Interaction Notes

Note 454

1 July 1985

On the Use of Electromagnetic Topology for the Decomposition of Scattering Matrices for Complex Physical Structures

Carl E. Baum
Air Force Weapons Laboratory

Abstract

Electromagnetic topology has qualitative and quantitative aspects, both of which are explored in this note. EM topology is developed in a hierarchical form and associated topological indices are defined. The dual graph or interaction sequence diagram is also discussed and used to structure the BLT equation describing signal transport through a system. Partitioning the BLT equation leads to a supervector/supermatrix description. This is solved under appropriate assumptions in a form known as the good-shielding approximation which is discussed in some detail.

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Prologue

"leave all hope, ye that enter."

Canto III (Inferno)

"Let us go for the length of way impels us." Thus he entered, and made me enter, into the first circle that girds the abyss.

Canto IV (Inferno)

Thus I descended from the first circle down into the second, which encompasses less space, and so much greated pain that it stings to wailing.

Canto V (Inferno)

I am in the Third Circle, that of the eternal, accursed, cold, and heavy rain; its law and quality is never new.

Canto VI (Inferno)

Thus we descended into the fourth concavity, taking in more of the dismal bank, which shuts up all the evil of the universe.

Canto VII (Inferno)

The Divine Comedy
by Dante Alighieri
Carlysle-Wicksteed Translation

Contents

I. Introduction

Electromagnetic topology is concerned with the ordering of the electromagnetic response properties of an electromagnetic scatterer (in general a quite complex scatterer) according to appropriate topological descriptions (decompositions) of the scatterer. This division is accomplished by the definition of volumes and boundary surfaces that partition the space occupied by the scatterer. Some of the surfaces (or parts thereof) are chosen to correspond to shield walls in the scatterer (system) [12,16], so that electromagnetic topology is also (at least in part) shielding topology. Corresponding to the volume/surface topology one can define a vertex/edge topology (a graph) which corresonds to the transport of signals between volumes through the surfaces; this graph is the interaction sequence diagram [12,1,13,16]. One can use such topological concepts to obtain guidelines for system hardening (with respect to undesired electromagnetic environments) by identifying penetrations and controlling them in the sense of controlling the passage of undesirable signals through the shielding surfaces [2,4,6,7].

These developments in electromagnetic topology have been primarily qualitative in nature in the sense that, while discrete topological entities are identified (with associated symbols and indexing), the quantitative electromagnetic response of the complex scatterer is not considered in this topological context. A previous paper [5] briefly discussed the use of EM topology for decomposing the system scattering matrix. The present paper goes into greater detail concerning this matter.

Beginning with the toplogical representation of a system in terms of volumes and surfaces, the hierarchical aspects of the topology are represented in terms of appropriate symbols with indices (subscripts). Then supermatrices and the BLT equation are introduced and applied to the topological description of an electromagnetic scatterer. Fundamental to this is the wave indexing on the interaction sequence diagram. This leads to a partitioning of the scattering matrix at five levels. This partitioning also leads to a supermatrix which is quite block sparse. This is used to define a good-shielding approximation; in this case the supermatrix equation has a useful approximate solution in which the control by the shields of the signals passing through the shields is exhibited. Norm concepts are then used to bound the internal signals by a product of terms with each term corresponding to the electromagnetic behavior of appropriate topological entities.

II. Volume/Surface Hierarchical Topology

A. Basic scatterer topology

Let us first review some of the topological concepts [16] which will be used in later developments. Consider dividing three dimensional Euclidean space E_3 into some set of volumes $\{V_\delta\}$ where δ is some alphanumeric index set which we choose in some convenient form(s). Let each V_δ be an open set, not including its surface boundary. Let V_δ^+ be the closure of V_δ, i.e., including its boundary surface to form a closed set. Then we require

$$E_3 = \bigcup_\delta V_\delta^+$$

$$V_\delta \bigcap V_{\delta'} = \begin{cases} V_\delta \text{ for } \delta = \delta' \\ 0 \text{ (the null set) for } \delta \neq \delta' \end{cases} \tag{2.1}$$

Associated with this volume decomposition of space we have a set of boundary surfaces $\{S_{\delta;\delta'}\}$. The closed set $S_{\delta;\delta'}^+$ (i.e., $S_{\delta;\delta'}$ with its boundary curve(s)) is given by

$$S_{\delta;\delta'}^+ \equiv V_\delta^+ \bigcap V_{\delta'}^+ = S_{\delta';\delta}^+ \text{ for } \delta \neq \delta' \tag{2.2}$$

One can expand the use of superscripts from + meaning the closure of a set (inclusion of the boundary) to - meaning the "opening" of a set in the sense of removing the boundaries. Note that such boundary removal is defined to remove only those points necessary to produce an open set. Thus we can write

$$S_{\delta;\delta'} = [S_{\delta;\delta'}^+]^- = [V_\delta^+ \bigcap V_{\delta'}^+]^- \tag{2.3}$$

One can define boundary operators for volumes as

$$\begin{aligned} S(V_\delta) &\equiv \text{boundary (surface) of } V_\delta \\ &= V_\delta^+ - V_\delta \\ &= S(V_\delta^+) \end{aligned} \tag{2.4}$$

and for surfaces as

$$
\begin{aligned}
C(S_{\delta;\delta'}) &\equiv \text{boundary (curve) of } S_{\delta;\delta'} \\
&= S^{+}_{\delta;\delta'} - S_{\delta;\delta'} \\
&= C(S^{+}_{\delta;\delta'})
\end{aligned}
\tag{2.5}
$$

where these boundaries are not necessarily connected, but may consist of sets of disjoint surfaces and curves respectively. Here, subtraction (-) is taken in the set theoretic sense which is also expressable as intersection with the complement (e.g., $-S_{\delta;\delta'} \equiv \bigcap \overline{S}_{\delta;\delta'}$), where complement is with respect to the Euclidean space, i.e.,

$$
\begin{aligned}
&A \bigcup \overline{A} = E_3 \quad \text{for any } A \subset E_3 \\
&A \bigcap \overline{A} = 0 \\
&\overline{E}_3 = 0 \quad \text{(the null set)}
\end{aligned}
\tag{2.6}
$$

Now we have

$$
\begin{aligned}
S(V_\delta) = S(V^{+}_\delta) &= \bigcup_{\delta' \neq \delta} S^{+}_{\delta;\delta'} \\
&= \bigcup_{\delta' \neq \delta} [V^{+}_\delta \bigcap V^{+}_{\delta'}] \\
&= V^{+}_\delta \bigcap [\bigcup_{\delta' \neq \delta} V^{+}_{\delta'}] \\
&= V^{+}_\delta \bigcap [E_3 - V_\delta] \\
&= V^{+}_\delta - V_\delta
\end{aligned}
\tag{2.7}
$$

giving several representations for the boundary of a volume. Note that $S(V_\delta)$ produces a closed surface (or set of closed surfaces) for any volume, i.e., $S(V_\delta)$ has no boundary curve

$$
C(S(V_\delta)) = 0 \tag{2.8}
$$

Similarly, if P represents the boundary operator for curves, giving the set of end points, we have

$$P(C(S_{\delta;\delta'})) = 0 \quad \text{for } \delta \neq \delta' \tag{2.9}$$

since the boundary of $S_{\delta;\delta'}$ is in general a set of closed curves (i.e., no end points).

These concepts are illustrated in fig. 2.1A where the index set δ takes on Roman letters a,b,c,... for this basic form of the scatterer topology. Remember that this is a two-dimensional representation of something that is three dimensional and can have a more complex clustering of volumes V_δ than is depicted here. If we let V_a be the scatterer exterior, then the outer boundary surface of the scatterer is

$$S(V_a) = S(V_a^+) = \bigsqcup_{\delta \neq a} S_{a;\delta}^+ \tag{2.10}$$

One way to construct a graph equivalent to our volume/surface topology is to make a transformation

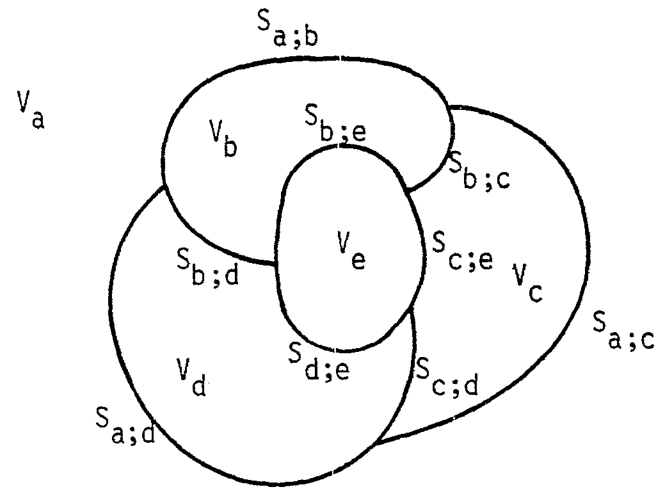
$$\begin{aligned} \text{volume} &\leftrightarrow \text{vertex} \\ \text{surface} &\leftrightarrow \text{edge} \end{aligned} \tag{2.11}$$

This leads to the graph in fig. 2.1B as a representation of the volume/surface topology in 2.1A.

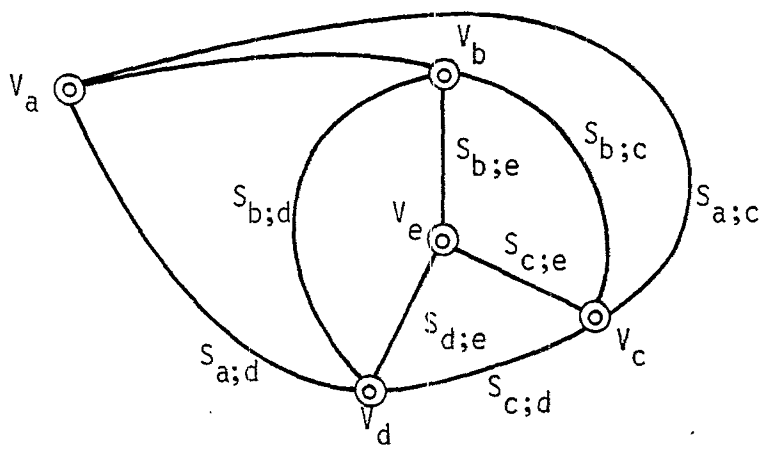
Another type of graph one can use is the bipartite graph [14,15] in fig. 2.1C in which we have the transformation

volume ↔ vertex type A
surface ↔ vertex type B (2.12)
All edges connect A vertices to B vertices
B vertices connect to exactly two edges

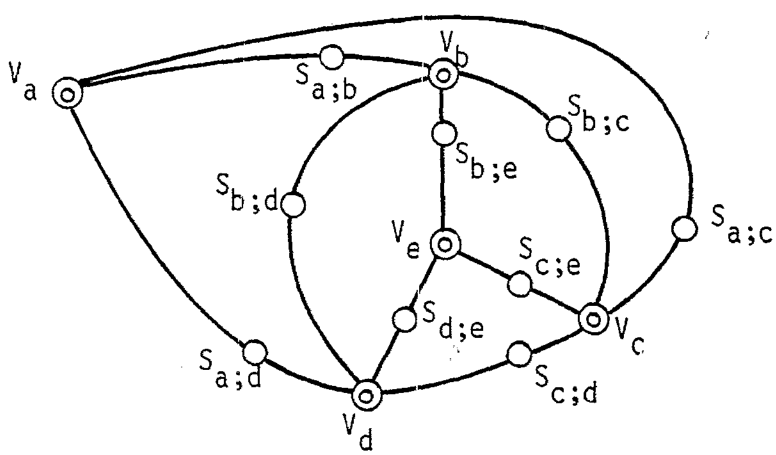
In effect the B vertices (surfaces) are "two sided" (per (2.2)). The edges (branches) represent the transfer (connection) of electromagnetic signals between volumes and surfaces, while the vertices represent the transfer of electromagnetic signals through the volumes and surfaces. Figure 2.1C does not have labels on the edges; each can be labelled by the vertices to which it is connected. Later a wave-indexing scheme with two "wave" indices for two directions (orientations) for each edge will be introduced.



A. Volume/surface topology



B. Related graph (interaction sequence diagram)



C. Related bipartite graph (interaction sequence diagram)

Fig. 2.1. Scatterer Topology

This bipartite form of the graph will be very useful for our later development. In these graphs the vertices and edges are labelled according to the appropriate volumes and surfaces. These graphs can be referred to as "interaction sequence diagrams" [12,1] thereby giving them a physical interpretation. It should be noted that the volume/surface topology and related graph topology are equivalent abstract representations of the same electromagnetic scattering problem. They can be considered mutually complementary.

B. Layers and shields

In extending the volume/surface topology into a hierarchical form let us being with layers and shields as illustrated in fig. 2.2A. Let us define

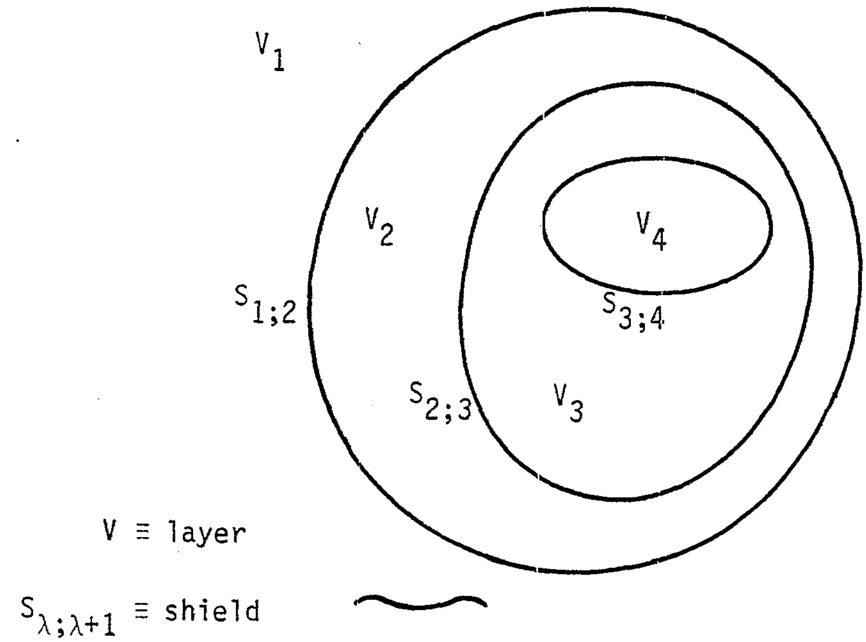
$$\begin{aligned}
V_\lambda &\equiv \lambda\text{th layer (or principal volume)} \\
\lambda &\equiv 1,2,\ldots,\lambda_{max} \equiv \text{layer index} \\
S_{\lambda;\lambda'} = S_{\lambda';\lambda} &\equiv \lambda;\lambda'\text{th shield (or principal surface)} \\
\lambda;\lambda' &\equiv \text{shield index set} \\
\lambda_{max} - 1 &\equiv \text{shielding order} \equiv \text{number of shields}
\end{aligned} \tag{2.13}$$

Note that

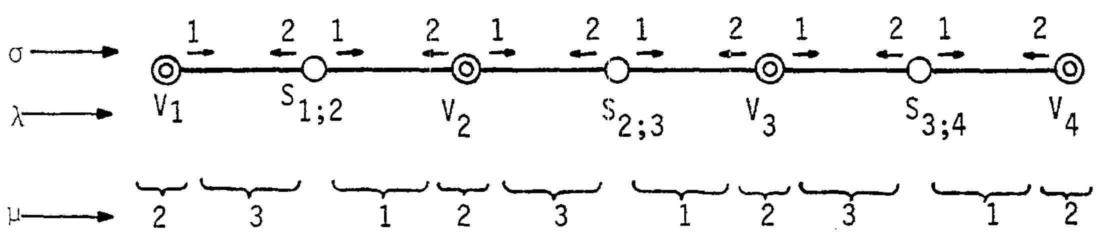
$$S_{\lambda;\lambda'} = \begin{cases} \text{shield, for } \lambda' = \lambda \pm 1 \\ 0 \text{ for } \lambda' \neq \lambda \pm 1 \end{cases} \tag{2.14}$$

This property, as illustrated in fig. 2.2A, is associated with the nested property of our definition of a shield. A shield is defined as a closed surface and no two shields are allowed to intersect, i.e.,

$$\begin{aligned}
S_{\lambda;\lambda+1} &= S^{+}_{\lambda;\lambda+1} = S^{-}_{\lambda;\lambda+1} = V_\lambda \bigcap V_{\lambda+1} \\
S_{\lambda_1;\lambda_1+1} \bigcap S_{\lambda_2;\lambda_2+1} &= \begin{cases} S_{\lambda_1;\lambda_1+1} & \text{for } \lambda_2 = \lambda_1 \\ 0 & \text{for } \lambda_2 \neq \lambda_1 \end{cases}
\end{aligned} \tag{2.15}$$



A. Volume/surface topology



This bipartite graph is a path.

B. Interaction sequence diagram

Fig. 2.2. Layers and Shields in Hierarchical Topology

One might relax this last requirement a little in that shields might be allowed to intersect with a zero measure (in a surface sense) as a set of curves and/or points. However, it is necessary that shields do not cross, i.e., one shield must be "inside" the other with no exterior points which is implied by (2.2) because at such a crossing of $S_{\lambda;\lambda+1}$ and $S_{\lambda+n;\lambda+n+1}$ with $n > 1$ we would have $V_{\lambda+n+1} \cap V_\lambda \neq 0$ including non-zero in a volume measure sense. So we require

$$\begin{aligned} &V_{\lambda+n+1} \cap V_\lambda = 0 \text{ (at least in volume measure)} \\ &n > 1 \\ &V_{\lambda+n} \text{ is inside } V_\lambda \\ &S_{\lambda+n} \text{ is inside } S_\lambda \end{aligned} \qquad (2.16)$$

where "inside" here means that $V_{\lambda+n}$ and $S_{\lambda+n}$ cannot be continuously deformed to infinity without intersecting V_λ and S_λ respectively.

Note that our notation here is slightly different from that in [16]. λ is started at 1 (instead of 0) since λ will later take the role of a vector- and matrix-element index. Also note that layers are bounded by precisely two shields, except for outermost and innermost layers, as

$$V_\lambda^+ \cap S_{\lambda+m-1;\lambda+m} = \begin{cases} S_{\lambda+m-1;\lambda+m} & \text{for } m = 0,1 \\ 0 & \text{otherwise} \end{cases} \qquad (2.17)$$

$$\lambda \neq 1, \lambda_{max}$$

Corresponding to the layer/shield topology there is an interaction sequence diagram which is illustrated in fig. 2.2B. This is a very special form of (bipartite) graph referred to as a <u>path</u> [14,15]. A path begins at a vertex and ends at a vertex with no vertex appearing twice in the sequence. Beginning from V_1 and ending with $V_{\lambda_{max}}$ there is only one path in this network. This follows from (2.15) and (2.17).

Note in fig. 2.2B that another index has been added. This layer-part index μ is used to divide a layer (principal volume) into three parts. As one progresses along the path in the direction of increasing λ, the value $\mu = 1$ is assigned to the edge just after the vertex (symbol ○) assigned to a shield (principal surface) $S_{\lambda-1;\lambda}$, the value $\mu = 2$ is assigned to the layer V_λ (symbol ◎) itself, and the value $\mu = 3$ is assigned to the edge connection to the next shield $S_{\lambda;\lambda+1}$.

One can also define a dual-wave index

$$\sigma = 1,2 \tag{2.18}$$

corresponding to the two different (opposite) directions on each edge. Various conventions are possible for which direction to assign $\sigma = 1$; fig. 2.2B has 1 to the right and 2 to the left. As the hierarchical topology is developed the choice is more complex. In later analysis this dual-wave index will correspond to waves propagating both directions in the system. For now $\sigma = 1$ corresponds to increasing λ. Note that the introduction of two orientations on the edges gives what might be defined as a <u>bidirectional graph</u>.

C. Sublayers and subshields

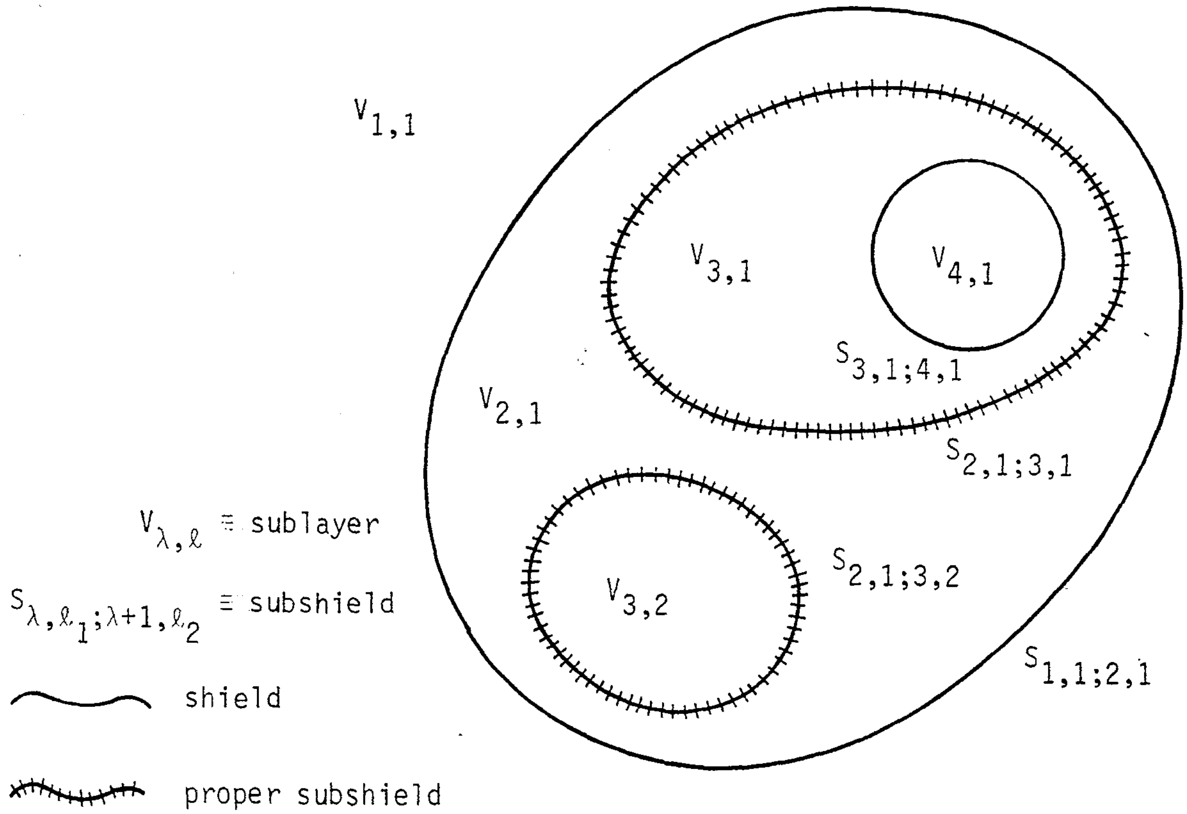
Now go to a more complex volume/surface topology in fig. 2.3A. Basically, the layers and shields are divided into disjoint sublayers and subshields respectively in which the basic layer and shield properties of successive containment in going from the outside to the inside are preserved.

Symbolize sublayers by a second index as

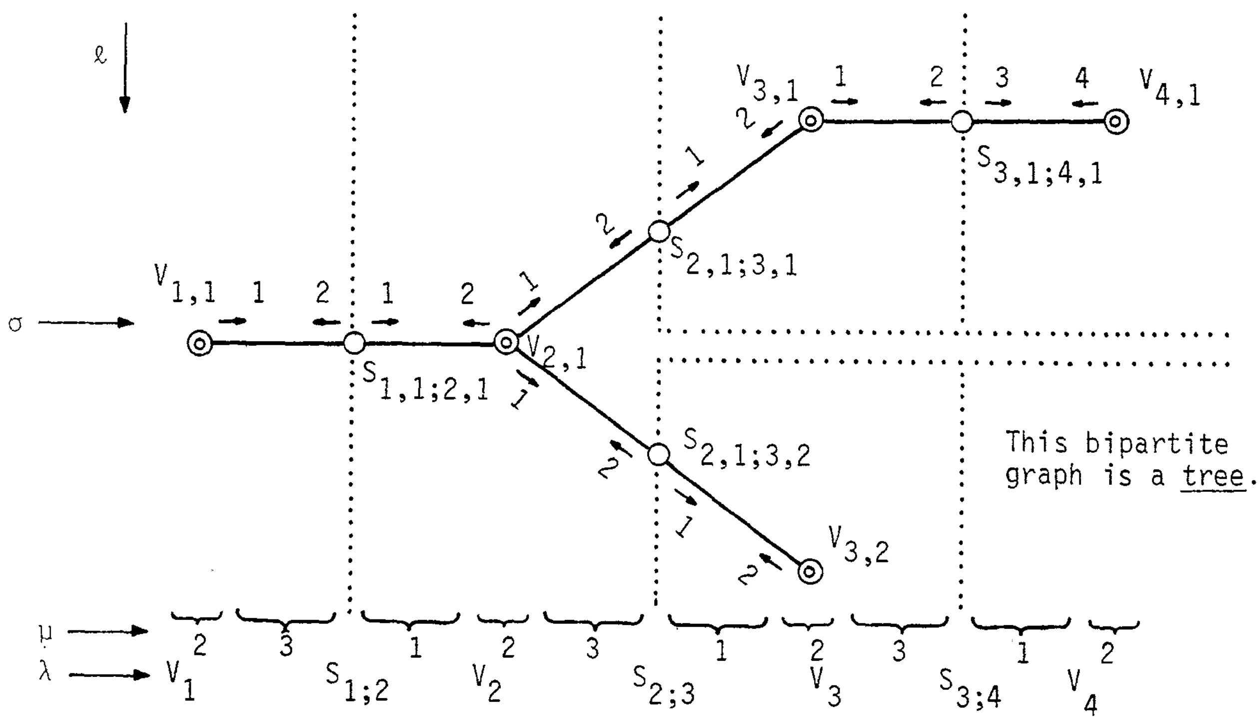
$$\begin{aligned} \ell &= 1,2,\ldots,\ell_{max}(\lambda) \equiv \text{sublayer index} \\ V_{\lambda,\ell} &\equiv \ell\text{th sublayer of the } \lambda\text{th layer} \end{aligned} \tag{2.19}$$

Note that ℓ_{max} is a function of λ (the layer under consideration). We have the λth layer as

$$\begin{aligned} V_\lambda &\equiv \bigcup_{\ell=1}^{\ell_{max}(\lambda)} V_{\lambda,\ell} \\ V_{\lambda,\ell_1} \bigcap V_{\lambda,\ell_2} &\equiv \begin{cases} V_{\lambda,\ell_1} & \text{for } \ell_2 = \ell_1 \\ 0 & \text{for } \ell_2 \neq \ell_1 \end{cases} \end{aligned} \tag{2.20}$$



A. Volume/surface topology



B. Interaction sequence diagram

Fig. 2.3. Sublayers and Subshields in Hierarchical Topology

Again it is conceivable that the requirement for zero intersection might be relaxed to zero volume measure in some cases. Note a special case for some λ

$$\begin{aligned} \ell_{max}(\lambda) &= 1 \\ V_{\lambda,\ell} &\equiv V_\lambda \end{aligned} \tag{2.21}$$

Hence a sublayer can be a layer in some cases. Sublayers which are not also (complete) layers by themselves are called proper sublayers, and for the corresponding λ have $\ell_{max}(\lambda) > 1$.

Note that for

$$1 \leq \lambda_1 < \lambda_2 \leq \lambda_{max} \tag{2.22}$$

$$V_{\lambda_2,\ell_2} \text{ is inside } V_{\lambda_1} \text{ (for some } \ell_1 = 1,2,\ldots,\ell_{1_{max}}(\lambda_1))$$

However, there is some flexibility in choosing the ℓ_2 sublayer index set for V_{λ_2,ℓ_2}, given the ℓ_1 sublayer index set for V_{λ_1,ℓ_1}. This ambiguity results from the property that sublayers in a given layer are effectively in "parallel" while the layers are in "series." Furthermore, the number of sublayers $\ell_{max}(\lambda)$ varies in general from layer to layer, and the sublayers of $V_{\lambda+1}$ do not in general correspond to those of V_λ, particularly in the sense of $V_{\lambda+1}$ sublayers being inside of V_λ sublayers in a one-to-one correspondence.

Now consider subshields as boundaries of sublayers according to

$$S_{\lambda_1,\ell_1;\lambda_2,\ell_2} = V^+_{\lambda_1,\ell_1} \bigcap V^+_{\lambda_2,\ell_2} \text{ for } (\lambda_1,\ell_1) \neq (\lambda_2,\ell_2)$$

$$= \begin{cases} \text{subshield, for } \lambda_2 = \lambda_1 \pm 1 \text{ and if } \ell_1 \text{ and } \ell_2 \text{ allow a common boundary} \\ 0, \text{ for } \lambda_2 \neq \lambda_1 \pm 1 \text{ or if } \ell_1 \text{ and } \ell_2 \text{ do not allow a common boundary} \end{cases} \tag{2.23}$$

Sublayers are bounded on the "outside" by exactly one subshield, except in the outermost layer, as

$$V^{+}_{\lambda,\ell_2} \bigcap S_{\lambda-1,\ell_1;\lambda,\ell_2} = \begin{cases} S_{\lambda-1,\ell_1;\lambda,\ell_2} & \text{for exactly one value of } \ell_1 \text{ if } \lambda > 1 \\ 0 & \text{for other } \ell_1 \text{ or if } \lambda = 1 \end{cases} \tag{2.24}$$

On the inside a sublayer may be bounded by any non-negative integer number of subshields related by

$$V^{+}_{\lambda,\ell_1} \bigcap S_{\lambda,\ell_1;\lambda+1,\ell_2} = \begin{cases} S_{\lambda,\ell_1;\lambda+1,\ell_2} & \text{for some set of } \ell_2 \text{ values, and if } \lambda < \lambda_{max} \\ 0 & \text{for } \ell_2 \text{ outside this set, or if } \lambda = \lambda_{max} \end{cases} \tag{2.25}$$

This relationship could be exhibited in the form of a table. The equivalent bipartite graph (such as the example in fig. 2.3B) shows another way to identify the subshield boundaries of a particular sublayer $V_{\lambda,\ell}$; the single edge connecting to a "surface vertex" to the "left" identifies the "outside" subshield boundary ($\lambda > 1$), while edges connecting to the "surface vertices" to the "right" ($\lambda < \lambda_{max}$) identify the "inside" subshield boundaries. Note the ordering of this bipartite graph from left to right according to the shields (and hence layers) by the use of "vertical" dividing lines (dotted).

The interaction sequence diagram (equivalent bipartite graph) corresponding to the sublayer/subshield topology is illustrated in fig. 2.3B. This special form of (bipartite) graph is referred to as a <u>tree</u> [14,15]. Such a graph has <u>no loops</u>. A path connecting two vertices in the graph is unique (noting that no vertices or edges are repeated in a path. In later analysis this path gives the important edges for signal flow in the graph (system) from one sublayer to another. Now the vertices are for sublayers (symbol ◎) and subshields (symbol ○).

d. Elementary volumes and elementary surfaces

Further extending the volume/surface decomposition in the hierarchical topology, the concept of elementary volumes and the associated elementary surfaces are added as in the example in fig. 2.4A. These do not have the same convenient properties of layers and shields with their longitudinal, "series," or "nesting" or "containment" properties, or of sublayers and subshields which extend the decomposition to transverse or "parallel" aspects.

Now each sublayer $V_{\lambda,\ell}$ is divided into elementary volumes $V_{\lambda,\ell,\tau}$ with

$$\begin{aligned} \tau &= 1,2,\ldots,\tau_{max}(\lambda,\ell) \equiv \text{elementary-volume index} \\ V_{\lambda,\ell,\tau} &\equiv \tau\text{th elementary volume of the } (\lambda,\ell)\text{th sublayer} \end{aligned} \tag{2.26}$$

The (λ,ℓ)th sublayer is given by (including boundary surfaces)

$$V^+_{\lambda,\ell} \equiv \bigcup_{\tau=1}^{\tau_{max}} V^+_{\lambda,\ell,\tau} \tag{2.27}$$

$$V_{\lambda,\ell,\tau} \bigcap V_{\lambda,\ell,\tau_2} \equiv \begin{cases} V_{\lambda,\ell,\tau} & \text{for } \tau_2 = \tau_1 \\ 0 & \text{for } \tau_2 \neq \tau_1 \end{cases}$$

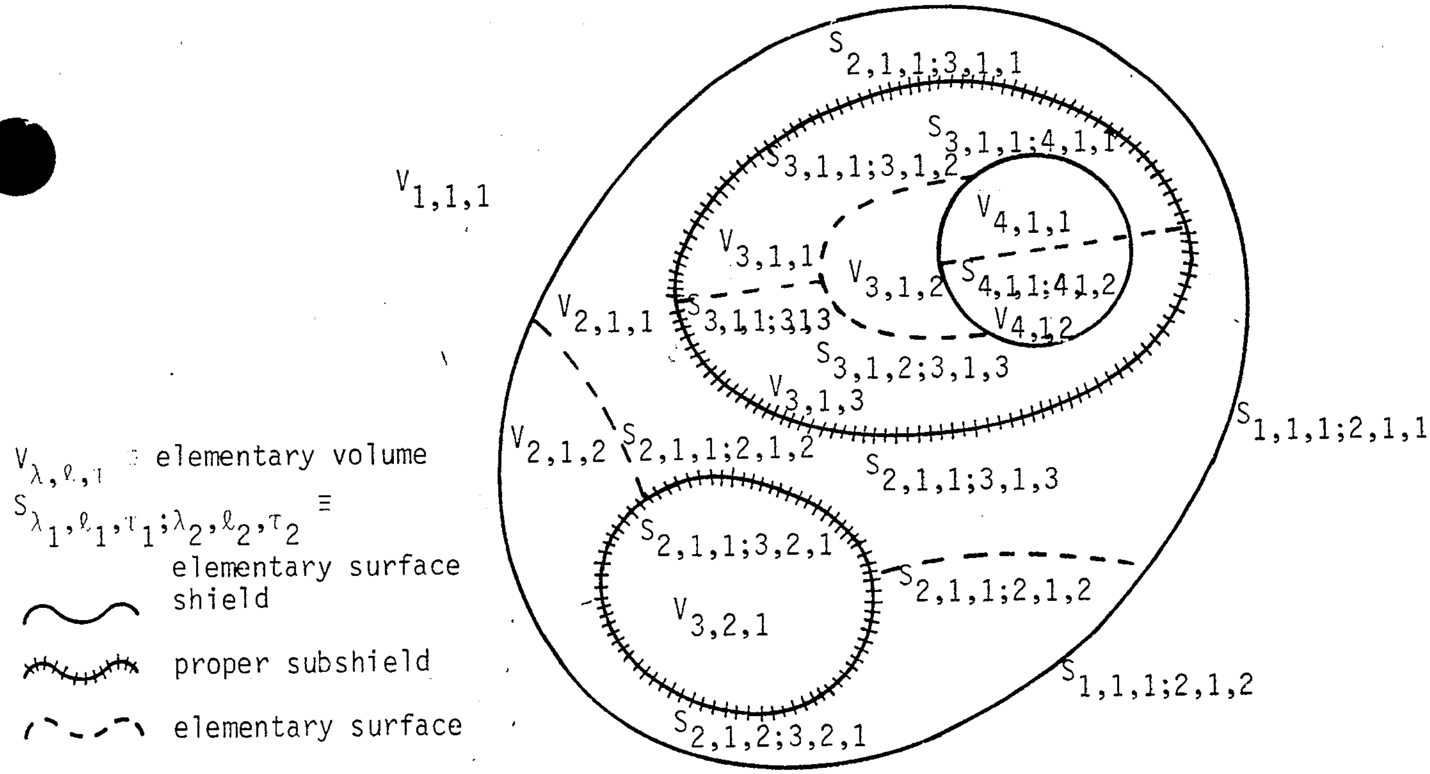
As a special case for some (λ,ℓ)

$$\begin{aligned} \tau_{max}(\lambda,\ell) &= 1 \\ V_{\lambda,\ell,\tau} &\equiv V_{\lambda,\ell} \end{aligned} \tag{2.28}$$

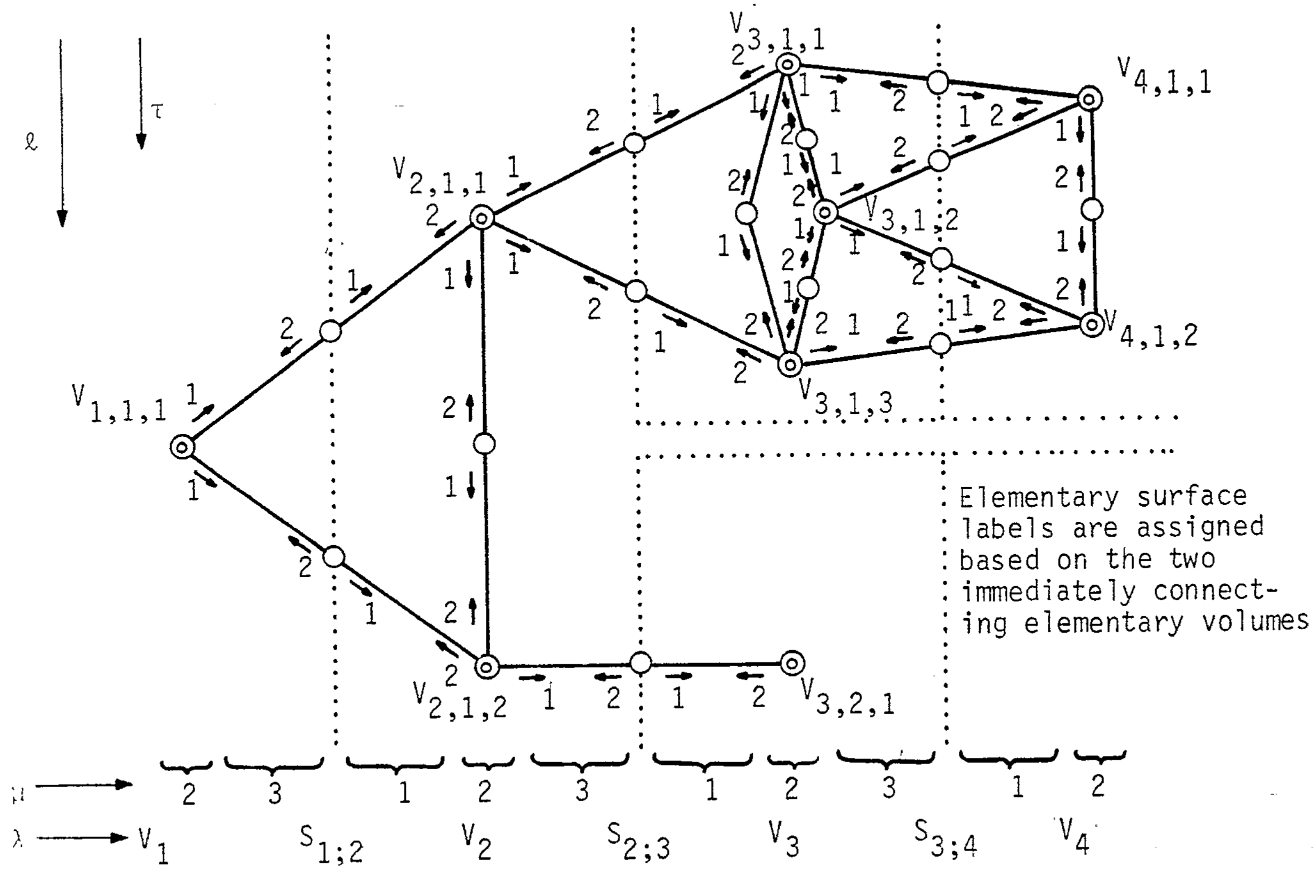
Hence an elementary volume can be a sublayer in some cases. Elementary volumes which are not also (complete) sublayers by themselves are called <u>proper elementary volumes</u>, and for the corresponding (λ,ℓ) have $\tau_{max}(\lambda,\ell) > 1$.

In the previous levels of topological decomposition, i.e., layer/shield and sublayer/subshield, there has been a certain ordering based on longitudinal (series) and transverse (parallel) decomposition respectively. We are left with decomposition within sublayers which can be arbitrary, as distinguished from the "nesting" relations as in (2.16) and (2.22).

In the usual form, elementary surfaces are defined by the common boundary of elementary volumes as



A. Volume/surface topology



B. Interaction sequence diagram

Fig. 2.4. Elementary Volumes and Elementary Surfaces in Hierarchical Topology

$$S^{+}_{\lambda_1,\ell_1,\tau_1;\lambda_2,\ell_2,\tau_2} = V^{+}_{\lambda_1,\ell_1,\tau_1} \bigcap V^{+}_{\lambda_2,\ell_2,\tau_2}$$

$$\text{for } (\lambda_1,\ell_1,\tau_1) \neq (\lambda_2,\ell_2,\tau_2) \qquad (2.29)$$

Inasmuch as elmentary volumes are subsets of sublayers, (2.23) can be applied to determine some combinations of $V^{+}_{\lambda_1,\ell_1}$ and $V^{+}_{\lambda_2,\ell_2}$ for which there is no common subshield boundary and hence no elementary surfaces which are subsets of the same. There are, in addition, other $V^{+}_{\lambda_1,\ell_1,\tau_1}$ and $V^{+}_{\lambda_2,\ell_2,\tau_2}$ which have no common boundary, at least in a non-zero surface measure sense.

The "interconnectivity" (common boundaries or elementary surfaces) between various elementary volumes can again be represented by an interaction sequence diagram as in fig. 2.4B. This type of bipartite graph, with elementary-volume vertices ◎ and elementary-surface vertices ○, directly specifies which combinations of $V^{+}_{\lambda_1,\ell_1,\tau_1}$ and $V^{+}_{\lambda_2,\ell_2,\tau_2}$ have common boundaries $S^{+}_{\lambda_1,\ell_1,\tau_1;\lambda_2,\ell_2,\tau_2}$. While the example in fig. 2.4B is a planar graph, in the general case the graph is non-planar (meaning some edges must cross each other in a planar representation).

Considering the layer-part index μ (=1,2,3), Note now that for $\mu = 2$ there are two such edges in the second layer, six such edges in the third layer, and two such edges in the fourth layer (there be ng no such edges in the first layer).

The dual-wave index σ now must be assigned to $\mu = 2$ edges. These edges do not correspond to increasing λ. For such edges one can use the elementary-volume index τ, e.g., $\sigma = 1$ can correspond to the direction of increasing τ.

Considering the surface/volume hierarchical topology and associated interaction sequence diagram in fig. 2.4, these diagrams can become rather complex when all the labelling is included. The degree of complexity corresponds to how far in detail one wishes to carry the topological decomposition of a given system.

III. Supervectors and Supermatrices

In order to efficiently utilize the topological concepts to decompose the system electromagnetic response, it is useful to introduce the concept of supervectors and supermatrices. These are vectors and matrices whose elements have been partitioned in a special ordered way. A previous note [3] has considered a single partitioning to give divectors and dimatrices.

A. Partitioning vectors to construct supervectors

Suppose that we have some vectors such as (a_{n_1}) of $N_1^{(0)}$ components with elements a_{n_1} for $n_1 = 1,2,\ldots,N_1^{(0)}$. Partition such a vector into N_1 parts, each part having $N_2^{(0)}(n_1)$ components in the form $((a_{n_2})_{n_1})$ as

$$(a_{n_2})_{n_1} \equiv \text{parts or blocks of } (a_{n_1}) \text{ or } ((a_{n_2})_{n_1})$$

$$n_1 = 1,2,\ldots,N_1$$

$$n_2 = 1,2,\ldots,N_2^{(0)}(n_1) \tag{3.1}$$

$$((a_{n_2})_{n_1}) = (a_{n_2})_1 \oplus (a_{n_2})_2 \oplus \cdots \oplus (a_{n_2})_{N_1}$$

$$= ((a_{n_2})_1, (a_{n_2})_2, \ldots, (a_{n_2})_{N_1})$$

$$\equiv \bigoplus_{n_1=1}^{N_1} (a_{n_2})_{n_1}$$

The individual components are

$$a_{n_2;n_1} \equiv \text{individual vector components}$$

$$N_1^{(0)} = \sum_{n_1=1}^{N_1} N_2^{(0)}(n_1) = \text{total number of individual vector components} \tag{3.2}$$

Note the use of the direct sum $\oplus$ to combine vectors in the form of a divector. This is also generalized to a continued direct sum $\oplus$ with the terms ordered in the order of increasing index (as n_1 in (3.1)). In form it is similar to addition + and continued summation $\sum$. However, note that the direct sum is non-commutative.

Thus far we have once-partitioned vectors or divectors. Let us consider a ν-vector with ν indices which takes the form

$$(((\cdots((a_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\cdots)_{n_2})_{n_1}) \equiv \text{a } \nu\text{-vector}$$

$$\nu \equiv \text{supervector order}$$

$$\nu - 1 \equiv \text{number of partitions or partition level}$$

$$a_{n_\nu;n_{\nu-1};n_{\nu-2};\ldots;n_2;n_1} \equiv \text{individual vector components} \tag{3.3}$$

$$n_p = 1,2,\ldots,N_p\ (n_1,n_2,\ldots,n_{p-1})$$
$$\text{for } p = 1,2,\ldots,\nu$$

One can refer to such supervectors and indicate their order by the use of Greek prefixes in the commonly used form as

monovector (or vector)	, $\nu = 1$
divector	, $\nu = 2$
trivector	, $\nu = 3$
quadrivector	, $\nu = 4$
pentavector	, $\nu = 5$
hexavector	, $\nu = 6$
etc.	

(3.4)

One can appreciate this form for a supervector by beginning with the partition of a monovector (a_{n_1}) to form a divector $((a_{n_2})_{n_1})$ as in (3.1). Continue by induction to transform a γ-vector to a (γ + 1)-vector as

$$(((\cdots((a_{n_\gamma})_{n_{\gamma-1}})_{n_{\gamma-2}}\cdots)_{n_2})_{n_1}) \rightarrow$$
$$(((\cdots(((a_{n_{\gamma+1}})_{n_\gamma})_{n_{\gamma-1}})_{n_{\gamma-2}}\cdots)_{n_2})_{n_1}) \tag{3.5}$$

by partitioning the individual blocks as

$$(a_{n_\gamma})_{n_{\gamma-1};n_{\gamma-2};\ldots;n_2;n_1} \rightarrow ((a_{n_{\gamma+1}})_{n_\gamma})_{n_{\gamma-1};n_{\gamma-2};\ldots;n_2;n_1} \tag{3.6}$$

where before partition

$$\begin{aligned} n_p &= 1,2,\ldots,N_p\ (n_1,n_2,\ldots,n_{p-1}) \quad \text{for } p = 1,2,\ldots,\gamma-1 \\ n_\gamma &= 1,2,\ldots,N_\gamma^{(0)}(n_1,n_2,\ldots,n_{\gamma-1}) \end{aligned} \tag{3.7}$$

and after partition

$$\begin{aligned} n_p &= 1,2,\ldots,N_p(n_1,n_2,\ldots,n_{p-1}) \text{ for } p = 1,2,\ldots,\gamma \\ n_{\gamma+1} &= 1,2,\ldots,N_{\gamma+1}^{(0)}(n_1,n_2,\ldots,n_\gamma) \end{aligned} \tag{3.8}$$

In the process of partitioning the $n_{\gamma-1}$ block a number of blocks less than or equal to the number of elements is formed as

$$1 \leq N_\gamma(n_1,n_2,\ldots,n_\gamma) \leq N_\gamma^{(0)}(n_1,n_2,\ldots,n_\gamma) \tag{3.9}$$

Hence the single partition in (3.1) can be extended to any number of partitions (a positive-integer number) by repeated partition of the innermost block (or vector). If one terminates this sequence of partitions at the (ν-1)th partition, thereby giving a ν-vector, then one can write

$$N_\nu^{(0)}(n_1,n_2,\ldots,n_{\nu-1}) \equiv N_\nu(n_1,n_2,\ldots,n_{\nu-1}) \tag{3.10}$$

indicating no further partition of the $N_\nu^{(0)}$ elements.

Note that as a special case the number of blocks $N_p(n_1,n_2,\ldots,n_{p-1})$ in the partition of $(a_{n_p})_{n_{p-1};\ldots;n_1}$ can be chosen to be 1 as desired, i.e.,

$$(a_{n_p})_{n_{p-1};\ldots;n_1} \rightarrow ((a_{n_{p+1}})_{n_p})_{n_{p-1};\ldots;n_1}$$

$$((a_{n_{p+1}})_{n_p})_{n_{p-1};\ldots;n_1} = ((a_{n_{p+1}})_1)_{n_{p-1};\ldots;n_1} \tag{3.11}$$

$$(a_{n_{p+1}})_{n_p;n_{p-1};\ldots;n_1} = (a_{n_{p+1}})_{1;n_{p-1};\ldots;n_1}$$

which corresponds to no partition of the particular block (vector).

Throughout this partitioning process the number of individual vector elements is conserved. They are merely partitioned in a certain hierarchical way involving successive partitions. This is expressed in the relations

$$N_p^{(0)}(n_1,n_2,\ldots,n_{p-1}) = \sum_{n_p=1}^{N_p(n_1,n_2,\ldots,n_{p-1})} \sum_{n_{p+1}=1}^{N_{p+1}(n_1,n_2,\ldots,n_p)} \cdots \sum_{n_{q-1}=1}^{N_{q-1}(n_1,n_2,\ldots,n_{q-2})} N_q^{(0)}(n_1,n_2,\ldots,n_{q-1})$$

$$\text{for } p = 1,2,\ldots,q-1$$

$$2 \leq q \leq \nu \tag{3.12}$$

which include the special case of $(p,q) = (1,\nu)$ which covers the full set of partitions.

B. Compatible order among supervectors

Supervectors, like ordinary vectors, can be combined by various operations. In ordinary vectors one must have a certain type of compatibility between two vectors in that, say for addition, they must have the same number of elements. In the case of supervectors, we must have the same number of elements at every level of partition for addition as

$$(((\ldots((a_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_2})_{n_1}) + (((\ldots((b_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_2})_{n_1}) =$$

$$(((\ldots((c_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_2})_{n_1}) \tag{3.13}$$

$$a_{n_\nu;n_{\nu-1};\ldots;n_1} + b_{n_\nu;n_{\nu-1};\ldots;n_1} = c_{n_\nu;n_{\nu-1};\ldots;n_1}$$

where now all three ν-vectors have the same number of partitions and the same partition structure as

$$n_p = 1,2,\ldots,N_p(n_1,n_2,\ldots,n_{p-1}) \text{ for } p = 1,2,\ldots,\nu \qquad (3.14)$$

which we call <u>compatible order</u> for supervectors.

Likewise, one can generalize dot multiplication (or contraction) between ordinary vectors to a generalized dot product indicated by ⊙ between supervectors as

$$(((\ldots((a_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_2})_{n_1}) \odot (((\ldots((b_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_2})_{n_1})$$

$$= \sum_{n_1=1}^{N_1} ((\ldots((a_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_2})_{n_1}) \odot ((\ldots((b_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}})\ldots)_{n_2})_{n_1}$$

$$= \sum_{n_1=1}^{N_1} \cdots \sum_{n_p=1}^{N_p(n_1,n_2,\ldots,n_{p-1})} ((\ldots((a_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_{p+1}})_{n_p;n_{p-1};\ldots;n_2;n_1}$$

$$\odot ((\ldots((b_{n_\nu})_{n_{\nu-1}})_{n_{\nu-2}}\ldots)_{n_{p+1}})_{n_p;n_{p-1};\ldots;n_2;n_1}$$

$$= \sum_{n_1=1}^{N_1} \cdots \sum_{n_\nu=1}^{N_\nu(n_1,n_2,\ldots,n_{\nu-1})} a_{n_\nu;n_{\nu-1};\ldots;n_1} b_{n_\nu;n_{\nu-1};\ldots;n_1}$$

$$= \text{a scalar} \qquad (3.15)$$

Again the vectors must be of compatible order as in (3.14) for the summations in (3.15) to be defined. The same type of compatible order for supervectors then applies to both addition and generalized dot multiplication.

C. Partitioning matrices to construct supermatrices

Now consider some matrices such as (A_{n_1,m_1}) with elements

$$A_{n_1,m_1} \text{ for } n_1 = 1,2,\ldots,N_1^{(0)} \qquad (3.16)$$
$$m_1 = 1,2,\ldots,M_1^{(0)}$$

Partition both the rows and columns of such a matrix in the same manner as vectors have already been partitioned. With ν-1 such partitions of both rows and columns we have

$$(((\ldots(A_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1}) \equiv \text{a } \nu\text{-matrix}$$

$\nu \equiv$ supermatrix order

ν-1 $\equiv$ number of partitions or partition order (3.17)

$A_{n_\nu,m_\nu;n_{\nu-1},m_{\nu-1};\ldots;n_2,m_2;n_1,m_1} \equiv$ individual matrix components

$$\left.\begin{aligned} n_p &= 1,2,\ldots,N_p(n_1,n_2,\ldots,n_{p-1}) \\ m_p &= 1,2,\ldots,M_p(m_1,m_2,\ldots,m_{p-1}) \end{aligned}\right\} \text{ for } p = 1,2,\ldots,\nu$$

As in the case of supervectors, supermatrices can be referred to in a manner which indicates their order by using Greek prefixes as

monomatrix (or matrix)	, ν = 1
dimatrix	, ν = 2
trimatrix	, ν = 3
quadrimatrix	, ν = 4
pentamatrix	, ν = 5
hexamatrix	, ν = 6
etc.	

(3.18)

The conservation of rows and columns in the partitioning process is expressed by (3.12) for both N_p and M_p indices, their unpartitioned forms being $N_p^{(0)}$ and $M_p^{(0)}$ respectively.

D. Compatible order among supermatrics

Supermatrices can be combined with other supermatrices and with supervectors by various operations. Again, there must be a compatibility of the numbers of elements and their partition(s) pertinent to the particular operation(s). In the case of supermatrices, addition is similar to supervectors as

$$(((\ldots(A_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{m_1,m_1})$$

$$+\ (((\ldots(B_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1})$$

$$= (((\ldots(C_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1}) \tag{3.19}$$

$$A_{n_\nu,m_\nu;n_{\nu-1},m_{\nu-1};\ldots;n_1,m_1} + B_{n_\nu,m_\nu;n_{\nu-1},m_{\nu-1};\ldots;n_1,m_1}$$

$$= C_{n_\nu,m_\nu;n_{\nu-1},m_{\nu-1};\ldots;n_1,m_1}$$

where the partitioning of the n_p and m_p indices is the same for all the above supermatrices and is given in (3.17). This is the rule for <u>compatible order for addition</u>. Note that addition for supermatrices is <u>commutative</u>.

For dot multiplication of supermatrices (<u>non-commutative</u> in general) one can generalize dot multiplication (or contraction) between ordinary matrices indicated by $\odot$ as

$$(((\ldots(A_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1})$$

$$\odot\ (((\ldots(B_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1})$$

$$= (((\ldots(C_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1})$$

$$((\ldots(C_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_{p+1},m_{p+1}})_{n'_p,m_p;n'_{p-1},m_{p-1};\ldots;n'_1,m_1}$$

$$= \sum_{n_1=1}^{N_1} \cdots \sum_{n_p=1}^{N_p(n_1,n_2,\ldots,n_{p-1})}$$

$$((\ldots(A_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_{p+1},m_{p+1}})_{n'_p,n_p;n'_{p-1},n_{p-1};\ldots;n'_1,n_1}$$

$$\odot\ ((\ldots(B_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_{p+1},m_{p+1}})_{n_p,m_p;n_{p-1},m_{p-1};\ldots;n_1,m_1} \tag{3.20}$$

$$C_{n'_\nu,m_\nu;\ldots;n'_1,m_1}$$

$$= \sum_{n_1=1}^{N_1} \cdots \sum_{n_\nu=1}^{N_\nu(n_1,n_2,\ldots,n_{\nu-1})} A_{n'_\nu,n_\nu;\ldots;n'_1,n_1} B_{n_\nu,m_\nu;\ldots;n_1,m_1}$$

In this type of multiplication the number of elements in each partition of the columns of the first matrix must equal the number of elements in each partition of the rows of the second matrix. For the first ν-matrix we have

$$A_{n'_\nu,m'_\nu;\ldots;n'_1,m'_1}\text{ , ν-matrix elements}$$
$$\left.\begin{array}{l} n'_p = 1,2,\ldots,N'_p(n'_1,n'_2,\ldots,n'_{p-1}) \\ m'_p = 1,2,\ldots,M'_p(m'_1,m'_2,\ldots,m'_{p-1}) \end{array}\right\} \text{ for } p = 1,2,\ldots,\nu \tag{3.21}$$

For the second ν-matrix we have

$$B_{n_\nu,m_\nu;\ldots;n_1,m_1}\text{ , ν-matrix elements}$$
$$\left.\begin{array}{l} n_p = 1,2,\ldots,N_p(n_1,n_2,\ldots,n_{p-1}) \\ m_p = 1,2,\ldots,M_p(m_1,m_2,\ldots,m_{p-1}) \end{array}\right\} \text{ for } p = 1,2,\ldots,\nu \tag{3.22}$$

The product is a ν-matrx as

$$C_{n'_\nu,m_\nu;\ldots;n'_1,m_1}\text{ , ν-matrix elements}$$
$$\left.\begin{array}{l} n'_p = 1,2,\ldots,N'_p(n'_1,n'_2,\ldots,n'_{p-1}) \\ m_p = 1,2,\ldots,M_p(m_1,m_2,\ldots,m_{p-1}) \end{array}\right\} \text{ for } p = 1,2,\ldots,\nu \tag{3.23}$$

with the constraint

$$\left.\begin{array}{l} m'_p = n_p \\ M'_p(m'_1,m'_2,\ldots,m'_{p-1}) = N_p(n_1,n_2,\ldots,n_{p-1}) \end{array}\right\} \text{ for } p = 1,2,\ldots,\nu \tag{3.24}$$

This constraint makes the two ν-matrices of compatible order for generalized dot multiplication in the particular order specified.

Analogous to sets of NxN square matrices we have supermatrices of symmetric compatible order. Such ν-matrices have the form

$$(((\ldots(D_{n_\nu,m_\nu})_{n_{\nu-1},m_{\nu-1}}\ldots)_{n_2,m_2})_{n_1,m_1}) \equiv \nu\text{-matrix}$$

$$D_{n_\nu,m_\nu;\ldots;n_1,m_1} \equiv \nu\text{-matrix elements} \tag{3.25}$$

$$\left.\begin{array}{l} n_p = 1,2,\ldots,N_p(n_1,n_2,\ldots,n_{p-1}) \\ m_p = 1,2,\ldots,N_p(m_1,m_2,\ldots,m_{p-1}) \end{array}\right\} \text{ for } p = 1,2,\ldots,\nu$$

so that the number of elements at each level of partition is the same for both rows and columns. Such supermatrices may be dot multiplied in either order to give supermatrices of the same symmetric compatible order as in (3.25). Of course, these supermatrices are in general non-commutative with each other with respect to generalized dot multiplication.

IV. Supermatrix Inverse

One of the important matrix operations is that of inverse. For supermatrices this operation still applies. For regular matrices, the inverse comes as the solution of

$$(A_{n,m}) \cdot (x_n) = (y_n)$$
$$(x_n) = (A_{n,m})^{-1} \cdot (y_n) \tag{4.1}$$

where (x_n) and (y_n) are N component vectors and $(A_{n,m})$ is an NxN matrix. Since this solution is to apply for all (y_n) (assuming a solution exists) then we have

$$(A_{n,m}) \cdot (A_{n,m})^{-1} = (A_{n,m})^{-1} \cdot (A_{n,m}) = (1_{n,m})$$

$$(1_{n,m}) \equiv \text{identity matrix} \tag{4.2}$$

$$1_{n,m} = \begin{cases} 1 \text{ for } n = m \\ 0 \text{ for } n \neq m \end{cases}$$

Here all the matrices, including $(A_{n,m})^{-1}$ and $(1_{n,m})$ are square NxN matrices.

Now it is well known that the inverse exists provided

$$\det((A_{n,m})) \neq 0 \tag{4.3}$$

If this determinant is zero, the matrix is said to be singular.

Let

$$(B_{n,m}) = (A_{n,m})^{-1} \tag{4.4}$$

assuming the inverse exists. Now partition $(A_{n,m})$ and $(B_{n,m})$ in symmetric compatible order to form two dimatrices.

For simplicity, consider first the case of dimatrices which are consisting of four blocks so that we can write

$$((A_{n_2,m_2})_{n_1,m_1}) = \begin{pmatrix} (A_{n_2,m_2})_{1,1} & (A_{n_2,m_2})_{1,2} \\ (A_{n_2,m_2})_{2,1} & (A_{n_2,m_2})_{2,2} \end{pmatrix}$$

$$((B_{n_2,m_2})_{n_1,m_1}) = \begin{pmatrix} (B_{n_2,m_2})_{1,1} & (B_{n_2,m_2})_{1,2} \\ (B_{n_2,m_2})_{2,1} & (B_{n_2,m_2})_{2,2} \end{pmatrix} = ((A_{n_2,m_2})_{n_1,m_1})^{-1} \tag{4.5}$$

where the partitioning has

$$\begin{aligned} n,m &= 1,2,\ldots,N_1^{(0)} \\ n_1,m_1 &= 1,2 \quad , \quad N_1 = 2 \\ n_2 &= 1,2,\ldots,N_2^{(0)}(n_1) \\ m_2 &= 1,2,\ldots,N_2^{(0)}(m_1) \\ N_1^{(0)} &= \sum_{n_1=1}^{2} N_2^{(0)}(n_1) \end{aligned} \tag{4.6}$$

This can be referred to as a binary partition since each index is split into two parts. Similarly, the vectors in (4.1) of $N_1^{(0)}$ components can be partitioned as

$$\begin{aligned} ((x_{n_2})_{n_1}) &= ((x_{n_2})_1 \; , \; (x_{n_2})_2) \\ ((y_{n_2})_{n_1}) &= ((y_{n_2})_1 \; , \; (y_{n_2})_2) \end{aligned} \tag{4.7}$$

with the same indices as in (4.6).

Now write (4.1) as a set of linear equations as

$$\begin{aligned} (A_{n_2,m_2})_{1,1} \cdot (x_{n_2})_1 + (A_{n_2,m_2})_{1,2} \cdot (x_{n_2})_2 &= (y_{n_2})_1 \\ (A_{n_2,m_2})_{2,1} \cdot (x_{n_2})_1 + (A_{n_2,m_2})_{2,2} \cdot (x_{n_2})_2 &= (y_{n_2})_2 \end{aligned} \tag{4.8}$$

Solve these linear equations by using the inverses of the diagonal (and hence square because of symmetric compatible order) blocks to give first

$$(x_{n_2})_1 + (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2} \cdot (x_{n_2})_2 = (A_{n_2,m_2})^{-1}_{1,1} \cdot (y_{n_2})_1$$
$$(A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1} \cdot (x_{n_2})_1 + (x_{n_2})_2 = (A_{n_2,m_2})^{-1}_{2,2} \cdot (y_{n_2})_2 \tag{4.9}$$

where we have assumed the diagonal blocks are non-singular. Now to eliminate $(x_{n_2})_1$ and $(x_{n_2})_2$ from the left sides of (4.9) one need only dot multiply each equation by the appropriate matrix product coefficient and subtract to give

$$[(1_{n_2,m_2})_{1,1} - (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}] \cdot (x_{n_2})_1$$
$$= (A_{n_2,m_2})^{-1}_{1,1} \cdot (y_{n_2})_1 - (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (y_{n_2})_2$$
$$\tag{4.10}$$
$$[(1_{n_2,m_2})_{2,2} - (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}] \cdot (x_{n_2})_2$$
$$= -(A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (y_{n_2})_1 + (A_{n_2,m_2})^{-1}_{2,2} \cdot (y_{n_2})_2$$

Here we have introduced the identity dimatrix with the same symmetrical compatible partitioning as in (4.5) and (4.6)

$$((1_{n_2,m_2})_{n_1,m_1}) = \begin{pmatrix} (1_{n_2,m_2})_{1,1} & (0_{n_2,m_2})_{1,2} \\ (0_{n_2,m_2})_{2,1} & (1_{n_2,m_2})_{2,2} \end{pmatrix} \tag{4.11}$$

Note that the off-diagonal blocks are zero matrices, i.e., (4.2) is generalized as

$$(1_{n_2,m_2})_{n_1,m_1} = \begin{cases} (1_{n_2,m_2})_{n_1,m_1} & \text{for } n_1 = m_1 \text{ (square)} \\ (0_{n_2,m_2})_{n_1,m_1} & \text{for } n_1 \neq m_1 \text{ (not in general square)} \end{cases} \tag{4.12}$$

Using the matrix inverse of the matrix coefficients on the left of equations (4.10) gives

$$(x_{n_2})_1 =$$

$$[(1_{n_2,m_2})_{1,1} - (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]^{-1}$$

$$\cdot[(A_{n_2,m_2})^{-1}_{1,1} \cdot (y_{n_2})_1 - (A^{-1}_{n_2,m_2})_{1,1} \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (y_{n_2})_2]$$

$$= [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]^{-1} \cdot (y_{n_2})_1$$

$$- [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]^{-1}$$

$$\cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (y_{n_2})_2$$

(4.13)

$$(x_{n_2})_2 =$$

$$[(1_{n_2,m_2})_{2,2} - (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}]^{-1}$$

$$\cdot [-(A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (y_{n_3})_1 + (A_{n_2,m_2})^{-1}_{2,2} \cdot (y_{n_2})_2$$

$$= -[(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}]^{-1}$$

$$\cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (y_{n_2})_1$$

$$+ [(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}]^{-1} \cdot (y_{n_2})_1$$

Writing these as one supermatrix equation gives

$$((x_{n_2})_{n_1}) = ((B_{n_2,m_2})_{n_1,m_1}) \cdot ((y_{n_2})_{n_1})$$

$$(B_{n_2,m_2})_{1,1} = [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1} \cdot (A_{n_2,m_2})_{2,1}]^{-1}$$

$$(B_{n_2,m_2})_{1,2} = -[(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1} \cdot (A_{n_2,m_2})_{2,1}]^{-1} \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1}$$

$$(4.14)$$

$$(B_{n_2,m_2})_{2,1} = -[(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2}]^{-1} \cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1}$$

$$(B_{n_2,m_2})_{2,2} = [(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2}]^{-1}$$

This is an explicit representation of the binary partitioned dimatrix in terms of its blocks.

There is an alternate representation of the supermatrix inverse. In particular, the off-diagonal blocks can be represented as

$$\begin{aligned}(B_{n_2,m_2})_{1,2} &= -[(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1} \cdot (A_{n_2,m_2})_{2,1}]^{-1} \\ &\quad \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1} \\ &\quad \cdot [(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2}] \\ &\quad \cdot [(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2}]^{-1} \\ &= -[(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1} \cdot (A_{n_2,m_2})_{2,1}]^{-1} \\ &\quad \cdot [(A_{n_2,m_2})_{1,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2} \\ &\quad -(A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})_{2,2}^{-1} \cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2}] \\ &\quad \cdot [(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})_{1,1}^{-1} \cdot (A_{n_2,m_2})_{1,2}]^{-1}\end{aligned}$$

$$= -(A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}$$

$$\cdot [(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}]^{-1} \quad (4.15)$$

$$(B_{n_2,m_2})_{2,1} = -[(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n\ ,m\ })_{2,1}]^{-1}$$

$$\cdot (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1}$$

$$\cdot [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]$$

$$\cdot [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]^{-1}$$

$$= -[(A_{n_2,m_2})_{2,2} - (A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2}]^{-1}$$

$$\cdot [(A_{n_2,m_2})_{2,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}$$

$$-(A_{n_2,m_2})_{2,1} \cdot (A_{n_2,m_2})^{-1}_{1,1} \cdot (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]$$

$$\cdot [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]^{-1}$$

$$= -(A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}$$

$$\cdot [(A_{n_2,m_2})_{1,1} - (A_{n_2,m_2})_{1,2} \cdot (A_{n_2,m_2})^{-1}_{2,2} \cdot (A_{n_2,m_2})_{2,1}]^{-1}$$

As can be verified with these results $(A_{n_2,m_2})_{n_1,m_1})$ and $((B_{r_2,m_2})_{n_1,m_1})$ can be dot multiplied in either order to give

$$((A_{n_2,m_2})_{n_1,m_1}) \odot ((B_{n_2,m_2})_{n_1,m_1}) = ((B_{n_2,m_2})_{n_1,m_1}) \odot ((A_{n_2,m_2})_{n_1,m_1})$$

$$= ((1_{n_2,m_2})_{n_1,m_1}) \quad (4.16)$$

The supermatrix inverse of matrices partitioned in a binary fashion at each level and in symmetric compatible order can be extended by induction to arbitrary orders of supermatrices. Take the results for the first partition in (4.14). In these results, we first require the inverse of the diagonal (and hence square) blocks $(A_{n_2,m_2})_{1,1}$ and $(A_{n_2,m_2})_{2,2}$. As diagonal blocks they are partitioned in symmetric compatible order, and have the same inverse in terms of their blocks using again the formulas in (4.14) (at this next level of partition). Similarly, the combinations of matrices to be inverted are square matrices, and are also partitioned in symmetric compatible order and the formulas of (4.14) can again be applied to the blocks at this level of partition. This process can in principle be applied indefinitely to higher and higher orders until all the matrix blocks are reduced to scalars (or 1x1 matrices).

V. BLT Equation for Transmission-Line Networks

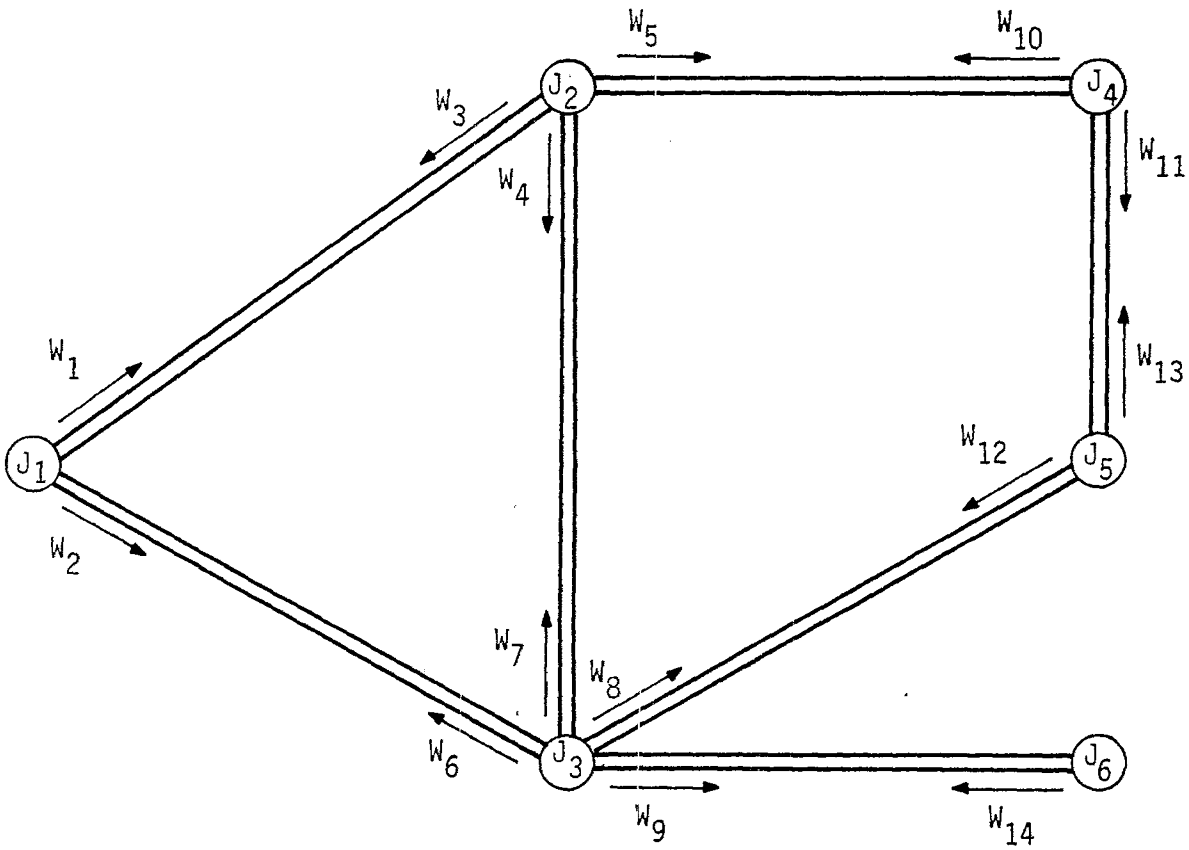
At this juncture let us consider the BLT equation, originally constructed to describe the behavior of transmission-line networks. The full development is in [3].

First consider a transmission-line network such as illustrated in fig. 5.1. This network is a graph in which we call the vertices as junctions and the edges as tubes. The junctions are described as general (linear) N-port networks which may be distributed as well as lumped; they are described by scattering (or impedance or admittance) matrices which are functions of the complex frequency s. The tubes are described as general N-wire (plus reference) transmission lines of various lengths.

As indicated in fig. 5.1, the junctions are numbered arbitrarily (in this case from 1 to 6). The tubes can be numbered by the junctions to which they connect [3]. Various matrices can also be defined which exhibit the interconnection of junctions and tubes with themselves or with each other. For our purposes, the interesting way to label things is according to the waves, two of which propagate (in opposite directions), on each tube. As indicated in fig. 5.1 assign a wave index w which takes two values on each tube. Each wave leaves one junction and arrives at another junction (or perhaps back at the same junction in the case of a self tube) so that each tube can be labelled by the waves on it, and each junction can be labelled by the waves leaving or entering it.

For present purposes, it is the wave-wave interconnection matrix which we will use. For fig. 5.1 this matrix is

$$(W_{u,v}) = \begin{pmatrix}
0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&1&0&0&0&0\\
1&0&0&0&0&0&1&0&0&1&0&0&0&0\\
1&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&1&0&1\\
0&1&0&1&0&0&0&0&0&0&0&1&0&1\\
0&1&0&1&0&0&0&0&0&0&0&1&0&1\\
0&1&0&1&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0
\end{pmatrix} \tag{5.1}$$



1,2,... Junctions

Tubes

$W_1, W_2, \ldots$ Waves (N-waves)

Fig. 5.1. Wave Indexing in Transmission-Line Networks

The elements of such a matrix have the property that

$$W_{u,v} = \begin{cases} 1 \text{ if wave } W_v \text{ scatters into the wave } W_u \text{ with the wave v incoming and wave w outgoing at some junction} \\ 0 \text{ otherwise} \end{cases} \quad (5.2)$$

$$W_{u,v} = \begin{cases} 1 \text{ for a self tube} \\ 0 \text{ otherwise (normal situation)} \end{cases}$$

In our transmission-line network we have

$$\begin{aligned} N_J &\equiv \text{number of junctions (6 in this case)} \\ N_T &\equiv \text{number of tubes (7 in this case)} \\ N_W &\equiv \text{number of N-waves (14 in this case)} \\ &= 2\ N_T \end{aligned} \quad (5.3)$$

This wave-wave interconnection matrix is important in that it describes the transport of waves through the junctions from one tube to another and thereby gives the structure of a scattering supermatrix for the junctions. In $(W_{u,v})$ an element which is 1 corresponds to a generally non-zero block of the scattering supermatrix $((\tilde{S}_{n,m}(s))_{u,v})$, while an element which is zero corresponds to a zero block of the scattering supermatrix, i.e.,

$$(\tilde{S}_{n,m}(s))_{u,v} = \begin{cases} (\tilde{S}_{n,m}(s))_{u,v} \text{ if } W_{u,v} = 1 \\ \qquad \text{(scattering vth N-wave } (N_u \text{ variables)} \\ \qquad \text{(uth N-wave } (N_v \text{ variables)} \\ (\tilde{0}_{n,m}(s))_{u,v} \text{ if } W_{u,v} = 0 \end{cases} \quad (5.4)$$

These scattering-matrix blocks can be computed from the scattering matrices (or impedance matrices or admittance matrices) of the junctions by techniques discussed in [3].

As developed in [3] the scattering supermatrix is one term in the BLT equation which has the form

$$[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{\Gamma}_{n,m}(s))_{u,v})] \odot ((\tilde{V}_n(0,s))_u)$$

$$= ((\tilde{S}_{n,m}(s))_{u,v}) \odot ((\tilde{\Lambda}_{n,m}(x,s;(\cdot)))_{u,v}) \odot ((\tilde{V}'_{s_n}(x,s))_u) \qquad (5.5)$$

In this equation there is also a delay supermatrix

$((\tilde{\Gamma}_{n,m}(s))_{u,v}) \equiv$ delay supermatrix

$$= \bigoplus_{u=1}^{N_W} (\Gamma_{n,m}(s))_{u,u} = \bigoplus_{u=1}^{N_W} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}L_u}$$

$$(\tilde{\Gamma}_{n,m}(s))_{u,v} = \begin{cases} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,v}L_u} & \text{for } u = v \\ (0_{n,m})_{u,v} & \text{for } u \neq v \end{cases}$$

$(\tilde{\gamma}_{c_{n,m}}(s))_{u,u} \equiv$ propagation matrix for uth N-wave

$$= [(\tilde{Z}'_{n,m}(s))_{u,u} \cdot (\tilde{Y}'_{n,m}(s))_{u,u}]^{1/2} \text{ (principal or p.r. value)} \qquad (5.6)$$

$(\tilde{Z}_{c_{n,m}}(s))_{u,u} \equiv$ characteristic impedance matrix for uth N-wave

$$= (\tilde{\gamma}_{c_{n,m}}(s))_{u,u} \cdot (\tilde{Y}'_{n,m}(s))_{u,u} = (\tilde{\gamma}_{c_{n,m}}(s))^{-1}_{u,u} \cdot (\tilde{Z}'_{n,m}(s))_{u,u}$$

$(\tilde{Y}_{c_{n,m}}(s))_{u,u} \equiv$ characteristic admittance matrix for uth N-wave

$$= (\tilde{Z}_{c_{n,m}}(s))^{-1}_{u,u}$$

$(\tilde{Z}'_{n,m}(s))_{u,u} \equiv$ longitudinal impedance-per-unit-length matrix for uth N-wave

$(\tilde{Y}'_{n,m}(s))_{u,u} \equiv$ transverse admittance-per-unit-length matrix for uth N-wave

$L_u \equiv$ length of tube on which the uth N-wave propagates

$\bigoplus \equiv$ direct sum (produces diagonal supermatrix, or block diagonal matrix)

We also have a supermatrix integral operator

$$((\tilde{\Lambda}_{n,m}(x,s;(\bullet)))_{u,v}) \equiv \text{supermatrix integral operator}$$

$$= \bigoplus_{u=1}^{N_W} (\tilde{\Lambda}_{n,m}(x_u,s;(\bullet)))_{u,u} = \bigoplus_{u=1}^{N_W} \int_0^{L_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}[L_u - x_u]} (\bullet)\, dx_u$$

$$(\tilde{\Lambda}_{n,m}(x,s;(\bullet)))_{u,v} = \begin{cases} \int_0^{L_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,v}[L_u - x_u]} (\bullet)\, dx_u & \text{for } u = v \\ (0_{n,m})_{u,v} & \text{for } u \neq v \end{cases} \tag{5.7}$$

x_u = coordinate (position) for uth N-wave along tube $(0 < x_u < L_u)$

Here $(\bullet)$ designates that the terms following the operator are to be inserted at the indicated position in the integral(s) with multiplication in the indicated sense (dot product in this case). Note that this operator operates over the range of each tube.

The response variables in the BLT equation are contained in the combined voltage supervector given by

$$((\tilde{V}_n(0,s))_u) \equiv \text{combined voltage supervector (for N-waves propagating into tubes (away from junctions)}$$

$$= ((\tilde{V}_n^{(0)}(0,s))_u) + ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) \odot ((\tilde{I}_n^{(0)}(0,s))_u)$$

$$((\tilde{Z}_{c_{n,m}}(s))_{u,v}) \equiv \text{characteristic impedance supermatrix}$$

$$= \bigoplus_{u=1}^{N_W} (\tilde{Z}_{c_{n,m}}(s))_{u,u}$$

$$(\tilde{Z}_{c_{n,m}}(s))_{u,v} = (0_{n,m})_{u,v} \text{ for } u \neq v \tag{5.8}$$

$((\tilde{V}_n^{(0)}(x_u,s))_u) \equiv$ voltage supervector for voltages at positions x_u along tubes

$((\tilde{I}_n^{(0)}(x_u,s))_u) \equiv$ current supervector for currents at positions x_u along tubes with positive current in directions of increasing x_u

$$((\tilde{Y}_{c_{n,m}}(s))_{u,v}) \equiv \text{characteristic admittance supermatrix}$$

$$= ((\tilde{Z}_{c_{n,m}}(s))_{u,v})^{-1} = \bigoplus_{u=1}^{N_W} (\tilde{Y}_{c_{n,m}}(s))_{u,u}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{u,u} = \text{characteristic impedance matrix for uth N-wave}$$

$$= (\tilde{Z}_{c_{n,m}}(s))^{-1}_{u,u}$$

The source variables in the BLT equation are contained in the combined voltage-per-unit-length supervector given by

$$((\tilde{V}'_{s_n}(x_u,s))_u) \equiv \text{combined voltage-per-unit-length supervector (a source distributed over the tubes)}$$

$$= ((\tilde{V}'^{(0)}_{s_n}(x_u,s))_u) + ((\tilde{Z}_{c_{n,m}}(s))_{u,v}) \odot ((\tilde{I}'^{(0)}_{s_n}(x_u,s))_u) \tag{5.9}$$

$$((\tilde{V}'^{(0)}_{s_n}(x_u,s))_u) \equiv \text{voltage-per-unit-length supervector (longitudinal voltage source per unit length)}$$

$$((\tilde{I}'^{(0)}_{s_n}(x_u,s))_u) \equiv \text{current-per-unit-length supervector (transverse current source per unit length)}$$

The BLT equation then can take a set of assumed sources along the tubes of a transmission-line network and determine the combined voltages leaving the junctions. These combined voltages can be converted back to regular voltages and currents, if desired. First, the combined voltage N-waves entering a tube can be transformed to any position along that tube via

$$(\tilde{V}_n(x_u,s))_u = e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}x_u} \cdot (\tilde{V}_n(0,s))_u$$

$$+ \int_0^{x_u} e^{-(\tilde{\gamma}_{c_{n,m}}(s))_{u,u}[x_u - x'_u]} \cdot (\tilde{V}'_{s_n}(x'_u,s))_u \, dx'_u \tag{5.10}$$

Now let u and v represent the two N-waves propagating in opposite directions on a given tube. (See fig. 5.1 for an example to see that this is a rather simple correlation.) Then (5.10) applies to both waves on a particular tube with

$$(\tilde{V}_n^{(0)}(x_u,s))_u = (\tilde{V}_n^{(0)}(x_v,s))_v = \frac{1}{2}\,[(\tilde{V}_n(x_u,s))_u + (\tilde{V}_n(x_v,s))_v]$$

$$(\tilde{I}_n^{(0)}(x_u,s))_u = -(\tilde{I}_n^{(0)}(x_v,s))_v = \frac{1}{2}\,(\tilde{Y}_{c_{n,m}}(s))_{u,u} \cdot [(\tilde{V}_n(x_u,s))_u - (\tilde{V}_n(x_v,s))_v] \tag{5.11}$$

$$x_u + x_v = L_u = L_v$$

This formalism allows for lumped sources as well as distributed sources by the introduction of δ functions at any particular x_u (= $L_u - x_v$) of interest. By interpreting any sources ascribed to the junctions as sources just inside the tubes (at each $x_u = 0+$), junction equivalent sources are also handled in this formalism.

VI. BLT Equation for the Case of No Tubes

In applying the results of transmission-line network theory (the BLT equation) to EM topology let us first shrink the lengths of the tubes (the L_u) to zero. In taking the limit as the L_u tend to zero (through positive values) gives

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) \to ((1_{n,m})_{u,v})$$

$$((\tilde{\Lambda}_{n,m}(x_u,s))_{u,v}) \to (\int_0^{L_u} (1_{n,m})_{u,v}\, dx_u) \tag{6.1}$$

$$((\tilde{V}'_{s_n}(x,s))_u) \to ((\tilde{V}_{s_n}(s))_u\ \delta(x_u - 0+))$$

where the distributed sources have been replaced by lumped sources (at x_u = 0+) in the tubes before the tubes are shrunk to zero. The BLT equation then becomes

$$[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v})] \odot ((\tilde{V}_n(s))_u)$$
$$= ((\tilde{S}_{n,m}(s))_{u,v}) \cdot ((\tilde{V}_{s_n}(s))_u) \tag{6.2}$$

One might think of this as the reduced, or simplified, BLT equation. Since the tubes are now of zero length the N-waves leaving particular junctions are also the waves arriving at adjacent junctions with the addition of the lumped sources. Then (5.10) and (5.11) become

$$(\tilde{V}_n(0+,s))_u = (\tilde{V}_n(0-,s))_u + (\tilde{V}_{s_n}(s))_u$$

$$(\tilde{V}_n^{(0)}(0-,s))_u = (\tilde{V}_n^{(0)}(0+,s))_v = \frac{1}{2}\,[(\tilde{V}_n(0-,s))_u + (\tilde{V}_n(0+,s))_v] \tag{6.3}$$

$$(\tilde{I}_n^{(0)}(0-,s))_u = -(\tilde{I}_n^{(0)}(0+,s))_v = \frac{1}{2}\,(\tilde{Y}_{c_{n,m}}(s))_{u,u} \cdot [(\tilde{V}_n(0-,s))_u - (\tilde{V}_n(0+,s))_v]$$

Note that if the sources are zero on a particular tube (with the uth and vth N-waves), then the above formulas further simplify since the variables evaluated at x_u and x_v equal 0- and 0+ both are evaluated at 0.

Now (6.2) can be implemented in various ways in the context of EM topology. With the tubes gone we can identify the junctions with various volumes since such volumes can be regarded as general N-port networks. Where the tubes had previously connected to these volumes (junctions) can now be regarded as ports or penetrations into (and out of) these volumes.

An alternate representation has both surfaces and volumes in the EM topology identified as junctions. In this way the penetrations through the surfaces can be assigned scattering matrices as in [9]. In effect the zero-length tubes now represent the connections of the surfaces to the volumes which they bound.

As discussed in previous papers [10,11], the normalizing impedance matrices combining voltage and current to form combined voltage variables (as in (5.9)) can be chosen in particularly convenient forms. In particular we can choose

$$\left(\left(\tilde{Z}_{c_{n,m}}(s)\right)_{u,v}\right) \equiv R\left(\left(1_{n,m}\right)_{u,v}\right)$$

$$\left(\left(\tilde{Y}_{c_{n,m}}(s)\right)_{u,v}\right) = R^{-1}\left(\left(1_{n,m}\right)_{u,v}\right) \tag{6.4}$$

$$R = \text{real constant} > 0 \text{ (and finite)}$$

This allows us to bound the 2-norm of the scattering matrix of any linear, passive, time-invariant network (system) between zero and one. In particular, applied to the present case these constraints imply

$$0 < \left\|\left(\left(\tilde{S}_{n,m}(j\omega)\right)_{u,v}\right)\right\|_2 < 1$$

$$0 < \left\|\left(\tilde{S}_{n,m}(j\omega)\right)_{u,u}\right\|_2 < 1 \quad \text{for } u = 1,2,\ldots,N_W \tag{6.5}$$

$$\omega \text{ real}$$

VII. General Form of the Good-Shielding Approximation

Let us take the reduced BLT equation in (6.2) and write it in the further simplified form

$$((\tilde{I}_{n,m}(s))_{p,q}) \odot ((\tilde{V}_n(s))_p) = ((\tilde{V}_n^{(s)}(s))_q)$$

$$((\tilde{I}_{n,m}(s))_{p,1}) \equiv \text{interaction supermatrix}$$

$$= ((1_{n,m})_{p,q}) - ((\tilde{S}_{n,m}(s))_{p,q}) \tag{7.1}$$

$$((\tilde{V}_n^{(s)}(s))_p) \equiv \text{equivalent source supervector}$$

$$= ((\tilde{S}_{n,m}(s))_{p,q}) \odot ((\tilde{V}_{s_n}(s))_p)$$

Our problem is then to invert the interaction supermatrix at least as an approximation. Note that the wave subscripts (dummy variables here) have been replaced by p,q to emphasize that at this stage no identification is made with particular wave indices which are to be later partitioned and identified at our convenience.

Letting

$$p,q = 1,2,\ldots,N \tag{7.2}$$

let us assume that the interaction matrix is block tridiagonal, i.e.,

$$(\tilde{I}_{n,m}(s))_{p,q} = (0_{n,m})_{p,q} \quad \text{for } |p-q| > 1 \tag{7.3}$$

Next let us assume that the off-diagonal blocks ($|p-q| = 1$) are in some sense small compared to the square diagonal blocks ($p = q$). This is used later when these off-diagonal blocks are identified with transmission of electromagnetic energy through the shields (and/or subshields) in the EM topology of interest.

To obtain an approximate solution of (7.1) under the above assumptions let us consider the supermatrix equation as a set of linear equations in which the sources are the vectors $(\tilde{V}_n^{(s)}(s))_q$, the responses are the vectors $(\tilde{V}_n(s))_q$, and the coefficients are the matrices $(\tilde{I}_{n,m}(s))_{p,q}$ for $|p-q| \le 1$. These equations are written out as

$$(\tilde{I}_{n,m}(s))_{1,1} \cdot (\tilde{V}_n(s))_1 + (\tilde{I}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(s))_2 = (\tilde{V}_n^{(s)}(s))_1$$

$$(\tilde{I}_{n,m}(s))_{2,1}\cdot(\tilde{V}_n(s))_1+(\tilde{I}_{n,m}(s))_{2,2}\cdot(\tilde{V}_n(s))_2+(\tilde{I}_{n,m}(s))_{2,3}\cdot(\tilde{V}_n(s))_3 = (\tilde{V}_n^{(s)}(s))_2$$

$$(\tilde{I}_{n,m}(s))_{3,2}\cdot(\tilde{V}_n(s))_2+(\tilde{I}_{n,m}(s))_{3,3}\cdot(\tilde{V}_n(s))_3 + (\tilde{I}_{n,m}(s))_{3,4}\cdot(\tilde{V}_n(s))_4 = (\tilde{V}_n^{(s)}(s))_3$$

$$\vdots \qquad (7.4)$$

$$(\tilde{I}_{n,m}(s))_{N-2,N-3}\cdot(\tilde{V}_n(s))_{N-3}+(\tilde{I}_{n,m}(s))_{N-2,N-2}\cdot(\tilde{V}_n(s))_{N-2} + (\tilde{I}_{n,m}(s))_{N-2,N-1}\cdot(\tilde{V}_n(s))_{N-1} = (\tilde{V}_n^{(s)}(s))_{N-2}$$

$$(\tilde{I}_{n,m}(s))_{N-1,N-2}\cdot(\tilde{V}_n(s))_{N-2}+(\tilde{I}_{n,m}(s))_{N-1,N-1}\cdot(\tilde{V}_n(s))_{N-1} + (\tilde{I}_{n,m}(s))_{N-1,N}\cdot(\tilde{V}_n(s))_N = (\tilde{V}_n^{(s)}(s))_{N-1}$$

$$(\tilde{I}_{n,m}(s))_{N,N-1}\cdot(\tilde{V}_n(s))_{N-1} + (\tilde{I}_{n,m}(s))_{N,N}\cdot(\tilde{V}_n(s))_N = (\tilde{V}_n^{(s)})_N$$

Next to simplify the results let us assume that the sources are only on the "outside," i.e.,

$$(\tilde{V}_n^{(s)}(s))_q = (0_n)_q \quad \text{for } q > 1 \qquad (7.5)$$

so that only the first equation of (7.4) has a source term. Then solve this set of linear equations beginning with the last as

$$(\tilde{V}_n(s))_N = -(\tilde{I}_{n,m}(s))_{N,N}^{-1} \cdot (\tilde{I}_{n,m}(s))_{N,N-1} \cdot (\tilde{V}_n(s))_{N-1} \qquad (7.6)$$

Moving to the general equation of the form

$$(\tilde{I}_{n,m}(s))_{p,p-1}\cdot(\tilde{V}_n(s))_{p-1}+(\tilde{I}_{n,m}(s))_{p,p}\cdot(\tilde{V}_n(s))_p+(\tilde{I}_{n,m})_{p,p+1}\cdot(\tilde{V}_n(s))_{p+1} = 0 \qquad (7.7)$$

we use an assumption that

$$||(\tilde{V}_n(s))_{p-1}|| >> ||(\tilde{V}_n(s))_p|| >> ||(\tilde{V}_n(s)||_{p+1} \tag{7.8}$$

which is related to the assumption that the off-diagonal blocks are small compared to the diagonal blocks which represent the transfer functions through the system and give smaller responses for increasing p. Rearrange (7.7) then as

$$(\tilde{V}_n(s))_p = -(\tilde{I}_{n,m}(s))^{-1}_{p,p}\cdot[(\tilde{I}_{n,m}(s))_{p,p-1}\cdot(\tilde{V}_n(s))_{p-1}+(\tilde{I}_{n,m}(s))_{p,p+1}\cdot(\tilde{V}_n(s))_{p+1}] \tag{7.9}$$

In norm sense then what we require is

$$\begin{aligned} &||(\tilde{I}_{n,m}(s))^{-1}_{p,p} \cdot (\tilde{I}_{n,m}(s))_{p,p-1} \cdot (\tilde{V}_n(s)_{p-1}|| \\ &\qquad >> ||(\tilde{I}_{n,m}(s))^{-1}_{p,p} \cdot (\tilde{I}_{n,m}(s))_{p,p+1} \cdot (\tilde{V}_n(s))_{p+1}|| \end{aligned} \tag{7.10}$$

In essence if we can state that

$$||(\tilde{I}_{n,m}(s))^{-1}_{p,p} \cdot (\tilde{I}_{n,m}(s))_{p,p+1} \cdot (\tilde{V}_n(s))_{p+1}|| << ||(\tilde{V}_n(s))_p|| \tag{7.11}$$

then we have

$$\begin{aligned} (\tilde{V}_n(s))_p &= -(\tilde{I}_{n,m}(s))^{-1}_{p,p} \cdot (\tilde{I}_{n,m}(s))_{p,p-1} \cdot (\tilde{V}_n(s))_{p-1} \\ &\qquad + \text{error} \\ \text{error} &= -(\tilde{I}_{n,m}(s))^{-1}_{p,p} \cdot (\tilde{I}_{n,m}(s))_{p,p+1} \cdot (\tilde{V}_n(s))_{p+1} \end{aligned} \tag{7.12}$$

with the error reduced from the principal term as $(\tilde{V}_n(s))_{p+1}$ is twice reduced from $(\tilde{V}_n(s))_{p-1}$. The off-diagonal blocks $(\tilde{I}_{n,m}(s))_{p,p-1}$ and $(\tilde{I}_{n,m}(s))_{p,p+1}$ are of course both assumed small to help in this result. Of course one requires that the diagonal blocks, the $(\tilde{I}_{n,m}(s))_{p,p}$, do have inverses (are non-singular).

There is one equation, the first one in (7.4), which has a source term and which is solved as

$$(\tilde{V}_n(s))_1 = (\tilde{I}_{n,m}(s))^{-1}_{1,1} \cdot [(\tilde{V}^{(s)}_n(s))_1 - (\tilde{I}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(s))_2] \tag{7.13}$$

Applying the same assumption that, in this case $(\tilde{V}_n(s))_2$ is small compared to $(\tilde{V}_n(s))_1$ and $(\tilde{I}_{n,m}(s))_{1,1}$ is small compared to one (in norm sense as usual) then we have

$$(\tilde{V}_n(s))_1 = (\tilde{I}_{n,m}(s))^{-1}_{1,1} \cdot (\tilde{V}^{(s)}_n(s))_1 + \text{error} \tag{7.14}$$

$$\text{error} = -(\tilde{I}_{n,m}(s))^{-1}_{1,1} \cdot (\tilde{I}_{n,m}(s))_{1,2} \cdot (\tilde{V}_n(s))_2$$

Working backwards from p = N to p = 1 we can solve for the responses at any level p within the system. Beginning with (7.6) we have

$$(\tilde{V}_n(s))_N = -(\tilde{I}_{n,m}(s))^{-1}_{N,N} \cdot (\tilde{I}_{n,m}(s))_{N,N-1} \cdot (\tilde{V}_n(s))_{N-1}$$

$$\simeq (\tilde{I}_{n,m}(s))_{N,N} \cdot (\tilde{I}_{n,m}(s))_{N,N-1} \cdot (\tilde{I}_{n,m}(s))^{-1}_{N-1,N-1} \cdot (\tilde{I}_{n,m}(s))_{N-1,N-2} \cdot (\tilde{V}_n(s))_{N-2}$$

$$\simeq (-1)^{N-p} \left\{ \underset{p'=p}{\overset{N-1}{\odot}} \left[(\tilde{I}_{n,m}(s))^{-1}_{N+p-p',N+p-p'} \cdot (\tilde{I}_{n,m}(s))_{N+p-p',N-1+p-p'}\right]\right\} \cdot (\tilde{V}_n(s))_p \quad (\text{for } 1 \leq \ell \leq N)$$

$$\simeq (-1)^{N-1} \left\{ \underset{p'=1}{\overset{N-1}{\odot}} \left[(\tilde{I}_{n,m}(s))^{-1}_{N+1-p',N+1-p'} \cdot (\tilde{I}_{n,m}(s))_{N+1-p',N-p'}\right]\right\} \cdot (\tilde{I}_{n,m}(s))^{-1}_{1,1} \cdot (\tilde{V}^{(s)}_n(s))_1 \tag{7.15}$$

This gives the signals in the "deepest" part of the system in terms of the sources "outside" and the matrix blocks connecting the two.

One need not concentrate on the signals at the "deepest" part of the system. Starting at $n = M \leq N$ and working back up gives

$$
\begin{aligned}
(\tilde{V}_n(s))_M &\simeq -(\tilde{I}_{n,m}(s))^{-1}_{M,M} \cdot (\tilde{I}_{n,m}(s))_{M,M-1} \cdot (\tilde{V}_n(s))_{M-1} \\
&\simeq (-1)^{M-p} \left\{ \bigodot_{p'=p}^{M-1} [(\tilde{I}_{n,m}(s))^{-1}_{M+p-p',M+p-p'} \cdot (\tilde{I}_{n,m}(s))_{M+p-p',M-1+p-p'}] \right\} \\
&\qquad \cdot (\tilde{V}_n(s))_p \quad (\text{for } 1 \leq p \leq M) \\
&\simeq (-1)^{M-1} \left\{ \bigodot_{p'=1}^{M-1} [(\tilde{I}_{n,m}(s))^{-1}_{M+1-p',M+1-p'} \cdot (\tilde{I}_{n,m}(s))_{M+1-p',M-p'}] \right\} \\
&\qquad \cdot (\tilde{I}_{n,m}(s))^{-1}_{1,1} \cdot (\tilde{V}_n^{(s)}(s))_1
\end{aligned}
\tag{7.16}
$$

This gives the signals at any level (or layer) in the system in terms of the sources "outside" and the matrix blocks connecting the two.

The approximate solution (the good-shielding approximation) in (7.16) has reduced the supermatrix equation (7.1) to an equation involving the smaller matrix blocks. One can go another step in simplifying this result by the use of norms (vector norms and their associated matrix norms). Since the norm of a product is less than or equal to the product of the norms we have

$$
\begin{aligned}
\|(\tilde{V}_n(s))_M\| &\lessapprox \left\{ \prod_{\ell'=1}^{M-1} \|(\tilde{I}_{n,m}(s))^{-1}_{M+1-\ell',M+1-\ell'}\| \; \|(\tilde{I}_{n,m}(s))_{M+1-\ell',M-\ell'}\| \right\} \\
&\qquad \|(\tilde{I}_{n,m}(s))^{-1}_{1,1}\| \; \|(\tilde{V}_n^{(s)}(s))_1\| \\
&\qquad 1 \leq M \leq N
\end{aligned}
\tag{7.17}
$$

In this form the particular norm to be used is not specified. Various norms can be used, all of which in replacing a vector or matrix by a non-negative real number have bounded that vector or matrix or some sense. So in this bounding formula we are concerned with maximum signal or transfer function, maximum power, etc.

The terms in this approximate bounding formula represent bounds on the transfer of signals from level to level within the system. In going from $\ell - 1$ to ℓ we have from (7.12)

$$||(\tilde{V}_n(s))||_\ell \simeq ||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell} \cdot (\tilde{I}_{n,m}(s))_{\ell,\ell-1} \cdot (\tilde{V}_n(s))_{\ell-1}||$$

$$\tilde{<} \; ||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell} \cdot (\tilde{I}_{n,m}(s))_{\ell,\ell-1}|| \; ||(\tilde{V}_n(s))_{\ell-1}||$$

$$\tilde{<} \; ||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell}|| \; ||(\tilde{I}_{n,m}(s))_{\ell,\ell-1}|| \; ||(\tilde{V}_n(s))_{\ell-1}|| \quad (7.18)$$

So in designing a system the good-shielding approximation shows that it is the above product of matrix norms which is to be kept small to attenuate (to some desired degree) the signals propagating into the system. This is consistent with the requirement that the off-diagonal blocks of the interaction supermatrix be small (in norm now) so that the good-shielding approximation is valid.

Reexamining the fundamental approximation in (7.12) we require

$$\text{error} = -(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell} \cdot (\tilde{I}_{n,m}(s))_{\ell,\ell+1} \cdot (\tilde{V}_n(s))_{\ell+1}$$

$$\simeq (\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell} \cdot (\tilde{I}_{n,m}(s))_{\ell,\ell+1} \cdot (\tilde{I}_{n,m}(s))^{-1}_{\ell+1,\ell+1}$$

$$\cdot \, (\tilde{I}_{n,m}(s))_{\ell+1,\ell} \cdot (\tilde{V}_n(s))_\ell \quad (7.19)$$

In norm sense then

$$||\text{error}|| \; \tilde{<} \; ||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell} \cdot (\tilde{I}_{n,m}(s))_{\ell,\ell+1} \cdot (\tilde{I}_{n,m}(s))^{-1}_{\ell+1,\ell+1} \cdot (\tilde{I}_{n,m}(s))_{\ell+1,\ell}||$$

$$||(\tilde{V}_n(s))_\ell|| \quad (7.20)$$

Since the error should be small compared to the approximation of $(\tilde{V}_n(s))_\ell$ we have the requirement

$$||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell} \cdot (\tilde{I}_{n,m}(s))_{\ell,\ell+1} \cdot (\tilde{I}_{n,m}(s))^{-1}_{\ell+1,\ell+1} \cdot (\tilde{I}_{n,m}(s))_{\ell+1,\ell}|| << 1 \quad (7.21)$$

to assure the validity of (7.12).

Expanding (7.12) via norms a more severe requirement for the validity of the good-shielding approximation is

$$||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell}|| \; ||(\tilde{I}_{n,m}(s))_{\ell,\ell+1}|| \; ||(\tilde{I}_{n,m}(s))^{-1}_{\ell+1,\ell+1}||$$

$$||(\tilde{I}_{n,m}(s))_{\ell+1,\ell}|| \ll 1 \tag{7.22}$$

Provided that the diagonal-block terms are bounded of order one, i.e. (for all ℓ)

$$||(\tilde{I}_{n,m}(s))^{-1}_{\ell,\ell}|| \simeq 1 \tag{7.23}$$

or even a small number not much greater than one, then if we can require for all the off-diagonal blocks

$$\begin{aligned} ||(\tilde{I}_{n,m}(s))_{\ell,\ell+1}|| &\ll 1 \\ ||(\tilde{I}_{n,m}(s))_{\ell+1,\ell}|| &\ll 1 \end{aligned} \tag{7.24}$$

the good-shielding approximation is assured.

VIII. Partitioning the Terms in the BLT Equation for Application of the Good-Shielding Approximation

Let us now organize the wave indices in the BLT equation according to the hierarchical EM topology in a manner which makes the good-shielding approximation applicable. In this case shields (and subshields) are assumed to be the elements which provide significant attenuation of the waves in their travel from V_1 to $V_{\lambda_{max}}$.

In accordance with the previous section then let us associate the scattering transfer function through a shield as that of an off-diagonal block of the scattering (or interaction) supermatrix. This is accomplished by making the diagonal blocks contain all the terms associated with the layers V_λ. The off-diagonal blocks then contain all the terms corresponding to transfer of signals from one layer to an adjacent layer.

This suggests that we associate the p,q indices in section 7 to the topological indices in section 2B (corresponding to a hierarchical topology defined to the level of layers and shields, but not yet further) with the correspondences

$$\begin{aligned} p &= (\sigma,\mu,\lambda) \\ q &= (\sigma',\mu',\lambda') \end{aligned} \tag{8.1}$$

This makes the BLT equation (7.1) take the form

$$((\tilde{I}_{n,m}(s))_{\sigma,\mu,\lambda,\sigma',\mu',\lambda'}) \odot ((\tilde{V}_n(s))_{\sigma,\mu,\lambda}) = ((\tilde{V}_n^{(s)}(s))_{\sigma,\mu,\lambda}) \tag{8.2}$$

However, this has to be understood in a special way in that the 4th through 6th subscripts of the matrix blocks are summed in the product with the three subscripts of the vector blocks (partitions). Another form this can take is

$$((((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}) \odot ((((\tilde{V}_n(s))_\sigma)_\mu)_\lambda) = ((((\tilde{V}_n^{(s)}(s))_\sigma)_\mu)_\lambda) \tag{8.3}$$

consistent with the matrix/vector partitioning scheme in section 3. In this latter form (as quadravectors and quadramatrices) matrix/vector partitioning is successively accomplished according to the topological indices λ, μ, and σ.

First view this ordering according to the interaction sequence diagram in fig. 8.1. This is a path which reflects the nesting property in the hierarchical EM topology with signals going from layer to shield to layer to shield, etc., in going from the outside to the "deepest" layer $V_{\lambda_{max}}$ (or in the opposite order) encountering every layer and shield along the path. Note that the layer-part index μ only takes on values of 1 and 3 here. This is because the layers in this case have not been divided into elementary volumes (as in fig. 2.4), for which case the value $\mu = 2$ would be used. Also for $\lambda = 1$ only $\mu = 3$ is used and for $\lambda = \lambda_{max}$ only $\mu = 1$ is used.

Corresponding to this ordering and path the interaction matrix is exhibited in (8.4). Note the block tridiagonal character of this matrix as in (7.3). The point is that the ordering of the interaction matrix as a supermatrix makes the form of the equation amenable to approximate solution via the good-shielding approximation. As (8.4) exhibits the interaction matrix is quite sparse and the partitioning of this matrix as a supermatrix orders this sparseness on various levels of partitioning.

Note that for these results we require $\lambda_{max} > 2$ so that the EM topology is non-trivial as well as the corresponding interaction sequence diagram in fig. 8.1.

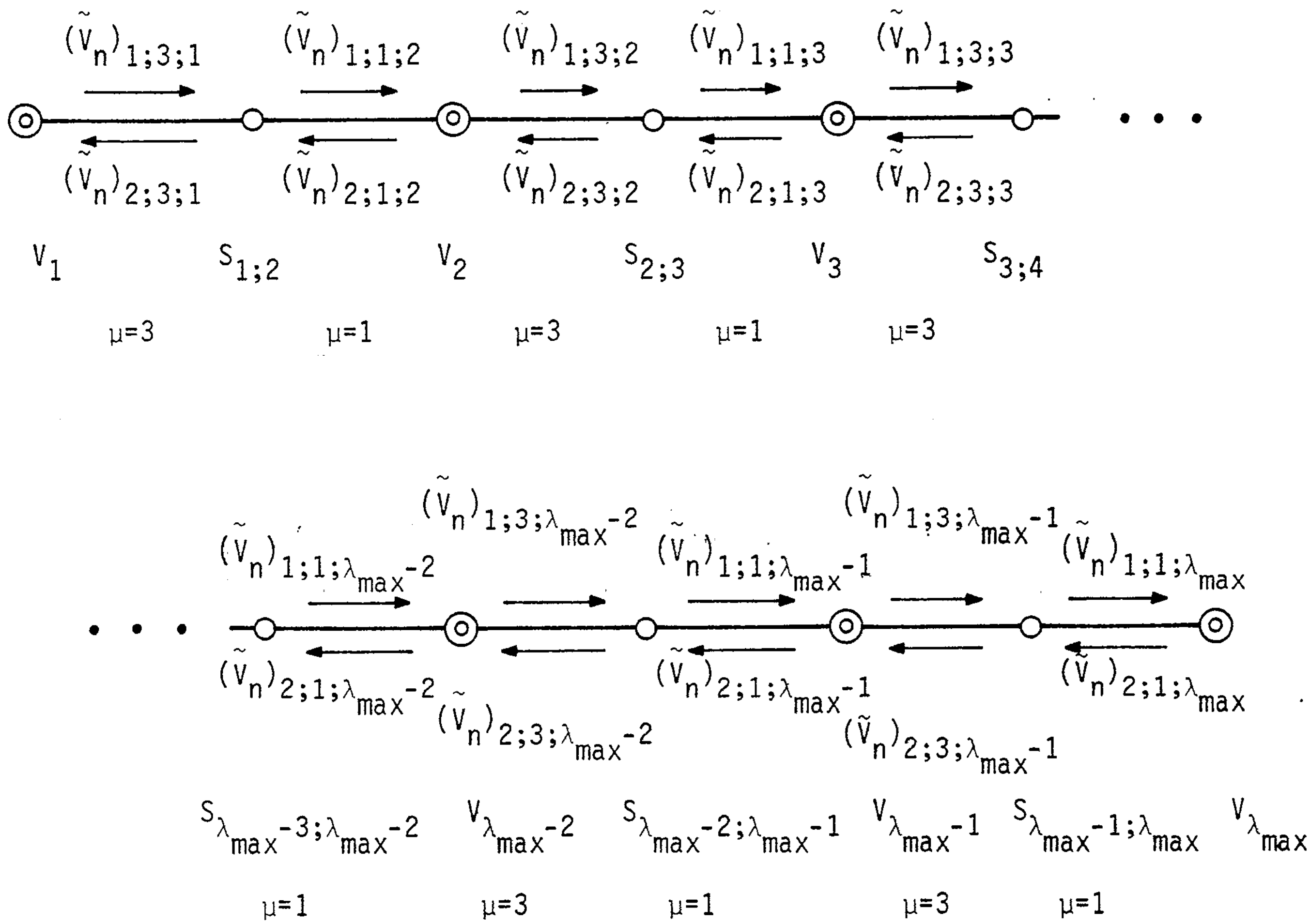


Fig. 8.1. Interaction Sequence Diagram (Path) Corresponding to a Layer Shield Decomposition in a Hierarchical EM Topology

$$((((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}) = ((((1_{n,m})_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}) - ((((\tilde{S}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}) =$$

$$\begin{bmatrix}
\begin{bmatrix}\begin{bmatrix}(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1}) \quad (1_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{1,1} &
\begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{1,2} & & \\
\begin{bmatrix}\begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{2,1} &
\begin{bmatrix}\begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{2,2} &
\begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{2,3} & \\
& \begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{3,2} &
\begin{bmatrix}\begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{3,3} & \ddots \\
& \ddots & \ddots & \ddots
\end{bmatrix}$$

$$\begin{bmatrix}
\ddots & \begin{bmatrix}\begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{\lambda_{max}-2,\lambda_{max}-2} &
\begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,1}\end{bmatrix}_{1,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{\lambda_{max}-2,\lambda_{max}-1} & \\
& \begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,3}\end{bmatrix}_{\lambda_{max}-1,\lambda_{max}-2} &
\begin{bmatrix}\begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{1,3} \\ \begin{bmatrix}-(\tilde{S}_{n,m})_{2,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{3,1} & \begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{1,1}\end{bmatrix}_{3,3}\end{bmatrix}_{\lambda_{max}-1,\lambda_{max}-1} &
\begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,1} \\ \begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2}\end{bmatrix}_{3,1}\end{bmatrix}_{\lambda_{max}-1,\lambda_{max}} \\
& & \begin{bmatrix}\begin{bmatrix}(0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,1} & \begin{bmatrix}-(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{2,1} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2}\end{bmatrix}_{1,3}\end{bmatrix}_{\lambda_{max},\lambda_{max}-1} &
\begin{bmatrix}\begin{bmatrix}(1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2}\end{bmatrix}_{1,1}\end{bmatrix}_{\lambda_{max},\lambda_{max}}
\end{bmatrix} \tag{8.4}$$

IX. The Off-Diagonal Blocks

Referring to (8.4) note that the general form of the off-diagonal blocks used in the good-shielding approximation in (7.16) connecting layers λ-1 and λ is

$$(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1} =$$

$$\begin{bmatrix} \begin{bmatrix} (0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{bmatrix}_{1,1} & \begin{bmatrix} -(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{bmatrix}_{1,3} \\ \begin{bmatrix} (0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{bmatrix}_{3,1} & \begin{bmatrix} (0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{bmatrix}_{3,3} \end{bmatrix}_{\lambda,\lambda-1} \tag{9.1}$$

For $\lambda = 1$ and $\lambda = \lambda_{max}$ only parts of this matrix block (specifically two of the blocks at the μ,μ' partition) remain to match the size of the $\lambda,\lambda' = 1,1$ and $\lambda,\lambda' = \lambda_{max},\lambda_{max}$ diagonal blocks in either row or column sense as appropriate.

Note the sparsity of this matrix. At the μ,μ' level of partition only one block, the 1,3 block, is not a zero matrix. Furthermore, only one of the blocks of this matrix at the σ,σ' level of partition, the 1,1 block, is not a zero matrix. Roughly speaking only 1/16 of the matrix in (9.1) consists of generally non-zero elements; this is tempered by noting that the various matrix blocks can in general contain varying numbers of elements consistent with the partitioning scheme.

One can directly use (9.1) for the off-diagonal blocks appearing in (7.16) (the good-shielding approximation), and take advantage of the matrix sparsity in the matrix multiplication process. However, our interest here is in the matrix norms such as used in (7.17), the bound form of the good-shielding approximation. In appendix A it is shown that the p-norm of a supermatrix with only one non-zero block is the same as the p-norm of that block. Hence we have

$$||(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1}||_p = ||((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{1,3;\lambda,\lambda-1}||_p \tag{9.2}$$

$$((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{1,3;\lambda,\lambda-1} = \begin{bmatrix} -(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{bmatrix}_{1,3;\lambda,\lambda-1}$$

The supermatrix in (9.2) can be treated the same way since it has only one non-zero block, giving

$$\begin{aligned} ||(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1}||_p &= ||(\tilde{I}_{n,m}(s))_{1,1;1,3;\lambda,\lambda-1}||_p \\ &= ||(\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda,\lambda-1}||_p \end{aligned} \tag{9.3}$$

Referring back to (8.4) note that this result applies for $\lambda = 1$ and $\lambda = N$ as well since these off-diagonal blocks have the same single non-zero block as in (9.3).

Interpreting this result, the p-norm of an off-diagonal block in the good-shielding approximation is merely the p-norm of the scattering matrix describing the transport of signals through a shield. The large off-diagonal blocks can be reduced to something more tractable. Note that this scattering matrix in (9.3) is precisely the scattering matrix discussed in [9], where procedures are given for measuring the matrix elements and finding the appropriate norms.

X. The Diagonal Blocks

Referring to (8.4) note that the general form of the diagonal blocks used in the good-shielding approximation in (7.16) for layer λ is

$$(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda} =$$

$$\begin{bmatrix} \begin{bmatrix} (1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2} \end{bmatrix}_{1,1} & \begin{bmatrix} (0_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & -(\tilde{S}_{n,m})_{2,2} \end{bmatrix}_{1,3} \\ \begin{bmatrix} -(\tilde{S}_{n,m})_{1,1} & (0_{n,m})_{1,2} \\ (0_{n,m})_{2,1} & (0_{n,m})_{2,2} \end{bmatrix}_{3,1} & \begin{bmatrix} (1_{n,m})_{a,a} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2} \end{bmatrix}_{3,3} \end{bmatrix}_{\lambda,\lambda} \tag{10.1}$$

Note that this block is square as are the diagonal blocks $((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu';\lambda,\lambda}$ and $(1_{n,m})_{\sigma,\sigma;\mu,\mu;\lambda,\lambda}$ resulting from further partition. For $\lambda = 1$ and $\lambda = \lambda_{max}$ only parts of the matrix block in (10.1) are present in (8.4), but these are the same as the μ,μ blocks in (10.1).

In taking the norm of the inverse of this matrix block one can use [8]

$$||(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})^{-1}_{\lambda,\lambda}||$$

$$= ||[(((1_{n,m})_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'} - (((\tilde{S}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}]^{-1}||$$

$$\leq \frac{1}{1 - ||(((\tilde{S}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}||}$$

$$\text{if } ||(((\tilde{S}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}|| < 1 \tag{10.2}$$

Since the 2-norm of a passive scattering matrix has [10]

$$0 \leq ||(((\tilde{S}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda'}||_2 \leq 1 \tag{10.3}$$

then we can use (10.2) to provide some bound on the norm of the inverse of the diagonal blocks provided. For this to be interesting, however, we need that the 2-norm of the scattering matrix be bounded away from (below) 1 and preferably small compared to 1.

Now since the norm of a sum of matrices is less than or equal to the sum of the norms, and since from appendix A the p-norm of a matrix with one block is equal to the p-norm of that block, then the p-norm of a matrix is less than or equal to the sum of the p-norms of the blocks. Noting the 6 scattering matrix blocks in (10.1) this bound is not very useful if the 2-norm of any of these blocks is near 1.

Referring back to fig. 8.1 note that the scattering matrix blocks in the λ layer should have the property that only certain of these blocks need have small norms to make the transfer of signals through the layer small. The above result would then seem to be overly pessimistic, i.e., giving an overly loose bound. So let us try an alternate approach in the next section.

XI. The General Term in the Good-Shielding Approximation in Terms of Elementary Matrix Blocks

Write out the general recursion (7.12) that occurs in the good-shielding approximation using the terms in (8.3) as

$$(((\tilde{V}_n(s))_\sigma)_\mu)_\lambda \simeq -(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})^{-1}_{\lambda,\lambda} \odot (((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1}$$

$$\odot \; (((\tilde{V}_n(s))_\sigma)_\mu)_{\lambda-1} \tag{11.1}$$

Instead of inverting $(((\tilde{I}_{n,m})_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda}$ directly, let us use the sparsity of $(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1}$ to reduce the complexity of the problem and perhaps obtain a tighter bound when norms are applied. First rewrite (11.1) as

$$(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda} \odot (((\tilde{V}_n(s))_\sigma)_\mu)_\lambda$$

$$\simeq -(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1} \odot (((\tilde{V}_n(s))_\sigma)_\mu)_{\lambda-1} \tag{11.2}$$

Essentially this equation represents the λth row in (8.4) with the $(\lambda,\lambda+1)$ term neglected as being small compared to the terms in (11.2).

The right side of (11.2) is found from (9.1) as

$$-(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda-1} \odot (((\tilde{V}_n(s))_\sigma)_\mu)_{\lambda-1}$$

$$= \begin{bmatrix} (\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1} \\ (0_n)_{1;3;\lambda-1} \\ (0_n)_{2;1;\lambda-1} \\ (0_n)_{2;3;\lambda-1} \end{bmatrix} \tag{11.3}$$

Note that only one of the four blocks is non-zero.

The left side of (11.2) is found from (10.1) as

$$(((\tilde{T}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda,\lambda} \odot (((\tilde{V}_n(s))_\sigma)_\mu)_\lambda$$

$$= \begin{bmatrix} (\tilde{V}_n)_{1;1;\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{2;1;\lambda} \\ -(\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda} + (\tilde{V}_n)_{2;1;\lambda} - (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{V}_{2;3;\lambda} \\ -(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda} + (\tilde{V}_n)_{1;3;\lambda} - (\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{2;3;\lambda} \\ -(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda} + (\tilde{V}_n)_{2;3;\lambda} \end{bmatrix} \tag{11.4}$$

Here each of the four blocks are non-zero, but there are only two or three terms in each block.

Referring to fig. 11.1, the various terms in (11.3) and (11.4) are illustrated graphically, indicating the connections among the terms (waves and scattering matrices). This gives the structure of these equations, including the sparseness of the associated supermatrices.

Comparing (11.3) and (11.4) note that only the first block is non-zero, giving

$$(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1} =$$

$$(\tilde{V}_n)_{1;1;\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{2;1;\lambda} \tag{11.5}$$

This equation gives the link from the λ-1 layer to the other combined voltages in the λ layer. Note that the corresponding combined voltage to $(\tilde{V}_n)_{1;3;\lambda-1}$ in the λ-1 layer is $(\tilde{V}_n)_{1;3;\lambda}$ in the λ layer. Let us remove the other terms in (11.3) and (11.4) to give a relation between these two.

First eliminate the backward propagating (σ = 2) waves in favor of the forward propagating (σ = 1) waves. The last three equations in (11.4) can be solved for $(\tilde{V}_n)_{2;3;\lambda}$ as

$$(\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{2;3;\lambda} = -(\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda} + (\tilde{V}_n)_{2;1;\lambda}$$

$$(\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{2;3;\lambda} = -(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda} + (\tilde{V}_n)_{1;3;\lambda}$$

$$(\tilde{V}_n)_{2;3;\lambda} = (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda} \tag{11.6}$$

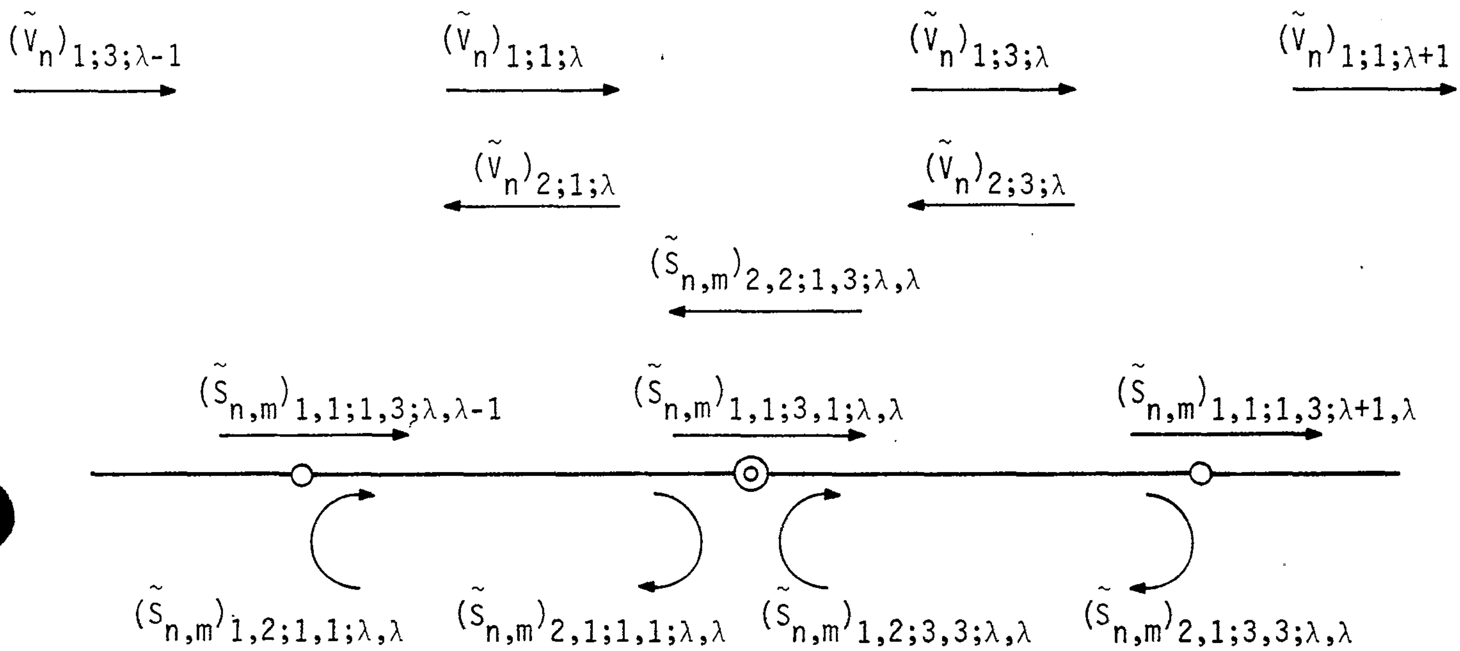


Fig. 11.1. Interaction Sequence Diagram Corresponding to One Layer in a Hierarchical EM Topology for the Good-Shielding Approximation

Substituting the third of these into the second gives

$$(\tilde{V}_n)_{1;3;\lambda} \simeq (\tilde{S}_n)_{1,3;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda} + (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda} \quad (11.7)$$

$$(\tilde{V}_n)_{1;3;\lambda} \simeq [(1_{n,m})_{1,1;3,3;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}]^{-1} \cdot (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda}$$

This last result then gives us $(\tilde{V}_n)_{1;3;\lambda}$ in terms of $(\tilde{V}_n)_{1;1;\lambda}$.

Substituting for $(\tilde{V}_n)_{2;3;\lambda}$ from the last of (11.6) in the first of (11.6) gives

$$(\tilde{V}_n)_{2;1;\lambda} \simeq (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda} + (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda} \quad (11.8)$$

This is in turn substituted in (11.5) to give

$$\begin{aligned}(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1} &\simeq (\tilde{V}_n)_{1;1;\lambda} \\ &- (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot [(\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda} \\ &\quad + (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;1;\lambda}] \\ &= [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}] \cdot (\tilde{V}_n)_{1;1;\lambda} \\ &- (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda}\end{aligned} \quad (11.9)$$

Now (11.9) can be solved for $(\tilde{V}_n)_{1;1;\lambda}$ to give

$$\begin{aligned}(\tilde{V}_n)_{1;1;\lambda} \simeq{} & [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1} \\ & \cdot [(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1} \\ & + (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda}]\end{aligned} \quad (11.10)$$

Substituting this in (11.7) gives

$$[(1_{n,m})_{1,1;3,3;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}](\tilde{V}_n)_{1;3;\lambda}$$
$$\simeq (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}$$
$$\cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}$$
$$\cdot [(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1}$$
$$+ (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} \cdot (\tilde{V}_n)_{1;3;\lambda}] \quad (11.11)$$

which is rearranged as

$$(\tilde{V}_n)_{1;3;\lambda} \simeq \{[(1_{n,m})_{1,1;3,3;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}]$$
$$-(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}$$
$$\cdot (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\}^{-1}$$
$$\cdot (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}$$
$$\cdot (\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1} \quad (11.12)$$

This last formula relates $(\tilde{V}_n)_{1;3;\lambda}$ to $(\tilde{V}_n)_{1;3;\lambda-1}$ with a matrix coefficient that should be small in some sense for the good-shielding approximation to apply. One term in the product of matrices is $(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1}$ which gives the transfer of signals from layer $\lambda-1$ to layer λ; this is the term which is first identified as one which should be kept small for the shielding (due to shield $(\lambda-1,\lambda)$) to be effective. The other terms in the matrix coefficient involve only scattering matrices which correspond to signal transport within layer λ; this part of the matrix coefficient should also be kept small or at least bounded so that it does not overcome the shielding afforded by $(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1}$. Note the presence of the matrix inverse which can be thought of as representing possible resonances (which might be suppressed) in layer λ.

Applying norms to (11.12) we first have

$$||(\tilde{V}_n)_{1;3;\lambda}|| \lessapprox ab||(\tilde{V}_n)_{1;3;\lambda-1}|| \tag{11.13}$$

where

$$b \equiv ||(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1}||$$

$$= \text{contribution from shield } S_{\lambda-1,\lambda} \tag{11.14}$$

$$a \equiv \text{contribution from layer } V_\lambda$$

The complicated term is a, which is

$$a \equiv ||\{[(1_{n,m})_{1,1;3,3;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}$$

$$-(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{1}$$

$$\cdot (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\}^{-1}$$

$$\cdot (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}$$

$$\cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}|| \tag{11.15}$$

Now consider the norms of certain terms in (11.15). First consider the term

$$a^{(1)} \equiv ||[(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}||$$

$$< [1 - ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}||]^{-1}$$

$$\text{if } ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}|| < 1 \tag{11.16}$$

Now

$$||(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}||$$

$$< ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}|| \; ||(\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}|| \tag{11.17}$$

Note that these two terms, as indicated in fig. 11.1, represent reflections in V_λ between the node representing $S_{\lambda-1;\lambda}$ and that of V_λ itself. As such energy conservation requires for the 2-norm (see appendix C)

$$\begin{aligned} 0 &< \| (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \|_2 < 1 \\ 0 &< \| (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda} \|_2 < 1 \end{aligned} \qquad (11.18)$$

which in turn implies

$$\begin{aligned} 0 &< \|(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}\|_2 \|(\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}\|_2 < 1 \\ 0 &< \|(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}\|_2 < 1 \end{aligned} \qquad (11.19)$$

These bounds can be improved on by our design of the system such that the norms of one or both of these scattering matrices are constrained to be small (i.e., much less than 1). Note that $(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}$ represents the scattering of waves in V_λ off $S_{\lambda-1;\lambda}$, or specifically off the penetrations into V_λ through $S_{\lambda-1;\lambda}$. If we make no constraints on the electromagnetic properties of V_λ, other than linearity and passivity, then by controlling $(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}$ we control the size of $a^{(1)}$. Thus, let us design the penetrations of $S_{\lambda-1;\lambda}$ (i.e., the protection networks, etc.) such tha the reflections in V_λ at $S_{\lambda-1;\lambda}$ are negligible, i.e., let us assume that

$$0 < \|(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}\| \ll 1 \qquad (11.20)$$

which in turn gives

$$a^{(1)} \simeq 1 \qquad (11.21)$$

With these results (11.15) can be reduced, in norm sense, to

$$\begin{aligned} a \tilde{\le} \|\{[(1_{n,m})_{1,1;3,3;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}] \\ -(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1} \\ \cdot (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\}^{-1}\| \end{aligned}$$

$$||(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}||$$

$$\lessapprox \{1 - ||(\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} + (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}$$

$$\cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}$$

$$\cdot (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}||\}^{-1}$$

$$||(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}||$$

$$\equiv \{1 - a^{(2)}\}^{-1}||(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}||$$

$$\text{if } 0 < a^{(2)} < 1 \qquad (11.22)$$

Let us next consider the term $a^{(2)}$ as

$$a^{(2)} \equiv ||(\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda} + (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}$$

$$\cdot [(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}$$

$$\cdot (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}||$$

$$\leq ||(\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda}||\ ||(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}||$$

$$+ ||(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}||\ ||[(1_{n,m})_{1,1;1,1;\lambda,\lambda} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda,\lambda}]^{-1}||$$

$$||(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}||\ ||(\tilde{S}_{n,m})_{2,2;1,3;\lambda,\lambda}||\ ||(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}|| \qquad (11.23)$$

Using (11.20) and (11.21) together with the relations in 2-norm sense (see appendix C)

$$0 < \|(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}\|_2 < 1$$
$$0 < \|(\tilde{S}_{n,m})_{2,1;1,3;\lambda,\lambda}\|_2 < 1 \tag{11.24}$$
$$0 < \|(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\|_2 < 1$$
$$0 < \|(\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda}\|_2 < 1$$

we next have

$$a^{(2)} \lessapprox \|(\tilde{S}_{n,m})_{1,2;3,3;\lambda,\lambda}\| \; \|(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\| + \|(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}\| \tag{11.25}$$

As when considering $a^{(1)}$ in (11.16), let us constrain

$$0 < \|(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\| \ll 1 \tag{11.26}$$

This represents the reflection of waves in V_λ at $S_{\lambda;\lambda+1}$, or specifically the penetrations (i.e., the protection networks, etc.) through $S_{\lambda;\lambda+1}$. Then using again (11.24) we have

$$a^{(2)} \lessapprox \|(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}\| + \|(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}\| \tag{11.27}$$

Since both of these terms are small compared to 1 (per (11.20) and (11.26)) we then have

$$0 < a^{(2)} \ll 1 \tag{11.28}$$

Substituting this result in (11.22) gives

$$a \mathrel{\tilde{<}} \|(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}\| \tag{11.29}$$

which with (11.13) gives

$$\|(\tilde{V}_n)_{1;3;\lambda}\| \mathrel{\tilde{<}} \|(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}\| \; \|(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1}\| \; \|(\tilde{V}_n)_{1;3;\lambda-1}\| \tag{11.30}$$

This product of matrix norms is small since (see appendix C)

$$0 < ||(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}||_2 < 1$$
$$0 < ||(\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1}|| \ll 1 \qquad (11.31)$$

where the second inequality ($\ll 1$) is a fundamental assumption for the validity of the good-shielding approximation (as in section 7). Note that $(\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda}$ represents a transfer-function matrix through V_λ, and extra shielding is obtained if this term is also small. Recollect that we have required that the reflections at the penetrations of both $S_{\lambda-1;\lambda}$ and $S_{\lambda;\lambda+1}$ are assumed to be kept small for the above results to hold, i.e., we have constrained

$$0 < ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda,\lambda}|| \ll 1$$
$$0 < ||(\tilde{S}_{n,m})_{2,1;3,3;\lambda,\lambda}|| \ll 1 \qquad (11.32)$$

Using these same assumptions in (11.32) the results of (11.30) can be posed somewhat stronger from (11.12) as

$$(\tilde{V}_n)_{1;3;\lambda} \simeq (\tilde{S}_{n,m})_{1,1;3,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m})_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n)_{1;3;\lambda-1} \qquad (11.33)$$

XII. The Good-Shielding Approximation for $\lambda = \lambda_{max}$

In the case of $\lambda = \lambda_{max}$ (8.4) and fig. 8.1 show some differences from the general case discussed in section 11. In particular, $V_{\lambda_{max}}$ is described by a set of scattering matrices which include signal transport from $V_{\lambda_{max}-1}$, but to no V_λ for $\lambda > \lambda_{max}$. Now one could define the scattering matrices in $V_{\lambda_{max}}$ in a manner which mimicked those in the more general V_λ by adding another node, perhaps representing some set of selected terminals in V_λ, the signals reaching there being of particular interest. In this form the results of section 11 (specifically (11.30)) apply to the transfer of signals from $V_{\lambda_{max}-1}$ to $V_{\lambda_{max}}$.

If one uses the form of the equations in (8.4) and fig. 8.1, then the procedure in section 11 can be applied for the transport of signals from $V_{\lambda_{max}-1}$ to $V_{\lambda_{max}}$. The right side of (11.2) becomes

$$- (((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})_{\lambda_{max},\lambda_{max}-1} \odot (((\tilde{V}_n(s))_\sigma)_\mu)_{\lambda_{max}-1}$$

$$= \begin{bmatrix} (\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1} \cdot (\tilde{V}_n)_{1;3;\lambda_{max}-1} \\ (0_n)_{1;3;\lambda_{max}-1} \end{bmatrix} \tag{12.1}$$

while the left side becomes

$$(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu;})_{\lambda_{max},\lambda_{max}} \odot (((\tilde{V}_n(s))_\sigma)_\mu)_{\lambda_{max}}$$

$$= \begin{bmatrix} (\tilde{V}_n)_{1;1;\lambda_{max}} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{V}_n)_{2;1;\lambda_{max}} \\ -(\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{V}_n)_{1;1;\lambda_{max}} + (\tilde{V}_n)_{2;1;\lambda_{max}} \end{bmatrix} \tag{12.2}$$

Between (12.1) and (12.2) we have two matrix equations. The second of these equations gives

$$(\tilde{V}_n)_{2;1;\lambda_{max}} = (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{V}_n)_{1;1;\lambda_{max}} \tag{12.3}$$

The first equation is

$$(\tilde{V}_n)_{1;1;\lambda_{max}} \simeq (\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{V}_n)_{2;1;\lambda_{max}}$$

$$+ (\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1} \cdot (\tilde{V}_n)_{1;3;\lambda_{max}-1} \quad (12.4)$$

Substituting from (12.3) gives

$$(\tilde{V}_n)_{1;1;\lambda_{max}} \simeq$$

$$[(1_{n,m})_{1,1;1,1;\lambda_{max},\lambda_{max}} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}]^{-1}$$

$$\cdot (\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1} \cdot (\tilde{V}_n)_{1;3;\lambda_{max}-1} \quad (12.5)$$

Comparing this result to (11.12) shows the simpler form of the result for the transport of the signals in $V_{\lambda_{max}}$ from $V_{\lambda_{max}-1}$.

In norm sense we have

$$||(\tilde{V}_n)_{1;1;\lambda_{max}}|| \tilde{<} ab||(\tilde{V}_n)_{1;3;\lambda_{max}-1}|| \quad (12.6)$$

where

$$b \equiv ||(\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1}||$$

$$= \text{contribution from shield } S_{\lambda_{max}-1,\lambda_{max}} \quad (12.7)$$

$$a \equiv \text{contribution from layer } V_{\lambda_{max}}$$

The term a is

$$a \equiv ||[(1_{n,m})_{1,1;1,1;\lambda_{max},\lambda_{max}} - (\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}}$$

$$\cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}]^{-1}|| < [1 - ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}}$$

$$\cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}||]^{-1}$$

$$\text{if } ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}|| < 1 \quad (12.8)$$

As before

$$||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}||$$

$$\leq ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}}|| \; ||(\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}|| \qquad (12.9)$$

Again energy conservation requires (see appendix C)

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}}||_2 \leq 1$$

$$0 \leq ||(\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}||_2 \leq 1 \qquad (12.10)$$

which in turn implies

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}}||_2 \; ||(\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}||_2 \leq 1$$

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}} \cdot (\tilde{S}_{n,m})_{2,1;1,1;\lambda_{max},\lambda_{max}}||_2 \leq 1 \qquad (12.11)$$

If, as before, we design the penetrations of $S_{\lambda_{max}-1,\lambda_{max}}$ such that the reflections in $V_{\lambda_{max}}$ at $S_{\lambda_{max}-1,\lambda_{max}}$ are negligible, i.e. let us assume that

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;1,1;\lambda_{max},\lambda_{max}}|| \ll 1 \qquad (12.12)$$

then in turn we have

$$a \simeq 1 \qquad (12.13)$$

This reduces (12.6) to

$$||(\tilde{V}_n)_{1;1;\lambda_{max}}|| \lessapprox ||(\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1}||$$

$$||(\tilde{V}_n)_{1;3;\lambda_{max}-1}|| \qquad (12.14)$$

The matrix norm is small, i.e.

$$0 \leq ||(\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1}|| \ll 1 \qquad (12.15)$$

as the fundamental assumption of the good-shielding approximation. Note that (12.14) has only one scattering matrix; this can be compared to (11.30) in the general case which has two scattering matrices.

Using the assumption in (12.12) the result of (12.14) can be improved somewhat in that (12.5) becomes

$$(\tilde{V}_n)_{1;1;\lambda_{max}} \simeq (\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1} \cdot (\tilde{V}_n)_{1;3;\lambda_{max}-1} \qquad (12.16)$$

XIII. The Good-Shielding Approximation for $\lambda = 1$

The case of $\lambda = 1$ is also special as indicated in (7.14), (8.4), and fig. 8.1. For this special case (7.14) becomes

$$(((\tilde{V}_n(s))_\sigma)_\mu)_1 = (((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})^{-1}_{1,1} \odot (((\tilde{V}_n^{(s)}(s))_\sigma)_\mu)_1 \quad (13.1)$$

From appendix B we have

$$(((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})_{\mu,\mu'})^{-1}_{1,1}$$

$$= \begin{bmatrix} (1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2} \end{bmatrix}^{-1}_{3,3;1,1}$$

$$= \begin{bmatrix} [(1_{n,m})_{1,1}-(\tilde{S}_{n,m})_{1,2}\cdot(\tilde{S}_{n,m})_{2,1}]^{-1} & [(1_{n,m})_{1,1}-(\tilde{S}_{n,m})_{1,2}\cdot(\tilde{S}_{n,m})_{2,1}]^{-1}\cdot(\tilde{S}_{n,m})_{1,2} \\ [(1_{n,m})_{2,2}-(\tilde{S}_{n,m})_{2,1}\cdot(\tilde{S}_{n,m})_{1,2}]^{-1}\cdot(\tilde{S}_{n,m})_{2,1} & [(1_{n,m})_{2,2}-(\tilde{S}_{n,m})_{2,1}\cdot(\tilde{S}_{n,m})_{1,2}]^{-1} \end{bmatrix}_{3,3;1,1}$$

(13.2)

The supervectors in (13.1) can be reduced to two blocks, the first of $(((\tilde{V}_n(s))_\sigma)_\mu)_1$ being the one of interest giving

$$(\tilde{V}_n)_{1;3;1} =$$

$$[(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}_{3,3;1,1} \cdot (\tilde{V}_n^{(s)})_{1;3;1}$$

$$+ [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}_{3,3;1,1}) \cdot (\tilde{S}_{n,m})_{1,2;3,3;1,1} \cdot (\tilde{V}_n^{(s)})_{2;3;1}$$

(13.3)

Applying norms we first have

$$||[(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}_{3,3;1,1}||$$

$$\leq [1 - ||(\tilde{S}_{n,m})_{1,2;3,3;1,1} \cdot (\tilde{S}_{n,m})_{2,1;3,3;1,1}||]^{-1}$$

$$\text{if } ||(\tilde{S}_{n,m})_{1,2;3,3;11} \cdot (\tilde{S}_{n,m})_{2,1;3,3;11}|| < 1$$

(13.4)

Again

$$||(\tilde{S}_{n,m})_{1,2;3,3;1,1} \cdot (\tilde{S}_{n,m})_{2,1;3,3;11}||$$

$$\leq ||(\tilde{S}_{n,m})_{1,2;3,3;1,1}|| \; ||(\tilde{S}_{n,m})_{2,1;3,3;1,1}|| \tag{13.5}$$

and energy conservation requires (see appendix C)

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;3,3;1,1}||_2 \leq 1$$

$$0 \leq ||(\tilde{S}_{n,m})_{2,1;3,3;1,1}||_2 \leq 1 \tag{13.6}$$

implying

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;3,3;1,1}||_2 \; ||(\tilde{S}_{n,m})_{2,1;3,3;1,1}||_2 \leq 1$$

$$0 \leq ||(\tilde{S}_{n,m})_{1,2;3,3;1,1} \cdot (\tilde{S}_{n,m})_{2,1;3,3;1,1}||_2 \leq 1 \tag{13.7}$$

Since $(\tilde{S}_{n,m})_{2,1;3,3;1,1}$ represents reflections back into V_1 from $S_{1;2}$, then by design of the penetrations of $S_{1;2}$ this term can be made negligible, i.e. we can have

$$0 \leq ||(\tilde{S}_{n,m})_{2,1;3,3;1,1}|| << 1 \tag{13.8}$$

in which case

$$||(\tilde{V}_n)_{1;3;1}|| \lessapprox ||(\tilde{V}_n^{(s)})_{1;3;1}|| + ||(\tilde{S}_{n,m})_{1,2;3,3;1,1}|| \; ||(\tilde{V}_n^{(s)})_{2;3;1}||$$

$$\lessapprox ||(\tilde{V}_n^{(s)})_{1;3;1}|| + ||(\tilde{V}_n^{(s)})_{2;3;1}|| \tag{13.9}$$

Using the assumption of (13.8) the result in (13.9) can be made stronger by substitution in (13.3) giving

$$(\tilde{V}_n)_{1;3;1} \simeq (\tilde{V}_n^{(s)})_{1;3;1} + (\tilde{S}_{n,m})_{1,2;3,3;1,1} \cdot (\tilde{V}_n^{(s)})_{2;3;1} \tag{13.10}$$

XIV. Solution for the Good-Shielding Approximation

Combining the results of sections 11 through 13 in the general form of section 7, specifically (7.16), we have

$$(\tilde{V}_n(s))_{1;3;\lambda} \approx (\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda,\lambda} \cdot (\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda,\lambda-1} \cdot (\tilde{V}_n(s))_{1;3;\lambda-1}$$

$$\approx \{ \bigodot_{\lambda'=1}^{\lambda-1} [(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda+1-\lambda',\lambda+1-\lambda'} \cdot (\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda+1-\lambda',\lambda-\lambda'}]\}$$

$$\cdot (\tilde{V}_n(s))_{1;3;1}$$

$$\approx \{ \bigodot_{\lambda'=0}^{\lambda-2} [(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda-\lambda',\lambda-\lambda'} \cdot (\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda-\lambda',\lambda-1-\lambda'}]\}$$

$$\cdot (\tilde{V}_n(s))_{1;3;1} \qquad (14.1)$$

for $\lambda = 1,2,\cdots,\lambda_{max} - 1$

Here (13.10) can be used to give an alternate representation of $(\tilde{V}_n(s))_{1;3;1}$, but perhaps the above form is simpler.

For $\lambda = \lambda_{max}$ one can organize $V_{\lambda_{max}}$ $\ni$ there is a $(\tilde{V}_n(s))_{1;3;\lambda_{max}}$ and (14.1) applies. Alternately, one can use (12.5) to put the results in terms of $(\tilde{V}_n(s))_{1;1;\lambda_{max}}$ as

$$(\tilde{V}_n(s))_{1;1;\lambda_{max}} \approx (\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda_{max},\lambda_{max}-1} \cdot$$

$$\{ \bigodot_{\lambda'=2}^{\lambda_{max}-1} [(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda_{max}+1-\lambda',\lambda_{max}+1-\lambda'} \cdot (\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda_{max}+1-\lambda',\lambda_{max}-\lambda'}]$$

$$\cdot (\tilde{V}_n(s))_{1;3;1}$$

$$= (\tilde{S}_{n,m})_{1,1;1,3;\lambda_{max},\lambda_{max}-1} \cdot$$

$$\{ \bigodot_{\lambda'=1}^{\lambda_{max}-2} [(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda_{max}-\lambda',\lambda_{max}-\lambda'} \cdot (\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda_{max}-\lambda',\lambda_{max}-1-\lambda'}]\}$$

$$\cdot (\tilde{V}_n(s))_{1;3;1} \qquad (14.2)$$

These results can be readily cast in norm form. From (14.1) we have (for $s=j\omega$ here and following)

$$||(\tilde{V}_n(s))_{1;3;\lambda}|| \lesssim$$

$$\{\prod_{\lambda'=0}^{\lambda-2} [||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda-\lambda',\lambda-\lambda'}|| \; ||(\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda-\lambda',\lambda-1-\lambda'}||]\}$$

$$||(\tilde{V}_n(s))_{1;3;1}||$$

$$= \{\prod_{\lambda'=2}^{\lambda} [||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}|| \; ||(\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda',\lambda'-1}||]\}$$

$$||(\tilde{V}_n(s))_{1;3;1}|| \tag{14.3}$$

for $\lambda = 1,2,\cdots,\lambda_{max}-1$

From (14.2) we have

$$||(\tilde{V}_n(s))_{1;1;\lambda_{max}}|| \lesssim ||(\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda_{max},\lambda_{max}-1}||$$

$$\{\prod_{\lambda'=1}^{\lambda_{max}-2} [||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda_{max}-\lambda',\lambda_{max}-\lambda'}|| \, ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda_{max}-\lambda',\lambda_{max}-1-\lambda'}||]\}$$

$$||(\tilde{V}_n(s))_{1;3;1}||$$

$$= \{\prod_{\lambda'=2}^{\lambda_{max}-1} [||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}|| \; ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'-1}||]\}$$

$$||(\tilde{S}_{n,m}(s))_{1,1;1,3;\lambda_{max},\lambda_{max}-1}|| \; ||(\tilde{V}_n(s))_{1;3;1}|| \tag{14.4}$$

If we now specialize our results to the 2-norm then, of course, (14.3) and (14.4) still apply. In appendix C it is shown that the 2-norm of any block of a passive 2-port network or system is bounded by 1. Here we can apply this result to the matrix blocks for transmission through a volume, i.e.

$$0 < ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}||_2 < 1 \tag{14.5}$$

for $\lambda' = 2,3,\cdots,\lambda_{max}-1$

Then (14.3) reduces to

$$||(\tilde{V}_n(s))_{1;3;\lambda}||_2 \lessapprox \{ \prod_{\lambda'=2}^{\lambda} ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'-1}||_2$$

$$||(\tilde{V}_n(s))_{1;3;1}||_2 \tag{14.6}$$

for for $\lambda = 2,3,\cdots,\lambda_{max}-1$

and (14.4) reduces to

$$||(\tilde{V}_n(s))_{1;1;\lambda_{max}}||_2 \lessapprox \{ \prod_{\lambda'=2}^{\lambda_{max}} ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'-1}||_2\}$$

$$||(\tilde{V}_n(s))_{1;3;1;}||_2 \tag{14.7}$$

These results reduce the complexity of the characterization of the shielding to $\lambda_{max}-1$ non-negative scalar terms, each of which characterizes the $\lambda_{max}-1$ (≥ 1) shields. Each of these terms is assumed small compared to 1 as is required by the good-shielding approximation.

The 2-norm is related to power. In this case it is related to the maximum power entering V_λ. Other norms are also of interest. In particular the ∞-norm is a maximum-signal norm. This norm can be used then to bound the maximum signal in V_λ. In this case the ∞-norm can be directly applied to (14.3) and (14.4). A previous paper [8] has related the ∞-norm to the 2-norm as

$$||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}||_\infty \leq M_{\lambda'}^{1/2} ||\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}||_2 \tag{14.8}$$

$M_{\lambda'} \equiv$ number of columns of $(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}$

Using (14.5) we have

$$0 \leq ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'}||_\infty \leq M_{\lambda'}^{1/2} \tag{14.9}$$

Applying this result to (14.3) gives

$$||(\tilde{V}_n(s))_{1;3;\lambda}||_\infty \lessapprox \{\prod_{\lambda'=2}^{\lambda} [M_{\lambda'}^{1/2} ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'-1}||_\infty]\}$$

$$||(\tilde{V}_n(s))_{1;3;1}||_\infty \tag{14.10}$$

for $\lambda = 1,2,\cdots,\lambda_{max}-1$

and to (14.4) gives

$$||(\tilde{V}_n(s))_{1;1;\lambda_{max}}||_\infty \lessapprox \{\prod_{\lambda'=2}^{\lambda_{max}} [M_{\lambda'}^{1/2} ||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'-1}||_\infty]\}$$

$$||(\tilde{V}_n(s))_{1;3;1}||_\infty \tag{14.11}$$

For this to be useful let us require

$$0 < M_{\lambda'}^{1/2}||(\tilde{S}_{n,m}(s))_{1,1;3,1;\lambda',\lambda'-1}||_\infty << 1 \tag{14.12}$$

as a generalized requirement for the good-shielding approximation. Note that in using the ∞-norm for bounding signals in V_λ it is helpful to have the $M_{\lambda'}$ (for for $\lambda' = 2,3,\cdots,\lambda$) kept as small as possible. Physically the M_λ' represent the number of ports leading in to the $V_{\lambda'}$ from the $S_{\lambda'-1;\lambda'}$.

XV. Summary

Well this note has covered quite a lot of ground! It has gone into some detail concerning some points raised in an earlier paper [5] and the theory somewhat further. Referring to the table of contents one can see the progression of topics beginning with the qualitative (or descriptive) aspects of electromagnetic topology, and leading to quantitative aspects involving supervectors/supermatrices, and the BLT equation evolving from transmission-line networks to EM topology. This leads to the major results of this note involving a detailed treatment of the good-shielding approximation.

This is not the last word concerning EM topology. There are other aspects such as the implications of sublayers on the good-shielding approximation, as well as other implications involving elementary volumes. Furthermore many of the results need consideration in time domain as well as frequency domain.

The good-shielding approximation imposes certain requirements concerning the smallness (in norm sense) of certain scattering-matrix blocks. This smallness needs to be achieved in practice by the design of various networks at the shields $S_{\lambda;\lambda+1}$ to control the passage of unwanted signals through these shields as well as the reflection of such signals from these shields. These are stated as assumptions in section 11; these assumptions are in general realizable and can be discussed in future notes.

Appendix A: Norms of Supervectors and Supermatrices with One Non-Zero Block

In considering the norms of sparse matrices one would like to relate their norms to the norms of the non-zero blocks.

Define supervectors

$$((a_n)_u) \tag{A.1}$$

with blocks and elements as

$$(a_n)_u \quad , \quad a_{n;u}$$
$$n = 1,2,\cdots,M_u \tag{A.2}$$
$$u = 1,2,\cdots,M$$

For norms consider the p-norm defined by

$$||((a_n)_u)||_p \equiv \{\sum_{u=1}^{M} \sum_{n=1}^{M_u} |a_{n;u}|^p\}^{1/p}$$
$$= \{\sum_{u=1}^{M} ||(a_n)_u||_p^p\}^{1/p}$$
$$= ||(||(a_n)_1||_p, ||(a_n)_2||_p, \cdots, ||(a_n)_M||_p)||_p$$
$$p > 0 \tag{A.3}$$

which is also discussed in [8]. In words the p-norm of a supervector is the p-norm of a vector whose components are the p-norms of the blocks.

Keeping the same partitioning for the various supervectors let us consider a supervector which is the same as $((a_n)_u)$ for one block, but zero otherwise as

$$(x_n)_u = \begin{cases} (a_n)_u & \text{for } u = v_1 \\ (0_n)_u & \text{for } u \neq v_1 \end{cases}$$
$$1 \leq v_1 \leq M \tag{A.4}$$

$$((y_n)_u) \equiv ((a_n)_u) - ((x_n)_u)$$

$$(y_n)_u = \begin{cases} (0_n)_u & \text{for } u = v_1 \\ (a_n)_u & \text{for } u \neq v_1 \end{cases}$$

The p-norm of this special supervector is

$$||((x_n)_u)||_p = \{\sum_{u=1}^{M} ||(x_n)_u||_p^p\}^{1/p}$$

$$= \{||(a_n)_{v_1}||_p^p\}^{1/p}$$

$$= ||(a_n)_{v_1}||_p \tag{A.5}$$

Thus the p-norm of a supervector with only one non-zero block is the same as the p-norm of that block.

Next consider the supermatrix

$$((A_{n,m})_{u,v}) \tag{A.6}$$

with blocks and elements as

$$(A_{n,m})_{u,v} \quad , \quad A_{n,m;u,v}$$

$$n = 1,2,\cdots,N_u$$

$$m = 1,2,\cdots,M_v \tag{A.7}$$

$$u = 1,2,\cdots,N$$

$$v = 1,2,\cdots,M$$

This partitioning allows us to write the product

$$((A_{n,m})_{u,v}) \odot ((a_n)_u) \tag{A.8}$$

since the terms have compatible order for generalized dot multiplication. Let the supermatrix now have only one non-zero block as

$$(A_{n,m})_{u,v} = \begin{cases} (A_{n,m})_{u,v} \text{ for } (u,v) = (u_1,v_1) \\ (0_{n,m})_{u,v} \text{ for } (u,v) \neq (u_1,v_1) \end{cases} \tag{A.9}$$

Writing out the product in (A.8)

$$((A_{n,m})_{u,v}) \odot ((a_n)_{u,v}) = ((A_{n,m})_{u,v}) \odot ((x_n)_u) + ((A_{n,m})_{u,v}) \odot ((y_n)_u)$$

$$((A_{n,m})_{u,v}) \odot ((x_n)_u) = (\sum_{v=1}^{N_v} (A_{n,m})_{u,v} \cdot (x_n)_v)$$

$$= ((A_{n,m})_{u,v_1} \cdot (x_n)_{v_1}$$

$$= \{(0_n)_1, \cdots, (0_n)_{u_1-1}, (A_{n,m})_{u_1,v_1} \cdot (x_n)_{v_1}, (0_n)_{u_1+1}, \cdots, (0_n)_N\} \quad \text{(A.10)}$$

$$((A_{n,m})_{u,v}) \odot ((y_n)_u) = (\sum_{v=1}^{N_v} (A_{n,m})_{u,v} \cdot (y_n)_v)$$

$$= ((0_n)_v)$$

Thus it is only the $((x_n)_u)$ part of $((a_n)_u)$, and the single non-zero block of $((A_{n,m})_{u,v})$ that contributes to this dot product, leaving a supervector with only the $u = u_1$ block non zero.

The associated norm of a supermatrix with one non-zero block is then

$$||((A_{n,m})_{u,v})|| = \sup_{((a_n)_u \neq ((0_n)_u)} \frac{||((A_{n,m})_{u,v}) \odot ((a_n)_u)||}{||((a_n)_u)||}$$

$$= \sup_{((a_n)_u \neq ((0_n)_u)} \frac{||((A_{n,m})_{u,v}) \odot ((x_n)_u)||}{||((a_n)_u)||} \quad \text{(A.11)}$$

Now interpret the norm as a p-norm. Note that only a "part" of $((a_n)_u)$, namely $((x_n)_u)$, appears in the numerator. The demonimator is (from (A.3) through (A.5))

$$||((a_n)_u)||_p^p = \sum_{u=1}^{M} ||(a_n)_u||_p^p$$

$$\geq ||(a_n)_{v_1}||_p^p = ||((x_n)_u)||_p^p \quad \text{(A.12)}$$

$$||((a_n)_u)||_p \geq ||((x_n)_u)||_p$$

with equality of course when $((a_n)_u) = ((x_n)_u)$.

For any particular $((x_n)_u) \neq ((0_n)_u)$ chosen the denominator in (A.12) is minimized by the choice of

$$((a_n)_u) = ((x_n)_u)$$

$$((y_n)_u) = ((0_n)_u) \tag{A.13}$$

giving

$$\|((A_{n,m})_{u,v})\|_p = \sup_{(x_n)_{v_1} \neq (0_n)_{v_1}} \frac{\|((A_{n,m})_{u,v}) \odot ((x_n)_u)\|_p}{\|((x_n)_u)\|_p} \tag{A.14}$$

Now from (A.5) we have

$$\|((x_n)_u)\|_p = \|(x_n)_{v_1}\|_p \tag{A.15}$$

and from (A.10) and (A.3) we have

$$\|((A_{n,m})_{u,v}) \odot ((x_n)_u)\|_p^p$$

$$= \|(0_n)_1\|_p^p + \cdots + \|(0_n)_{u_1-1}\|_p^p + \|(A_{n,m})_{u_1,v_1} \cdot (x_n)_{v_1}\|_p^p$$

$$+ \|(0_n)_{u_1+1}\|_p^p + \cdots + \|(0_n)_N\|_p^p$$

$$= \|A_{n,m})_{u_1,v_1}) \cdot (x_n)_{v_1}\|_p^p \tag{A.16}$$

$$\|((A_{n,m})_{u,v}) \odot ((x_n)_u)\|_p = \|(A_{n,m})_{u_1,v_1} \cdot (x_n)_{v_1}\|_p$$

Combining these results we have

$$\|((A_{n,m})_{u,v})\|_p = \sup_{(x_n)_{v_1} \neq (0_n)_{v_1}} \frac{\|(A_{n,m})_{u_1,v_1} \cdot (x_n)_{v_1}\|_p}{\|(x_n)_{v_1}\|_p}$$

$$= \|(A_{n,m})_{u_1,v_1}\|_p \tag{A.17}$$

Thus the p-norm of a supermatrix with only one non-zero block is the same as the p-norm of that block.

Appendix B. Supermatrix Inverse of Special Matrix Blocks

Considering the diagonal blocks at various levels of partition we first have

$$(1_{n,m})^{-1}_{\sigma,\sigma;\mu,\mu;\lambda,\lambda} = (1_{n,m})_{\sigma,\sigma;\mu,\mu;\lambda,\lambda} \qquad \text{(B.1)}$$

since the identity is its own inverse.

Next consider the μ,μ diagonal blocks

$$((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})^{-1}_{\mu,\mu;\lambda,\lambda}$$

$$= \begin{bmatrix} (1_{n,m})_{1,1} & -(\tilde{S}_{n,m})_{1,2} \\ -(\tilde{S}_{n,m})_{2,1} & (1_{n,m})_{2,2} \end{bmatrix}^{-1}_{\mu,\mu;\lambda,\lambda}$$

$$\equiv \begin{bmatrix} (B_{n,m})_{1,1} & (B_{n,m})_{1,2} \\ (B_{n,m})_{2,1} & (B_{n,m})_{2,2} \end{bmatrix}_{\mu,\mu;\lambda,\lambda} \qquad \text{(B.2)}$$

Using the results of section 4 this supermatrix inverse can be found. Temporarily dropping the μ and λ indices the blocks $(B_{n,m})_{\sigma,\sigma'}$ can be found from the blocks of the original matrix which can be denoted as the $(A_{n,m})_{\sigma,\sigma'}$ consistent consistent with section 4. Using (4.14) and (4.15) we have

$$(B_{n,m})_{1,1}$$

$$= [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (1_{n,m})^{-1}_{2,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}$$

$$= [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}$$

$$(B_{n,m})_{1,2}$$

$$= [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (1_{n,m})^{-1}_{2,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1} \cdot (S_{n,m})_{1,2} \cdot (1_{n,m})^{-1}_{2,2}$$

$$= [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1} \cdot (\tilde{S}_{n,m})_{1,2}$$

$(B_{n,m})_{1,2}$

$$= (1_{n,m})^{-1}_{1,1} \cdot (\tilde{S}_{n,m})_{1,2} \cdot [(1_{n,m})_{2,2} - (\tilde{S}_{n,m})_{2,1} \cdot (1_{n,m})^{-1}_{1,1} \cdot (\tilde{S}_{n,m})_{1,2}]^{-1}$$

$$= (\tilde{S}_{n,m})_{1,2} \cdot [(1_{n,m})_{2,2} - (\tilde{S}_{n,m})_{2,1} \cdot (\tilde{S}_{n,m})_{1,2}]^{-1} \quad \text{(B.3)}$$

$(B_{n,m})_{2,1}$

$$= [(1_{n,m})_{2,2} - (\tilde{S}_{n,m})_{2,1} \cdot (1_{n,m})^{-1}_{1,1} \cdot (\tilde{S}_{n,m})_{1,2}]^{-1} \cdot (\tilde{S}_{n,m})_{2,1} \cdot (1_{n,m})^{-1}_{1,1}$$

$$= [(1_{n,m})_{2,2} - (\tilde{S}_{n,m})_{2,1} \cdot (\tilde{S}_{n,m})_{1,2}]^{-1} \cdot (\tilde{S}_{n,m})_{2,1}$$

$(B_{n,m})_{2,1}$

$$= (1_{n,m})^{-1}_{2,2} \cdot (\tilde{S}_{n,m})_{2,1} \cdot [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (1_{n,m})^{-1}_{2,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}$$

$$= (\tilde{S}_{n,m})_{2,1} \cdot [(1_{n,m})_{1,1} - (\tilde{S}_{n,m})_{1,2} \cdot (\tilde{S}_{n,m})_{2,1}]^{-1}$$

$(B_{n,m})_{2,2}$

$$= [(1_{n,m})_{2,2} - (\tilde{S}_{n,m})_{2,1} \cdot (1_{n,m})^{-1}_{1,1} \cdot (\tilde{S}_{n,m})_{1,2}]^{-1}$$

$$= [(1_{n,m})_{2,2} - (\tilde{S}_{n,m})_{2,1} \cdot (\tilde{S}_{n,m})_{1,2}]^{-1}$$

Note that the off-diagonal blocks have two different-looking but equivalent expressions. These substituted in (B.2) give an explicit expression for the inverse of the μ,μ blocks as

$$((\tilde{I}_{n,m}(s))_{\sigma,\sigma'})^{-1}_{\mu,\mu;\lambda,\lambda}$$

$$= \begin{bmatrix} [(1_{n,m})_{1,1}-(\tilde{S}_{n,m})_{1,2}\cdot(\tilde{S}_{n,m})_{2,1}]^{-1} & [(1_{n,m})_{1,1}-(\tilde{S}_{n,m})_{1,2}\cdot(\tilde{S}_{n,m})_{2,1}]^{-1}\cdot(\tilde{S}_{n,m})_{1,2} \\ [(1_{n,m})_{2,2}-(\tilde{S}_{n,m})_{2,1}\cdot(\tilde{S}_{n,m})_{1,2}]^{-1}\cdot(\tilde{S}_{n,m})_{2,1} & [(1_{n,m})_{2,2}-(\tilde{S}_{n,m})_{2,1}\cdot(\tilde{S}_{n,m})_{1,2}]^{-1} \end{bmatrix}_{\mu,\mu;\lambda,\lambda}$$

$$= \begin{bmatrix} [(1_{n,m})_{1,1}-(\tilde{S}_{n,m})_{1,2}\cdot(\tilde{S}_{n,m})_{2,1}]^{-1} & (\tilde{S}_{n,m})_{1,2}\cdot[(1_{n,m})_{2,2}-(\tilde{S}_{n,m})_{2,1}\cdot(\tilde{S}_{n,m})_{1,2}]^{-1} \\ (\tilde{S}_{n,m})_{2,1}\cdot[(1_{n,m})_{1,1}-(\tilde{S}_{n,m})_{1,2}\cdot(\tilde{S}_{n,m})_{2,1}]^{-1} & [(1_{n,m})_{2,2}-(\tilde{S}_{n,m})_{2,1}\cdot(\tilde{S}_{n,m})_{1,2}]^{-1} \end{bmatrix}_{\mu,\mu;\lambda,\lambda} \quad \text{(B.4)}$$

In this form the alternate expressions for the off-diagonal blocks are also exhibited. Furthermore if we set $(\mu,\mu;\lambda,\lambda) = (3,3;1,1)$ this formula gives the inverse of the first diagonal block in (8.4); if we set $(\mu,\mu;\lambda,\lambda) = (1,1;\lambda_{max},\lambda_{max})$ this formula gives the inverse of the last diagonal block in (8.4).

Appendix C. 2-Norm of Blocks of Scattering Matrices

Consider the linear, passive, time invariant, reciprocal N-port network (or distributed system) indicated in fig. C.1. Two previous notes have considered the 2-norm of the corresponding scattering matrix [10,11], and have shown that under the above assumptions (for $s = j\omega$ which is assumed throughout this appendix)

$$0 \leq \left\| (\tilde{S}_{n,m}(s) \right\|_2 \leq 1 \tag{C.1}$$

provided the normalizing impedance (or admittance) matrix for the wave variables is merely a positive constant times the identity matrix, i.e.

$$(\tilde{V}_n(s))_{in} = (\tilde{V}_n(s)) + R(\tilde{I}_n(s)) \qquad \text{(incoming wave)}$$

$$(\tilde{V}_n(s))_{out} = (\tilde{V}_n(s)) - R(\tilde{I}_n(s)) \qquad \text{(outgoing wave)} \tag{C.2}$$

$$R > 0 \text{ (normalizing frequency-independent resistance)}$$

Of course, by definition, the scattering matrix is given by

$$(\tilde{V}_n(s))_{out} \equiv (\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_n(s))_{in} \tag{C.3}$$

In fig. C.1 we also have a vector of voltage sources to provide the excitation. Note the termination resistors R on each of the N-ports so that the outgoing wave is terminated (no reflection at the source). Matching boundary conditions at the source we have

$$(\tilde{V}_{s_n}(s)) - R(\tilde{I}_n(s)) = (\tilde{V}_n(s))$$

$$(\tilde{V}_n(s)) = \frac{1}{2}\left[(\tilde{V}_n(s))_{in} + (\tilde{V}_n(s))_{out}\right] \tag{C.4}$$

$$R(\tilde{I}_n(s)) = \frac{1}{2}\left[(\tilde{V}_n(s))_{in} - (\tilde{V}_n(s))_{out}\right]$$

Combining these results gives

$$(\tilde{V}_n(s))_{in} = (\tilde{V}_{s_n}(s)) \tag{C.5}$$

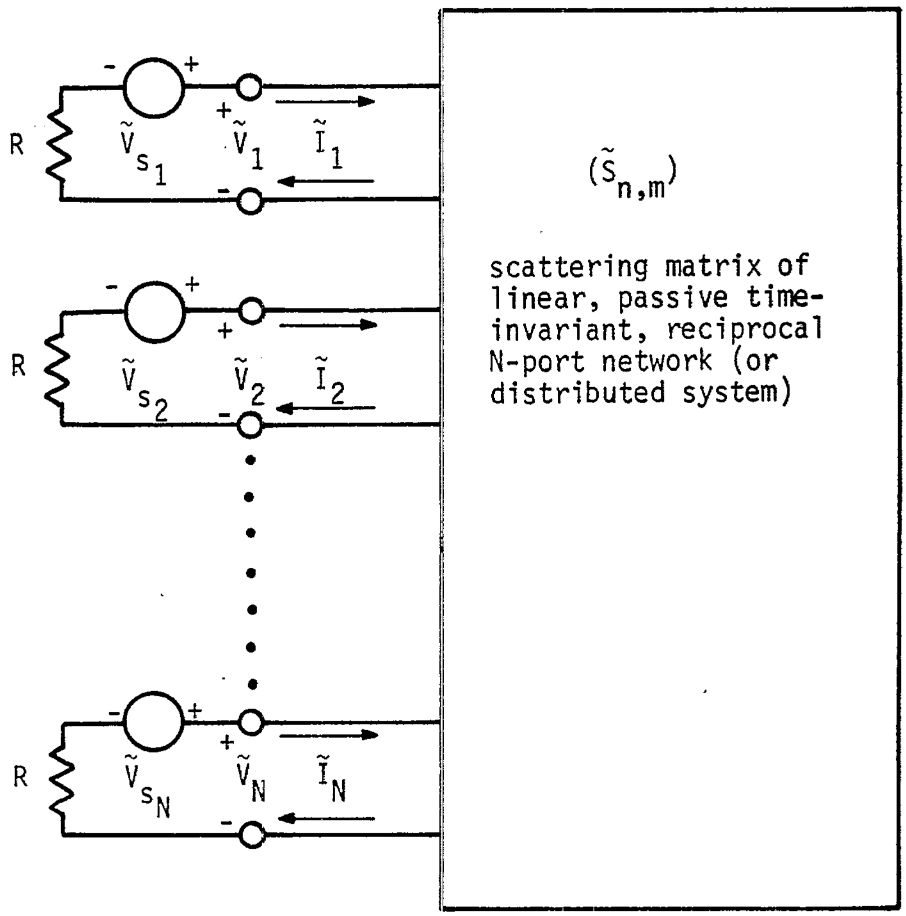


Fig. C.1. Equivalent Circuit for Scattering at N-Port which when combined with (C.3) gives the complete solution for both incoming and outgoing waves.

Now let us consider in some more detail the energy properties of our assumed passive scattering matrix. The constraints in [10] allow us to write an energy relation in the wave variables as

$$(\tilde{V}_n(s))^*_{in} \cdot (\tilde{V}_n(s))_{in} > (\tilde{V}_n(s))^*_{out} \cdot (\tilde{V}_n(s))_{out} > 0 \tag{C.6}$$

The scattering matrix and source vector can be inserted to give

$$(\tilde{V}_{s_n}(s))^* \cdot (\tilde{V}_{s_n}(s)) > (\tilde{V}_{s_n}(s))^* \cdot (\tilde{S}_{n,m}(s))^\dagger \cdot (\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_{s_n}(s)) > 0 \tag{C.7}$$

In terms of the 2-norm this is

$$||(\tilde{V}_{s_n}(s))||_2 > ||(\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_{s_n}(s)||_2 > 0 \tag{C.8}$$

Since the source vector can be selected to suit our purposes and since (C.8) applies for all $(\tilde{V}_{s_n}(s))$ (with $s = j\omega$), let us choose this source vector to have a number of zero elements, e.g. like

$$(\tilde{V}_{s_n}(s)) = (0, a_2, 0, 0, a_5, a_6, \cdots\cdots) \tag{C.9}$$

So let us define a vector $(a_n) \ni$

$$a_n = 0 \quad \text{for } n \not\in A \tag{C.10}$$

$A \equiv$ set of distinct integers selected from $1, 2, \cdots, N$

For those $n \in A$ the a_n are in general nonzero and (C.8) becomes

$$||(a_n)||_2 > ||(\tilde{S}_{n,m}) \cdot (a_n)||_2 > 0 \tag{C.11}$$

Normalizing by the left hand term

$$1 > \frac{||(\tilde{S}_{n,m}) \cdot (a_n)||_2}{||(a)||_2} > 0 \tag{C.12}$$

for $(a_n) \neq (0)$

The middle term of C.12 looks like a scattering-matrix norm. Note however this applies for all $(a_n) \neq (0_n)$. In particular choosing the a_n as in (C.10) we have

$$(\tilde{S}_{n,m}) \cdot (a_n) = (b_n)$$

$$b_n = \sum_{m \in A} \tilde{S}_{n,m} a_m$$

For $m \notin A$ in this sum the matrix elements $\tilde{S}_{n,m}$ do not contribute to the b_n. Then we have

$$\|(b_n)\|_2 = \{\sum_{n=1}^{N} |b_n|^2\}^{1/2} = \{\sum_{n=1}^{N} \left| \sum_{m \in A} \tilde{S}_{n,m} a_m \right|^2 \}^{1/2}$$
$$= \|(\tilde{S}_{n,m}) \cdot (a_n)\|_2 \quad (C.14)$$

Now suppose that we split (b_n) into two parts where

$$(b_n) = (c_n) + (d_n)$$
$$c_n = 0 \quad \text{for} \quad n \notin B \quad (C.15)$$
$$d_n = 0 \quad \text{for} \quad n \in B \, .$$

$B \equiv$ set of distinct integers selected from $1, 2, \cdots N$ (independently chosen from the set A)

Then we have

$$\|(b_n)\|_2 = \{\|(c_n)\|_2^2 + \|(d_n)\|_2^2\}^{1/2}$$

$$\|(c_n)\|_2 < \|(b_n)\|_2 \quad (C.16)$$

$$\|(d_n)\|_2 < \|(b_n)\|_2$$

which implies

$$\|(b_n)\|_2 > \|(d_n)\|_2 = \{\sum_{n \in B} |b_n|^2\}^{1/2}$$
$$= \{\sum_{n \in B} | \sum_{m \in A} \tilde{S}_{n,m} a_m |^2\}^{1/2} \quad (C.17)$$

Hence we have

$$1 < \frac{\{\sum_{n\in B}|\sum_{m\in A} \tilde{S}_{n,m}a_m|^2\}^{1/2}}{||(a_n)||_2} > 0 \qquad (C.18)$$

Define another scattering matrix as

$(\tilde{S}_{n,m}) \equiv$ a matrix consisting of rows $\in B$ and columns $\in A$ of $(\tilde{S}_{n,m})$

This matrix constructed by deleting selected rows and columns of $(\tilde{S}_{n,m})$ has a 2-norm

$$||(\tilde{S}_{n,m})||_2 \equiv \sup_{(a_n)\neq(0_n)} \frac{\{\sum_{n\in B}|\sum_{m\in A} \tilde{S}_{n,m}a_m|^2\}^{1/2}}{||(a_n)||_2} \qquad (C.20)$$

so that we have

$$1 > ||(\tilde{S}_{n,m}')||_2 > 0 \qquad (C.21)$$

Summarizing the results then, a matrix formed by deleting any rows and columns of $(\tilde{S}_{n,m})$ has a 2-norm bounded by 1. Specifically then, a corollary is that any block of $(\tilde{S}_{n,m})$ has a 2-norm bounded by 1.

Commenting on these results note that the numbering of the ports of the N-port network or distributed system in fig. C.1 is arbitrary. Port 1 could be renumbered as port 3, etc. This renumbering is equivalent to transforming $(\tilde{S}_{n,m})$ by a permutation matrix. In this new form the scattering matrix still has all the properties discussed above. By such a permutation a block of $(\tilde{S}_{n,m})$ can be transformed into a matrix with various rows and columns deleted, and conversely.

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