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Scattering of Transient Plane Waves

Carl E. Baum
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Abstract

The classical forward-scattering theorem is studied here in some detail. It is generalized to complex frequencies and to somewhat arbitrary incident pulsed plane waves. Considering a step-function plane wave, an ambiguity is noted concerning the results depending on the order in which limits are taken concerning the pulse width and the radius of the sphere of integration tending to ∞. The difference is accounted for in terms of low-frequency backscattering.

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CONTENTS

I. Introduction

A classical problem in electromagnetic theory is the evaluation of the energy absorbed and scattered from a plane wave by a finite-sized scatterer in free space. There is a well known result known as the forward-scattering theorem (or "optical" theorem) [7, 12]. This relates the absorbed plus scattered real power in frequency domain ($s = j\omega$) to the imaginary part of the scattering amplitude in the forward direction, i.e. the direction of propagation of the incident plane wave.

In this paper we generalize the forward-scattering theorem to complex frequency domain defined via the Laplace transform (two sided) as

$$\tilde{g}(s) \equiv \int_{-\infty}^{\infty} g(t) e^{-st}\, dt$$

$$g(t) = \frac{1}{2\pi j} \int_{Br} \tilde{g}(s) e^{st}\, ds \tag{1.1}$$

Br ≡ Bromwich contour in strip of convergence parallel to $j\omega$ axis
$\sim \Rightarrow$ Laplace−transformed quantity
$s = \Omega + j\omega \equiv$ Laplace−transform variable
$\equiv$ complex frequency

While this can be considered as an analytic-continuation procedure, there are some advantages in considering this problem via the time domain.

A recent paper [8] extended the forward-scattering theorem to the case of a step-function incident wave via a Hilbert transform. The present paper generalizes the forward-scattering theorem to essentially arbitrary incident pulse plane waves. This is accomplished by first assuming that the incident wave is a time-limited pulse so that the integral of the product of incident and scattered fields on the sphere at ∞ occurs only near the forward-scattering direction. Through Laplace transforms this is related to the usual scattering amplitude, but now as a function of the complex frequency.

Extending the pulse width to ∞ reproduces the step-function results in [8]. However, this result disagrees with a result next obtained by considering an incident step-function wave for which the product of incident and scattered fields must be integrated over the entire sphere at ∞. This discrepancy is related to the order of two limits, letting the pulse width $\rightarrow\infty$ and letting the sphere of integration $\rightarrow\infty$. The difference in the two results is given by a combination of electric and magnetic dipoles which does not contribute in the forward-scattering direction, but contributes strongly in the backscattering direction near $s = 0$.

II. The Poynting-Vector Theorem in Time Domain

Write the Maxwell equations in time domain as

$$\nabla \times \vec{E}(\vec{r},t) = \vec{J}_h(\vec{r},t) - \frac{\partial}{\partial t}\vec{B}(\vec{r},t) \equiv -\vec{J}_{h_t}(\vec{r},t)$$
$$\equiv - \text{total magnetic current density}$$

$$\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t) \equiv \vec{J}_t(\vec{r},t)$$
$$\equiv \text{total electric current density} \tag{2.1}$$

These current densities can take various forms by imposing various constitutive relations, simple ones being of the form

$$\begin{aligned}
\vec{J}_h(\vec{r},t) &= \overleftrightarrow{\sigma}_h(\vec{r}) \bullet \vec{H}(\vec{r},t) \\
\vec{B}(\vec{r},t) &= \overleftrightarrow{\mu}(\vec{r}) \bullet \vec{H}(\vec{r},t) \\
\vec{J}_{h_t}(\vec{r},t) &= [\overleftrightarrow{\sigma}_h(\vec{r}) + \overleftrightarrow{\mu}(\vec{r})\frac{\partial}{\partial t}] \bullet \vec{H}(\vec{r},t) \\
\vec{J}(\vec{r},t) &= \overleftrightarrow{\sigma}(\vec{r}) \bullet \vec{E}(\vec{r},t) \\
\vec{D}(\vec{r},t) &= \overleftrightarrow{\in}(\vec{r}) \bullet \vec{E}(\vec{r},t) \\
\vec{J}_t(\vec{r},t) &= [\overleftrightarrow{\sigma}(\vec{r}) + \overleftrightarrow{\in}(\vec{r})\frac{\partial}{\partial t}] \bullet \vec{H}(\vec{r},t)
\end{aligned} \tag{2.2}$$

Of course the constitutive parameters can be functions of frequency which makes convolutions appear in time domain as in (2.2). The introduction of the magnetic current density gives symmetry to (2.1), and while $\overleftrightarrow{\sigma}_h$ (at least in a zero-frequency sense) is not known to exist, a general form of $\overleftrightarrow{\mu}(\vec{r})$ with frequency dependence (including losses) achieves the same thing. For the special (but important) case of free space we have

$$\begin{aligned}
\vec{J}_{h_t}(\vec{r},t) &= \mu_o \frac{\partial}{\partial t}\vec{H}(\vec{r},t) \\
\vec{J}_t(\vec{r},t) &= \in_o \frac{\partial}{\partial t}\vec{E}(\vec{r},t)
\end{aligned} \tag{2.3}$$

Use a vector identity [18]

$$\nabla \bullet [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] = -\vec{E}(\vec{r},t) \bullet [\nabla \times \vec{H}(\vec{r},t)] + \vec{H}(\vec{r},t) \bullet [\nabla \times \vec{E}(\vec{r},t)]$$
$$= -\vec{E}(\vec{r},t) \bullet \vec{J}_t(\vec{r},t) - \vec{H}(\vec{r},t) \bullet \vec{J}_{h_t}(\vec{r},t) \tag{2.4}$$

Applying the Gauss divergence theorem to some volume V within an exterior boundary surface S we have

$$\int_S [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] \bullet \vec{1}_S ds = -\int_V [\vec{E}(\vec{r},t) \bullet \vec{J}_t(\vec{r},t) + \vec{H}(\vec{r},t) \bullet \vec{J}_{h_t}(\vec{r},t)] dV$$

$$\vec{1}_S \equiv \text{outward pointing normal on S} \tag{2.5}$$

If this form of the Poynting-vector theorem doesn't look familiar, consider the current-density terms in the volume integral as

$$\vec{E}(\vec{r},t) \bullet \vec{J}_t(\vec{r},t) = \vec{E}(\vec{r},t) \bullet \vec{J}(\vec{r},t) + \vec{E}(\vec{r},t) \bullet \frac{\partial}{\partial t}\vec{D}(\vec{r},t)$$
$$\vec{H}(\vec{r},t) \bullet \vec{J}_{h_t}(\vec{r},t) = \vec{H}(\vec{r},t) \bullet \vec{J}_h(\vec{r},t) + \vec{H}(\vec{r},t) \bullet \frac{\partial}{\partial t}\vec{D}(\vec{r},t) \tag{2.6}$$

In this form one often identifies the first terms as power loss to the medium and the second terms as the time rate of change of the electromagnetic energy density. In free space the above simplifies to

$$\vec{E}(\vec{r},t) \bullet \vec{J}_t(\vec{r},t) + \vec{H}(\vec{r},t) \bullet \vec{J}_{h_t}(\vec{r},t) = \frac{\partial}{\partial t}U(\vec{r},t)$$
$$U(\vec{r},t) = \frac{1}{2} \epsilon_o \vec{E}(\vec{r},t) \bullet \vec{E}(\vec{r},t) + \frac{1}{2}\mu_o \vec{H}(\vec{r},t) \bullet \vec{H}(\vec{r},t) \tag{2.7}$$

with U now being the energy density in the fields.

For a transient problem with V taken as free space we have

$$\int_S [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] \bullet \vec{1}_S dS = -\int_V \frac{\partial}{\partial t} U(\vec{r},t) dV \tag{2.8}$$

and integrating over all time gives

$$\int_{-\infty}^{\infty} \int_{S} [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] \bullet \vec{1}_S \, dt = - \int_{V} U(\vec{r},\infty) dV$$

$$U(\vec{r},\infty) = \frac{1}{2} \in_o \vec{E}(\vec{r},\infty) \bullet \vec{E}(\vec{r},\infty) + \frac{1}{2} \mu_o \vec{H}(\vec{r},\infty) \bullet \vec{H}(\vec{r},\infty) \tag{2.9}$$

Here the fields in V are assumed to be initially zero and the final values to be bounded (a reasonable finite-energy condition).

III. The Poynting-Vector Theorem in Complex-Frequency Domain

In complex-frequency domain the Maxwell equations are

$$\nabla \times \tilde{\vec{E}}(\vec{r},s) = -\tilde{\vec{J}}_h(\vec{r},s) - s\,\tilde{\vec{B}}(\vec{r},s) \equiv -\tilde{\vec{J}}_{h_t}(\vec{r},s)$$

$$\nabla \times \tilde{\vec{H}}(\vec{r},s) = \tilde{\vec{J}}(\vec{r},s) + s\,\tilde{\vec{D}}(\vec{r},s) \equiv \tilde{\vec{J}}_t(\vec{r},s) \tag{3.1}$$

Allowing the constitutive parameters to be functions of s we have

$$\tilde{\vec{J}}_h(\vec{r},s) = \tilde{\overleftrightarrow{\sigma}}_h(\vec{r},s) \bullet \tilde{\vec{H}}(\vec{r},s)$$

$$\tilde{\vec{B}}(\vec{r},s) = \tilde{\overleftrightarrow{\mu}}(\vec{r},s) \bullet \tilde{\vec{H}}(\vec{r},s)$$

$$\tilde{\vec{J}}_{h_t}(\vec{r},s) = [\,\tilde{\overleftrightarrow{\sigma}}_h(\vec{r},s) + s\,\tilde{\overleftrightarrow{\mu}}(\vec{r},s)\,] \bullet \tilde{\vec{H}}(\vec{r},s)$$

$$\tilde{\vec{J}}(\vec{r},s) = \tilde{\overleftrightarrow{\sigma}}(\vec{r},s) \bullet \tilde{\vec{E}}(\vec{r},s) \tag{3.2}$$

$$\tilde{\vec{D}}(\vec{r},s) = \tilde{\overleftrightarrow{\epsilon}}(\vec{r},s) \bullet \tilde{\vec{E}}(\vec{r},s)$$

$$\tilde{\vec{J}}_t(\vec{r},s) = [\,\tilde{\overleftrightarrow{\sigma}}(\vec{r},s) + s\,\tilde{\overleftrightarrow{\epsilon}}(\vec{r},s)\,] \bullet \tilde{\vec{E}}(\vec{r},s)$$

In complex-frequency form one can then consider $\overleftrightarrow{\sigma}_h + s\,\overleftrightarrow{\mu}$ and $\overleftrightarrow{\sigma} + s\,\overleftrightarrow{\epsilon}$ as some combined or effective constitutive parameters. In time domain such become convolution operators, generalizing the form in (2.2).

Using the same vector identity as in (2.4) we have

$$\nabla \bullet [\,\tilde{\vec{E}}(\vec{r},s) \times \tilde{\vec{H}}(\vec{r},-s)\,] = -\tilde{\vec{E}}(\vec{r},s) \bullet [\nabla \times \tilde{\vec{H}}(\vec{r},-s)\,] - \tilde{\vec{H}}(\vec{r},-s) \bullet [\nabla \times \tilde{\vec{E}}(\vec{r},s)\,]$$

$$= -\tilde{\vec{E}}(\vec{r},-s) \bullet \tilde{\vec{J}}_t(\vec{r},-s) - \tilde{\vec{H}}(\vec{r},-s) \bullet \tilde{\vec{J}}_{h_t}(\vec{r},s) \tag{3.3}$$

Again applying the Gauss divergence theorem we have

$$\int_S [\, \tilde{\vec{E}}(\vec{r},s) \times \tilde{\vec{H}}(\vec{r},s) \,] \bullet \vec{1}_S dS$$

$$= -\int_V [\, \tilde{\vec{E}}(\vec{r},s) \times \tilde{\vec{J}}_t(\vec{r},-s)] + \tilde{\vec{H}}(\vec{r},-s) \times \tilde{\vec{J}}_{h_t}(\vec{r},s)]dV \qquad (3.4)$$

This combination of electric field evaluated at s and magnetic field evaluated at -s is one of the several forms considered in [5]. This is the form of the Poynting-vector theorem in complex-frequency domain. Note that in the volume integral both terms have the <u>same sign</u> just like in time domain in (2.5). The difference in form is given by the combination of functions of s with functions of -s.

Relating time to frequency domains is conveniently done for energy per unit area as (per appendix A)

$$\int_{-\infty}^{\infty} \tilde{\vec{E}}(\vec{r},t) \times \tilde{\vec{H}}_t(\vec{r},t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{E}}(\vec{r},s) \times \tilde{\vec{H}}(\vec{r},-s)ds$$

$$= \frac{1}{2\pi j} \int_{Br} \tilde{\vec{E}}(\vec{r},-s) \times \tilde{\vec{H}}(\vec{r},s)ds \qquad (3.5)$$

Similarly for energy per unit volume (including stored and dissipated) there is the electric part

$$\int_{-\infty}^{\infty} \tilde{\vec{E}}(\vec{r},t) \bullet \tilde{\vec{J}}_t(\vec{r},t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{E}}(\vec{r},s) \bullet \tilde{\vec{J}}_t(\vec{r},-s)ds$$

$$= \frac{1}{2\pi j} \int_{Br} \tilde{\vec{E}}(\vec{r},-s) \bullet \tilde{\vec{J}}_t(\vec{r},s)ds \qquad (3.6)$$

and the magnetic part

$$\int_{-\infty}^{\infty} \vec{\tilde{H}}(\vec{r},t) \bullet \vec{\tilde{J}}_{h_t}(\vec{r},t)\,dt = \frac{1}{2\pi j}\int_{Br} \vec{\tilde{H}}(\vec{r},s) \bullet \vec{\tilde{J}}_{h_t}(\vec{r},-s)ds$$

$$= \frac{1}{2\pi j}\int_{Br} \vec{\tilde{H}}(\vec{r},-s) \bullet \vec{\tilde{J}}_{h_t}(\vec{r},s)ds \tag{3.7}$$

When integrating over all time which is equivalent to integrating over s on the Bromwich contour, note the symmetry between s and -s (or between t and -t for that matter). At least then in this integral form the Poynting-vector theorems in time domain (2.5) and frequency domain (3.7) are equivalent.

Now letting V be characterized as free space (3.6) becomes (using (2.9)).

$$\int_{-\infty}^{\infty} \vec{E}(\vec{r},t) \bullet [\in_o \frac{\partial}{\partial t}\vec{E}(\vec{r},t)]dt = \frac{1}{2\pi j}\int_{Br+} \vec{\tilde{E}}(\vec{r},s) \bullet [-s \in_o \vec{\tilde{E}}(\vec{r},-s)\ ]ds$$

$$= \frac{1}{2\pi j}\int_{Br-} \vec{\tilde{E}}(\vec{r},-s) \bullet [s \in_o \vec{\tilde{E}}(\vec{r},s)\ ]ds$$

$$= \frac{1}{2} \in_o \vec{E}(\vec{r},\infty) \bullet \vec{E}(\vec{r},\infty) \tag{3.8}$$

and (3.7) becomes

$$\int_{-\infty}^{\infty} \vec{H}(\vec{r},t) \bullet [\mu_o \frac{\partial \vec{H}}{\partial t}(\vec{r},t)]dt = \frac{1}{2\pi j}\int_{Br+} \vec{\tilde{H}}(\vec{r},s) \bullet [-s\mu_o\ \vec{\tilde{H}}(\vec{r},-s)\ ]ds$$

$$= \frac{1}{2\pi j}\int_{Br-} \vec{\tilde{H}}(\vec{r},-s) \bullet [s\mu_o\ \vec{\tilde{H}}(\vec{r},s)]ds$$

$$= \frac{1}{2}\mu_o \vec{H}(\vec{r},\infty) \bullet \vec{H}(\vec{r},\infty) \tag{3.9}$$

One may think that these integrals must be zero since in s domain each equals the negative of itself, or equivalently, the integrands are odd in s. However, this only applies if there is a Bromwich contour for the common convergence of the two Laplace transforms (one of s, the other of -s). If in time domain there exist non-zero fields at $t = \infty$, the Laplace transforms have poles at s=0 and the Bromwich contour cannot squeeze between two such poles. The above results have to be understood in some limiting sense, properly derived from the time-domain form which utilizes the time-domain energy density as in (2.7). Considered more carefully note that the terms from the electric and magnetic current density involve $\pm s$ times the field transform and so do not give a pole at s=0. So the two forms are not the same in that the Bromwich contour goes to the right of s=0 in the first case (first term a function of s) and to the left of s=0 in the second case (first term a function of-s). One could distinguish these contours as Br+ and Br- (if desired) as indicated in (3.8) and (3.9).

Going back to (3.4) notice that in the case of free space the frequency-domain Poynting vector theorem becomes

$$\int_S [\, \tilde{\vec{E}}(\vec{r}, s) \times \tilde{\vec{H}}(\vec{r}, -s)\,] \bullet \vec{1}_S \, dS = \int_V [\, \epsilon_0 \, s \, \tilde{\vec{E}}(\vec{r}, s) \bullet \tilde{\vec{E}}(\vec{r}, -s) - \mu_0 \, s \, \tilde{\vec{H}}(\vec{r}, -s) \bullet \tilde{\vec{H}}(\vec{r}, s)\,] dV \tag{3.10}$$

While one can note the sign difference of the two terms in the volume integral and the fact that this is different from the case of time domain as in (2.8) and (2.7)[10], the basic reason concerns the requirement to reverse the sign on s of the electric or magnetic field term (and the corresponding magnetic- or electric-current-density term). This is required by the generalized Parseval theorem (appendix A) relating such energy-like constructions in time and frequency domains. Note that while (3.10) uses -s in $\vec{H}$, it could equally well have been done in $\vec{E}$ instead with all s replaced by -s and conversely in (3.4) and (3.10).

IV. Scattering of a Plane Wave

As in Fig. 4.1 consider S as being in two parts, with

$$
\begin{aligned}
S_s &\equiv \text{scatterer surface with outward pointing normal } \vec{1}_{S_s} \\
&\equiv \text{outer boundary of } V_s \ (\equiv \text{scatterer volume}) \\
S_\infty &\equiv \text{sphere of radius } r_\infty \text{ (large)} \\
&\quad \text{with outward pointing normal } \vec{1}_r \\
V_{ext} &\equiv \text{volume bounded by } S_s \cup S_\infty \\
&\equiv \text{free space}
\end{aligned}
\tag{4.1}
$$

The total fields are given by

$$
\begin{aligned}
\vec{E}(\vec{r}, t) &= \vec{E}^{(inc)}(\vec{r}, t) + \vec{E}^{(sc)}(\vec{r}, t) \\
\vec{H}(\vec{r}, t) &= \vec{H}^{(inc)}(\vec{r}, t) + \vec{H}^{(sc)}(\vec{r}, t)
\end{aligned}
\tag{4.2}
$$

The incident fields are specified as a plane wave with linear polarization as

$$
\begin{aligned}
\vec{E}^{(inc)}(\vec{r},t) &= \vec{1}_2 E_o f(t - \vec{1}_1 \bullet \vec{r}/c) + \vec{E}^{(sc)}(\vec{r},t) \\
\tilde{\vec{E}}^{(inc)}(\vec{r},s) &= \vec{1}_2 E_o \tilde{f}(s) e^{-\gamma \vec{1}_1 \bullet \vec{r}} \\
\vec{H}^{(inc)}(\vec{r},t) &= \vec{1}_3 H_o f(t - \vec{1}_1 \bullet \vec{r}/c) \\
\tilde{\vec{H}}^{(inc)}(\vec{r},s) &= \vec{1}_3 H_o \tilde{f}(s) e^{-\gamma \vec{1}_1 \bullet \vec{r}}
\end{aligned}
\tag{4.3}
$$

$$
E_o = Z_o H_o
$$

$$
Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \equiv \text{wave impedance of free space}
$$

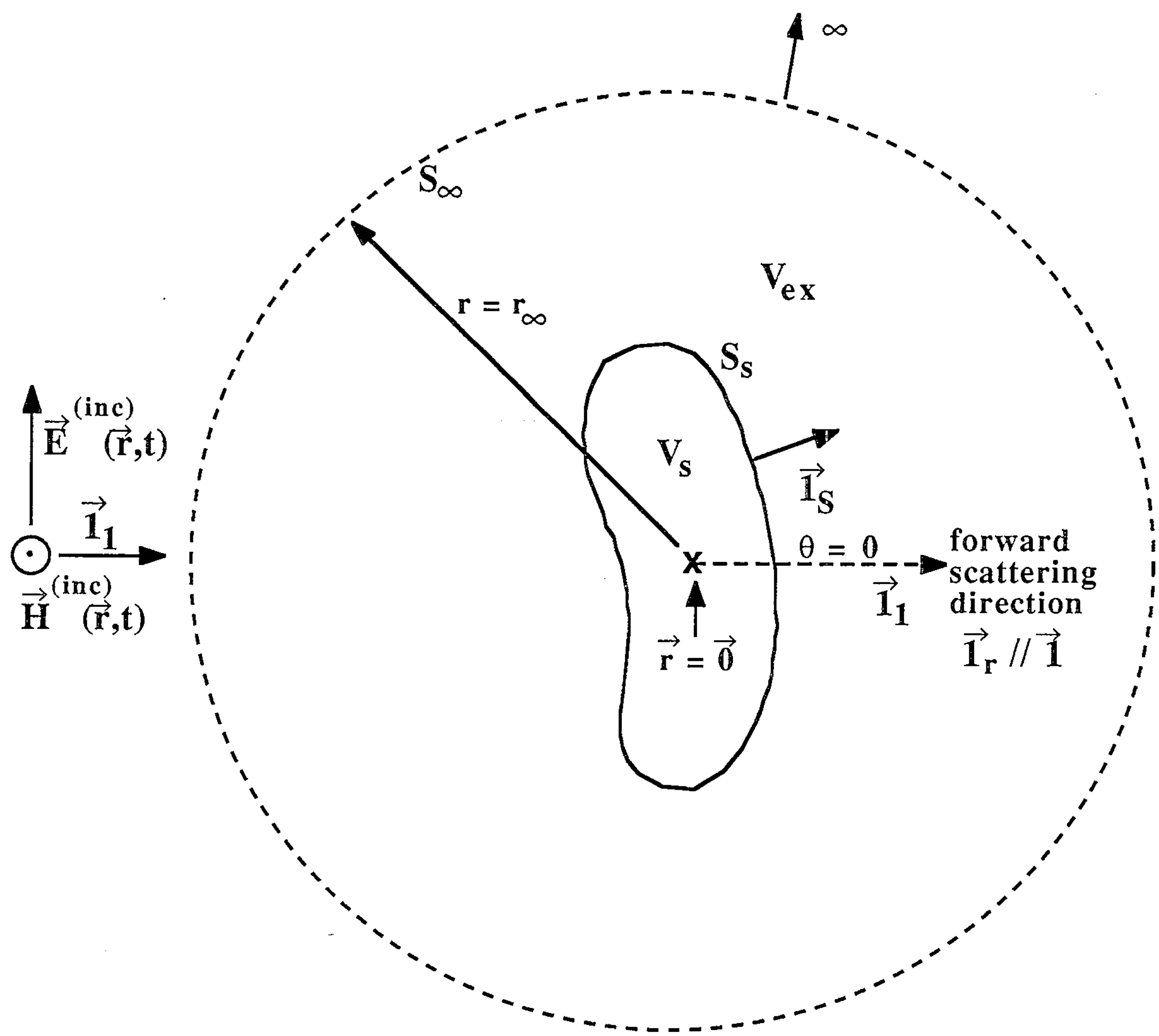

Figure 4.1 Scattering of a plane wave.

$c = \frac{1}{\sqrt{\mu_o \epsilon_o}} \equiv$ speed of light

$\gamma \equiv \frac{s}{c} \equiv$ propagation constant

$f(t) \equiv$ waveform (normalized by E_o)

$\vec{1}_1 \times \vec{1}_2 = \vec{1}_3, \; \vec{1}_2 \times \vec{1}_3 = \vec{1}_1, \; \vec{1}_3 \times \vec{1}_1 = \vec{1}_2$

The scattered fields are much more complicated but have asymptotic forms for large r as

$$\tilde{\vec{E}}^{(sc)}(\vec{r}, s) = \left[\frac{1}{r}\tilde{\vec{V}}_f(\theta, \phi, s) + 0(r^{-2})\right]e^{-\gamma r}$$

$$\tilde{\vec{H}}^{(sc)}(\vec{r}, s) = \left[\frac{1}{r}\tilde{\vec{I}}_f(\theta, \phi, s) + 0(r^{-2})\right]e^{-\gamma r}$$

$$\tilde{\vec{V}}_f(\theta, \phi, s) \bullet \vec{1}_r(\theta, \phi, s) \equiv 0, \; \tilde{\vec{I}}_f(\theta, \phi, s) \bullet \vec{1}_r(\theta, \phi) = 0$$

$$\vec{1}_r(\theta, \phi) \times \tilde{\vec{V}}_f(\theta, \phi, s) = Z_o \tilde{\vec{I}}_f(\theta, \phi, s), \; Z_o \vec{1}_r(\theta, \phi) \times \tilde{\vec{I}}_f(\theta, \phi, s) = \tilde{\vec{V}}_f(\theta, \phi, s)$$

$$\tilde{\vec{V}}_f(\theta, \phi, s) \equiv E_o \tilde{f}(s) \tilde{\vec{v}}_f(\theta, \phi, s) \qquad (4.4)$$

$$\tilde{\vec{I}}_f(\theta, \phi, s) \equiv H_o \tilde{f}(s) \vec{1}_r(\theta, \phi) \times \tilde{\vec{v}}_f(\theta, \phi, s)$$

$$\tilde{\vec{v}}_f(\theta, \phi, s) \equiv \text{normalized far field}$$

In time domain the far field is given by a convolution of $\vec{v}_f(\theta, \phi, t)$ with f(t-r/c) where now retarded time is t-r/c as distinguished from the retarded time for the incident plane wave which is $t - (\vec{1}_1 \bullet \vec{r}) / c$. Note that $\vec{r} = \vec{0}$ is defined somewhere in the immediate vicinity of the scatterer, perhaps even in V_s.

For S_∞ we have (with convention of outward real power positive)

$$P_\infty(t) \equiv \int_{S_\infty} [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] \bullet \vec{1}_r dS$$

$$\tilde{p}_\infty(s) \equiv \int_{S_\infty} \left[\tilde{\vec{E}}(\vec{r},s) \times \tilde{\vec{H}}(\vec{r},-s)\right] \bullet \vec{1}_r dS \tag{4.5}$$

$$W_\infty = \int_{-\infty}^{\infty} P_\infty(t)dt = \frac{1}{2\pi j}\int_{Br} \tilde{p}_\infty(s)ds = \frac{1}{2\pi j}\int_{Br} \tilde{p}_\infty(-s)ds$$

Note that r_∞ is taken as some large, but not infinite, radius. It is only allowed to tend to ∞ after the integals have been performed. Note for the above

$$\begin{aligned} &Re[\tilde{p}_\infty(j\omega)] \leq 0 \\ &W_\infty \leq 0 \end{aligned} \tag{4.6}$$

which expresses the fact that the scatterer is passive and cannot emit real power for any $s=j\omega$. Note also that in each case $\tilde{p}_n(s)$ is not the Laplace transform of $P_n(t)$, this fact being denoted by the use of lower case in s domain but capital in t domain.

Consider now the scatterer for which we have (with convention of positive real power into the scatterer)

$$P_s(t) \equiv -\int_{S_s} [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] \bullet \vec{1}_S dS$$

$$\tilde{p}_s(s) \equiv -\int_{S_s} \left[\tilde{\vec{E}}(\vec{r},s) \times \tilde{\vec{H}}(\vec{r},s)\right] \bullet \vec{1}_S dS \tag{4.7}$$

$$W_s \equiv \int_{-\infty}^{\infty} P_s(t)dt = \frac{1}{2\pi j}\int_{Br} \tilde{p}_s(s)ds = \frac{1}{2\pi j}\int_{Br} \tilde{p}_s(-s)ds$$

Again

$$\begin{aligned} &Re[\tilde{p}_s(j\omega)] \geq 0 \\ &W_s \geq 0 \end{aligned} \tag{4.8}$$

In V_{ex} we have (with convention of positive real power into V_{ex})

$$P_{ex}(t) = \frac{\partial}{\partial t} \int_{V_{ex}} U(\vec{r},t) dV$$

$$U(\vec{r},t) = \frac{1}{2} \epsilon_o \vec{E}(\vec{r},t) \bullet \vec{E}(\vec{r},t) + \frac{1}{2} \mu_o \vec{H}(\vec{r},t) \bullet \vec{H}(\vec{r},t)$$

$$\tilde{p}_{ex}(s) = s \int_{V_{ex}} \left[-\epsilon_o \tilde{\vec{E}}(\vec{r},s) \bullet \tilde{\vec{E}}(\vec{r},-s) + \mu_o \tilde{\vec{H}}(\vec{r},-s) \bullet \tilde{\vec{H}}(\vec{r},s) \right] dV$$

(4.9)

$$W_{ex} = \int_{-\infty}^{\infty} P_{ex}(t) dt$$

$$= \frac{1}{2\pi j} \int_{Br+} \int_{V_{ex}} \left[-s \epsilon_o \tilde{\vec{E}}(\vec{r},s) \bullet \tilde{\vec{E}}(\vec{r},-s) \right] dV \, ds$$

$$+ \frac{1}{2\pi j} \int_{Br-} \int_{V_{ex}} \left[s\mu_o \tilde{\vec{H}}(\vec{r},-s) \bullet \tilde{\vec{H}}(\vec{r},s) \right] dV \, ds$$

$$= \int_{V_{ex}} \left[\frac{1}{2} \epsilon_o \vec{E}(\vec{r},\infty) \bullet \vec{E}(\vec{r},\infty) + \frac{1}{2} \mu_o \vec{H}(\vec{r},\infty) \bullet \vec{H}(\vec{r},\infty) \right] dV$$

with other possible forms as in (3.9). Note that W_{ex} is zero if the late-time fields are zero. Now we have

$$Re[\tilde{p}_{ex}(j\omega)] = 0 \qquad \text{except possibly for } \omega = 0$$

(4.10)

$$W_{ex} \geq 0$$

In terms of the terms defined here we have the conservation equations

$$P_{\infty}(t) + P_s(t) + P_{ex}(t) = 0$$

$$\tilde{p}_{\infty}(s) + \tilde{p}_s(s) + \tilde{p}_{ex}(s) = 0$$

(4.11)

$$W_{\infty} + W_s + W_{ex} = 0$$

Now the contribution at S_{∞} can be divided into three terms based on the incident and scattered fields as

$$P_\infty^{(inc)}(t) \equiv \int_{S_\infty} \left[\vec{E}^{(inc)}(\vec{r},t) \times \vec{H}^{(inc)}(\vec{r},t) \right] \bullet \vec{1}_r \, dS$$

$$P_\infty^{(inc)}(t) \equiv \int_{S_\infty} \left[\vec{E}^{(sc)}(\vec{r},t) \times \vec{H}^{(sc)}(\vec{r},t) \right] \bullet \vec{1}_r \, dS$$

$$P_\infty^{(mix)}(t) \equiv \int_{S_\infty} \left[\vec{E}^{(inc)}(\vec{r},t) \times \vec{H}^{(sc)}(\vec{r},t) + \vec{E}^{(sc)}(\vec{r},t) \times \vec{H}^{(inc)}(\vec{r},t) \right] \bullet \vec{1}_r \, dS$$

$$P_\infty(t) = P_\infty^{(inc)}(t) + P_\infty^{(sc)}(t) + P_\infty^{(mix)}(t)$$

$$\tilde{p}_\infty^{(inc)}(s) \equiv \int_{S_\infty} \left[\tilde{\vec{E}}^{(inc)}(\vec{r},s) \times \tilde{\vec{H}}^{(inc)}(\vec{r},-s) \right] \bullet \vec{1}_r \, dS$$

$$\tilde{p}_\infty^{(sc)}(s) \equiv \int_{S_\infty} \left[\tilde{\vec{E}}^{(sc)}(\vec{r},s) \times \tilde{\vec{H}}^{(sc)}(\vec{r},-s) \right] \bullet \vec{1}_r \, dS \tag{4.12}$$

$$\tilde{p}_\infty^{(mix)}(s) \equiv \int_{S_\infty} \left[\tilde{\vec{E}}^{(inc)}(\vec{r},s) \times \tilde{\vec{H}}^{(sc)}(\vec{r},-s) + \tilde{\vec{E}}^{(sc)}(\vec{r},s) \times \tilde{\vec{H}}^{(inc)}(\vec{r},-s) \right] \bullet \vec{1}_r \, dS$$

$$\tilde{p}_\infty(s) = \tilde{p}_\infty^{(inc)}(s) + \tilde{p}_\infty^{(sc)}(s) + \tilde{p}_\infty^{(mix)}(s)$$

$$W_\infty^{(inc)} = \int_{-\infty}^{\infty} P_\infty^{(inc)}(t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(inc)}(s)\, ds = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(inc)}(-s)\, ds$$

$$W_\infty^{(sc)} = \int_{-\infty}^{\infty} P_\infty^{(sc)}(t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(sc)}(s)\, ds = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(sc)}(-s)\, ds$$

$$W_\infty^{(mix)} = \int_{-\infty}^{\infty} P_\infty^{(mix)}(t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(sc)}(s)\, ds = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(mix)}(-s)\, ds$$

$$W_\infty = W_\infty^{(inc)} + W_\infty^{(sc)} + W_\infty^{(mix)}$$

V. Cross Sections

In complex-frequency domain it is convenient to normalize the fields as

$$\tilde{\vec{e}}(\vec{r},s) = \frac{1}{E_0\tilde{f}(s)}\tilde{\vec{E}}(\vec{r},t)\ ,\ \tilde{\vec{h}}(\vec{r},s) = \frac{1}{H_0\tilde{f}(s)}\tilde{\vec{H}}(\vec{r},s) \tag{5.1}$$

so that

$$\begin{aligned}
&\tilde{\vec{e}}^{(inc)}(\vec{r},s) = \vec{1}_2\, e^{-\gamma\vec{1}_1\bullet\vec{r}} \quad , \quad \tilde{\vec{h}}^{(inc)}(\vec{r},s) = \vec{1}_3\, e^{-\gamma\vec{1}_1\bullet\vec{r}} \\
&\tilde{\vec{e}}^{(sc)}(\vec{r},s) = \left[\frac{1}{r}\tilde{\vec{v}}_f(\theta,\phi,s) + 0(r^{-2})\right]e^{-\gamma r} \\
&\tilde{\vec{h}}^{(sc)}(\vec{r},s) = \left[\frac{1}{r}\tilde{\vec{1}}_r(\theta,\phi)\times\tilde{\vec{v}}_f(\theta,\phi,s) + 0(r^{-2})\right]e^{-\gamma r}
\end{aligned} \tag{5.2}$$

In this form the incident fields have unit amplitude (for $s=j\omega$). In time domain this corresponds to a delta-function incident plane wave.

By a cross section we mean the ratio of some power to a power per unit area. In the above normalized form

$$\tilde{\vec{e}}^{(inc)}(\vec{r},s)\times\tilde{\vec{h}}^{(inc)}(\vec{r},-s) = \vec{1}_1 \tag{5.3}$$

so all we need is to place the normalized fields in the power expressions to obtain cross sections.

Corresponding to S_∞, S_S, and V_{ex} we then have

$$\begin{aligned}
\tilde{A}_\infty(s) &\equiv \int_{S_\infty}[\,\tilde{\vec{e}}(\vec{r},s)\times\tilde{\vec{h}}(\vec{r},-s)\,]\bullet\vec{1}_r\,dS \\
\tilde{A}_S(s) &\equiv -\int_{S_S}[\,\tilde{\vec{e}}(\vec{r},s)\times\tilde{\vec{h}}(\vec{r},-s)\,]\bullet\vec{1}_S\,dS
\end{aligned} \tag{5.4}$$

$$\tilde{A}_{ex}(s) \equiv \gamma \int_{V_{ex}} [- \tilde{\vec{e}}(\vec{r},s) \times \tilde{\vec{e}}(\vec{r},-s)] + \tilde{\vec{h}}(\vec{r},-s) \bullet \tilde{\vec{h}}(\vec{r},s)]dV$$

$$\tilde{A}_{\infty}(s) + \tilde{A}_s(s) + \tilde{A}_{ex}(s) \equiv 0$$

$$Re[\tilde{A}_s(j\omega)] \geq 0$$

$$Re[\tilde{A}_{ex}(j\omega)] = 0$$

In a more general sense note that

$$\tilde{A}_{ex}(s) = - \tilde{A}_{ex}(-s) \tag{5.5}$$

i.e. is an odd function of s. This gives

$$[\tilde{A}_{\infty}(s) + \tilde{A}_{\infty}(-s)] + [\tilde{A}_s(s) + \tilde{A}_s(-s)] = 0 \tag{5.6}$$

i.e. the even part of $\tilde{A}_{\infty}$ is just minus the even part of $\tilde{A}_s$. Note that $-\tilde{A}_{\infty}$ is often referred to as the extinction cross section [7, 12]

Consider now $\tilde{A}_s$ which might be defined as the (complex) absorption cross section of the scatterer, representing the complex power flow through S_s into V_s. In a previous paper [6] the absorption cross section was defined for a single port in a scatterer by defining transfer functions for voltage and current with respect to the incident electric and magnetic fields. Assume that the scatterer has tangential electric field zero on S_s except at N ports where the electric field can be integrated in a quasi static sense to give a voltage and the magnetic field similarly give a current. An example of this for a single port (easily generalized to give N ports) is given in [17]. Then we define voltage and current vectors with associated transfer functions as

$$(\tilde{V}_n(s)) \equiv (\tilde{T}_{V_n}(s)) E_o \tilde{f}(s)$$
$$(\tilde{I}_n(s)) \equiv (\tilde{T}_{I_n}(s)) H_o \tilde{f}(s) \tag{5.7}$$

For this case the absorption cross section becomes

$$\tilde{A}_s(s) = (\tilde{T}_{V_n}(s)) \bullet (\tilde{T}_{I_n}(-s)) \tag{5.8}$$

For a passive scatterer the voltage and current vectors are related by

$$\begin{aligned}
(\tilde{V}_n(s)) &= (\tilde{Z}_{n,m}(s)) \bullet (\tilde{I}_n(s)) \\
(\tilde{I}_n(s)) &= (\tilde{Y}_{n,m}(s)) \bullet (\tilde{V}_n(s)) \\
(\tilde{Y}_{n,m}(s)) &\equiv \text{admittance matrix} = (\tilde{Z}_{n,m}(s))^{-1} \\
(\tilde{Z}_{n,m}(s)) &\equiv \text{impedance matrix}
\end{aligned} \tag{5.9}$$

These matrices are positive real (or p.r.) in the sense that their eigenvalues have non-negative real parts for s=jω and throughout the right half s plane. Another form $\tilde{A}_s$ takes comes from applying section 3 to V_s giving

$$\begin{aligned}
\tilde{A}_s(s) &= \int_{V_s} \left[\tilde{\vec{e}}(\vec{r},s) \bullet \tilde{\vec{j}}_t(\vec{r},-s) + \tilde{\vec{h}}(\vec{r},-s) \bullet \tilde{\vec{j}}_{h_t}(r,s) \right] dV \\
\tilde{\vec{j}}_t(\vec{r},s) &= \frac{1}{H_o \tilde{f}(s)} \tilde{\vec{J}}_t(\vec{r},s) \equiv \text{normalized total current density} \\
\tilde{\vec{j}}_{h_t}(\vec{r},s) &= \frac{1}{E_o \tilde{f}(s)} \tilde{\vec{J}}_{h_t}(\vec{r},s) \equiv \text{normalized total magnetic current density}
\end{aligned} \tag{5.10}$$

In this form we can think of V_s as some electrical network with $\vec{j}_t$ representing the currents in resistors and capacitors (including fringe capacitance) and $\vec{e}$ the voltage across these, and with $\vec{j}_{h_t}$ representing magnetic currents (such as time rate of change of flux or voltage, including losses) in inductors (including fringe inductance) and $\vec{h}$ the current through these.

In [6] there is also defined an effective volume V_e as the ratio of the energy absorbed to the energy density in a step-function incident wave. In terms of our present variables this can be written for the entire scatterer as

$$V_e = -\frac{c}{2\pi j}\int_{Br}\frac{\tilde{A}_s(s)}{s^2}\,ds = \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{A}_s(j\omega)\,d\omega}{\omega^2} = \frac{c}{\pi}\int_0^{\infty}\frac{\mathrm{Re}[\tilde{A}_s(j\omega)]}{\omega^2}\,d\omega$$
$$= \text{effective volume} \tag{5.11}$$

Any of the forms for $\tilde{A}_s$ in (5.4), (5.8), and (5.10) can be substituted in (5.11). Note that strictly speaking the Bromwich contour threads between two poles at s=0, one from $\tilde{f}(s) = (s - s_o)^{-1}$ and the other from $\tilde{f}(-s) = (-s - s_o)^{-1}$ with the limit taken as s_o (real and positive) $\rightarrow 0$. This is aided by the fact that $\tilde{A}_s(s)$ has a zero (generally from both $\tilde{T}_V$ and $\tilde{T}_I$) at s=0.

Using (4.12) $\tilde{A}_\infty$ can also be decomposed as

$$\tilde{A}_\infty^{(inc)}(s) \equiv \int_{S_\infty}\left[\tilde{\vec{e}}^{(inc)}(\vec{r}, s)\times\tilde{\vec{h}}^{(inc)}(\vec{r}, -s)\right]\bullet\vec{1}_r\,dS$$
$$\tilde{A}_\infty^{(sc)}(s) \equiv \int_{S_\infty}\left[\tilde{\vec{e}}^{(sc)}(\vec{r}, s)\times\tilde{\vec{h}}^{(sc)}(\vec{r}, -s)\right]\bullet\vec{1}_r\,dS \tag{5.12}$$
$$\tilde{A}_\infty^{(mix)}(s) \equiv \int_{S_\infty}\left[\tilde{\vec{e}}^{(inc)}(\vec{r}, s)\times\tilde{\vec{h}}^{(sc)}(\vec{r}, -s) + \tilde{\vec{e}}^{(sc)}(\vec{r}, s)\times\tilde{\vec{h}}^{(inc)}(\vec{r}, -s)\right]\bullet\vec{1}_r\,dS$$
$$\tilde{A}_\infty(s) = \tilde{A}_\infty^{(inc)}(s) + \tilde{A}_\infty^{(sc)}(s) + \tilde{A}_\infty^{(mix)}(s)$$

The incident part of this is zero, since from (5.2)

$$\tilde{A}_\infty^{(inc)}(s) = \int_{S_\infty}\vec{1}_1\bullet\vec{1}_r\,dS = r_\infty^2\int_0^{2\pi}\int_0^{\pi}\cos(\theta)\sin(\theta)\,d\theta\,d\phi$$
$$= 0 \tag{5.13}$$

This merely expresses the fact that all the incident power both enters and leaves through S_∞. Note that this is the only term for the case of no scatterer present. Also from (5.2) we have the scattering cross section as

$$\tilde{A}_\infty^{(sc)}(s) = \int_{S_\infty} \left\{ \left[\frac{1}{r_\infty} \vec{\tilde{v}}_f(\theta,\phi,s) + O(r_\infty^{-2}) \right] \times \left[\frac{1}{r_\infty} \vec{1}_r(\theta,\phi) \times \vec{\tilde{v}}_f(\theta,\phi,-s) + O(r_\infty^{-2}) \right] \right\} \bullet \vec{1}_r(\theta,\phi)\, dS$$

$$= \int_0^{2\pi} \int_0^{\pi} \left[\vec{\tilde{v}}_f(\theta,\phi,s) \bullet \vec{\tilde{v}}_f(\theta,\phi,-s) + O(r_\infty^{-1}) \right] \sin(\theta)\, d\theta\, d\phi$$

$$= \int_0^{2\pi} \int_0^{\pi} \vec{\tilde{v}}_f(\theta,\phi,s) \bullet \vec{\tilde{v}}_f(\theta,\phi,-s) \sin(\theta)\, d\theta\, d\phi \quad \text{as } r_\infty \to \infty \tag{5.14}$$

Clearly this is even in s and real and non-negative on the $j\omega$ axis, i.e.,

$$\begin{aligned} &\tilde{A}_\infty^{(sc)}(s) = \tilde{A}_\infty^{(sc)}(-s) \\ &\tilde{A}_\infty^{(sc)}(j\omega) \ge 0 \end{aligned} \tag{5.15}$$

This in turn gives for the odd part of $\tilde{A}_\infty$

$$\begin{aligned} \left[\tilde{A}_\infty(s) - \tilde{A}_\infty(-s)\right] &= \left[\tilde{A}_\infty^{(mix)}(s) - \tilde{A}_\infty^{(mix)}(-s)\right] \\ &= -\left[\tilde{A}_s(s) - \tilde{A}_s(-s)\right] - 2\,\tilde{A}_{ex}(s) \end{aligned} \tag{5.16}$$

For the even part we have

$$\begin{aligned} \left[\tilde{A}_\infty(s) + \tilde{A}_\infty(-s)\right] &= 2\,\tilde{A}_\infty^{(sc)}(s) + \left[\tilde{A}_\infty^{(mix)}(s) + \tilde{A}_\infty^{(mix)}(-s)\right] \\ &= -\left[\tilde{A}_s(s) + \tilde{A}_s(-s)\right] \end{aligned} \tag{5.17}$$

These last two equations can be interpreted as equations for $\tilde{A}_s$ so that $\tilde{A}_\infty^{(sc)}$ does not enter into the odd part and $\tilde{A}_{ex}$ does not enter into the even part.

VI. Forward Scattering in Complex-Frequency Domain

As one might surmize the "mixed" cross section term involving both the incident and scattered fields at S_∞ is the key to the forward scattering theorem [7, 12]. This mixed cross section is evaluated from (5.2) and (5.12) as

$$\begin{aligned}\tilde{A}_\infty^{(mix)}(s) &= \int_{S_\infty} \left\{ e^{-\gamma \vec{1}_1 \bullet \vec{r}}\, \vec{1}_2 \times \left[\vec{1}_r(\theta,\phi) \times \tilde{\vec{v}}_{p(\theta,\phi,-s)} + O(r^{-1}) \right] \frac{e^{\gamma r}}{r} \right. \\ &\quad \left. + \frac{e^{-\gamma r}}{r} \left[\tilde{\vec{v}}_f(\theta,\phi,s) + O(r^{-1}) \right] \times \vec{1}_3 e^{\gamma \vec{1}_1 \bullet \vec{r}} \right\} \bullet \vec{1}_r(\theta,\phi) dS \\ &= \int_{S_\infty} \left\{ \left[\vec{1}_2 \bullet \tilde{\vec{v}}_f(\theta,\phi,-s) + O(r^{-1}) \right] \frac{e^{\gamma r - \gamma \vec{1}_1 \bullet \vec{r}}}{r} \right. \\ &\quad \left. + \left[[\vec{1}_3 \times \vec{1}_r(\theta,\phi)] \bullet \tilde{\vec{v}}_f(\theta,\phi,s) + O(r^{-1}) \right] \frac{e^{-\gamma r + \gamma \vec{1}_1 \bullet \vec{r}}}{r} \right\} dS \end{aligned} \tag{6.1}$$

This formula indicates some of the difficulty involved. On S_∞ we have

$$\gamma r - \gamma \vec{1}_1 \bullet \vec{r} = \gamma r_\infty [1 - \cos(\theta)] \tag{6.2}$$

As this is the argument of the exponential then as $r_\infty \to \infty$ this allows an asymptotic expansion. In the traditional derivation [7,12] with s=jω the stationary-phase technique is used. Here we can be a little more general.

Defining

$$\begin{aligned}\tilde{A}_\infty^{(mix)}(s) &\equiv \tilde{A}_1(s) + \tilde{A}_2(s) \\ \tilde{A}_1(s) &\equiv r_\infty \int_0^{2\pi} \int_0^{\pi} \left[\vec{1}_2 \bullet \tilde{\vec{v}}_f(\theta,\phi,-s) + O(r^{-1}) \right] e^{\gamma r_\infty [1-\cos(\theta)]} \sin(\theta) d\theta d\phi \end{aligned} \tag{6.3}$$

$$\tilde{A}_2(s) \equiv r_\infty \int_0^{2\pi} \int_0^{\pi} \left[[\vec{1}_3 \times \vec{1}_r(\theta,\phi)] \bullet \tilde{\vec{v}}_f(\theta,\phi,s) + O(r_\infty^{-1}) \right] e^{\gamma r_\infty [1-\cos(\theta)]} \sin(\theta) d\theta d\phi$$

and remembering that $\theta = 0$ is defined in the $\vec{1}_1$ direction let us consider what is happening near $\theta = 0$. Here we have

$$\alpha \equiv 1 - \cos(\theta) \quad , \quad d\alpha = \sin(\theta)\, d\theta$$

$$\tilde{A}_1(s) = \vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\theta,-s) r_\infty \int_0^{2\pi} \int_0^{2} e^{\gamma r_\infty \alpha} [1 + O(r_\infty^{-1})] d\alpha d\phi$$

$$+ O\left(r_\infty \int_0^2 \alpha e^{\gamma r_\infty \alpha} d\alpha \right) \qquad \text{as } r_\infty \to \infty \text{ with } \operatorname{Re}[\gamma] < 0$$

$$= -\frac{2\pi}{\gamma} \vec{1}_2 \bullet \vec{v}_f(0,\phi,-s) + O(r_\infty^{-1}) \qquad \text{as } r_\infty \to \infty \text{ with } \operatorname{Re}[\gamma] < 0 \tag{6.4}$$

$$\tilde{A}_2(s) = \left[\vec{1}_3 \times \vec{1}_1 \right] \bullet \tilde{\vec{v}}_f(\theta,\phi,s) r_\infty \int_0^{2\pi} \int_0^{2} e^{-\gamma r_\infty \alpha} [1 + O(r_\infty^{-1})] d\alpha d\phi$$

$$+ O\left(r_\infty \int_0^2 \alpha e^{-\gamma r_\infty \alpha} d\alpha \right) \qquad \text{as } r_\infty \to \infty \text{ with } \operatorname{Re}[\gamma] > 0$$

$$= \frac{2\pi}{\gamma} \vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\phi,s) + O(r_\infty^{-1}) \qquad \text{as } r_\infty \to \infty \text{ with } \operatorname{Re}[\gamma] > 0$$

Note that by evaluating the integrals with γ bounded away from the $j\omega$ axis as indicated the integrands exponentially decay away from $\theta = 0$ giving the order of error as indicated. Also $\tilde{\vec{v}}_f$ is assumed to have a bounded angular derivative near $\theta = 0$. Alternately utilizing the technique of stationary phase [13] the above results apply to the $j\omega$ axis as well except that the contribution near $\theta = \pi$ is also needed as another stationary point. The contribution near $\theta = \pi$ is given by

$$\alpha' = 1 + \cos(\theta) = 2 - \alpha \quad , \quad d\alpha' = -\sin(\theta)\, d\theta$$

$$\tilde{A}_1(s) = -\vec{1}_2 \bullet \vec{\tilde{v}}_f\,(\pi,\phi,-s)r_\infty e^{2\gamma r_\infty}\int_0^{2\pi}\int_2^0 e^{-\gamma r_\infty \alpha'}[1+O(r_\infty^{-1})]d\alpha' d\phi$$

$$+O\left(r_\infty e^{2\gamma r_\infty}\int_2^0 \alpha' e^{-\gamma r_\infty \alpha'} d\alpha'\right) \qquad \text{as } r_\infty \to \infty \text{ with } \mathrm{Re}[\gamma] > 0$$

$$= \frac{2\pi}{\gamma}\vec{1}_2 \bullet \vec{\tilde{v}}_f\,(\pi,\phi,-s)\,e^{2\gamma r_\infty} + O(r_\infty^{-1} e^{2\gamma r_\infty}) \qquad \text{as } r_\infty \to \infty \text{ with } \mathrm{Re}[\gamma] > 0 \tag{6.5}$$

$$\tilde{A}_2(s) = [\vec{1}_3 \times \vec{1}_1] \bullet \vec{\tilde{v}}_f\,(\pi,\phi,s)r_\infty e^{-2\gamma r_\infty}\int_0^{2\pi}\int_2^0 e^{\gamma r_\infty \alpha'}[1+O(r_\infty^{-1})]d\alpha' d\phi$$

$$+O\left(r_\infty e^{-2\gamma r_\infty}\int_2^0 \alpha' e^{\gamma r_\infty \alpha'} d\alpha'\right) \qquad \text{as } r_\infty \to \infty \text{ with } \mathrm{Re}[\gamma] > 0$$

$$= \frac{2\pi}{\gamma}\vec{1}_2 \bullet \vec{\tilde{v}}_f\,(\pi,\phi,s)\,e^{-2\gamma r_\infty} + O(r_\infty^{-1} e^{-2\gamma r_\infty}) \qquad \text{as } r_\infty \to \infty \text{ with } \mathrm{Re}[\gamma] > 0$$

This is somewhat perplexing in that in order to make an asymptotic evaluation near $\theta = \pi$ we get terms which blow up exponentially as $r_\infty \to \infty$. However, if we let $\mathrm{Re}[\gamma] = 0$ and consider the result as a stationary phase evaluation the result does not blow up but is still oscillatory. The terms corresponding to forward scattering are much better behaved, but converge in different half planes.

Combining the results gives (on the $j\omega$ axis)

$$\tilde{A}_\infty^{(mix)}(s) = \frac{2\pi}{\gamma}\left\{\vec{1}_2 \bullet \vec{\tilde{v}}_f\,(0,\phi,s) - \vec{1}_2 \bullet \vec{\tilde{v}}_f\,(0,\phi,-s)\right\}$$

$$+\frac{2\pi}{\gamma}\left\{\vec{1}_2 \bullet \vec{\tilde{v}}_f\,(\pi,\phi,s)e^{-2\gamma r_\infty} + \vec{1}_2 \bullet \vec{\tilde{v}}_f\,(\pi,\phi,-s)e^{2\gamma r_\infty}\right\}$$

$$+O(r_\infty^{-1}) \qquad \text{as } r_\infty \to \infty \text{ with } \mathrm{Re}[\gamma] = 0 \tag{6.6}$$

In order to remove the oscillatory terms note that this part is odd in s, so we form

$$\tilde{A}_{\infty}^{(mix)}(s) + \tilde{A}_{\infty}^{(mix)}(-s) = \frac{4\pi}{\gamma}\left\{\vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\theta,s) - \vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\theta,-s)\right\}$$

$$+ O(r_{\infty}^{-1}) \qquad \text{as } r_{\infty} \to \infty \text{ with } Re[\gamma] = 0$$

$$= [\tilde{A}_{\infty}(s) + \tilde{A}_{\infty}(-s)] - 2A_{\infty}^{(sc)}(s) \tag{6.7}$$

$$= -[\tilde{A}_s(s) + \tilde{A}_s(-s)] - 2\tilde{A}_{\infty}^{(sc)}(s)$$

where we have recalled the results of (5.17). Here $\tilde{A}_s(s) + \tilde{A}_s(-s)$ is from the absorption and $2\tilde{A}_{\infty}^{(sc)}$ (sc) is from the scattering. This may be considered the appropriate form of the forward scattering theorem, but its extension into the complex s plane is still problematical due to the asymptotic evaluation of the integrals.

VII. Forward Scattering for Time-Limited Pulses

As indicated in Fig. 7.1 the forward-scattering computation is at least conceptually simpler in time domain. Let us suppose that the incident wave takes the form of a time-limited pulse, i.e. that f(t) is zero after some finite time. Then on S_∞ for large r_∞ there is an overlap region of the incident with the scattered field only near $\theta = 0$, i.e. in the forward-scattering or $\vec{1}_1$ direction. It is only this region then that will contribute to the "mixed" term. There is no contribution from near $\theta = \pi$ contributing the oscillatory terms in the previous section. Actually for these results the pulse need not be zero after some time, but can asymptotically approach zero at late time with some sufficient rapidity.

The scattered field is also of limited time duration. Basically the late-time scattered far field is dominated by one of the natural frequencies of the scatterer with smallest damping. This gives a late-time waveform dominated by a decaying exponential, at least for incident waveforms of simple structure (or ones of limited duration). This is a fundamental transient scattering result associated with the singularity expansion method [16].

From Section 4 we then have

$$W_{ex} = 0 \tag{7.1}$$

since the late-time fields are zero throught V_{ex}. This gives

$$W_s = -W_\infty \geq 0 \tag{7.2}$$

which in more classical terminology is absorption equals extinction. So for our time-limited excitation, in terms of energy (power integrated over all time) there is a fundamental simplification in our formula. For further reference we have

$$\begin{aligned} W_s &= \int_{-\infty}^{\infty} P_s(t)dt = \frac{1}{2\pi j}\int_{Br} \tilde{p}_s(s)ds \\ &= -\frac{1}{2\pi j}\int_{Br}\int_{S_s}\left[\vec{\tilde{E}}(\vec{r},s)\times\vec{\tilde{H}}(\vec{r},-s)\right]\bullet\vec{1}_S\,dS\,ds \end{aligned} \tag{7.3}$$

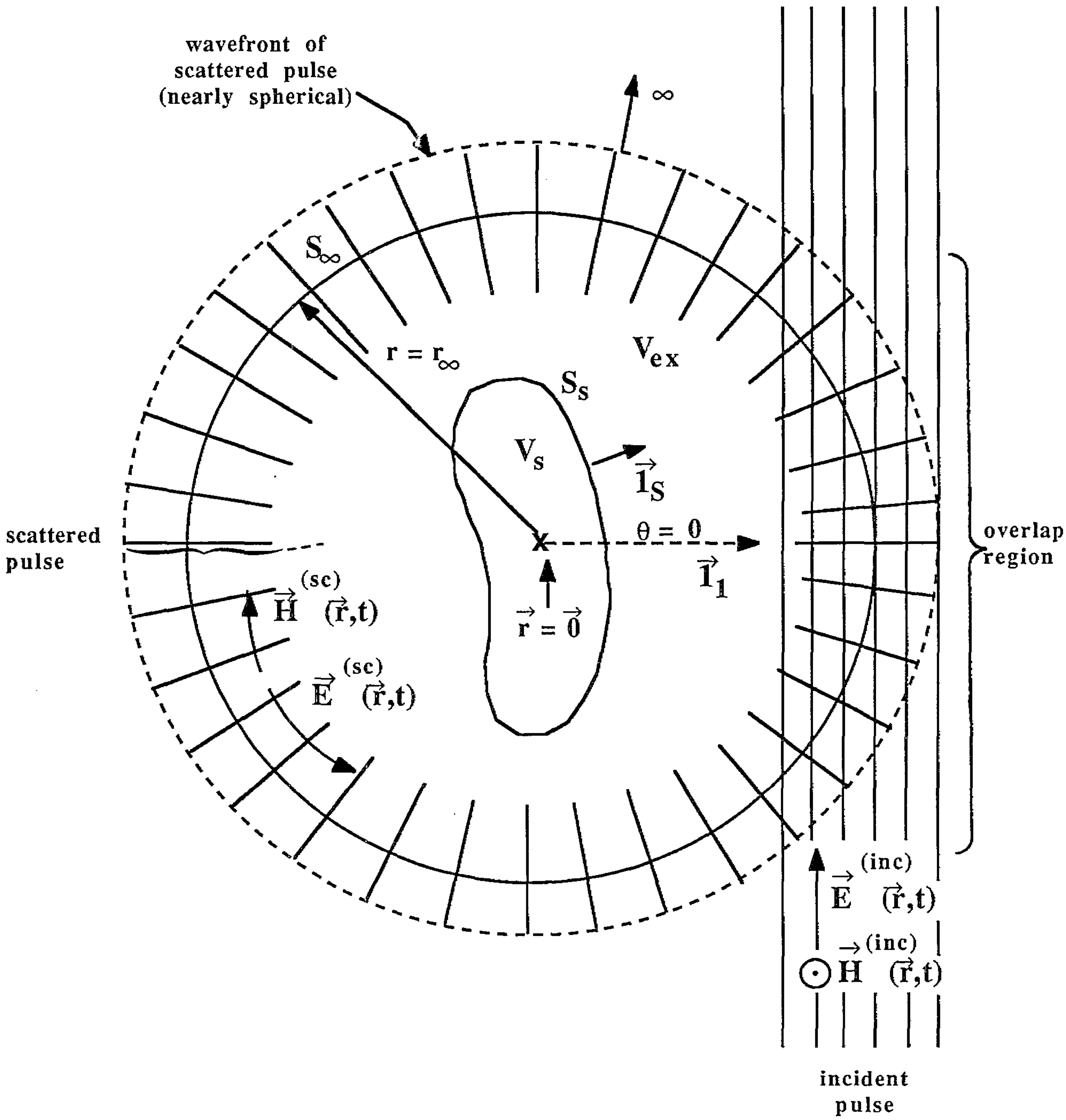

Figure 7.1 Forward scattering in time domain..

$$= -\frac{1}{2\pi j} E_o H_o \int_{Br} \tilde{f}(s)\, \tilde{f}(-s) \left\{ \int_{S_s} \left[\tilde{\vec{e}}(\vec{r}, s) \times \tilde{\vec{h}}(\vec{r}, -s) \right] \bullet \vec{1}_S \, dS \right\} ds$$

$$= \frac{1}{2\pi j} E_o H_o \int_{Br} \tilde{f}(s)\, \tilde{f}(-s)\; \tilde{A}_s(s)\, ds$$

In decomposing W_∞ we have the evident result

$$W_\infty^{(inc)} = \int_{-\infty} P_\infty^{(inc)}(t) dt = 0 \tag{7.4}$$

which merely expresses the fact that a time limited pulse passes through S_∞ in both directions leaving no energy behind if there are no remaining fields. Refer to (4.9) and consider $-W_{ex}$ for the case that V_{ex} is extended to include V_s, except with free space instead of the scatterer.

Another term in W_∞ is the scattered energy given by

$$W_\infty^{(sc)} = \int_{-\infty}^{\infty} P_\infty^{(sc)}(t) dt = \frac{1}{2\pi j} \int_{Br} \tilde{p}_\infty^{(sc)}(s) ds$$

$$= \frac{1}{2\pi j} \int_{Br} \int_{S_\infty} \left[\tilde{\vec{E}}^{(sc)}(\vec{r}, s) \times \tilde{\vec{H}}^{(sc)}(\vec{r}, -s) \right] \bullet \vec{1}_r \, dS\, ds$$

$$= \frac{1}{2\pi j} E_o H_o \int_{Br} \tilde{f}(s)\, \tilde{f}(-s) \left\{ \int_{S_s} \left[\tilde{\vec{e}}^{(sc)}(\vec{r}, s) \times \tilde{\vec{h}}^{(sc)}(\vec{r}, s) \right] \bullet \vec{1}_S \, dS \right\} ds$$

$$= \frac{1}{2\pi j} E_o H_o \int_{Br} \tilde{f}(s)\, \tilde{f}(-s)\, \tilde{A}_\infty^{(sc)}(s) ds$$

$$\geq 0 \tag{7.5}$$

The scattering cross section is also expressible in terms of the far-field as in (5.14).

Last (but not least, in magnitude anyway) we have

$$W_{\infty}^{(mix)} = \int_{-\infty}^{\infty} P_{\infty}^{(mix)}(t)\,dt$$
$$= \int_{-\infty}^{\infty} \int_{S_\infty} \left[\vec{E}^{(inc)}(\vec{r},t) \times \vec{H}^{(sc)}(\vec{r},t) + \vec{E}^{(sc)}(\vec{r},t) \times \vec{H}^{(inc)}(\vec{r},t)\right] \bullet \vec{1}_r \, dS\, dt \tag{7.6}$$

Expressing this as a time-domain integral the overlap-region near $\theta = 0$ in Fig. 7.1 is applicable. Writing out the fields gives

$$W_{\infty}^{(mix)} = E_o H_o \int_{-\infty}^{\infty} \int_{S_\infty} \Big\{ f(t - (\vec{1}_1 - \vec{r})/c)\vec{1}_2 \times \left[\vec{1}_r(\theta,\phi) \times \vec{V}_f(\theta,\phi,t - \tfrac{r}{c}) + O(r^{-1})\right]\tfrac{1}{r}$$
$$+ \tfrac{1}{r}\left[\vec{V}_f(\theta,\phi,t - \tfrac{r}{c}) + O(r^{-1})\right] \times \vec{1}_3 f(t - (\vec{1}_1 - \vec{r})/c) \Big\} \bullet \vec{1}_r(\theta,\phi)\, dS\, dt \tag{7.7}$$

Now shift to retarded time as

$$t_r = t - r/c \quad , \quad dt_r = dt$$

$$t - (\vec{1}_1 \bullet \vec{r})/c = t_r + \tfrac{1}{c}\left[r - \vec{1}_1 \bullet \vec{r}\right] = t_r + \tfrac{r}{c}[1 - \cos(\theta)]$$
$$= t_r + \tfrac{r}{c}\alpha$$
$$= t_r + t'$$

$$\alpha = 1 - \cos(\theta) \quad , \quad d\alpha = \sin(\theta)\, d\theta$$

$$t' = \tfrac{r}{c}\alpha \quad , \quad dt' = \tfrac{r}{c}\, d\alpha \tag{7.8}$$

Then for convenience divide $W_{\infty}^{(mix)}$ as

$$W_{\infty}^{(mix)} \equiv W_1 + W_2$$

$$W_1 = E_o H_o \int_{-\infty}^{\infty} \int_{S_\infty} \left\{ \frac{1}{r} f(t_r + t') \vec{1}_2 \times \left[\vec{1}_r(\theta, \phi) \times \vec{V}_f(\theta, \phi, t_r) + O(r^{-1}) \right] \right\} \bullet \vec{1}_r(\theta, \phi)\, dS\, dt_r$$

$$W_2 = E_o H_o \int_{-\infty}^{\infty} \int_{S_\infty} \left\{ \frac{1}{r} f(t_r + t') \left[\vec{V}_f(\theta, \phi, t_r) + O(r^{-1}) \right] \times \vec{1}_3 \right\} \bullet \vec{1}_r(\theta, \phi)\, dS\, dt_r \tag{7.9}$$

Evaluating W_1 we have

$$W_1 = E_o H_o \int_{-\infty}^{\infty} \int_{S_\infty} \frac{1}{r} f(t_r + t') \vec{1}_2 \left[\vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) + O(r^{-1}) \right] dS\, dt_r$$

$$= E_o H_o \int_{-\infty}^{\infty} r_\infty \left\{ \int_0^{2\pi} \int_0^{2} f(t_r + t') \left[\vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) + O(r_\infty^{-1}) \right] \sin(\theta) d\theta\, d\phi \right\} dt_r$$

$$= E_o H_o \int_{-\infty}^{\infty} r_\infty \left\{ \int_0^{2\pi} \int_0^{2} f(t_r + t') \left[\vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) + O(r_\infty^{-1}) \right] d\alpha\, d\phi \right\} dt_r$$

$$= E_o H_o \int_{-\infty}^{\infty} c \left\{ \int_0^{2\pi} \int_0^{2r_\infty/c} f(t_r + t') \left[\vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) + O(r_\infty^{-1}) \right] dt'\, d\phi \right\} dt_r \tag{7.10}$$

Now letting $r_\infty \rightarrow \infty$ note that for fixed t'

$$\alpha = \frac{ct'}{r_\infty} = O(r_\infty^{-1})$$

$$\theta = O(\alpha^{1/2}) = O(r_\infty^{-1/2})$$

$$\vec{V}_f(\theta, \phi, t_r) = \vec{V}_f(\theta, \phi, t_r) + O(r_\infty^{-1/2}) \tag{7.11}$$

provided $\partial \vec{V}_f / \partial \theta$ is bounded near $\theta = 0$. Further assuming that $\vec{V}_f$ is zero before some t_r (causality, based on $f(t_r)$ being zero before some t_r) and that $f(t_r+t')$ is zero for t' greater than some t' (with t_r not less than that for $\vec{V}_f$ to be zero) we have been taking r_∞ to ∞

$$W_1 = E_o H_o \int_{-\infty}^{\infty} c \left\{ \int_0^{2\pi} \int_0^{\infty} f(t_r + t') \vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r)\, dt'\, d\phi \right\} dt_r$$

$$= 2\pi c E_o H_o \int_{-\infty}^{\infty} \int_0^{\infty} f(t_r + t') \vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r)\, dt'\, dt_r \tag{7.12}$$

Similarly evaluating W_2 we have

$$W_2 = E_o H_o \int_{-\infty}^{\infty} r_\infty \left\{ \int_0^{2\pi} \int_0^{\pi} f(t_r + t') \left[[\vec{V}_f(\theta,\phi,t_r) + O(r_\infty^{-1})] \times \vec{1}_3 \right] \bullet \vec{1}_r(\theta,\phi) \sin(\theta)\, d\theta\, d\phi \right\} dt_r$$

$$= E_o H_o \int_{-\infty}^{\infty} r_\infty \left\{ \int_0^{2\pi} \int_0^{\pi} f(t_r + t') \left[[\vec{V}_f(\theta,\phi,t_r) + O(r_\infty^{-1})] \times \vec{1}_3 \right] \bullet \vec{1}_r(\theta,\phi)\, d\alpha\, d\phi \right\} dt_r$$

$$= E_o H_o \int_{-\infty}^{\infty} r_\infty \left\{ \int_0^{2\pi} \int_0^{2 r_\infty \pi / c} f(t_r + t') \left[[\vec{V}_f(\theta,\phi,t_r) + O(r_\infty^{-1})] \times \vec{1}_3 \right] \bullet \vec{1}_r(\theta,\phi)\, dt'\, d\phi \right\} dt_r \tag{7.13}$$

Then letting $r_\infty \to \infty$ with the results of (7.11) including restrictions we have

$$W_2 = E_o H_o \int_{-\infty}^{\infty} c \left\{ \int_0^{2\pi} \int_0^{\pi} f(t_r + t') \left[\vec{V}_f(\theta, \phi, t_r) \times \vec{1}_3 \right] \bullet \vec{1}_1 dt' d\phi \right\} dt_r$$

$$= 2\pi c E_o H_o \int_{-\infty}^{\infty} \int_0^{\infty} f(t_r + t') \vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) dt' dt_r$$

$$= W_1 \qquad (7.14)$$

Thus the two mixed terms (incident electric with scattered magnetic and incident magnetic with scattered electric) give the same results.

Now we have

$$W_\infty^{(mix)} = 4\pi c E_o H_o \int_{-\infty}^{\infty} \int_0^{\infty} f(t_r + t') \vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) dt' dt_r \qquad (7.15)$$

Alternately shifting t' to

$$t'' = t_r + t' \qquad (7.16)$$

we have

$$W_\infty^{(mix)} = 4\pi c E_o H_o \int_{-\infty}^{\infty} \int_{t_r}^{\infty} f(t'') \vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) dt'' dt_r$$

$$= 4\pi c E_o H_o \int_{-\infty}^{\infty} \vec{1}_2 \bullet \vec{V}_f(\theta, \phi, t_r) \left\{ \int_{t_r}^{\infty} f(t'') dt'' \right\} dt_r \qquad (7.17)$$

Defining

$$F(t) \equiv \int_{-\infty}^{t} f(\tau)\, d\tau$$

$$\vec{v}_I(0,\phi,t) = \int_{-\infty}^{t} \vec{v}_f(0,\phi,\tau)\, d\tau \tag{7.18}$$

and noting

$$\tilde{f}(0) = \int_{-\infty}^{\infty} f(\tau)\, d\tau$$

$$\int_{t}^{\infty} f(\tau)\, d\tau = \tilde{f}(0) - F(t) \tag{7.19}$$

we have

$$W_{\infty}^{(mix)} = 4\pi c E_o H_o \int_{-\infty}^{\infty} \left[\tilde{f}(0) - F(t_r)\right]\vec{1}_2 \bullet \vec{V}_f(0,\phi,t_r)\, dt_r \tag{7.20}$$

Considering the first term we have

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{f}(0)\vec{1}_2 \bullet \vec{V}_f(0,\phi,t_r)\, dt_r &= \tilde{f}(0) \int_{-\infty}^{\infty} \vec{1}_2 \bullet \vec{V}_f(0,\phi,t_r)\, dt_r \\ &= \tilde{f}(0)\vec{1}_2 \bullet \tilde{\vec{V}}_f\,(0,\phi,0) \\ &= \tilde{f}(0)\vec{1}_2 \bullet \left[\tilde{f}(0)\ \tilde{\vec{v}}_f\,(0,\phi,0)\right] \\ &= \tilde{f}^2(0)\vec{1}_2 \bullet \tilde{\vec{v}}_f\,(0,\phi,0) \\ &= 0 \end{aligned} \tag{7.21}$$

since $\tilde{f}(0)$ is bounded (finite area (or impulse) of incident waveform) and $\tilde{\vec{v}}_f\,(0,\ \phi,\ 0)$ is zero (no radiation or scattering to the far field at zero frequency). Thus we have

$$W_{\infty}^{(mix)} = -4\pi c E_o H_o \int_{-\infty}^{\infty} F(t_r)\vec{1}_2 \bullet \vec{V}_f(0, \phi, t_r)\, dt_r \tag{7.22}$$

Alternate representations are found using the generalized Parseval theorem (Appendix A) giving

$$\begin{aligned} W_{\infty}^{(mix)} &= -4\pi c E_o H_o \frac{1}{2\pi j} \int_{Br} \tilde{F}(s)\vec{1}_2 \bullet \tilde{\vec{V}}_f(0, \phi, -s)\, ds \\ &= -4\pi c E_o H_o \frac{1}{2\pi j} \int_{Br} \frac{\tilde{f}(s)}{s} \tilde{f}(-s)\vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s)\, ds \end{aligned} \tag{7.23}$$

From (4.8), (4.11), (4.12), (5.15), (7.1), (7.2), (7.4), (7.5) and (7.23) we have

$$\begin{aligned} & W_s + W_{\infty}^{(sc)} = -W_{\infty}^{(mix)} \\ & W_s \geq 0 \\ & W_{\infty}^{(sc)} \geq 0 \end{aligned} \tag{7.24}$$

giving

$$\begin{aligned} & \frac{1}{2\pi j} E_o H_o \int_{Br} \tilde{f}(s)\tilde{f}(-s)\tilde{A}_s(s)ds + \frac{1}{2\pi j} E_o H_o \int_{Br} \tilde{f}(s)\tilde{f}(-s)\tilde{A}_{\infty}^{(sc)}(s)ds \\ & \quad = \frac{1}{2\pi j} 4\pi c E_o H_o \int_{Br} \frac{\tilde{f}(s)}{s} \tilde{f}(-s)\vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \theta, -s)ds \end{aligned} \tag{7.25}$$

all three terms of which are real and non-negative. Noting that the symmetry of the Bromwich contour means that only the even parts of the integrands contribute to the integrals, then (7.25) can be considered as an integral form of (5.17) with weight $\tilde{f}(s)f(-s)$. Note also that the even part of $\tilde{A}_{\infty}^{(mix)}$ as in (6.7) is consistent with (7.25). Furthermore, this property of using only the even parts of the cross sections means that the backscattered contributions in (6.6) do not enter the results for the total energy in the three contributions.

For completeness (7.25) can be stated using time-domain quantities as

$$\int_{-\infty}^{\infty} P_s(t)dt + \int_{-\infty}^{\infty} P_{\infty}^{(sc)}(t)dt = -\int_{-\infty}^{\infty} P_{\infty}^{(mix)}(t)dt$$

$$P_s(t) = -\int_{S_s} [\vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)] \bullet \vec{1}_S dS \geq 0$$

$$P_{\infty}^{(sc)}(t) = \int_{S_{\infty}} \left[\vec{E}^{(sc)}(\vec{r},t) \times \vec{H}^{(sc)}(\vec{r},t)\right] \bullet \vec{1}_r dS \geq 0$$

$$-\int_{-\infty}^{\infty} P_{\infty}^{(mix)}(t) = -W_{\infty}^{(mix)} = 4\pi c E_o H_o \int_{-\infty}^{\infty} F(t)\vec{1}_2 \bullet \vec{V}_f(0,\phi,t)dt$$

$$= 4\pi c E_o H_o \int_{-\infty}^{\infty} \left\{\int_0^t f(t')dt'\right\} f(t) o \left[\vec{1}_2 \bullet \vec{V}_f(0,\phi,t)\right]dt$$

o $\equiv$ convolution (with respect to time)

(7.26)

with various alternative representations also possible.

VIII. Limiting Case of Step-Function Incident Pulse

Now let us take from before

$$W_s + W_\infty^{(sc)} = 4\pi c E_o H_o \frac{1}{2\pi j} \int_{Br} \frac{\tilde{f}(s)}{s} \tilde{f}(-s)\vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s)ds \tag{8.1}$$

and see what happens as f(t) assumes the character of a step function. So now let

$$f(t) \equiv e^{s_o t} u(t) \quad , \quad s_o < 0$$
$$\tilde{f}(s) = \frac{1}{s - s_o} \tag{8.2}$$

This is not strictly a time-limited pulse. However, referring to the diagram Fig. 7.1 notice that the trailing edge of such a pulse is exponentially small. While the overlap region is not limited as before the portion away from $\theta = 0$ includes only this exponentially small weight. Then taking $r_\infty \to \infty$ the contribution from this trailing part should be asymptotically zero.

As indicated in Fig. 8.1 the Bromwich contour goes to the right of the pole at x=0 as well as singularities of $\tilde{f}(s)$, these being in the left half plane. Here $\tilde{f}(s) / s$ is the transform of the integral of f(t) and is hence causal. However, $\tilde{f}(-s)\vec{1} \bullet \tilde{\vec{v}}_f(0, \theta, -s)$ is not causal, having singularities in the right half plane. The Bromwich contour goes to the left of these.

Now in the left half plane as $s \to \infty$, $\tilde{f}(s)\tilde{f}(-s)/s$ goes as s^{-3} while $\tilde{\vec{v}}_f(0, \theta, s)$ is bounded. So close the contour at ∞ in the left half plane with a semicircular contour as indicated in Fig. 8.1, this semicircular contour giving zero additional contribution to the integral. Shrink the contour or equivalently use the residue theorem to give contributions from the poles at 0 and s_o as

$$W_s + W_\infty^{(sc)} = 4\pi c E_o H_o \left\{ \frac{1}{s_o^2} \vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\phi,0) - \frac{1}{2s_o^2} \vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\phi,-s_o) \right\} \tag{8.3}$$

Now from appendix B and [1] the low-frequency delta-function response is

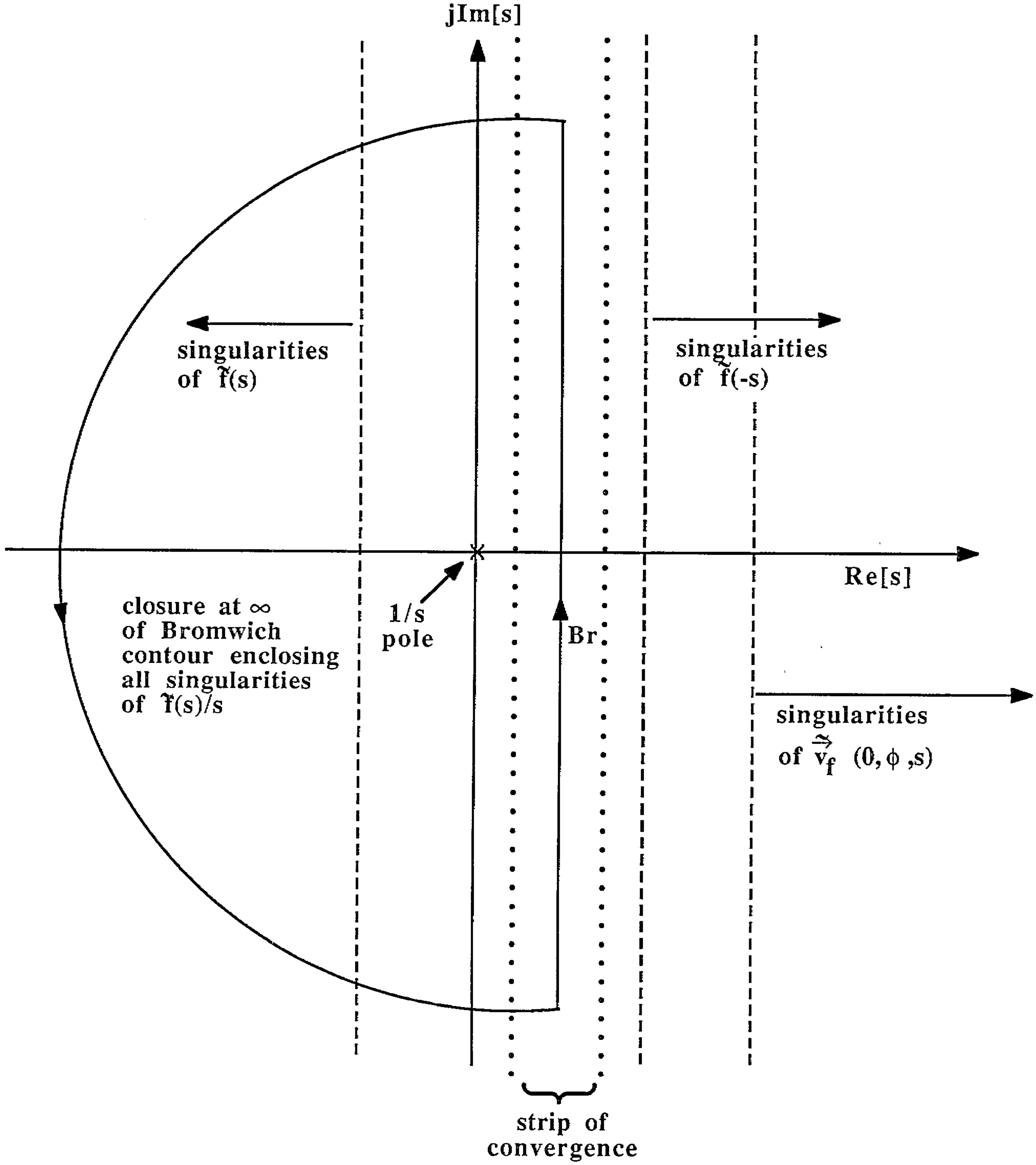

Figure 8.1 Integration in complex plane for forward scattering.

$$\tilde{\vec{v}}_f(\theta,\phi,s) = \frac{1}{E_o} s^2 \left\{ -\frac{\mu_o}{4\pi}\left[\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{p}_\infty + \frac{\mu_o}{4\pi c}\vec{1}_r \times \vec{m}_\infty + O(s) \right\} \tag{8.4}$$

This being zero at s=0 give

$$W_s + W_\infty^{(sc)} = 4\pi c E_o H_o \left\{ -\frac{1}{2s_o^2}\vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\phi,-s_o) \right\} \tag{8.5}$$

Now evaluating $\vec{1}_r$ for forward scattering gives

$$\begin{aligned}
&\vec{1}_r = \vec{1}_1 \\
&\overleftrightarrow{1} - \vec{1}_r\vec{1}_r = \vec{1}_2\vec{1}_2 + \vec{1}_3\vec{1}_3 \\
&\tilde{\vec{v}}_f(0,\phi,s) = \frac{1}{E_o} s^2 \left\{ -\frac{\mu_o}{4\pi}\left[\vec{1}_2\vec{1}_2 + \vec{1}_3\vec{1}_3\right] \bullet \vec{p}_\infty + \frac{\mu_o}{4\pi c}\vec{1}_1 \times \vec{m}_\infty + O(s) \right\} \\
&\vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\phi,s) = \frac{1}{E_o} s^2 \frac{\mu_o}{4\pi_o} \left\{ -\vec{1}_2 \bullet \vec{p}_\infty - \frac{1}{c}\vec{1}_3 \bullet \vec{m}_\infty + O(s) \right\}
\end{aligned} \tag{8.6}$$

Applying the zero-frequency polarizabilities from appendix B gives

$$\vec{1}_2 \bullet \tilde{\vec{v}}_f(0,\phi,s) = -\frac{s^2}{4\pi c^2} \left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 + O(s) \right\} \tag{8.7}$$

Evaluating at s_o gives

$$W_s + W_\infty^{(sc)} = \frac{E_o H_o}{c} \frac{1}{2} \left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 + O(s_o) \right\} \tag{8.8}$$

While strictly s_o should be negative we see that if $s_o \to 0$ we have the step response

$$\begin{aligned}
W_s + W_\infty^{(sc)} &= -E_o H_o \frac{1}{2\pi j}\int_{Br} \frac{\tilde{A}_s(s)}{s^2} ds - E_o H_o \frac{1}{2\pi j}\int_{Br} \frac{\tilde{A}_\infty^{(sc)}(s)}{s^2} ds \\
&= \frac{E_o H_o}{c} V_e + W_\infty^{(sc)} \\
&= \frac{E_o H_o}{c} \frac{1}{2} \left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \right\}
\end{aligned} \tag{8.9}$$

This agrees with [8]. Note that we require that W_s and $W_\infty^{(sc)}$ when considered as functions of s_o be continuous as $s_o \to 0$ so that the limit applies. In normalized form we have

$$V_e + \frac{c}{E_o H_o} W_\infty^{(sc)} = V_e - \frac{c}{2\pi j} \int_{Br} \frac{\tilde{A}_\infty^{(sc)}(s)}{s^2} ds$$
$$= \frac{1}{2}\left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \bullet + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \right\} \tag{8.10}$$

This gives the three terms as volumes so we could define

$$V_\infty^{(sc)} \equiv \frac{c}{E_o H_o} W_\infty^{(sc)} = -\frac{c}{2\pi j} \int_{Br} \frac{\tilde{A}_\infty^{(sc)}(s)}{s^2} ds$$
$$= \frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{A}_\infty^{(sc)}(j\omega)}{\omega^2} d\omega = \frac{c}{\pi} \int_0^{\infty} \frac{\mathrm{Re}[\tilde{A}_\infty^{(sc)}(j\omega)]}{\omega^2} d\omega$$
$$V_\infty^{(mix)} \equiv -\frac{c}{E_o H_o} W_\infty^{(mix)} = \frac{1}{2}\left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \right\} \tag{8.11}$$

with now

$$V_e + V_\infty^{(sc)} = V_\infty^{(mix)} \tag{8.12}$$

It has long been known that polarizabilities have dimension volume [19]. Here they bound the effective volume (normalized step absorption). There is also now a scattering volume $V_\infty^{(sc)}$ (normalized step scattering to ∞) it also bounds. Note now

$$V_e \geq 0, \quad V_\infty^{(sc)} \geq 0, \quad V_\infty^{(mix)} \geq 0 \tag{8.13}$$

Since $V_\infty^{(mix)}$ is non negative we have

$$\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \geq -\vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \geq 0 \tag{8.14}$$

for all choices of $\vec{1}_2$ and $\vec{1}_3$ with the constraint $\vec{1}_2 \bullet \vec{1}_3 = 0$. When diagonalizing these 3×3 matrices we have all non-negative elements in the electric case and all non-positive elements in the magnetic case.

Going further suppose we reverse the direction of the incident wave. Keeping the same electric field polarization we reverse $\vec{1}_1$ and thus $\vec{1}_3$. But this changes nothing in $V_\infty^{(mix)}$. Hence this result is invariant to reversal of wave direction even though the scatterer is not symmetric with respect to such a reversal (i.e. not symmetric with respect to reflection in a plane perpendicular to $\vec{1}_1$ (or equivalently a plane parallel to $\vec{1}_2$ and $\vec{1}_3$)). More generally we have

$$V_\infty^{(mix)}(\vec{1}_2, \vec{1}_3) = V_\infty^{(mix)}(-\vec{1}_2, \vec{1}_3) = V_\infty^{(mix)}(\vec{1}_2, -\vec{1}_3) = V_\infty^{(mix)}(-\vec{1}_2, -\vec{1}_3) \tag{8.15}$$

which is a high order of symmetry. So we may wish to think of this as something other than forward scattering when dealing with step-function scattering.

IX. Scattering of a Step-Function Plane Wave

A. Decomposition of Step Response

As in section 4 let us decompose the scattering by energy terms. From (4.11) we have

$$W_s + W_\infty + W_{ex} = 0 \tag{9.1}$$

based on the energy absorbed by the scatterer, radiated to ∞, and in the external fields. From (4.12) we have a decomposition of the energy at ∞ as

$$W_\infty = W_\infty^{(inc)} + W_\infty^{(mix)} + W_\infty^{(sc)} \tag{9.2}$$

Furthermore similarly decompose the external fields as

$$W_{ex} = W_{ex}^{(inc)} + W_{ex}^{(mix)} + W_{ex}^{(sc)} \tag{9.3}$$

Considering the incident fields to be a step-function wave by selecting f(t) in (4.3) as u(t) there is a residual energy in the fields in V_{ex} which is non zero so W_{ex} is non zero involving respectively the incident fields, the scattered fields, and the mixed terms involving both incident and scattered fields. Rearranging terms we have

$$\left[W_s + W_\infty^{(sc)}\right] + \left[W_\infty^{(inc)} + W_{ex}^{(inc)}\right] + \left[W_{ex}^{(mix)} + W_{ex}^{(sc)}\right] + W_\infty^{(mix)} = 0 \tag{9.4}$$

Note the grouping of the terms which we now consider.

B. $W_s + W_\infty^{(sc)}$

This is the basic term we wish to evaluate, the sum of the absorbed energy plus that scattered to ∞. As discussed in sections 5 and 8 these basic terms take the form

$$W_s + W_\infty^{(sc)} = \frac{E_o H_o}{c} V_\infty^{(mix)} = \frac{E_o H_o}{c}\left[V_e + V_\infty^{(sc)}\right]$$

$$V_e = -\frac{c}{2\pi j}\int_{Br}\frac{\tilde{A}_s(s)}{s^2}\,ds = \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{A}_s(j\omega)}{\omega^2}\,d\omega \tag{9.5}$$

$$V_\infty^{(sc)} = -\frac{c}{2\pi j}\int_{Br}\frac{\tilde{A}_\infty^{(sc)}(s)}{s^2}\,ds = \frac{c}{2\pi}\int_{-\infty}^{\infty}\frac{\tilde{A}_\infty^{(sc)}(j\omega)}{\omega^2}\,d\omega$$

We have required that as $s \to 0$ there is no stored energy in V_S, and no power radiated to ∞ so that the above integrands are well behaved as $s \to 0$.

C. $W_\infty^{(inc)} + W_{ex}^{(inc)}$

Now from (4.12) we have

$$P_\infty^{(inc)}(t) = \int_{S_\infty}\left[\vec{E}^{(inc)}(\vec{r}, t) \times \vec{H}^{(inc)}(\vec{r}, t)\right] \bullet \vec{1}_r\, dS$$

$$W_\infty^{(inc)} = \int_{-\infty}^{\infty} P_\infty^{(inc)}(t)\, dt \tag{9.6}$$

This gives the energy flowing out of V_∞ due to the incident fields. Since this corresponds to the Poynting - vector theorem with no scatterer present we can use (2.9) to give

$$W_\infty^{(inc)} = -\left\{\frac{\epsilon_o}{2}\vec{E}^{(inc)}(\vec{r}, \infty) \bullet \vec{E}^{(inc)}(\vec{r}, \infty) + \frac{\mu_o}{2}\vec{H}^{(inc)}(\vec{r}, \infty) \bullet \vec{H}^{(inc)}(\vec{r}, \infty)\right\} V_\infty$$

$$= -\left\{\frac{\epsilon_o}{2}E_o^2 + \frac{\mu_o}{2}H_o^2\right\} V_\infty$$

$$= -\epsilon_o E_o^2 V_\infty \tag{9.7}$$

$$V_\infty = \frac{4}{3}\pi r_\infty^3$$

This diverges as $r_\infty \to \infty$.

From (4.9) we have

$$W_{ex}^{(inc)} = \left\{ \frac{\in_o}{2} \vec{E}^{(inc)}(\vec{r}, \infty) \bullet \vec{E}^{(inc)}(\vec{r}, \infty) + \frac{\mu_o}{2} \vec{H}^{(inc)}(\vec{r}, \infty) \bullet \vec{H}^{(inc)}(\vec{r}, \infty) \right\} V_{ex}$$

$$= \left\{ \frac{\in_o}{2} E_o^2 + \frac{\mu_o}{2} H_o^2 \right\} V_{ex}$$

$$= \in_o E_o^2 V_{ex} \tag{9.8}$$

This is the energy in the late-time incident fields, but only in V_{ex}. This also diverges as $r_\infty \rightarrow \infty$.

Noting that

$$V_\infty = V_{ex} + V_s \tag{9.9}$$

we have

$$W_\infty^{(inc)} + W_{ex}^{(inc)} = - \in_o E_o^2 V_s \tag{9.10}$$

which is well behaved as $r_\infty \rightarrow \infty$.

D. $W_{ex}^{(mix)} + W_{ex}^{(sc)}$

The details of these terms are treated in appendices C and D for the electric and magnetic parts respectively. This is the energy of the static scattered and mixed late-time fields in V_{ex}. So take the results for V_∞ in the appendices and remove the energy terms corresponding to V_s.

The mixed term is

$$W_{ex}^{(mix)} = \int_{V_{ex}} \left\{ \in_o \vec{E}^{(inc)}(\vec{r}, \infty) \bullet \vec{E}^{(sc)}(\vec{r}, \infty) + \mu_o \vec{H}^{(inc)}(\vec{r}, \infty) \bullet \vec{H}^{(sc)}(\vec{r}, \infty) \right\} dV \tag{9.11}$$

The electric term is evaluated from appendix C as

$$\epsilon_o \int_{V_{ex}} \vec{E}^{(inc)}(\vec{r}, \infty) \bullet \vec{E}^{(sc)}(\vec{r}, \infty)\, dV = W_e^{(mix)} + \epsilon_o E_o^2 V_s$$

$$= -\epsilon_o E_o^2 \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \epsilon_o E_o^2 V_s \qquad (9.12)$$

and the magnetic term is evaluated from appendix D as

$$\mu_o \int_{V_{ex}} \vec{H}^{(inc)}(\vec{r}, \infty) \bullet \vec{H}^{(sc)}(\vec{r}, \infty)\, dV = W_h^{(mix)} + \mu_o H_o^2 V_s$$

$$= -\mu_o H_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 + \mu_o H_o^2 V_s \qquad (9.13)$$

Combining these we have

$$W_{ex}^{(mix)} = -\epsilon_o E_o^2 \left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \right\} + 2 \epsilon_o E_o^2 V_s \qquad (9.14)$$

The scattered term is

$$W_{ex}^{(sc)} = \frac{1}{2} \int_{V_{ex}} \left\{ \epsilon_o \vec{E}^{(sc)}(\vec{r}, \infty) \bullet \vec{E}^{(sc)}(\vec{r}, \infty) + \mu_o \vec{H}^{(sc)}(\vec{r}, \infty) \bullet \vec{H}^{(sc)}(\vec{r}, \infty) \right\} dV$$

$$(9.15)$$

The electric term is evaluated from appendix C as

$$\frac{1}{2} \epsilon_o \int_{V_{ex}} \vec{E}^{(sc)}(\vec{r}, \infty) \bullet \vec{E}^{(sc)}(\vec{r}, \infty) dV = W_e^{(sc)} - \frac{1}{2} \epsilon_o E_o^2 V_s$$

$$= \frac{1}{2} \epsilon_o E_o^2 \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 - \frac{1}{2} \epsilon_o E_o^2 V_s \qquad (9.16)$$

and the magnetic term is evaluated from appendix D as

$$\frac{1}{2}\mu_o \int_{V_{ex}} \vec{H}^{(sc)}(\vec{r}, \infty) \bullet \vec{H}^{(sc)}(\vec{r}, \infty)\, dV = W_h^{(sc)} - \frac{1}{2}\mu_o H_o^2 V_s$$

$$= \frac{1}{2}\mu_o H_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 - \frac{1}{2}\mu_o H_o^2 V_s \tag{9.17}$$

Combining these we have

$$W_{ex}^{(sc)} = \frac{1}{2} \epsilon_o E_o^2 \left\{ \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \right\} - \epsilon_o E_o^2 V_s \tag{9.18}$$

Both terms being the same except for different coefficients we have

$$W_{ex}^{(mix)} + W_{ex}^{(sc)} = -\frac{1}{2} \epsilon_o E_o^2 \left\{ \vec{1}_2 \bullet \overleftrightarrow{P} \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \right\} + \epsilon_o E_o^2 V_s \tag{9.19}$$

E. $W_\infty^{(mix)}$

From (4.12) we have

$$P_\infty^{(mix)}(t) = \int_{S_\infty} \left[\vec{E}^{(inc)}(\vec{r}, t) \times \vec{H}^{(sc)}(\vec{r}, t) + \vec{E}^{(sc)}(\vec{r}, t) \times \vec{H}^{(inc)}(\vec{r}, t) \right] \bullet \vec{1}_r \, dS$$

$$W_\infty^{(mix)} = \int_{-\infty}^{\infty} P_\infty^{(mix)}(t)\, dt \tag{9.20}$$

Substituting for the plane-wave incident field from (4.3), note that behind the wavefront the incident fields are uniform for the case of a step function. Furthermore causality indicates that at any $\vec{r}$ the incident field must arrive there before the scattered field. In a product of an incident and a scattered field the product is non zero only after the scattered field "turns on". Looking back at Fig. 7.1 the scattered field interacts with a uniform field everywhere the scattered field exists. Substituting for the uniform incident field gives

$$P_{\infty}^{(mix)}(t) = \int_{S_\infty} \left[E_o \vec{1}_2 \times \vec{H}^{(sc)}(\vec{r}, t) - H_o \vec{1}_3 \times \vec{E}^{(sc)}(\vec{r}, t) \right] \bullet \vec{1}_r \, dS$$

$$= - E_o \vec{1}_2 \bullet \int_{S_\infty} \left[\vec{1}_r \times \vec{H}^{(sc)}(\vec{r}, t) \right] dS$$

$$+ H_o \vec{1}_3 \bullet \int_{S_\infty} \left[\vec{1}_r \times \vec{E}^{(sc)}(\vec{r}, t) \right] dS \tag{9.21}$$

using identities for the scalar triple product [18].

Converting the surface integrals to volume integrals via a form of "Gauss' theorem" [18] we have

$$P_{\infty}^{(mix)}(t) = - E_o \vec{1}_2 \bullet \int_{V_\infty} \nabla \times \vec{H}^{(sc)}(\vec{r}, t) \, dV + H_o \vec{1}_3 \bullet \int_{V_\infty} \nabla \times \vec{E}^{(sc)}(\vec{r}, t) \, dV$$

$$= - E_o \vec{1}_2 \bullet \int_{V_s} \vec{J}(\vec{r}, t) \, dV - \epsilon_o E_o \vec{1}_2 \bullet \int_{V_\infty} \frac{\partial}{\partial t} \vec{E}^{(sc)}(\vec{r}, t) \, dV$$

$$- \mu_o H_o \vec{1}_3 \bullet \int_{V_\infty} \frac{\partial}{\partial t} \vec{H}^{(sc)}(\vec{r}, t) \, dV \tag{9.22}$$

Integrating over all time

$$W_{\infty}^{(mix)} = \int_{-\infty}^{\infty} P(t) dt$$

$$= - E_o \vec{1}_2 \bullet \int_{-\infty}^{\infty} \int_{V_s} \vec{J}(\vec{r}, t) dV dt - \epsilon_o E_o \vec{1}_2 \bullet \int_{V_\infty} \vec{E}^{(sc)}(\vec{r}, \infty) dV$$

$$- \mu_o H_o \vec{1}_3 \bullet \int_{V_\infty} \vec{H}^{(sc)}(\vec{r}, \infty) dV \tag{9.23}$$

Substituting

$$\int_{V_s} \vec{J}(\vec{r}, t)dV = \int_{S_s} \vec{J}_s(\vec{r}, t)dS = \frac{\partial}{\partial t}\vec{p}(t)$$

$$E_o \vec{1}_2 \bullet \int_{-\infty}^{\infty} \int_{V_s} \vec{J}(\vec{r}, t)dVdt = E_o \vec{1}_2 \bullet \vec{p}(\infty) = \epsilon_o E_o^2 \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \tag{9.24}$$

and using the results of appendices C and D gives

$$W_\infty^{(mix)} = \mu_o H_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 = \epsilon_o E_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \tag{9.25}$$

F. Combining Terms

Now (9.4) can be evaluated to give

$$\begin{aligned} W_s + W_\infty^{(sc)} &= -[W_\infty^{(inc)} + W_{ex}^{(inc)}] - [W_{ex}^{(mix)} + W_{ex}^{(sc)}] - W_\infty^{mix)} \\ &= \epsilon_o E_o^2 V_s \\ &\quad + \frac{1}{2}\epsilon_o E_o^2 \left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} - \epsilon_o E_o^2 V_s \\ &\quad - \epsilon_o E_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \\ &= \frac{1}{2}\epsilon_o E_o^2 \left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} \\ &= \frac{E_o H_o}{c} \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} \end{aligned} \tag{9.26}$$

Note that this differs from (8.9) by a minus sign on the magnetic polarizability term. Of course since as is well known this term is negative the minus sign converts it to a positive quantity making the result in (9.26) larger than that in (8.9). The question is: "Which is correct?" The present results also change the result for $V_\infty^{(mix)}$ in (8.11) and following equations.

X. Backscattering Due to Induced Electric and Magnetic Dipoles

In order to account for the difference between (8.9) and (9.26) let us first consider the assumptions in the two cases.

The first result is based on the forward scattering theorem (use subscripts fs as needed). For this we have

$$\left[W_s + W_\infty^{(sc)}\right]_{fs} = \epsilon_0 E_0^2 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_0 \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_0 \bullet \vec{1}_3\right\}$$
$$= E_0 \frac{1}{2}\left\{\vec{1}_2 \bullet \vec{p}(\infty) + \frac{1}{c}\vec{1}_3 \bullet \vec{m}(\infty)\right\} \tag{10.1}$$

To obtain this result we considered a time-limited pulse. The integral at S_∞ for the mixed energy involved only the forward scattering $(\vec{1}_1)$ region on this sphere. Then the pulse width was extended to t=∞ as a step function. However, notice that $r_\infty \to \infty$ first, followed by pulse width $\to \infty$ (or in effect $s \to 0$).

The second result is based on the scattering of a step-function pulse (use subscripts sf as needed). For this we have

$$\left[W_s + W_\infty^{(sc)}\right]_{sf} = \epsilon_0 E_0^2 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_0 \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_0 \bullet \vec{1}_3\right\}$$
$$= E_0 \frac{1}{2}\left\{\vec{1}_2 \bullet \vec{p}(\infty) - \frac{1}{c}\vec{1}_3 \bullet \vec{m}(\infty)\right\} \tag{10.2}$$

To obtain this result we considered a step-function incident wave. The integral at S_∞ for the mixed energy involved an integral over all S_∞ because the incident field was present everywhere that the scattered field existed. So, notice that the pulse width $\to \infty$ first (or in effect $s \to 0$), followed by $r_\infty \to \infty$.

Thus it would seem that there is some possible difference between the fs and sf cases. Might there be some scattering term which does not contribute in the forward scattering direction?

Let us now consider the fields from dipoles at low frequencies. As in appendix B and [1] the far fields are

$$\tilde{\vec{E}}_f(\vec{r}, s) = e^{-\gamma r} s^2 \frac{\mu_o}{4\pi r} \left\{ \left[\vec{1}_r \vec{1}_r - \overleftrightarrow{1} \right] \bullet \tilde{\vec{p}}(s) + \frac{1}{c} \vec{1}_r \times \tilde{\vec{m}}(s) \right\}$$

$$Z_o \tilde{\vec{H}}_f(\vec{r}, s) = e^{-\gamma r} s^2 \frac{\mu_o}{4\pi r} \left\{ \frac{1}{c} \left[\vec{1}_r \vec{1}_r - \overleftrightarrow{1} \right] \bullet \tilde{\vec{m}}(s) - \vec{1}_r \times \tilde{\vec{p}}(s) \right\}$$

$$= \vec{1}_r \times \tilde{\vec{E}}_f(\vec{r}, s) \tag{10.3}$$

Consider the component of the electric field parallel to $(\vec{1}_2)$ (the polarization of the incident electric field) as

$$\vec{1}_2 \bullet \tilde{\vec{E}}_f(\vec{r}, s) = e^{-\gamma r} s^2 \frac{\mu_o}{4\pi r} \left\{ \left[\vec{1}_2 \bullet \vec{1}_r \vec{1}_r - \vec{1}_2 \right] \bullet \tilde{\vec{p}}(s) + \frac{1}{c} \left[\vec{1}_2 \times \vec{1}_r \right] \bullet \tilde{\vec{m}}(s) \right\} \tag{10.4}$$

with a similar result for the magnetic field. Now consider forward scattering

$$\vec{1}_r = \vec{1}_1$$

$$\vec{1}_2 \bullet \tilde{\vec{E}}_f(\vec{r}, s) = -e^{-\gamma r} s^2 \frac{\mu_o}{4\pi r} \left\{ \vec{1}_2 \bullet \tilde{\vec{p}}(s) + \frac{1}{c} \vec{1}_3 \bullet \tilde{\vec{m}}(s) \right\} \tag{10.5}$$

However, backward scattering has

$$\vec{1}_r = -\vec{1}_1$$

$$\vec{1}_2 \bullet \tilde{\vec{E}}_f(\vec{r}, s) = -e^{-\gamma r} s^2 \frac{\mu_o}{4\pi r} \left\{ \vec{1}_2 \bullet \tilde{\vec{p}}(s) - \frac{1}{c} \vec{1}_3 \bullet \tilde{\vec{m}}(s) \right\} \tag{10.6}$$

Note that the sign between the electric- and magnetic-dipole terms in (10.5) and (10.1) are the same (forward scattering), while those in (10.2) and (10.6) are the same (step-function scattering). So the step response seems to correspond to backscattering.

A picture of the quasi-static situation is given in Fig. 10.1. With the incident fields propagating to the right the electric and magnetic incident fields have the relative orientations as indicated so that $\vec{1}_1$ points to the right. On the perfectly conducting scatterer (at least perfectly

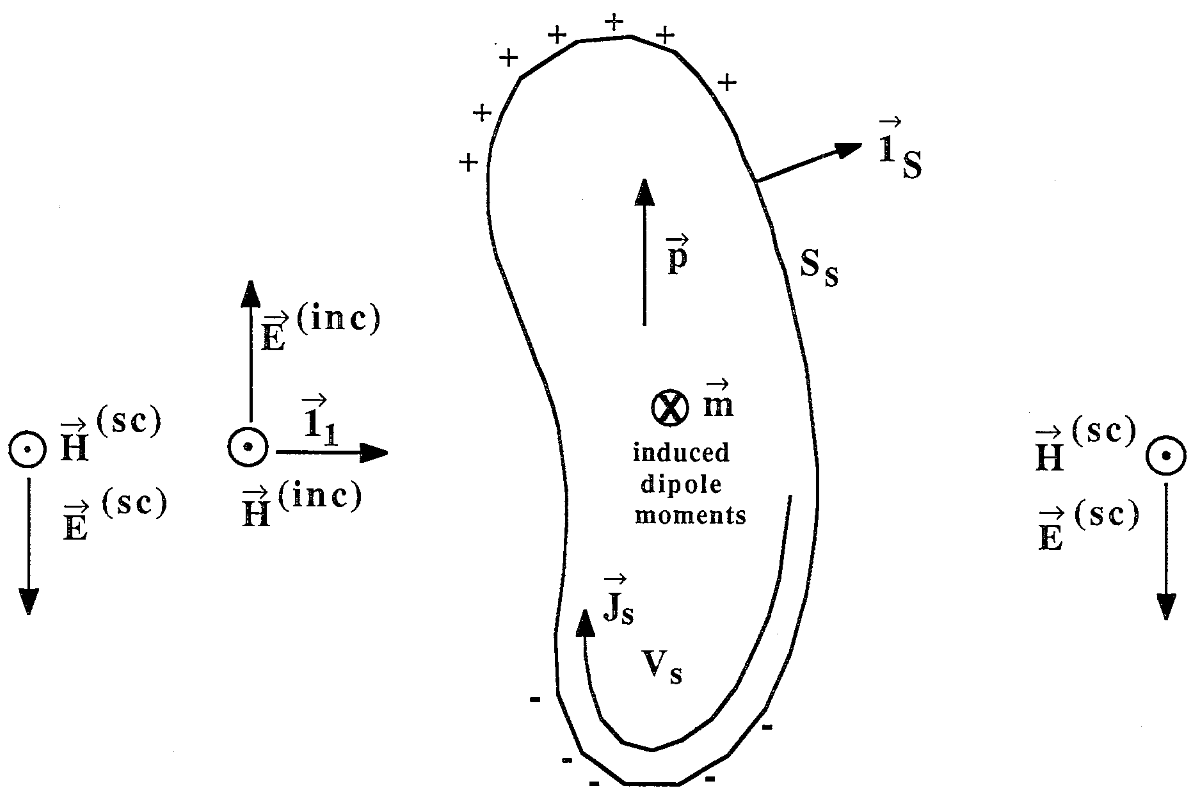

Figure 10.1 Low-frequency scattered fields.

conducting at s=0) the induced dipole moments are as indicated with $\vec{E}^{(inc)} \bullet \vec{p}$ positive and $\vec{H}^{(inc)} \bullet \vec{m}$ negative (for real-valued field vectors). The scattered fields are indicated to both right and left where $\vec{E}^{(inc)} \bullet \vec{E}^{(sc)}$ is negative and $\vec{H}^{(inc)} \bullet \vec{H}^{(sc)}$ is positive. Note that on both left and right sides of the scatterer $\vec{E}^{(sc)} \times \vec{H}^{(sc)}$ points to the left, which is oppositely directed to $\vec{E}^{(inc)} \times \vec{H}^{(inc)}$. Scattered power is predominantly backscattered.

This argument can be developed even further. As discussed in [1,2] there is a special type of radiator with balanced electric and magnetic dipole moments which can be referred to as MEDIUS or a $\vec{p} \times \vec{m}$ antenna. This is given by the constrained relations

$$\vec{p}(t) = p(t)\vec{1}_p \quad , \quad \vec{m}(t) = m(t)\vec{1}_m$$

$$\vec{1}_p \bullet \vec{1}_m = 0 \quad , \quad \vec{1}_p \times \vec{1}_m = \vec{1}_o$$

$$p(t) = \frac{1}{c}\, m(t) \tag{10.7}$$

Under these constraints the antenna radiates in the cardioid pattern centered around $\vec{1}_o$. In the $\vec{1}_o$ direction all the dipole terms (r^{-1}, r^{-2}, and r^{-3}) are balanced in that the electric and magnetic fields are perpendicular and related by Z_o for all frequencies and times. Furthermore in the back direction $(-\vec{1}_o)$ the dipole fields are zero to second order (r^{-1} and r^{-2} terms) but not to third order (r^{-3} term) which is the quasi-static field at s=0.

Let us decompose the 2 component of the induced electric dipole moment and the 3 component of the induced magnetic dipole moment into forward and backward components as

$$\vec{1}_2 \bullet \vec{p}(\infty) = \vec{1}_2 \bullet \vec{p}_f(\infty) + \vec{1}_2 \bullet \vec{p}_b(\infty)$$

$$\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 = \vec{1}_2 \bullet \overleftrightarrow{P}_f \bullet \vec{1}_2 + \vec{1}_2 \bullet \overleftrightarrow{P}_b \bullet \vec{1}_2$$

$$\vec{1}_3 \bullet \vec{m}(\infty) = \vec{1}_3 \bullet \vec{m}_f(\infty) + \vec{1}_3 \bullet \overleftrightarrow{m}_b(\infty)$$

$$\vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 = \vec{1}_3 \bullet \overleftrightarrow{M}_f \bullet \vec{1}_3 + \vec{1}_3 \bullet \overleftrightarrow{M}_b \bullet \vec{1}_3 \tag{10.8}$$

These components are constrained as

$$\vec{1}_2 \bullet \vec{p}_f(\infty) = \frac{1}{c}\vec{1}_3 \bullet \vec{m}_f(\infty)$$
$$\vec{1}_2 \bullet \overleftrightarrow{P}_f(\infty) \bullet \vec{1}_2 = \vec{1}_3 \bullet \overleftrightarrow{M}_f(\infty) \bullet \vec{1}_3$$
$$\vec{1}_2 \bullet \vec{p}_b(\infty) = -\frac{1}{c}\vec{1}_3 \bullet \vec{m}_b(\infty)$$
$$\vec{1}_2 \bullet \overleftrightarrow{P}_b \bullet \vec{1}_2 = -\vec{1}_3 \bullet \overleftrightarrow{M}_b \bullet \vec{1}_3 \qquad (10.9)$$

Here $\vec{1}_2 \bullet \vec{p}_f$ and $\vec{1}_3 \bullet \vec{m}_f$ are a pair of balanced dipoles radiating in the forward $(\vec{1}_1)$ direction, while $\vec{1}_2 \bullet \vec{p}_b$ and $\vec{1}_3 \bullet \vec{m}_b$ are a pair of balanced dipoles radiating in the backward $(-\vec{1}_1)$ direction.

Combining the above we have for forward scattering

$$\begin{aligned}\left[W_s + W_\infty^{(sc)}\right]_{fs} &= \epsilon_0 E_0^2 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\}\\ &= \epsilon_0 E_0^2 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_f \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_f \bullet \vec{1}_3\right\}\\ &= \epsilon_0 E_0^2\, \vec{1}_2 \bullet \overleftrightarrow{P}_f \bullet \vec{1}_2\\ &= -\epsilon_0 E_0^2\, \vec{1}_3 \bullet \overleftrightarrow{M}_f \bullet \vec{1}_3 \end{aligned} \qquad (10.10)$$

which shows only contributions from forward components. For step-function response we have

$$\begin{aligned}\left[W_s + W_\infty^{(sc)}\right]_{sf} &= \epsilon_0 E_0^2 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\}\\ &= \epsilon_0 E_0 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_b \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_b \bullet \vec{1}_3\right\}\\ &= \epsilon_0 E_0^2\, \vec{1}_2 \bullet \overleftrightarrow{P}_b \bullet \vec{1}_2\\ &= -\epsilon_0 E_0^2\, \vec{1}_3 \bullet \overleftrightarrow{M}_b \bullet \vec{1}_3 \end{aligned} \qquad (10.11)$$

which shows only contributions from backward components. From this we construct the forward components as

$$\vec{1}_2 \bullet \overleftrightarrow{P}_f \bullet \vec{1}_2 = \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} = \vec{1}_3 \bullet \vec{M}_f \bullet \vec{1}_3 \tag{10.12}$$

and the backward components as

$$\vec{1}_2 \bullet \overleftrightarrow{P}_b \bullet \vec{1}_2 = \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} = -\vec{1}_3 \bullet \vec{M}_b \bullet \vec{1}_3 \tag{10.13}$$

As is well known the multipole expansion of the radiated or scattered fields gives a complete orthogonal set on a sphere of constant r [9, 11]. The backward combination of dipoles cannot be seen (for r^{-1} and r^{-2} terms) in the forward direction and are not included in the forward scattering integral (near $\vec{1}_r = \vec{1}_1$). The forward combination of dipoles does appear in the forward scattering integral. The forward and backward parts are independent and so one does not directly imply the other. Energy considerations, however, do give bounds as

$$\begin{aligned}
&\vec{1}_2 \bullet \vec{P}_o \bullet \vec{1}_2 \geq -\vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \geq 0 \\
&\frac{1}{2}\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \geq \vec{1}_2 \bullet \overleftrightarrow{P}_f \bullet \vec{1}_2 = \vec{1}_3 \bullet \overleftrightarrow{M}_f \bullet \vec{1}_3 \geq 0 \\
&\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \geq \vec{1}_2 \bullet \overleftrightarrow{P}_b \bullet \vec{1}_2 = -\vec{1}_3 \bullet \overleftrightarrow{M}_b \bullet \vec{1}_3 \geq \frac{1}{2}\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2
\end{aligned} \tag{10.14}$$

XI. Bound on Absorption for Step Response

There is a difference in the result for $W_s + W_\infty^{(sc)}$ for a step-function incident wave depending on whether we take $r_\infty \to \infty$ for a time-limited pulse and then let the pulse tend to a step, or we first take a step function and then let $r_\infty \to \infty$. However, this should not affect the energy absorbed by the scatterer since from (5.4) and (8.9) we have

$$
\begin{aligned}
W_s &= -E_o H_o \frac{1}{2\pi j} \int_{Br} \frac{\tilde{A}_s(s)}{s^2} \, ds \\
&= E_o H_o \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{A}_s(j\omega)}{\omega^2} \, d\omega \\
&= \frac{E_o H_o}{c} V_e = \epsilon_o E_o^2 V_e \\
\tilde{A}_s(s) &= -\int_{S_s} \left[\tilde{\vec{e}}(\vec{r}, s) \times \tilde{\vec{h}}(\vec{r}, -s) \right] \bullet \vec{1}_S \, dS
\end{aligned}
\tag{11.1}
$$

Note that r_∞ does not enter this term at all. Then it makes no difference which of r_∞ and the pulse width goes to ∞ first.

This being the case then the difference between the two results (fs as in (10.1) and sf as in (10.2)) must be associated with $W_\infty^{(sc)}$ which does involve integrals over S_∞. Applying the appropriate subscripts to $W_\infty^{(sc)}$ in these two cases and subtracting the two results gives

$$
W_{\infty_{sf}}^{(sc)} - W_{\infty_{fs}}^{(sc)} = -\epsilon_o E_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \geq 0 \tag{11.2}
$$

In considering this difference one may note that the step-function response has contributions from all over S_∞, whereas the forward scattering response has contributions only near the forward scattering direction $\vec{1}_1$. As discussed in section 10 this can be explained by that portion of the dipole terms which does not appear in the forward direction.

Now since W_s does not depend on which case, fs or sf, is considered, then for a bound one may take the smaller of the two bounds which is associated with fs as

$$W_s \le \left[W_s + W_\infty^{(sc)}\right]_{fs} = \epsilon_o E_o^2 \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} \tag{11.3}$$

This agrees with [8].

Note that in section 8 various volumes are defined. While W_s and hence V_e does not depend on the difference in these two cases, $V_\infty^{(sc)}$ as in (8.11) does depend on this difference as does $A_\infty^{(sc)}(s)$, at least near s=0. This is based on

$$\left[V_e + V_\infty^{(sc)}\right]_{fs} = \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\}$$

$$\left[V_e + V_\infty^{(sc)}\right]_{sf} = \frac{1}{2}\left\{\vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 - \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3\right\} \tag{11.4}$$

The difference in $V_\infty^{(sc)}$ is given by

$$V_{\infty_{sf}}^{(sc)} - V_{\infty_{fs}}^{(sc)} = -\vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \ge 0 \tag{11.5}$$

XII. Damped Sinusoid Response

Another interesting kind of incident wave is a damped sinusoid. Following the procedure in section 8 let

$$f(t) \equiv \frac{1}{2}\left[e^{s_1 t + j\nu} + e^{s_1^* t - j\nu}\right] u(t)$$

$$\tilde{f}(s) = \frac{1}{2}\left[\frac{e^{j\nu}}{s - s_1} + \frac{e^{-j\nu}}{s - s_1^*}\right]$$

$$\mathrm{Im}[\nu] = 0 \quad , \quad \mathrm{Re}[s_1] \le 0 \quad , \quad \mathrm{Im}[s_1] > 0$$

$$* \equiv \text{complex} \quad \text{conjugate} \tag{12.1}$$

In this form ν is a phase-like parameter to be chosen for convenience. In another form

$$f(t) \equiv e^{\mathrm{Re}[s_1] t} \cos(\mathrm{Im}[s_1] + \nu)\, u(t) \tag{12.2}$$

so that if

$$|\mathrm{Re}[s_1]| << |\mathrm{Im}[s_1]| \tag{12.3}$$

then the peak of f(t) is nearly 1, and the peak electric field is nearly E_0.

If the incident field is only approximately a plane wave (near the scatterer) such as would come from a radiating antenna, then one may wish

$$\tilde{f}(0) = 0 \tag{12.4}$$

i.e., that the antenna does not radiate at zero frequency. In this case we have

$$e^{j 2 \nu} = -\frac{s_1}{s_1^*} \tag{12.5}$$

$$\nu = \frac{1}{2} \arg\left(- \frac{s_1}{s_1^*} \right) = \pm \frac{\pi}{2} + \arg(s_1)$$

If any (s_1) is just a little larger than $\pi/2$ it is appropriate to take

$$\nu = \arg(s_1) - \frac{\pi}{2} \tag{12.6}$$

so that ν is near zero and $e^{j\nu}$ is near one.

Now use the same procedure as in section 8 and close the Bromwich contour in the left half plane as in Fig. 8.1. Besides the pole at s=0 there are poles at s_1 and s_1^*. Using the residue theorem on (8.1) we have

$$\begin{aligned} W_s + W_\infty^{(sc)} &= 4\pi c E_o H_o \left\{ \tilde{f}^2(0) \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, 0) \right\} \\ &+ \frac{1}{s_1} \tilde{f}(-s_1) \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1) + \frac{1}{s_1^*} \tilde{f}(-s_1^*) \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1^*) \end{aligned} \tag{12.7}$$

The first term is zero, not only from (12.4), but also from (8.4) for the scattered field at s=0. Evaluating the remaining terms

$$\begin{aligned} W_s + W_\infty^{(sc)} &= 4\pi c E_o H_o \left\{ -\frac{1}{4 s_1} \left[\frac{e^{j\nu}}{s_1} + \frac{e^{-j\nu}}{\mathrm{Re}[s_1]} \right] \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1) \right. \\ &\quad \left. - \frac{1}{4 s_1^*} \left[\frac{e^{-j\nu}}{s_1^*} + \frac{e^{j\nu}}{\mathrm{Re}[s_1]} \right] \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1^*) \right\} \\ &= 2\pi c E_o H_o \,\mathrm{Re}\left\{ \frac{1}{s_1} \left[\frac{e^{j\nu}}{s_1} + \frac{e^{-j\nu}}{\mathrm{Re}[s_1]} \right] \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1) \right\} \\ &= -2\pi c E_o H_o \left\{ \mathrm{Re}\left[\frac{e^{j\nu}}{s_1^2} \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1) \right] \right. \\ &\quad \left. + \frac{1}{\mathrm{Re}[s_1]} \mathrm{Re}\left[\frac{e^{-j\nu}}{s_1} \vec{1}_2 \bullet \tilde{\vec{v}}_f(0, \phi, -s_1) \right] \right\} \end{aligned} \tag{12.8}$$

Going further than this requires some more detailed knowledge of $\overset{\sim}{\vec{v}}_f$. If the scatterer is electrically small at $s=s_1$ then a dipole approximation is appropriate as in (8.4). Then using (8.7) we have

$$
\begin{aligned}
W_s + W_\infty^{(sc)} &= \frac{E_o H_o}{c} \frac{1}{2}\left\{\vec{1}_2 \bullet \overset{\leftrightarrow}{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overset{\leftrightarrow}{M}_o \bullet \vec{1}_3\right\} \\
&\quad \left\{\cos(\nu) + \frac{1}{\mathrm{Re}[s_1]}\mathrm{Re}[s_1 e^{-j\nu}] + O(s_1) + \frac{O(s_1^2)}{\mathrm{Re}[s_1]}\right\} \\
&= \frac{E_o H_o}{c}\left\{\cos(\nu) + O(s_1) + \frac{O(s_1^2)}{\mathrm{Re}[s_1]}\right\}\frac{1}{2}\left\{\vec{1}_2 \bullet \overset{\leftrightarrow}{P}_o \bullet \vec{1}_2 + \vec{1}_3 \bullet \overset{\leftrightarrow}{M}_o \bullet \vec{1}_3\right\}
\end{aligned}
\tag{12.9}
$$

where we have used from (12.6)

$$
\begin{aligned}
\mathrm{Re}[s_1 e^{-j\nu}] &= |s_1| \, \mathrm{Re}[e^{j\arg(s_1)} e^{-j\nu}] \\
&= |s_1| \, \mathrm{Re}[e^{j\frac{\pi}{2}}] \\
&= 0
\end{aligned}
\tag{12.10}
$$

This shows where the low-frequency content of the incident wave is important. Note, however, that the result in (12.9) is essentially the same as the step response (in fs sense) in (8.8).

Of course, if we have some choice of s_1, we might try to maximize W_s. In a bound sense we could try to maximize $W_s + W_\infty^{(sc)}$ as in (12.9). Basically this involves maximizing things involving $\overset{\sim}{\vec{v}}_f\,(0, \phi, -s_1)$. One might look for resonances of this term with small values of $-\mathrm{Re}[s_1]$ where $\overset{\sim}{\vec{v}}_f$ is like the filter transfer function discussed in [4], while f is the excitation function. Basically one is trying to match $\mathrm{Im}[s_1]$ in the excitation to the imaginary part of a natural frequency of the filter and make $-\mathrm{Re}[s_1]$ as small as possible. This will maximize the coupling to an exterior resonance of the scatterer and hence the exterior scattering which will show up in

$W_{\infty}^{(sc)}$ as well as W_S. Considering the maximization of W_S one needs to consider interior natural frequencies as well.

XIII. Concluding Remarks

Well, there seems to be a lot contained in this forward scattering business. There is the question of the order of the limits (pulse width $\to\infty$, and $r_\infty \to \infty$). Somewhat arbitrary pulses can now be treated. This leads to concepts concerning optimal incident waveforms, such as appropriate choices of natural frequencies.

Perhaps now it is appropriate to apply these results to various example problems. One can study canonical scatterers with various kinds of loads to get further insight into optimization conditions.

Appendix A. Generalized Parseval Theorem

As discussed in previous papers [3, 6] the generalized Parseval Theorem is

$$\int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s) \tilde{f}_2(-s) ds = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(-s) \tilde{f}_2(s) ds \quad \text{(A.1)}$$

provided of course the integrals and transforms exist.

Here we merely point out a simple generalization to vectors, matrices, etc. Consider the dot product of two vectors for which we have

$$\int_{-\infty}^{\infty} \vec{f}_1(t) \bullet \vec{f}_2(t) dt = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{f}}_1(s) \bullet \tilde{\vec{f}}_2(-s) ds = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{f}}_1(-s) \bullet \tilde{\vec{f}}_2(s) ds \quad \text{(A.2)}$$

This is seen by writing out the dot product as a sum of products of the components and applying (A.1) to each. In this case the order can be reversed because the dot product of vectors commutes.

For the cross product, however, we have

$$\int_{-\infty}^{\infty} \vec{f}_1(t) \times \vec{f}_2(t) dt = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{f}}_1(s) \times \tilde{\vec{f}}_2(-s) ds = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{f}}_1(-s) \times \tilde{\vec{f}}_2(s) ds \quad \text{(A.3)}$$

Here order can be reversed, but with multiplication by -1. Again merely consider each component of the product.

For dyadic products of vectors we have

$$\int_{-\infty}^{\infty} \vec{f}_1(t) \vec{f}_2(t) dt = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{f}}_1(s) \tilde{\vec{f}}_2(-s) ds = \frac{1}{2\pi j} \int_{Br} \tilde{\vec{f}}_1(-s) \tilde{\vec{f}}_2(s) ds \quad \text{(A.4)}$$

Here in general the order cannot be reversed. In this case just consider each of the 9 components of the 3 x 3 dyad.

For general matrix dot products we have

$$\int_{-\infty}^{\infty} (f_{n,m}(t))_1 \bullet (f_{n,m}(t))_2 dt = \frac{1}{2\pi j} \int_{Br} (\tilde{f}_{n,m}(s))_1 \bullet (\tilde{f}_{n,m}(-s))_2 ds$$
$$= \frac{1}{2\pi j} \int_{Br} (\tilde{f}_{n,m}(-s))_1 \bullet (\tilde{f}_{n,m}(s))_2 \, ds \tag{A.5}$$

This is seen by writing

$$(f_{n,m}(t))_1 \bullet (f_{n,m}(t))_2 = \left(\sum_{m'=1}^{M} f_{n,m'}(t) f_{m',m}(t) \right) \tag{A.6}$$

and applying (A.1) to every term in the sum. Here M is the range of the second index of the first matrix and of the first index of the second matrix.

Note that (A.5) applies to the case that one of the matrices is replaced by an M component vector as well.

Often the Bromwich contour integral can be replaced by setting $s=j\omega$ for which

$$\frac{1}{2\pi j} \int_{Br} (\quad) ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\quad) d\omega \tag{A.7}$$

However the Bromwich form is more general.

Appendix B. Dipoles

Here we summarize some important formulae concerning dipoles. As discussed in [1] one can expand the fields from current and charge distributions for large distances (r) and low frequencies (s). First we have the low-frequency expansions

$$\tilde{\rho}(\vec{r}, s) = \tilde{f}_\rho(s)\rho_\infty(\vec{r}) + o(\tilde{f}_\rho(s))$$

$$\tilde{\vec{J}}(\vec{r}, s) = \tilde{f}_J(s)\vec{J}_\infty(\vec{r}) + o(\tilde{f}_J(s)) \tag{B.1}$$

$$\text{as } s \to 0$$

Here ρ_∞ and $\vec{J}_\infty$ are the low-frequency asymptotic forms of the charge and current densities. The subscript ∞ is indicative of their roles as late-time distributions such as appear in the response to step-function excitation, whether for antennas or scatterers. The coefficients f_ρ and $\tilde{f}_J$ are just s^{-1} for step excitation. The next terms in the expansion for small s are actually of order $s\tilde{f}_\rho$ and $s\,\tilde{f}_J$ if we make a Taylor series expansion of the response around s=0 giving

$$\tilde{\rho}(\vec{r}, s) = \tilde{f}_\rho(s)\rho_\infty(\vec{r}) + O(s\tilde{f}_\rho(s)) = \tilde{f}_\rho(s)\left[\rho_\infty(\vec{r}) + O(s)\right]$$

$$\tilde{\vec{J}}(\vec{r}, s) = \tilde{f}_J(s)\vec{J}_\infty(\vec{r}) + O(s\tilde{f}_J(s)) = \tilde{f}_J(s)\left[\vec{J}_\infty(\vec{r}) + O(s)\right]$$

$$\text{as } s \to 0 \tag{B.2}$$

Now the corresponding electric dipole moment is

$$\tilde{\vec{p}}(s) = \int_V \vec{r}\,\tilde{\rho}(\vec{r}, s)\,dV = f_\rho(s)\,\vec{p}_\infty + O(s\tilde{f}_p(s)) = \frac{1}{s}\int_V \tilde{\vec{J}}(\vec{r}, s)\,dV$$

$$\vec{p}_\infty = \int_V \vec{r}\,\tilde{p}_\infty(\vec{r})\,dV \tag{B.3}$$

The corresponding magnetic dipole moment is

$$\tilde{\vec{m}}(s) = \frac{1}{2} \int_V \vec{r} \times \tilde{\vec{J}}(\vec{r}, s)\, dV = \tilde{f}_J(s)\vec{m}_\infty + O(s\tilde{f}_J(s))$$

$$\vec{m}_\infty = \frac{1}{2} \int_V \vec{r} \times \vec{J}_\infty(\vec{r})\, dV \tag{B.4}$$

Dipoles produce fields. As $s \to 0$ for fixed r there is the usual asymptotic expansion. In addition, as in [1] these terms can be subsequently evaluated in an asymptotic sense as $r \to \infty$. Noting that $e^{-\gamma r}$ is first factored out, this gives for the electric field

$$\begin{aligned}
\tilde{\vec{E}}(\vec{r}, s) = e^{-\gamma r}\Big\{ & \tilde{f}_\rho(s) \{\frac{1}{4\pi\epsilon_0 r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{p}_\infty + O(r^{-4})\} \\
& + s\{\tilde{f}_\rho(s)\left[\frac{Z_0}{4\pi r^2}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{p}_\infty + O(r^{-3})\right] \\
& \quad + \tilde{f}_J(s)\left[\frac{\mu_0}{4\pi r^2}\vec{1}_r \times \vec{m}_\infty + O(r^{-3})\right]\} \\
& + s^2\{\tilde{f}_\rho(s)\left[\frac{\mu_0}{4\pi r}\left[\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{p}_\infty + O(r^{-2})\right] \\
& \quad + \tilde{f}_J(s)\left[\frac{\mu_0}{4\pi c r}\vec{1}_r \times \vec{m}_\infty + O(r^{-2})\right]\} \\
& + O(s^3\tilde{f}_\rho(s)) + O(s^3\tilde{f}_J(s))\Big\}
\end{aligned} \tag{B.5}$$

Similarly for the magnetic field we have

$$\begin{aligned}
\tilde{\vec{H}}(\vec{r}, s) = e^{-\gamma r}\Big\{ & \tilde{f}_J(s) \{\frac{1}{4\pi r^3}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{m}_\infty + O(r^{-4})\} \\
& + s\{\tilde{f}_J(s)\left[\frac{1}{4\pi c r^2}\left[3\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{m}_\infty + O(r^{-3})\right] \\
& \quad + \tilde{f}_\rho(s)\left[-\frac{1}{4\pi r^2}\vec{1}_r \times \vec{p}_\infty + O(r^{-3})\right]\} \\
& + s^2\{\tilde{f}_J(s)\left[\frac{1}{4\pi c^2 r}\left[\vec{1}_r\vec{1}_r - \overleftrightarrow{1}\right] \bullet \vec{m}_\infty + O(r^{-2})\right] \\
& \quad + \tilde{f}_\rho(s)\left[-\frac{1}{4\pi c r}\vec{1}_r \times \vec{p}_\infty + O(r^{-2})\right]\} \\
& + O(s^3\tilde{f}_J(s)) + O(s^3\tilde{f}_\rho(s))\Big\}
\end{aligned} \tag{B.6}$$

Now the dipoles are generated through a set of incident fields via tensor coefficients called polarizabilities [19]

$$\tilde{\vec{p}}(s) = \epsilon_0 \tilde{\vec{P}}(s) \bullet \tilde{\vec{E}}^{inc}(0, s) = \epsilon_0 E_0 \tilde{f}(s)\, \tilde{\overleftrightarrow{P}}(s) \bullet \vec{1}_2$$

$$\tilde{\vec{m}}(s) = \tilde{\vec{M}}(s) \bullet \tilde{\vec{H}}^{inc}(0, s) = H_0 \tilde{f}(s)\, \tilde{\overleftrightarrow{M}}(s) \bullet \vec{1}_3 \tag{B.7}$$

where the incident fields are evaluated at $\vec{r} = \vec{0}$, near which the scatterers are located (ideally centered). Now as $s \to 0$ we have

$$\overleftrightarrow{P}(s) = \overleftrightarrow{P}_0 + O(s)$$

$$\tilde{\vec{M}}(s) = \overleftrightarrow{M}_0 + O(s) \tag{B.8}$$

based on a Taylor expansion around s=0. Then we have

$$\tilde{\vec{p}}(s) = \tilde{f}(s)\left[\tilde{\vec{p}}_\infty + O(s)\right]$$
$$\tilde{\vec{m}}(s) = \tilde{f}(s)\left[\tilde{\vec{m}}_\infty + O(s)\right]$$
$$\vec{p}_\infty = \epsilon_0 E_0 \overleftrightarrow{P}_0 \bullet \vec{1}_2 = \epsilon_0 E_0 \vec{1}_2 \bullet \overleftrightarrow{P}_0 \tag{B.9}$$
$$\vec{m}_\infty = H_0 \overleftrightarrow{M}_0 \bullet \vec{1}_3 = H_0 \vec{1}_3 \bullet \overleftrightarrow{M}_0$$

The last properties are related to the symmetry of the polarizability tensors for the case of reciprocal media for which

$$\overleftrightarrow{P}(s)^T = \overleftrightarrow{P}(s)$$
$$\overleftrightarrow{M}(s)^T = \overleftrightarrow{M}(s) \tag{B.10}$$

For low frequencies we assume in general

$$\overleftrightarrow{P}_o \neq \overleftrightarrow{0}$$
$$\overleftrightarrow{M}_o \neq \overleftrightarrow{0} \tag{B.11}$$

which is appropriate to a perfectly conducting scatterer. Of course eventually the magnetic field penetrates any finitely conducting scatterer, but such an idealization is useful. This does not prevent the scatterer from absorbing energy through ports which transmit power to the interior for nonzero frequencies.

Appendix C. Energy Due to Perfectly Conducting Scatterer in Uniform Incident Electrostatic Field

Consider an incident uniform electrostatic field with associated scalar potential

$$\begin{aligned}\vec{E}^{(inc)} &= E_o \vec{1}_2 = -\nabla \Phi^{(inc)}(\vec{r}) \\ \Phi^{(inc)}(\vec{r}) &= -\vec{r} \bullet \vec{E}^{(inc)} = -E_o \vec{r} \bullet \vec{1}_2\end{aligned} \quad \text{(C.1)}$$

Given a scatterer with surface S_S, V_S inside, and V_{ex} outside we have a scattered field and potential

$$\vec{E}^{(inc)}(\vec{r}) = -\nabla \Phi^{(sc)}(\vec{r}) \quad \text{(C.2)}$$

with an induced surface charge density $\rho_s(\vec{r})$ on S_S (assumed perfectly conducting). Then in V_S and on S_S we have

$$\Phi^{(inc)}(\vec{r}) + \Phi^{(sc)}(\vec{r}) = 0 \quad \text{(C.3)}$$

where total charge on S_S is assumed zero and the origin of coordinates is chosen to make the constant in (C.3) zero for convenience. Also in V_S we have

$$\vec{E}^{(inc)} + \vec{E}^{(sc)}(\vec{r}) = 0 \quad \text{(C.4)}$$

Then following [15] we have the mixed term

$$\begin{aligned}W_e^{(mix)} &= \int_{S_s} \rho_s(\vec{r})\Phi^{(inc)}(\vec{r})dS_s = -\int_{S_s} \rho_s(\vec{r})\vec{r} \bullet \vec{E}^{(inc)} dS_s \\ &= -\vec{E}^{(inc)} \bullet \int_{S_s} \rho_s(\vec{r})\vec{r}\, dS_s \\ &= -\vec{E}^{(inc)} \bullet \vec{p} = -E_o \vec{1}_2 \bullet \vec{p}\end{aligned} \quad \text{(C.5)}$$

where $\vec{p}$ is now the induced electric dipole moment. Using the static electric polarizability from appendix B we have

$$W_e^{(mix)} = -\epsilon_o E_o^2 \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \tag{C.6}$$

An alternate expression for this term integrates over the fields as

$$W_e^{(mix)} = \epsilon_o \int_{V_s \cup V_{ex}} \vec{E}^{(inc)} \bullet \vec{E}^{(sc)}(\vec{r})\, dV = \epsilon_o E_o \int_{V_s \cup V_{ex}} \vec{1}_2 \bullet \vec{E}^{(sc)}(\vec{r})\, dV \tag{C.7}$$

Note that (with V_s as volume of scatterer)

$$\epsilon_o E_o \int_{V_s} \vec{1}_2 \bullet \vec{E}^{(sc)}(\vec{r})\, dV = -\epsilon_o E_o^2 V_s \tag{C.8}$$

Following [14] we have the self term for the scattered fields

$$\begin{aligned} W_e^{(sc)} &= \frac{1}{2} \int_{S_s} \rho_s(\vec{r}) \Phi^{(sc)}(\vec{r}) dS = -\frac{1}{2} \int_{S_s} \rho_s(\vec{r}) \Phi^{(inc)}(\vec{r}) dS \\ &= \frac{1}{2} \int_{S_s} \rho_s(\vec{r}) \vec{r} \bullet \vec{E}^{(inc)} dS = \frac{1}{2} \vec{E}^{(inc)} \bullet \int_{S_s} \rho_s(\vec{r}) \vec{r}\, dS \\ &= \frac{1}{2} \vec{E}^{(inc)} \bullet \vec{p} = \frac{1}{2} E_o \vec{1}_2 \bullet \vec{p} \end{aligned} \tag{C.9}$$

The 1/2 is the same as that for the energy stored in a capacitor. Then substituting as before

$$\begin{aligned} W_e^{(sc)} &= \frac{1}{2} \epsilon_o E_o^2 \vec{1}_2 \bullet \overleftrightarrow{P}_o \bullet \vec{1}_2 \\ &= -\frac{1}{2} W_e^{(mix)} \end{aligned} \tag{C.10}$$

An alternate expression for this term integrates over the fields as

$$W_e^{(sc)} = \frac{1}{2} \in_o \int_{V_s \cup V_{ex}} \vec{E}^{(sc)}(\vec{r}) \bullet \vec{E}^{(sc)}(\vec{r})\, dV \tag{C.11}$$

Note that

$$\frac{1}{2} \in_o \int_{V_s} \vec{E}^{(sc)}(\vec{r}) \bullet \vec{E}^{(sc)}(\vec{r}) = \frac{1}{2} \in_o E_o^2 V_s \tag{C.12}$$

In computing $W_e^{(inc)}$ from

$$W_e^{(inc)} = \frac{1}{2} \in_o \int_{V_s \cup V_{ex}} \vec{E}^{(inc)} \bullet \vec{E}^{(inc)} dV = \frac{1}{2} \in_o E_o^2 V_\infty \tag{C.13}$$

$V_\infty \equiv$ volume of sphere of radius r_∞

$= V_s + V_{ex}$

this term is unbounded as $r_\infty \rightarrow \infty$ but $W_e^{(mix)}$ and $W_e^{(sc)}$ are well behaved.

Appendix D. Energy Due to Perfectly Conducting Scatterer in Uniform Incident Magnetostatic Field

Consider an incident uniform magnetostatic field with associated vector potential

$$\vec{H}^{(inc)} = H_o \vec{1}_3 = \frac{1}{\mu_o} \nabla \times \vec{A}^{(inc)}(\vec{r}) \tag{D.1}$$

The choice of $\vec{A}^{(inc)}(\vec{r})$ is not unique since one can add the gradient of any scalar (the curl of this being zero). One can derive an appropriate one as in [14]. For our purposes we can use

$$\vec{A}^{(inc)}(\vec{r}) = \frac{1}{2}\mu_o \vec{H}^{(inc)} \times \vec{r} = \frac{1}{2}\mu_o H_o \vec{1}_3 \times \vec{r} \tag{D.2}$$

One can derive this expression using

$$\oint_C \vec{A}(\vec{r}) \bullet d\vec{\ell} = \mu_o \int_S \vec{H}(\vec{r}) \bullet d\vec{S} \tag{D.3}$$

and assume symmetry about an axis parallel to $\vec{1}_3$ through $\vec{r} = \vec{0}$. One can verify the results using the dyadic formula [18].

$$\begin{aligned}\frac{1}{\mu_o} \nabla \times \vec{A}^{(inc)}(\vec{r}) &= \frac{1}{2} \nabla \times [\vec{H}^{(inc)} \times \vec{r}] \\ &= \frac{1}{2}\mu_o \Big\{ \vec{H}^{(inc)} \nabla \bullet \vec{r} - \vec{r} \nabla \bullet \vec{H}^{(inc)} \\ &\quad + (\vec{r} \bullet \nabla)\vec{H}^{(inc)} - (\vec{H}^{(inc)} \bullet \nabla)\vec{r} \Big\}\end{aligned} \tag{D.4}$$

with

$$\begin{aligned}&\nabla \bullet \vec{r} = 3 \\ &\nabla \bullet \vec{H}^{(inc)} = 0 \\ &\nabla \vec{H}^{(inc)} = \overleftrightarrow{0} \\ &\nabla \vec{r} = \overleftrightarrow{1}\end{aligned} \tag{D.5}$$

Given a scatterer with surface S_s, V_s inside, and V_{ex} outside we have scattered field and potential

$$\vec{H}^{(sc)}(\vec{r}) = \frac{1}{\mu_o}\nabla \times \vec{A}^{(sc)}(\vec{r}) \tag{D.6}$$

with an induced divergenceless surface current density $\vec{J}_s(\vec{r})$ on S_s (assumed perfectly conducting). Then in V_s and on S_s we have

$$\vec{A}^{(sc)}(\vec{r}) + \vec{A}^{(inc)}(\vec{r}) = \vec{0} \tag{D.7}$$

Of course this need not be zero but could be any constant vector plus the gradient of any scalar since we only need the curl to be zero. The above, however, is convenient and can be constructed. Also in V_s we have

$$\vec{H}^{(inc)} + \vec{H}^{(sc)}(\vec{r}) = \vec{0} \tag{D.8}$$

Then following [14] we have the mixed energy term

$$W_h^{(mix)} = \mu_o \int_V \int_{V'} \frac{\vec{J}(\vec{r}) \bullet \vec{J}^{(inc)}(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} dV' \, dV \tag{D.9}$$

This is a mutual energy corresponding to the mutual inductance between two loops. Substituting

$$\vec{A}^{(inc)}(\vec{r}) = \mu_o \int_{V'} \frac{\vec{J}^{(inc)}(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} dV' \tag{D.10}$$

and replacing $\vec{J}$ by a surface current density $\vec{J}_s$ give

$$W_h^{(mix)} = \int_{S_s} \vec{J}_s(\vec{r}) \bullet \vec{A}^{(inc)}(\vec{r})\, dS \tag{D.11}$$

Substituting our form of $\vec{A}^{(inc)}$ we have

$$\begin{aligned} W_h^{(mix)} &= \frac{1}{2}\mu_o \int_{S_s} \vec{J}_s(\vec{r}) \bullet [\vec{H}^{(inc)} \times \vec{r}]\, dS \\ &= \mu_o \vec{H}^{(inc)} \bullet \int_{S_s} \frac{1}{2}\vec{J}_s(\vec{r}) \times \vec{r}\; dS \\ &= -\mu_o \vec{H}^{(inc)} \bullet \vec{m} = -\mu_o \vec{H}_o \vec{1}_3 \bullet \vec{m} \end{aligned} \tag{D.12}$$

where $\vec{m}$ is now the induced dipole moment. Using the static magnetic polarizability from appendix B we have

$$W_h^{(mix)} = -\mu_o H_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \tag{D.13}$$

An alternate expression for this term integrates over the fields as

$$W_h^{(mix)} = \mu_o \int_{V_s \cup V_{ex}} \vec{H}^{(inc)} \bullet \vec{H}^{(sc)}(r)\, dV = \mu_o H_o \int_{V_s \cup V_{ex}} \vec{1}_3 \bullet \vec{H}^{(sc)}(\vec{r})\, dV \tag{D.14}$$

Note that (with V_s as volume of the scatterer)

$$\mu_o H_o \int_{V_s} \vec{1}_3 \bullet \vec{H}^{(sc)}(\vec{r})\, dV = -\mu_o H_o^2 V_s \tag{D.15}$$

Following[14] we have the self term for the scattered fields

$$W_h^{(sc)} = \frac{\mu_o}{2} \int_V \int_{V'} \frac{\vec{J}(\vec{r}) \bullet \vec{J}(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} dV' dV \tag{D.16}$$

This corresponds to the self inductance of an inductor, giving the factor of 1/2. Then analogous to before

$$\vec{A}^{(sc)}(\vec{r}) = \mu_o \int_{V'} \frac{\vec{J}(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} dV' = \mu_o \int_{S_s} \frac{\vec{J}_s(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} dS'$$

$$W_h^{(sc)} = \frac{1}{2} \int_{S_s} \vec{J}_s(\vec{r}) \bullet \vec{A}^{(sc)}(\vec{r}) dS \tag{D.17}$$

Next using (D.7) we have

$$\begin{aligned} W_h^{(sc)} &= -\frac{1}{4} \int_{S_s} \vec{J}_s(\vec{r}) \bullet \vec{A}^{(inc)}(\vec{r}) dS \\ &= -\frac{1}{4} \mu_o \int_{S_s} \vec{J}_s(\vec{r}) \bullet [\vec{H}^{(inc)} \times \vec{r}] dS \\ &= -\frac{1}{2} \mu_o \vec{H}^{(inc)} \bullet \int_{S_s} \frac{1}{2} \vec{J}_s(\vec{r}) \times \vec{r} dS \\ &= \frac{1}{2} \mu_o \vec{H}^{(inc)} \bullet \vec{m} = \frac{1}{2} \mu_o H_o \vec{1}_3 \bullet \vec{m} \end{aligned} \tag{D.18}$$

Then substituting for the magnetic dipole moment

$$\begin{aligned} W_h^{(sc)} &= \frac{1}{2} \mu_o \vec{H}_o^2 \vec{1}_3 \bullet \overleftrightarrow{M}_o \bullet \vec{1}_3 \\ &= -\frac{1}{2} W_h^{(mix)} \end{aligned} \tag{D.19}$$

An alternate expression for this term integrates over the fields as

$$W_h^{(sc)} = \frac{1}{2}\mu_o \int_{V_s \cup V_{ex}} \vec{H}^{(sc)}(\vec{r}) \bullet \vec{H}^{(sc)}(\vec{r})\, dV \tag{D.20}$$

Note that

$$\frac{1}{2}\mu_o \int_{V_s} \vec{H}^{(sc)}(\vec{r}) \bullet \vec{H}^{(sc)}(\vec{r})\, dV = \frac{1}{2}\mu_o H_o^2 V_s \tag{D.21}$$

In computing $W_h^{(inc)}$ from

$$W_h^{(inc)} = \frac{1}{2}\mu_o \int_{V_s \cup V_{ex}} \vec{H}^{(inc)} \bullet \vec{H}^{(inc)}\, dV = \frac{1}{2}\mu_o H_o^2 V_\infty \tag{D.22}$$

$V_\infty \equiv$ volume of sphere of radius r_∞

$= V_s + V_{ex}$

this term is unbounded as $r_\infty \to \infty$, but $W_h^{(mix)}$ and $W_h^{(sc)}$ are well behaved.

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