vec{1}_p e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle}{\left\langle \vec{1}_{p_o} e^{-\gamma_\alpha \vec{1}_{1_o} \cdot \vec{r}_s'} ; \vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle} \tag{5.3}$$

For η_{r_α} we have a maximum magnitude at angles $\vec{1}_{r_o}$ and $\vec{1}_{m_o}$ giving

$$\sup_{\vec{1}_r, \vec{1}_m} \left|\eta_{r\alpha}\left(\vec{1}_r, \vec{1}_m\right)\right| = \left|\eta_{r\alpha}\left(\vec{1}_{r_o}, \vec{1}_{m_o}\right)\right|$$

$$\eta_{r\alpha}^{(n)}\left(\vec{1}_r, \vec{1}_m\right) \equiv \frac{\eta_{r\alpha}\left(\vec{1}_r, \vec{1}_m\right)}{\eta_{r\alpha}\left(\vec{1}_{r_o}, \vec{1}_{m_o}\right)}$$

$$\sup_{\vec{1}_r, \vec{1}_m} \left|\eta_{r\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p\right)\right| = 1 = \eta_{r\alpha}^{(n)}\left(\vec{1}_{r_o}, \vec{1}_{m_o}\right)$$

$$\eta_{r\alpha}^{(n)}\left(\vec{1}_r, \vec{1}_m\right) = \frac{\vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s'} ; \vec{j}_{s\alpha}\left(\vec{r}_s'\right)\right\rangle}{\vec{1}_{m_o} \cdot \left\langle \vec{1}_{r_o} e^{\gamma_\alpha \vec{1}_{r_o} \cdot \vec{r}_s'} ; \vec{j}_{s\alpha}\left(\vec{r}_s'\right)\right\rangle}$$

$$= \frac{\left\langle \vec{1}_m e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s'} ; \vec{j}_{s\alpha}\left(\vec{r}_s'\right)\right\rangle}{\left\langle \vec{1}_{m_o} e^{\gamma_\alpha \vec{1}_{r_o} \cdot \vec{r}_s'} ; \vec{j}_{s\alpha}\left(\vec{r}_s'\right)\right\rangle} \tag{5.4}$$

Comparing (5.4) to (5.3) we have the simple result (for at least one choice of angles)

$$\vec{1}_{m_o} = \vec{1}_{p_o}, \vec{1}_{r_o} = -\vec{1}_{1_o}$$

$$\eta_{\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p\right) = \eta_{r\alpha}^{(n)}\left(-\vec{1}_1, \vec{1}_p\right) \tag{5.5}$$

which is found by simply substituting $\vec{1}_r \to -\vec{1}_1$ and $\vec{1}_m \to \vec{1}_p$ in (5.4) giving a formula identical to (5.3).

Except then for an appropriate interchange of direction vectors the normalized recoupling and coupling coefficients are the same. So, knowing one we know the other. In a general form we can normalize the far coupling coefficient in (5.1) as

$$\eta_{f\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p; \vec{1}_r, \vec{1}_m\right) = \eta_{\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p\right)\eta_{r\alpha}^{(n)}\left(\vec{1}_r, \vec{1}_p\right) \tag{5.6}$$

One symmetry to observe is the basic reciprocity relation

$$\eta_{f_\alpha}^{(n)}\left(-\vec{1}_r, \vec{1}_m; -\vec{1}_1, \vec{1}_p\right) = \eta_\alpha^{(n)}\left(-\vec{1}_r, \vec{1}_m\right)\eta_{r_\alpha}^{(n)}\left(-\vec{1}_1, \vec{1}_p\right)$$

$$= \eta_{r_\alpha}^{(n)}\left(\vec{1}_r, \vec{1}_m\right)\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right)$$

$$= \eta_{f_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p; \vec{1}_r, \vec{1}_m\right)$$

$$\eta_{f_\alpha}\left(-\vec{1}_r, \vec{1}_m; -\vec{1}_1, \vec{1}_p\right) = \eta_{f_\alpha}\left(\vec{1}_1, 1_p; \vec{1}_r, \vec{1}_m\right) \tag{5.7}$$

Noting also

$$\eta_\alpha\left(\vec{1}_1, -\vec{1}_p\right) = -\eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) \quad , \quad \eta_\alpha^{(n)}\left(\vec{1}_1, -\vec{1}_p\right) = -\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right)$$

$$\eta_{r_\alpha}\left(\vec{1}_r, -\vec{1}_m\right) = -\eta_{r_\alpha}\left(\vec{1}_r, \vec{1}_m\right) \quad , \quad \eta_{r_\alpha}^{(n)}\left(\vec{1}_r, -\vec{1}_m\right) = -\eta_{r_\alpha}^{(n)}\left(\vec{1}_r, \vec{1}_m\right) \tag{5.8}$$

we can note that there are at least two choices each for $\vec{1}_{p_o}$ and $\vec{1}_{m_o}$ which results from

$$\eta_\alpha^{(n)}\left(\vec{1}_{1_o}, -\vec{1}_{p_o}\right) = -\eta_\alpha^{(n)}\left(\vec{1}_{1_o}, \vec{1}_{p_o}\right) = -1$$

$$\eta_{r_\alpha}^{(n)}\left(\vec{1}_{r_o}, -\vec{1}_{m_o}\right) = -\eta_{r_\alpha}^{(n)}\left(\vec{1}_{r_o}, \vec{1}_{m_o}\right) = -1 \tag{5.9}$$

which are of course now both of magnitude 1 on inversion of $\vec{1}_{p_o}$ and $\vec{1}_{m_o}$. Applying (5.8) in (5.7) we also have the symmetry

$$\eta_{f_\alpha}^{(n)}\left(-\vec{1}_r, -\vec{1}_m; -\vec{1}_1, -\vec{1}_p\right) = \eta_{f_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p; \vec{1}_r, \vec{1}_m\right) \tag{5.10}$$

For the special case of backscattering with measurement parallel to the incident field we have

$$\vec{1}_r \equiv -\vec{1}_1, \vec{1}_m \equiv \vec{1}_p$$

$$\eta_{b_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p\right) \equiv \eta_{f_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p; -\vec{1}_1, \vec{1}_p\right)$$

$$= \eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right)\eta_{r_\alpha}^{(n)}\left(-\vec{1}_1, \vec{1}_p\right)$$

$$\left[\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right)\right]^2 \tag{5.11}$$

Got that? The normalized backscattering coupling coefficient is exactly the square of the normalized coupling coefficient (a complex function of $\vec{1}_1$ and $\vec{1}_p$). Furthermore we have

$$\sup_{\vec{1}_1, \vec{1}_p} \left| \eta_{b_\alpha}^{(n)}(\vec{1}_1, \vec{1}_p) \right| = \sup_{\vec{1}_1, \vec{1}_p} \left| \eta_{\alpha}^{(n)}(\vec{1}_1, \vec{1}_p) \right|^2$$

$$= 1$$

$$= \eta_{b_\alpha}^{(n)}\left(\vec{1}_{1_o}, \vec{1}_{p_o}\right) \tag{5.12}$$

From the symmetry concerning inversion of $\vec{1}_p$ in (5.8) we also have

$$\eta_{b_\alpha}^{(n)}(\vec{1}_1, -\vec{1}_p) = \left[\eta_{\alpha}^{(n)}(\vec{1}_1, -\vec{1}_p)\right]^2$$

$$= \left[\eta_{\alpha}^{(n)}(\vec{1}_1, \vec{1}_p)\right]^2$$

$$= \eta_{b_\alpha}^{(n)}(\vec{1}_1, \vec{1}_p)$$

$$\eta_{b_\alpha}^{(n)}\left(\vec{1}_{1_o}, -\vec{1}_{p_o}\right) = \eta_{b_\alpha}^{(n)}\left(\vec{1}_{1_o}, \vec{1}_{p_o}\right) = 1 \tag{5.13}$$

So what this says is that measurements on the scatterer to obtain the normalized coupling coefficient for the natural-mode surface current density as a function of $\vec{1}_1$ and $\vec{1}_p$ are simply related to backscatter coupling coefficients (normalized) by a simple square. Note that $\vec{r} = \vec{0}$ needs to be maintained while rotating $\vec{1}_1$ and $\vec{1}_p$ (or equivalently the scatterer orientation) for these results to apply. The reference time for the backscatter signal is not taken from the first backscatter signal to arrive from a backscatter pulse, but the time for the backscatter signal to arrive from some reference position on the scatterer, such as the center of the minimum circumscribing sphere in fig. 2.1. Of course one could use the time of the first backscatter signal provided an angular-dependent correction is made.

The backscattered electric field can be considered as having two components, one parallel to $\vec{1}_p$, and one paralle to $\vec{1}_1 \times \vec{1}_p$. Defining

$$\vec{1}_c \equiv \vec{1}_1 \times \vec{1}_p \tag{5.14}$$

then we can have the cross polarized backscattering coupling coefficient as

$$\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p, \vec{1}_c\right) \equiv \tilde{\eta}_{f_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p; -\vec{1}_1, \vec{1}_c\right)$$

$$= \eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right) \eta_{r_\alpha}^{(n)}\left(-\vec{1}_1, \vec{1}_c\right)$$

$$= \eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right) \eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_c\right)$$

$$= -\eta_\alpha^{(n)}\left(\vec{1}_1, -\vec{1}_p\right) \eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_c\right)$$

$$= -\eta_\alpha^{(n)}\left(\vec{1}_1, \vec{1}_p\right) \eta_\alpha^{(n)}\left(\vec{1}_1, -\vec{1}_c\right)$$

$$= \eta_\alpha^{(n)}\left(\vec{1}_1, -\vec{1}_p\right) \eta_\alpha^{(n)}\left(\vec{1}_1, -\vec{1}_c\right) \tag{5.15}$$

This gives the symmetries

$$\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p, \vec{1}_c\right) = \eta_{c_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_c, \vec{1}_p\right) =$$

$$\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, -\vec{1}_p, -\vec{1}_c\right) = \eta_{c_\alpha}^{(n)}\left(\vec{1}_1, -\vec{1}_c, -\vec{1}_p\right) =$$

$$-\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, -\vec{1}_p, \vec{1}_c\right) = -\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_p, -\vec{1}_c\right) =$$

$$-\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, \vec{1}_c, -\vec{1}_p\right) = -\eta_{c_\alpha}^{(n)}\left(\vec{1}_1, -\vec{1}_c, \vec{1}_p\right) \tag{5.16}$$

Note that in general we have the inequality

$$\left|\eta_{c\alpha}^{(n)}(\vec{1}_1,\vec{1}_p,\vec{1}_c)\right| = \left|\eta_{\alpha}^{(n)}(\vec{1}_1,\vec{1}_p)\right| \left|\eta_{\alpha}^{(n)}(\vec{1}_1,\vec{1}_c)\right|$$

$$\leq \left\{\sup_{\vec{1}_p} \left|\eta_{\alpha}^{(n)}(\vec{1}_1,\vec{1}_p)\right|\right\} \left\{\sup_{\vec{1}_c} \left|\eta_{\alpha}^{(n)}(\vec{1}_1,\vec{1}_c)\right|\right\}$$

$$= \left\{\sup_{\vec{1}_p} \left|\eta_{\alpha}^{(n)}(\vec{1}_1,\vec{1}_p)\right|\right\}^2$$

$$= \sup_{\vec{1}_p} \left|\eta_{b\alpha}^{(n)}(\vec{1}_1,\vec{1}_p)\right| \tag{5.17}$$

which holds for all $\vec{1}_c$ for every choice of $\vec{1}_1$. So the cross polarized backscattering coupling coefficient is bounded in magnitude by the maximum of the backscattering (parallel polarized) coupling coefficient for every $\vec{1}_1$.

One can also express the far coupling coefficient in dyadic terms, anaolgous to the EEM form in section 4, as

$$\vec{1}_m \cdot \tilde{\vec{E}}_f(\vec{r}_s, s) = \frac{-E_o s \mu_o}{4\pi r} e^{-\gamma r} \sum_{\alpha} \tilde{f}(s_\alpha) \left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\vec{\vec{Z}}}_t(\vec{r}_s, \vec{r}_s\,'; s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle^{-1}$$

$$\vec{1}_m \cdot \vec{\vec{C}}_{f_\alpha}(\vec{1}_r, \vec{1}_1) \cdot \vec{1}_p [s - s_\alpha]^{-1}$$

$$+ \text{ singularities of } \tilde{f}(s)$$

$$+ \text{ possible entire function}$$

$$\vec{\vec{C}}_{f_\alpha}(\vec{1}_r, \vec{1}_1) = \vec{C}_{r_\alpha}(\vec{1}_r) \vec{C}_\alpha(\vec{1}_1)$$

$$\vec{C}_\alpha(\vec{1}_1) = \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s\,'}; \vec{j}_{s_\alpha}(\vec{r}_s\,') \right\rangle$$

$$\vec{C}_{r_\alpha}(\vec{1}_r) = \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s}; \vec{j}_{s_\alpha}(\vec{r}_s) \right\rangle$$

$$\eta_{f_\alpha}(\vec{1}_i;\vec{1}_p;\vec{1}_r,\vec{1}_m) = -s_\alpha \mu_o \left\langle \vec{j}_{s_\alpha}(\vec{r}_s); \frac{\partial}{\partial s} \tilde{\tilde{Z}}_t(\vec{r}_s,\vec{r}_s';s)\Big|_{s=s_\alpha}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle^{-1} \vec{1}_m \cdot \tilde{C}_{f_\alpha}(\vec{1}_r,\vec{1}_1)\cdot \vec{1}_p$$

$$\vec{1}_m \cdot C_{f_\alpha}(\vec{1}_r,\vec{1}_1)\cdot \vec{1}_p = \left[\vec{1}_m \cdot \left\langle \vec{1}_r e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s}; \vec{j}_{s_\alpha}(\vec{r}_s) \right\rangle\right] \left[\vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle\right]$$

$$= \left\langle \vec{1}_m e^{\gamma_\alpha \vec{1}_r \cdot \vec{r}_s}; \vec{j}_{s_\alpha}(\vec{r}_s) \right\rangle \left\langle \vec{1}_p e^{-\gamma_\alpha \vec{1}_1 \cdot \vec{r}_s'}; \vec{j}_{s_\alpha}(\vec{r}_s') \right\rangle \tag{5.18}$$

Note that the natural modes are not necessarily normalized in the same manner as the orthonormalized eigenmodes in section 4. In current form, however, the various pole terms (class 1) are not functions of the complex frequency.

Note that $\tilde{C}_{r_\alpha}$ and $\tilde{C}_\alpha$ are simply related via

$$\tilde{C}_{r_\alpha}(\vec{1}_r) = \tilde{C}_\alpha(-\vec{1}_r) \tag{5.19}$$

In dyadic form then we have

$$\tilde{C}_{f_\alpha}(\vec{1}_r,\vec{1}_1) = \tilde{C}_{r_\alpha}(\vec{1}_r)\tilde{C}_\alpha(\vec{1}_1)$$

$$= \tilde{C}_\alpha(-\vec{1}_r)\tilde{C}_{r_\alpha}(-\vec{1}_1)$$

$$= \tilde{C}_{f_\alpha}^T(-\vec{1}_1,-\vec{1}_r) \tag{5.20}$$

which is again reciprocity in terms of only the directions of incidence and scattering.

The present analysis is related to the earlier concept of far natural modes [1]. Except for a constant scaling factor these are the same as the $\tilde{C}_{r_\alpha}(\vec{1}_r)$. In the present analysis the relation of these to the recoupling coefficient is interesting due to the exhibition of various symmetries.

For backscattering we have in dyadic form

$$\vec{1}_r = -\vec{1}_1$$

$$\tilde{C}_{r_\alpha}\left(\vec{1}_r\right) = \tilde{C}_\alpha\left(\vec{1}_1\right)$$

$$\tilde{C}_{f_\alpha}\left(-\vec{1}_1,\vec{1}_1\right) = \tilde{C}_\alpha\left(\vec{1}_1\right)\tilde{C}_\alpha\left(\vec{1}_1\right)$$

$$\tilde{C}_{f_\alpha}^T\left(-\vec{1}_1,\vec{1}_1\right) \tag{5.21}$$

In scalar form we have for in-line or parallel polarization

$$\vec{1}_m = \vec{1}_p, \vec{1}_r = -\vec{1}_1$$

$$\vec{1}_m \cdot \tilde{C}_{f_\alpha}\left(\vec{1}_r,\vec{1}_1\right) \cdot \vec{1}_p = \vec{1}_p \cdot \tilde{C}_{f_\alpha}\left(-\vec{1}_1,\vec{1}_1\right) \cdot \vec{1}_p$$

$$= \left[\vec{1}_p \cdot \tilde{C}_\alpha\left(\vec{1}_1\right)\right]^2 \tag{5.22}$$

In contradistinction to the EEM results in section 4, the SEM results here when considered pole by pole at each natural frequency s_α do allow the factorization by $\vec{1}_1$ and $\vec{1}_p$.

VI. Application to Forward Scattering

For forward scattering with measurement parallel to the incident field we have

$$\vec{1}_r = \vec{1}_1, \vec{1}_m = \vec{1}_p \tag{6.1}$$

In this case (3.5) becomes

$$\vec{1}_p \cdot \tilde{\vec{E}}_f(\vec{r}, s) = \frac{-E_o \tilde{f}(s) s\mu_o}{4\pi r} e^{-\gamma r} \vec{1}_p \cdot \left\langle \vec{1}_1 e^{\gamma \vec{1}_1 \cdot \vec{r}_s}; \tilde{\overleftrightarrow{Z}}_t^{-1}\left(\vec{r}_s, \vec{r}_s'; s\right); \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_p$$

$$= \frac{-E_o \tilde{f}(s) s\mu_o}{4\pi r} e^{-\gamma r} \vec{1}_p \cdot \left\langle \vec{1}_1 e^{-\gamma \vec{1}_1 \cdot \vec{r}_s}; \tilde{\overleftrightarrow{Z}}_t^{-1}\left(\vec{r}_s, \vec{r}_s'; s\right); \vec{1}_1 e^{\gamma \vec{1}_1 \cdot \vec{r}_s'} \right\rangle \cdot \vec{1}_p \tag{6.2}$$

which is invariant to inversion of $\vec{1}_1$ (i.e., to interchanging coming and going). In EEM form the terms in (4.1) become

$$\tilde{\overleftrightarrow{C}}_\beta\left(\vec{1}_1, s\right) = \tilde{\overleftrightarrow{C}}_{r_\beta}\left(-\vec{1}_1, s\right)$$

$$\tilde{\overleftrightarrow{C}}_{f_\beta}\left(\vec{1}_1, \vec{1}_1; s\right) = \tilde{\overleftrightarrow{C}}_{r_\beta}\left(\vec{1}_1, s\right) \tilde{\overleftrightarrow{C}}_\beta\left(\vec{1}_1, s\right) = \tilde{\overleftrightarrow{C}}_\beta\left(-\vec{1}_1, s\right) \tilde{\overleftrightarrow{C}}_\beta\left(\vec{1}_1, s\right)$$

$$\tilde{\overleftrightarrow{C}}_{f_\beta}^T\left(\vec{1}_1, -\vec{1}_1; s\right) \tag{6.3}$$

In SEM form we have $\vec{C}_\alpha$ results as in (6.3) and for the coupling coefficients

$$\eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_p; \vec{1}_1, \vec{1}_p\right) = \eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) \eta_{r_\alpha}\left(\vec{1}_1, \vec{1}_p\right)$$

$$= \eta_{r_\alpha}\left(-\vec{1}_1, \vec{1}_p\right) \eta_\alpha\left(-\vec{1}_1, \vec{1}_p\right)$$

$$= \eta_{f_\alpha}^{(n)}\left(-\vec{1}_1, \vec{1}_p; -\vec{1}_1, \vec{1}_p\right) \tag{6.4}$$

For the normalized coupling coefficients this is

$$\begin{aligned}
\eta_{f_{\alpha}}^{(n)}\left(\vec{1}_1,\vec{1}_p;\vec{1}_1,\vec{1}_p\right) &= \eta_{\alpha}^{(n)}\left(\vec{1}_1,\vec{1}_p\right)\eta_{r_{\alpha}}^{(n)}\left(\vec{1}_1,\vec{1}_p\right) \\
&= \eta_{\alpha}^{(n)}\left(\vec{1}_1,\vec{1}_p\right)\eta_{\alpha}^{(n)}\left(-\vec{1}_1,\vec{1}_p\right) \\
&= \eta_{r_{\alpha}}^{(n)}\left(-\vec{1}_1,\vec{1}_p\right)\eta_{r_{\alpha}}^{(n)}\left(\vec{1}_1,\vec{1}_p\right) \\
&= \eta_{f_{\alpha}}^{(n)}\left(-\vec{1}_1,\vec{1}_p;-\vec{1}_1,\vec{1}_p\right) \qquad (6.5)
\end{aligned}$$

Now in what is known as the forward-scattering theorem [7, 9, 15], the forward-scattered fields are related to the absorption and scattering cross sections, i.e. the total power removed from the incident wave (extinction cross section). Here we have found that forward scattering is invariant to inversion of $\vec{1}_1$. Therefore, the extinction cross section (sum of absorption and scattering cross sections) is also invariant to inversion of $\vec{1}_1$.

In [7] the forward-scattering theorem is extended into time domain for arbitrary incident transient waveforms with some limitations concerning boundedness and late-time behavior. It was observed that for a step-function incident wave one had to be careful concerning the order of taking r to infinity and t to infinity. The result was (in present notation)

$$\begin{aligned}
W_s + W_{\infty}^{(sc)} &\equiv \text{absorbed plus scattered energy} \\
&= \frac{1}{2}\epsilon_0 E_0^2\left\{\vec{1}_p\cdot\overleftrightarrow{P}_0\cdot\vec{1}_p \pm \left[\vec{1}_1\times\vec{1}_p\right]\cdot\overleftrightarrow{M}_0\cdot\left[\vec{1}_1\times\vec{1}_p\right]\right\} \\
E_0 &\equiv \text{magnitude of incident step-function electric field} \\
\overleftrightarrow{P}_0 &\equiv \text{low-frequency electric polarizability of scatterer} \\
\overleftrightarrow{M}_0 &\equiv \text{low-frequency electric polarizability of scatterer} \\
+ &\Leftrightarrow r\to\infty \text{ before } t\to\infty \\
- &\Leftrightarrow t\to\infty \text{ before } r\to\infty \qquad (6.6)
\end{aligned}$$

The thing to note here is that this formula is also invariant to inversion of $\vec{1}_1$. In any event due to the fact that forward scattering is invariant to inversion of $\vec{1}_1$ and applies to all frequencies (and hence transients) then time-domain absorbed plus scattered energy is invariant to inversion of $\vec{1}_1$. Note that if the scatterer is lossless then scattered energy is also invariant to inversion of $\vec{1}_1$.

VII. Inclusion of Reflection Symmetry in Scatterer

Besides the symmetries inherent in reciprocal scattering one can have geometric symmetries in the scatterer. A common such symmetry is reflection symmetry with respect to a symmetry plane P as in fig. 7.1. Here a typical aircraft is (at least approximately) such a symmetrical scatterer. As discussed in [3,5] all electromagnetic parameters (including incident and scattered fields, currents on the scatterer, etc.) can be divided into two separate parts which do not couple to each other; these two parts are designated symmetric (sy) and antisymmetric (as).

Defining the coordinate center ($\vec{r} = \vec{0}$) as the center of the minimum circumscribing sphere (radius a) let there be a cartesian coordinate system as indicated. With

$$\begin{aligned}\vec{1}_z = \vec{1}_p &\equiv \text{ unit normal vector to P} \\ &= \vec{1}_x \times \vec{1}_y\end{aligned} \tag{7.1}$$

we have a reflection dyad

$$\vec{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y - \vec{1}_z\vec{1}_z \tag{7.2}$$

which associates with every position

$$\vec{r} = x\vec{1}_x + y\vec{1}_y + z\vec{1}_z \tag{7.3}$$

a mirror position

$$\vec{r}_m \equiv \vec{R}\cdot\vec{r} = x\vec{1}_x + y\vec{1}_y - z\vec{1}_z \tag{7.4}$$

For a perfectly conducting object with surface S this means that for every position $\vec{r} \in S$ then also $\vec{r}_m \in S$. This applies to non-perfectly-conducting objects as well by applying the symmetry requirement to the constitutive parameters ε, μ, and σ, and even if they take the form of 3×3 matrices (tensors) [3,5].

The fields, currents, etc., are decomposed into symmetric and antisymmetric parts, some examples of which are [3,5]

$$\vec{E}_{\substack{sy \\ as}}(\vec{r},t) = \frac{1}{2}\left\{\vec{E}(\vec{r},t) \pm \vec{R}\cdot\vec{E}(\vec{r}_m,t)\right\}$$

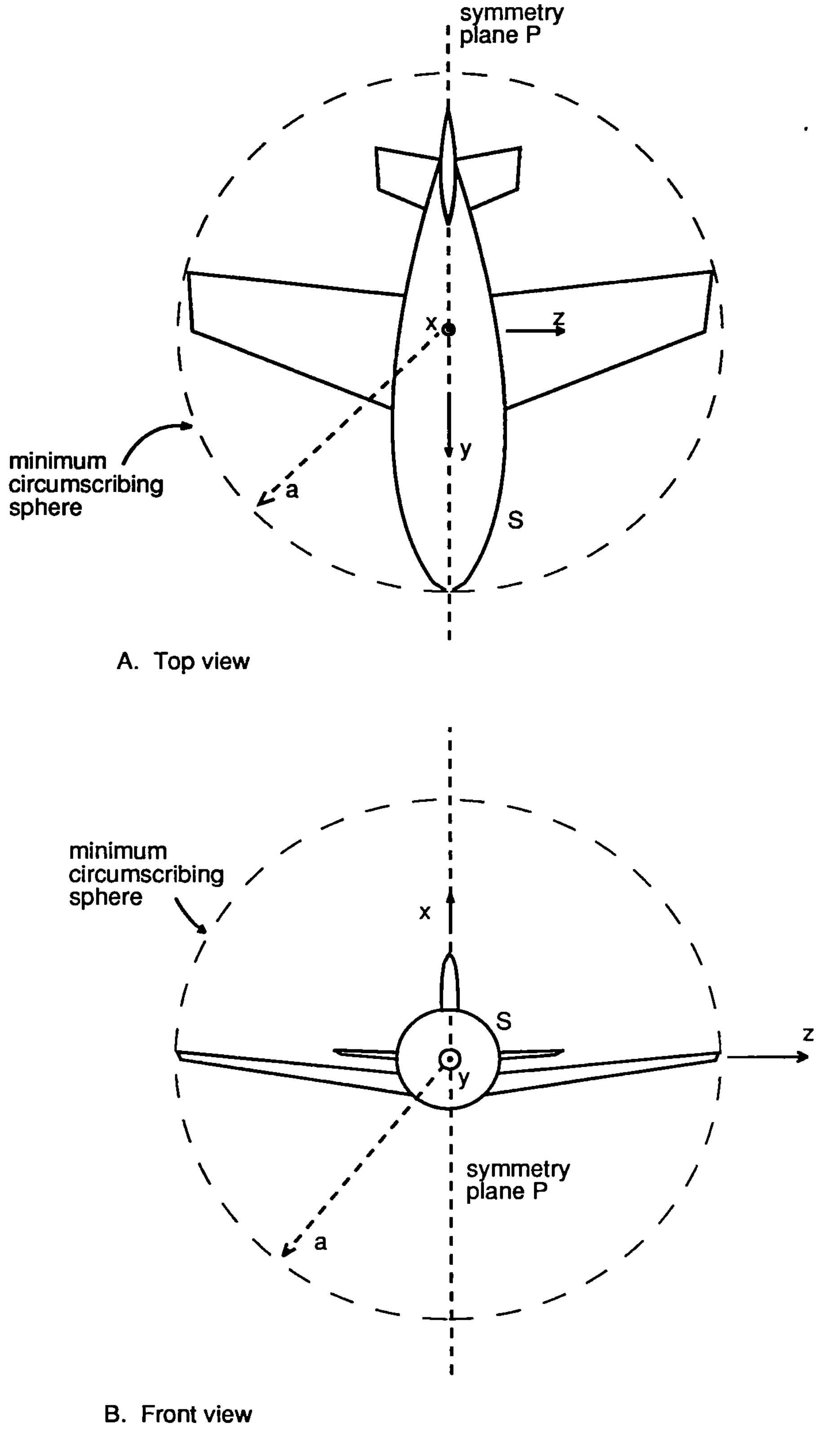

Fig. 7.1. Typical Aircraft with Symmetry Plane

$$\vec{H}_{\substack{sy\\as}}(\vec{r},t)=\frac{1}{2}\left\{\vec{H}(\vec{r},t)\mp\vec{R}\cdot\vec{H}(\vec{r}_m,t)\right\}$$

$$\vec{J}_{s_{\substack{sy\\as}}}(\vec{r},t)=\frac{1}{2}\left\{\vec{J}_s(\vec{r},t)\pm\vec{R}\cdot\vec{J}_s(\vec{r}_m,t)\right\}$$

$$\rho_{s_{\substack{sy\\as}}}(\vec{r},t)=\frac{1}{2}\left\{\tilde{\rho}_s(\vec{r},t)\pm\tilde{\rho}_s(\vec{r}_m,t)\right\} \tag{7.5}$$

$$k_{\substack{sy\\as}}(\vec{r},t)=\frac{1}{2}\left\{k(\vec{r},t)\mp k(\vec{r}_m,t)\right\}$$

where k is the equivalent magnetic charge density on S [2]. Note "sy" takes the upper sign and "as" takes the lower sign.

Let us consider the various terms that appear in the modal expansions. The eigenmodes have the symmetries as in (7.5) so that the modes in (2.6) can be divided as

$$\vec{j}_{s_{\substack{sy\\as}},\beta'}(\vec{r}_s,s)=\pm\vec{R}\cdot\vec{j}_{s_{\substack{sy\\as}},\beta'}(\vec{r}_{s_m},s)$$

$$\beta=\left\{\substack{sy\\as},\beta'\right\}\equiv \text{ partitioned index set for eigenmodes} \tag{7.6}$$

and similarly for the natural modes in (2.9) as

$$\vec{j}_{s_{\substack{sy\\as}},\alpha'}(\vec{r}_s)=\pm\vec{R}\cdot\vec{j}_{s_{\substack{sy\\as}},\alpha'}(\vec{r}_{s_m})$$

$$\alpha=\left\{\substack{sy\\as},\alpha'\right\}\equiv \text{ partitioned index set for natural modes} \tag{7.7}$$

Referring to fig. 7.2 let the incident wave be characterized by the incidence direction $\vec{1}_1$ with two orthogonal polarization vectors $\vec{1}_v$ and $\vec{1}_h$ with

$$\vec{1}_h // (y,z) \text{ plane}$$

$$\vec{1}_h\cdot\vec{1}_x=0\ ,\ \vec{1}_h\cdot\vec{1}_1=0$$

$$\vec{1}_v\cdot\vec{1}_h=0\ ,\ \vec{1}_v\cdot\vec{1}_1=0 \tag{7.8}$$

$$\vec{1}_1\times\vec{1}_v=\vec{1}_h\ ,\ \vec{1}_v\times\vec{1}_h=\vec{1}_1\ ,\ \vec{1}_h\times\vec{1}_1=\vec{1}_v$$

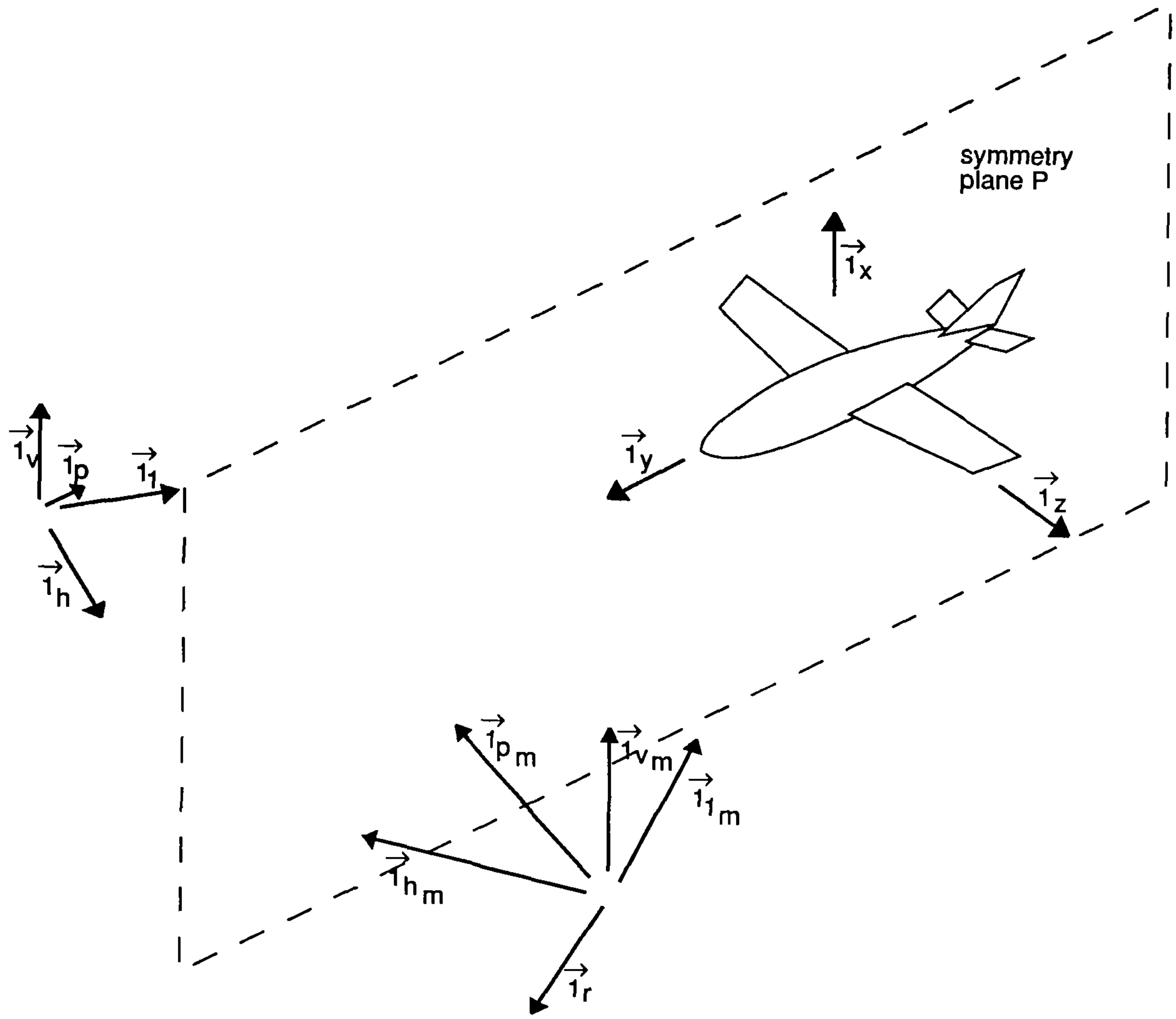

Fig. 7.2. Incident and Scattered Fields at Symmetric Positions and Orientations with Respect to Symmetry Plane

$\vec{1}_h \equiv$ horizontal polarization

$\vec{1}_v \equiv$ *"vertical" polarization - not exactly vertical, but perpendicular to* $\vec{1}_h$

Here the term vertical is used loosely, and is defined as above. Note that there is a set of mirror unit vectors (using the "electric" reflection rule in (7.5)) as

$$\vec{1}_{1_m} \equiv \vec{\vec{R}} \cdot \vec{1}_1 \ , \ \ \vec{1}_{1_v} \equiv \vec{\vec{R}} \cdot \vec{1}_v \ , \ \ \vec{1}_{h_m} \equiv \vec{\vec{R}} \cdot \vec{1}_h \tag{7.9}$$

Note that this is a left-handed system due to the reflection as

$$\vec{1}_{1m} \times \vec{1}_{v_m} = -\vec{1}_{h_m} \ , \ \ \vec{1}_{v_m} \times \vec{1}_{h_m} = -\vec{1}_{1m} \ , \ \ \vec{1}_{h_m} \times \vec{1}_{1m} = -\vec{1}_{v_m} \tag{7.10}$$

The transverse dyads reflect as

$$\begin{aligned} \vec{\vec{1}}_{1_m} &= \vec{\vec{1}} - \vec{1}_{1_m} \vec{1}_{1_m} = \vec{\vec{1}} - \left(\vec{\vec{R}} \cdot \vec{1}_1\right)\left(\vec{\vec{R}} \cdot \vec{1}_1\right) \\ &= \vec{\vec{1}} - \vec{\vec{R}} \cdot \vec{1}_1 \vec{1}_1 \cdot \vec{\vec{R}} \\ &= \vec{\vec{R}} \cdot \vec{\vec{1}}_1 \cdot \vec{\vec{R}} \end{aligned} \tag{7.11}$$

The other previously used general polarization vectors $\vec{1}_p$ and $\vec{1}_m$ also reflect as

$$\vec{1}_{p_m} = \vec{\vec{R}} \cdot \vec{1}_p \ , \ \ \vec{1}_{m_m} = \vec{\vec{R}} \cdot \vec{1}_m \tag{7.12}$$

Now let us apply the bistatic scattering formulae for the SEM coupling coefficients in section 5 to the case that

$$\vec{1}_r = -\vec{1}_{1_m} = -\vec{\vec{R}} \cdot \vec{1}_1 \ , \ \ \vec{1}_m = -\vec{1}_{p_m} = \vec{\vec{R}} \cdot \vec{1}_p \tag{7.13}$$

i.e., with scattering direction such that the receiver is in a mirror position to the transmitter and the receiver polarization is mirror to the transmitter polarization. This case can be referred to as mirror scattering. Then let us define a mirror coupling coefficient as

$$\begin{aligned} \eta_{m_\alpha}\left(\vec{1}_1, \vec{1}_p\right) &\equiv \eta_{f_\alpha}\left(\vec{1}_1, \vec{1}_p; -\vec{1}_{1_m}, \vec{1}_{p_m}\right) \\ &= \eta_\alpha\left(\vec{1}_1, \vec{1}_p\right) \eta_{r_\alpha}\left(-\vec{1}_{1_m}, \vec{1}_{p_m}\right) \\ &= \eta_\alpha\left(\vec{1}_{1_m}, \vec{1}_{p_m}\right) \eta_{r_\alpha}\left(-\vec{1}_1, \vec{1}_p\right) \\ &= \eta_{m_\alpha}\left(\vec{1}_{1_m}, \vec{1}_{p_m}\right) \end{aligned} \tag{7.14}$$

This result follows on the simple interchange of the roles of $\vec{1}_1 \leftrightarrow \vec{1}_{1_m}$ and $\vec{1}_p \leftrightarrow \vec{1}_{p_m}$ in (5.1) with the choices in (7.13). It is also just a statement of reciprocity for the chosen directions. In normalized form this is

$$\begin{aligned}\eta_{m_\alpha}^{(n)}\left(\vec{1}_1,\vec{1}_p\right) &= \eta_{f_\alpha}^{(n)}\left(\vec{1}_1,\vec{1}_p;-\vec{1}_{1_m},\vec{1}_{p_m}\right)\\ &= \eta_\alpha^{(n)}\left(\vec{1}_1,\vec{1}_p\right)\eta_\alpha^{(n)}\left(\vec{1}_{1_m},\vec{1}_{p_m}\right)\\ &= \eta_{m_\alpha}^{(n)}\left(\vec{1}_{1_m},\vec{1}_{p_m}\right)\end{aligned} \tag{7.15}$$

This better exhibits the symmetry between coupling and recoupling coefficients.

Now consider the symmetry in the coupling coefficients because of the symmetry of the natural modes in (7.7). Consider the recoupling coefficient in (5.1) applied to the present case giving

$$\begin{aligned}\eta_{r_\alpha}\left(-\vec{1}_{1_m},\vec{1}_{p_m}\right) &= -s_\alpha\mu_0\vec{1}_{p_m}\cdot\left\langle\vec{1}_{1_m}e^{-\gamma_\alpha\vec{1}_{1_m}\cdot\vec{r}_s'};\vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle\\ &= -s_\alpha\mu_0\left\langle\vec{1}_{p_m}e^{-\gamma_\alpha\vec{1}_{1_m}\cdot\vec{r}_s'};\vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle\\ &= -s_\alpha\mu_0\left\langle\vec{1}_p\cdot\overset{\leftrightarrow}{R}e^{-\gamma_\alpha\vec{1}_1\cdot\overset{\leftrightarrow}{R}\cdot\vec{r}_s'};\pm\overset{\leftrightarrow}{R}\cdot\vec{j}_{s_{\substack{sy\\as}},\alpha}\left(\vec{r}_{s_m}'\right)\right\rangle\\ &= \mp s_\alpha\mu_0\left\langle\vec{1}_p\,e^{-\gamma_\alpha\vec{1}_1\cdot\vec{r}_{s_m}'};\vec{j}_{s_{\substack{sy\\as}},\alpha}\left(\vec{r}_{s_m}'\right)\right\rangle\\ &= \mp s_\alpha\mu_0\vec{1}_p\cdot\left\langle\vec{1}_1 e^{-\gamma_\alpha\vec{1}_1\cdot\vec{r}_{s_m}'};\vec{j}_{s_{\substack{sy\\as}},\alpha}\left(\vec{r}_{s_m}'\right)\right\rangle\end{aligned} \tag{7.16}$$

Regarding the variable of integration over S as $\vec{r}_{s_m}'$ then this is proportional to $\pm$ the numerator for the coupling coefficient $\eta_\alpha\left(\vec{1}_1,\vec{1}_p\right)$. Noting that the three-term symmetric-product denominator is just a complex number we now have

$$\begin{aligned}\eta_{\substack{sy\\as},\alpha}\left(\vec{1}_1,\vec{1}_p\right) &= \pm\eta_{r_{\substack{sy\\as}},\alpha}\left(-\vec{1}_{1_m},\vec{1}_{p_m}\right)\frac{1}{-s_\alpha\mu_0}\left\langle\vec{j}_{s_\alpha}\left(\vec{r}_s\right);\frac{\partial}{\partial s}\overset{\leftrightarrow}{Z}_t\left(\vec{r},\vec{r}_s';s\right)\Big|_{s=s_\alpha};\vec{j}_{s_\alpha}\left(\vec{r}_s'\right)\right\rangle^{-1}\\ &= \pm\eta_{\substack{sy\\as},\alpha}\left(\vec{1}_{1_m},\vec{1}_{p_m}\right)\end{aligned} \tag{7.17}$$

In normalized form this is just

$$\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_p\right)=\pm\,\eta^{(n)}_{r\,\substack{sy\\as},\alpha'}\left(-\vec{1}_{1m},\vec{1}_{pm}\right)$$

$$=\pm\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_{1m},\vec{1}_{pm}\right) \tag{7.18}$$

These results are just the obvious extension of those for the natural modes in (7.7). Noting from (5.8) a symmetry on inversion of the polarization we also have

$$\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_p\right)=\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_{1m},\pm\vec{1}_{pm}\right) \tag{7.19}$$

Thus an alternate way to look at coupling to antisymmetric modes is to reverse $\vec{1}_{p_m}$.

Applying these results to (7.15) gives

$$\eta^{(n)}_{m\,\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_p\right)=\eta^{(n)}_{m\,\substack{sy\\as},\alpha'}\left(\vec{1}_{1m},\vec{1}_{pm}\right)$$

$$=\pm\left[\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_p\right)\right]^2$$

$$=\pm\left[\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_{1m},\vec{1}_{pm}\right)\right]^2 \tag{7.20}$$

So the mirror coupling coefficient is also expressible as the square of a coupling coefficient, reminiscent of the result in (5.11) for the backscattering coupling coefficient. There is a difference in the signs, however, depending on whether symmetric or antisymmetric modes are being considered. In terms of the backscattering coupling coefficient in (5.11) the mirror coupling coefficient is

$$\eta^{(n)}_{m\,\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_p\right)=\pm\eta^{(n)}_{b\,\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_p\right)=\pm\eta^{(n)}_{b\,\substack{sy\\as},\alpha'}\left(\vec{1}_{1m},\vec{1}_{pm}\right) \tag{7.21}$$

This shows that mirror scattering is simply related to backscattering with a sign dependent on mode type.

To understand this a little better consider the orientation of the polarization $\vec{1}_p$ in fig. 7.2. This can be taken successively as one of two orthogonal polarizations $\vec{1}_v$ and $\vec{1}_h$. Consider the case that $\vec{1}_1$ is parallel to P as in fig. 7.3, i.e.,

$$\vec{1}_1 \cdot \vec{1}_z = 0 \ , \ \vec{1}_{1_m} = \vec{1}_1 \qquad (7.22)$$

$$\vec{1}_v = \vec{1}_{v_m} \ , \ \vec{1}_h = -\vec{1}_{h_m}$$

noting the opposite ways that $\vec{1}_v$ and $\vec{1}_h$ reflect. From (7.18), (7.19) and (5.8) we have

$$\begin{aligned} \eta^{(n)}_{as,\alpha'}(\vec{1}_1,\vec{1}_v) &= -\eta^{(n)}_{as,\alpha'}(\vec{1}_1,\vec{1}_v) = 0 \\ \eta^{(n)}_{sy,\alpha'}(\vec{1}_1,\vec{1}_h) &= -\eta^{(n)}_{sy,\alpha'}(\vec{1}_1,\vec{1}_h) = 0 \end{aligned} \qquad (7.23)$$

So vertical only couples to symmetric (and only scatters symmetric) and horizontal only couples to antisymmetric (and only scatters antisymmetric) provided $\vec{1}_1 // P$. This is summarized in the backscattering formulae (as in (5.11))

$$\begin{aligned} \eta^{(n)}_{b_{\substack{sy\\as}},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{v\\h}}\right) &= \left[\eta^{(n)}_{\substack{sy\\as},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{v\\h}}\right)\right]^2 \\ &= \eta^{(n)}_{f_{\substack{sy\\as}},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{v\\h}};-\vec{1}_1,\vec{1}_{\substack{v\\h}}\right) \\ &= \pm\eta^{(n)}_{m_{\substack{sy\\as}},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{v\\h}}\right) \end{aligned}$$

$$\begin{aligned} \eta^{(n)}_{f_{\substack{sy\\as}},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{v\\h}};-\vec{1}_1,\vec{1}_{\substack{h\\v}}\right) &= \eta^{(n)}_{f_{\substack{sy\\as}},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{h\\v}};-\vec{1}_1,\vec{1}_{\substack{v\\h}}\right) \\ &= \eta^{(n)}_{f_{\substack{sy\\as}},\alpha'}\left(\vec{1}_1,\vec{1}_{\substack{v\\h}};-\vec{1}_1,\vec{1}_{\substack{h\\v}}\right) \\ &= 0 \end{aligned} \qquad (7.24)$$

Note that in the various pairs of symbols the upper symbols go together as do the lower symbols.

Consider for $\vec{1}_1 // P$ that first one uses vertical polarization. The backscattered field is not only vertically polarized, it also has only symmetric natural frequencies $s_{sy,\alpha'}$. Now second use horizontal

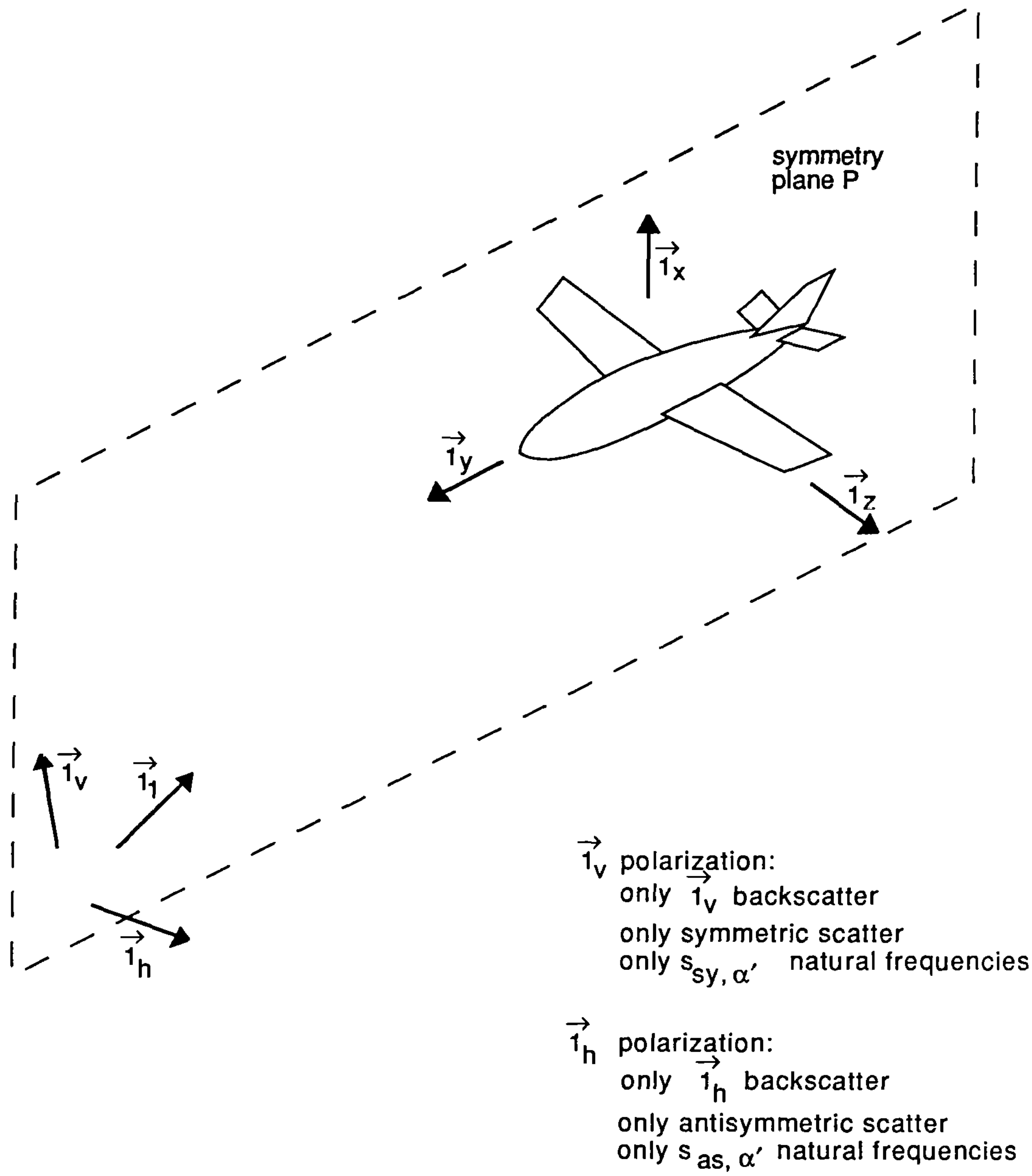

Fig. 7.3. Incident Fields Propagating Parallel
to Symmetry Plane

polarization. The backscattered field is not only horizontally polarized, it also has only antisymmetric natural frequencies $s_{as,\alpha'}$. This gives a technique for separating the natural frequencies of a scatterer into two separate sets in the complex frequency plane. In an experimental configuration this can be used to more accurately determine the s_α and help identify the scatterer by having two separate pole patterns at which to look.

Now there is the question of accuracy of alignment of $\vec{1}_v // P$ and $\vec{1}_h \perp P$. To the extent that there is some rotation of these unit vectors with $\vec{1}_1$ as the axis of rotation, this can be detected in the backscattered signals by cross polarization, i.e., $\vec{1}_v$ transmission scattering into $\vec{1}_h$ reception, and conversely. Suppose this angle of rotation is ϕ then from (7.23) we have the definition of $\phi = 0$ by the angle require to make the cross coupling zero. Since the coupling coefficients are proportional to the dot product of $\vec{1}_v$ and $\vec{1}_h$ with a vector then we have $\sin(\phi)$ and $\cos(\phi)$ variation of the various $\eta_\alpha^{(n)}$. The vertical and horizontal (in-line) backscattering goes like $\cos^2(\phi)$ while the cross polarized part goes like $\cos(\phi)\sin(\phi)$, or to first order like ϕ. So one could correct the data by numerically adjust ϕ to rotate the coordinates to make the cross-polarized part zero and separate out the symmetric and antisymmetric parts and thereby the $s_{sy,\alpha'}$ and $s_{as,\alpha'}$.

Concerning deviations of $\vec{1}_1$ from being parallel to P let ξ be the angle between $\vec{1}_1$ and P. Assuming that the coupling coefficients have bounded derivatives with respect to $\vec{1}_1$ variation, then the deviations away from the nulls in (7.23) are at most first order in ξ. This means that cross polarized backscattering is first order in ξ. Furthermore, in-line backscattering (for both vertical and horizontal polarizations) deviates from the $\xi = 0$ result by an error which is of order ξ^2 (due to the square for the backscattering coupling coefficient $\eta_b^{(n)}$ in (7.24)).

So for deviations of both direction of incidence and polarization that are small with respect to their ideal orientations with respect to P as in fig. 7.3, the errors are also small. This experimental configuration may then prove of practical significance.

VIII. Concluding Remarks

Symmetry then is a powerful concept in simplifying otherwise (more) complex problems. Reciprocity is a fundamental symmetry in the Maxwell equations for the case of suitably simple media (symmetric constitutive matrices). This leads to symmetry between incident and scattered far fields with special results for backscattering and forward scattering. Geometrical symmetry in a scatterer gives additional simplification to scattering, such as for the case that incidence and scattering directions are parallel to a symmetry plane.

These results can be used to guide scattering experiments. For SEM poles the coupling coefficients for surface currents are simply related to backscattering coupling coefficients. Furthermore a symmetry plane in a scatterer can be used to separate the natural frequencies into two sets in the backscattering.

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