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A Priori Application of Results of Electromagnetic Theory to the Analysis of Electromagnetic Interaction Data

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Abstract

Since the introduction of the singularity expansion method (SEM) for the representation of transient and broadband electromagnetic interaction with general objects, there has been considerable attention given to associated analysis of electromagnetic-response experimental data to find the natural frequencies. Usually this has considered only single waveforms or frequency spectra for a parameter such as the surface current density at a particular position on the object under some particular excitation such as a particular direction of incidence with a particular polarization.

This paper explores several concepts for advancing the analysis of interaction data to obtain the various SEM and EEM (eigenmode expansion method) parameters. Basically the various properties of natural modes and eigenmodes are explored for application to the problem of the taking and analyzing of experimental data. Various techniques are explored including the enforcement of the SEM pole factors in multiple data records, separation of the modes by object symmetry, separation of the modes into E and H modes by measurement of surface charge density and equivalent magnetic charge density, and separation of the natural modes and natural frequencies by association with eigenmodes and use of eigenmode orthogonality. Basically these concepts involve application of a priori physics to the design of experiments and data analysis.

I. Introduction

Since the introduction of the singularity expansion method (SEM) [8] as a characterization of (linear) electromagnetic interaction with scatterers, there has been much development of the general theory with numerous references. Here we mention only some of the general reviews [13,21,22] and a special issue with bibliography [18].

As noted in the beginning, the natural frequencies and modes of an object are characteristic of the object, and not some particular electromagnetic formulation such as a particular integral equation. As such they are in principle experimentally observable. There has been much work aimed at analyzing transient waveforms and frequency spectra for electromagnetic responses with the aim of accurately determining natural frequencies from such data [15,16,17].

The aim of this paper is to propose some ideas about the recovering of SEM parameters from experimental data. In this endeavor let us look at some results and speculations concerning electromagnetic theory, not from the point of view of theoretical developments per se, but from the point of view that EM theory has something to say about experiment, including what is to be measured, how it is to be measured, and what is to be done with the data once one has it. Let us attempt to apply as much a priori physics to the data-analysis problem as possible so as to best extract the requisite information.

We assume that the data can be represented by a sum of SEM terms corrupted by noise. This noise may be often thought of as additive, but not necessarily so. It should be made clear that the goal of our endeavor is not to best fit some individual or set of waveforms (or spectra in frequency domain). The goal is to best determine fundamental EM parameters from the data to characterize the EM interaction in an efficient and accurate way which allows us to best understand EM interaction phenomena over ranges of parameters of interest. Furthermore one can hopefully use this understanding to control the EM interaction process in ways useful to engineering.

In this paper we discuss the application of various properties of the SEM pole terms (natural frequencies, natural modes, and coupling coefficients) and of the eigenmodes (of EM integral equations) to the analysis of experimental data. These are lumped into four categories discussed in the following sections.

II. Factoring of SEM Pole Terms

Consider the pole terms in the singularity expansion. Write an SEM form of the surface current density on a finite-dimension, perfectly conducting object in free space in time domain as

$$\vec{J}_s(\vec{r}_s,t) = E_o \sum_\alpha \tilde{f}_p(s_\alpha)\, \eta_\alpha^{(max)} \eta_\alpha^{(1)}(\vec{1}_1,\vec{1}_p)\, \vec{j}_{s_\alpha}(\vec{r}_s)\, e^{s_\alpha t}\, u(t - t_o)$$

$$+ \text{ other SEM terms } + \text{ noise}$$

$$t_o \equiv \text{turn-on time} \tag{2.1}$$

$\vec{j}_{s_\alpha}(\vec{r}_s) \equiv$ natural mode (appropriately normalized)

$s_\alpha \equiv$ natural frequency

$\eta_\alpha^{(1)} \equiv$ coupling coefficient (appropriately normalized)

$\eta_\alpha^{(max)} \equiv$ normalization constant (for coupling coefficient)

$\vec{1}_1 \equiv$ direction of incidence

$\vec{1}_p \equiv$ polarization vector

$\tilde{f}_p(s) \equiv$ Laplace transform of incident waveform $f_p(t)$

$E_o \equiv$ scaling constant for incident wave (in V/m)

$\vec{r}_s \equiv$ coordinate on the surface S of the object

Here we have assumed an incident plane wave as the excitation. However, other forms of excitation can also be used.

Let the object have a surface S. On S let us suppose that we make measurements at N positions $\vec{r}_{s_n}$ for n = 1,2,...,N. At each of these positions measure the component of the surface current density in some particular direction $\vec{1}_n$ parallel at that $\vec{r}_{s_n}$. Furthermore let us postulate M different incident waves characterized by directions of incidence $\vec{1}_{1_m}$ and polarization $\vec{1}_{p_m}$ for $m = 1,2,\ldots,M$. The incident waveforms could also be different if convenient or if dictated by experimental conditions. The individual waveform measurements are then scalars given by

$$\vec{1}_n \cdot \vec{J}_s(\vec{r}_{s_n},t) =$$

$$E_0 \sum_\alpha \tilde{f}_{p_m}(s_\alpha)\, \eta_\alpha^{(\max)} \eta_\alpha(\vec{1}_{1_m},\vec{1}_{p_m})\, \vec{1}_n \cdot \vec{J}_{s_\alpha}(\vec{r}_{s_n})\, e^{s_\alpha t}\, u(t - t_0)$$

$$+ \text{ other SEM terms } + \text{ noise} \tag{2.2}$$

In general the time is also sampled, say for t_ℓ with $\ell = 1,2,\ldots,L$. Here we have written the formulas in time domain, but they could equally well be written in frequency domain, say with L samples ω_ℓ on the $j\omega$ axis of the s plane, corresponding to frequency-domain measurements.

There are various ways one might try to determine the SEM pole parameters from experimental data. First, one might determine some set of s_α from some combination (one or more) of the experimental waveforms (or frequency spectra). Second, fixing the incident wave (including direction of incidence and polarization), one might look at the variation of the appropriate pole residues over the object and use this to determine the corresponding natural modes (appropriately normalized). Third, fixing the measurement location and orientation, one might look at the variation of the pole residues with respect to the direction of incidence and polarization, and use this to determine the corresponding coupling coefficients (again appropriately normalized) and the normalization constants $\eta_\alpha^{(\max)}$.

An alternate approach might attempt to utilize the redundancy in a larger set of response waveforms (or frequency spectra). In this approach one fits (2.2) to the entire set of data with a common set of SEM pole parameters. Consider as discussed previously that there are N position/orientation combinations on the body and M incidence/polarization combinations for the incidence wave, and that waveforms for all NxM combinations of the above are obtained. With L samples of the waveform (or frequency spectrum) then NML pieces of information (real or complex) are available.

Assume that there are say K independent (not counting complex conjugate pairs) natural frequencies, together with the corresponding natural modes and coupling coefficients, that are significant and of interest in the data. After normalization there are N-1 independent position/orientation samples for the natural modes, M-1 independent incidence/polarization samples for the coupling coefficients, and 1 corresponding normalization constant (for each mode and corresponding coupling coefficient). Assuming no mode degeneracy (or allowing for such degeneracy by counting the relevant natural frequency the appropriate number of times) then we have (N + M - 1)K pieces of SEM information we are trying to recover.

Comparing the NML pieces of available information to the desired (N + M - 1)K pieces of SEM information, one has the possibility of improving the accuracy of the SEM information, particularly as N and M both become large in some sense. Here the basic idea is to enforce not only the commonality of the natural frequencies throughout the data, but to enforce the factorization of each of the residues into the product of a natural mode and a coupling coefficient. Of course, there is still the noise to be dealt with, as well as other SEM terms which may contribute (such as an entire function, particularly at early times).

III. Modal Filtering According to Symmetry

In order to "identify" the various natural modes one could try to separate them in various ways such that the experiment allowed only some of them to appear in the data. This would in effect increase the signal-to-noise ratio as far as identifying the modes in the remaining set is concerned. One can think of this as partitioning the natural modes into various subsets for separate identification. This is a kind of spatial filtering (instead of filtering according to frequencies) which we can call modal filtering.

One method for such partition involves the use of the symmetries of the object of concern. There are many kinds of such symmetry. Let us illustrate the concept here in the case of an object with a symmetry plane. This applies to a typical aircraft as far as its external interaction (or scattering) properties are concerned. As illustrated in figure 3.1 a typical aircraft has a vertical plane of symmetry P (vertical with respect to the local aircraft coordinates if it is in flight).

One can separate the electromagnetic fields around an object with a symmetry plane into two parts designated symmetric and antisymmetric. The theory of this is discussed in detail in [7]. This is applied to the design of EM sensor platforms in [1,3], including the use of an aircraft as a sensor platform [4].

Briefly summarizing we have

$$
\begin{aligned}
\overleftrightarrow{R} &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \equiv \text{reflection dyad} \\
&= \overleftrightarrow{R}^{-1} \\
\vec{r} &\equiv x\vec{1}_x + y\vec{1}_y + z\vec{1}_z \equiv \text{position or coordinates} \\
\vec{r}_m &\equiv \overleftrightarrow{R} \cdot \vec{r} \equiv \text{mirror position or mirror coordinates} \\
&= x\vec{1}_x + y\vec{1}_y - z\vec{1}_z
\end{aligned}
\tag{3.1}
$$

Our object with a symmetry plane (or equivalently with reflection symmetry) has the property that every position $\vec{r}$ on S is in a one-to-one correspondence

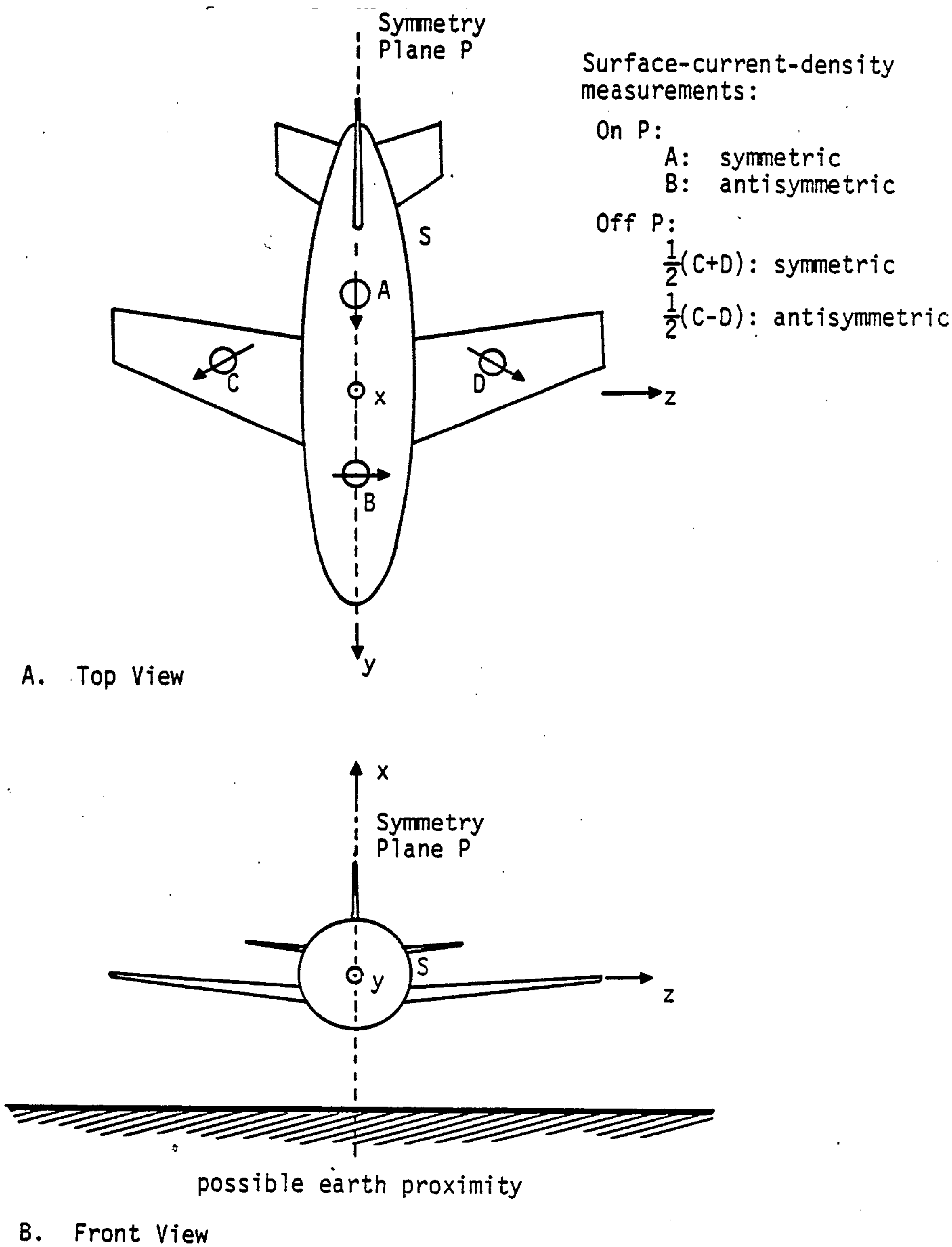

Fig. 3.1. Typical Aircraft with Symmetry Plane

with a mirror position $\vec{r}_m$ also on S. This concept can be extended to conductivities, permeabilities, and permittivities in scalar or dyadic form to give a more general definition [7]. For the present we are only concerned with the exterior of a perfectly conducting surface S which is symmetric in the sense above with respect to a plane P (the z = 0 plane).

For a given electromagnetic field distribution around our object we can find the symmetric parts (subscript sy) by

$$\begin{aligned}
\vec{E}_{sy}(\vec{r},t) &= \frac{1}{2}\{\vec{E}(\vec{r},t) + \overleftrightarrow{R} \cdot \vec{E}(\vec{r}_m,t)\} \\
\vec{H}_{sy}(\vec{r},t) &= \frac{1}{2}\{\vec{H}(\vec{r},t) - \overleftrightarrow{R} \cdot \vec{H}(\vec{r}_m,t)\} \\
\vec{J}_{s_{sy}}(\vec{r},t) &= \frac{1}{2}\{\vec{J}_s(\vec{r},t) + \overleftrightarrow{R} \cdot \vec{J}_s(\vec{r}_m,t)\} \\
\rho_{s_{sy}}(\vec{r},t) &= \frac{1}{2}\{\rho_s(\vec{r},t) + \rho_s(\vec{r}_m,t)\} \\
k_{sy}(\vec{r},t) &= \frac{1}{2}\{k(\vec{r},t) - k(\vec{r}_m,t)\}
\end{aligned} \tag{3.2}$$

where k is the equivalent magnetic charge density discussed in [5]. Similarly we have the antisymmetric parts (subscript as) as

$$\begin{aligned}
\vec{E}_{as}(\vec{r},t) &= \frac{1}{2}\{\vec{E}(\vec{r},t) - \overleftrightarrow{R} \cdot \vec{E}(\vec{r}_m,t)\} \\
\vec{H}_{as}(\vec{r},t) &= \frac{1}{2}\{\vec{H}(\vec{r},t) + \overleftrightarrow{R} \cdot \vec{H}(\vec{r}_m,t)\} \\
\vec{J}_{s_{as}}(\vec{r},t) &= \frac{1}{2}\{\vec{J}_s(\vec{r},t) - \overleftrightarrow{R} \cdot \vec{J}_s(\vec{r}_m,t)\} \\
\rho_{s_{as}}(\vec{r},t) &= \frac{1}{2}\{\rho_s(\vec{r},t) - \rho_s(\vec{r}_m,t)\} \\
k_{as}(\vec{r},t) &= \frac{1}{2}\{k(\vec{r},t) + k(\vec{r}_m,t)\}
\end{aligned} \tag{3.3}$$

The above can be interpreted in terms of mirror quantities as

$$\vec{E}_m(\vec{r},t) = \overleftrightarrow{R} \cdot \vec{E}(\vec{r}_m,t) \qquad \text{(electric field)}$$

$$\vec{H}_m(\vec{r},t) = -\overleftrightarrow{R} \cdot \vec{H}(\vec{r}_m,t) \qquad \text{(magnetic field)}$$

$$\vec{J}_{s_m}(\vec{r},t) = \overleftrightarrow{R} \cdot \vec{J}_s(\vec{r}_m,t) \qquad \text{(surface current density on S)} \tag{3.4}$$

$$\rho_{s_m}(\vec{r},t) = \rho_s(\vec{r}_m,t) \qquad \text{(surface charge density on S)}$$

$$k_m(\vec{r},t) = -k(\vec{r}_m,t) \qquad \text{(equivalent magnetic charge density on S)}$$

Note that some mirror quantities are defined with a plus sign; these correspond to electric quantities. Other mirror quantities are defined with a minus sign; these correspond to magnetic quantities. In effect electric and magnetic quantities reflect with opposite signs. The symmetric and antisymmetric parts are merely 1/2 the sum or difference of a quantity with its mirror quantity. This concept can be extended to all electromagnetic quantities including various kinds of potentials.

Now let us apply these concepts to natural modes. From (2.1) we have an SEM representation of the surface current density. Suppose that the incident field is purely symmetric (as in (3.2)). Then we would expect the scattered field to be purely symmetric, since that is what is required to match the boundary conditions on a scatterer with a symmetry plane. Then only natural modes that were symmetric would be excited. Similarly if the incident field were antisymmetric, only antisymmetric natural modes would be excited.

Let us then divide the surface-current-density natural modes into two kinds as

$$\vec{j}_{s_{sy,\alpha'}}(\vec{r}) = \overleftrightarrow{R} \cdot \vec{j}_{s_{sy,\alpha'}}(\vec{r}_m) \qquad \text{(symmetric)}$$

$$\vec{j}_{s_{as,\alpha'}}(\vec{r}) = -\overleftrightarrow{R} \cdot \vec{j}_{s_{as,\alpha'}}(\vec{r}_m) \qquad \text{(antisymmetric)} \tag{3.5}$$

$$\alpha = \left(\begin{matrix} sy \\ as \end{matrix}, \alpha'\right) = \text{natural mode index set}$$

This result can also be deduced from an integral equation for the scattering. Let us write an E-field integral equation over S as

$$\left\langle \tilde{\vec{Z}}(\vec{r}_s,\vec{r}_s\,';s)\ ;\ \tilde{\vec{J}}_s(\vec{r}_s\,',s)\right\rangle = \vec{1}_t(\vec{r}_s)\cdot\vec{E}^{(inc)}(\vec{r}_s,s)$$

$$\vec{r}_s,\vec{r}_s\,'\in S \tag{3.6}$$

$$\vec{1}_t = \vec{1} - \vec{1}_S\vec{1}_S = \text{transverse dyad on } S$$

$$\vec{1} - \vec{1}_x\vec{1}_x + \vec{1}_y\vec{1}_y + \vec{1}_z\vec{1}_z = \text{identity dyad}$$

$$\vec{1}_S = \text{unit normal (outward) on } S$$

with the symmetric product notation denoting integration over the common coordinates $\vec{r}_s\,'$. We need not concern ourselves here with details of the kernel. Now the natural modes for surface current density (and natural frequencies) are defined by

$$\left\langle \tilde{\vec{Z}}(\vec{r}_s,\vec{r}_s\,';s_\alpha)\ ;\ \vec{J}_{s_\alpha}(\vec{r}_s\,')\right\rangle = \vec{0} \tag{3.7}$$

It is shown in [7] that, under the conditions of a symmetry plane such as we have here, any solution of Maxwell's equations admits a mirror solution. So corresponding to a natural mode there must be a mirror natural mode (with the same natural frequency) as

$$\vec{J}_{s_{m_\alpha}}(\vec{r}) = \vec{R}\cdot\vec{J}_{s_\alpha}(\vec{r}_m) \tag{3.8}$$

which also satisfies (3.7). Then linear combinations of $\vec{J}_{s_\alpha}$ with its mirror must satisfy (3.7). In particular the combinations that give symmetric and antisymmetric as in (3.2) and (3.3) and exhibited in (3.5) must satisfy (3.7). Hence all natural modes can be constructed in this way.

As to whether a mode can possibly be neither purely symmetric nor purely antisymmetric, one can construct such cases involving modal degeneracy. In effect if, for example, a symmetric and an antisymmetric natural mode were to have the same natural frequency, then a linear combination of the two is also a natural mode. One can find such cases if even higher order symmetry is present, as in the cases of bodies of revolution. However, this allows one to divide the natural modes into symmetric and antisymmetric modes with respect to the chosen symmetry plane.

Now the coupling coefficient can be expressed as [21] in class 1 form

$$\tilde{\eta}_\alpha(\vec{1}_1,\vec{1}_p) = \frac{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s) \; ; \; \tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) \right\rangle}{\left\langle \vec{j}_{s_\alpha}(\vec{r}_s) \; ; \; \frac{d}{ds} \tilde{\vec{Z}}\,(\vec{r}_s,\vec{r}\,'_s;s)\Big|_{s=s_\alpha} \; ; \; \vec{j}_{s_\alpha}(\vec{r}\,'_s) \right\rangle} \tag{3.9}$$

Looking at the numerator of this expression, let us invoke the symmetry of the object surface S which is the domain of integration. If $\vec{j}_{s_\alpha}$ is symmetric and $\tilde{\vec{E}}^{(inc)}$ is antisymmetric this integral will be zero, and similarly if $\vec{j}_{s_\alpha}$ is antisymmetric and $\tilde{\vec{E}}^{(inc)}$ is symmetric. This reflects the observation that a symmetric incident field excites only symmetric natural modes, and an anti-symmetric incident field excites only antisymmetric natural modes.

The foregoing discussion indicates one way to separate the natural modes into two sets in an experiment. Let the incident field be first symmetric. Only symmetric natural modes and their corresponding natural frequencies will appear in the data. In measuring the natural frequencies and modes such an experiment reduces the number of significant natural frequencies and modes in the data before us, and thereby should increase the effective signal-to-noise ratio for obtaining these symmetric-mode parameters.

As a separate experiment let the incident field be antisymmetric. The corresponding natural frequencies and modes will be antisymmetric parameters. Again certain parameters (antisymmetric) will have been enhanced by suppressing others (symmetric).

The reader can consult [7] to give examples of excitation (incident fields) which exhibit the above. It is quite possible to design experiments which (within experimental error) do so separate the symmetric and anti-symmetric natural modes in the response.

An alternate approach allows the incident field to be arbitrary as long as the scattering is characterized by the requisite symmetry plane P. As discussed in [7] one can incorporate symmetry into the measurement scheme to separately measure the symmetric and antisymmetric parts of the response.

Consider first the case of measurements on P. This is discussed in some detail in [4]. Referring to fig. 3.1A note measurement locations and orientations on P designated A and B. Assuming the measurement is that of surface current density, then at A a measurement parallel to P (and S) is sensitive only to symmetric natural modes. Conversely at B a measurement perpendicular to P (but parallel to S) is sensitive only to antisymmetric natural modes.

Next consider measurements off P. As illustrated in fig. 3.1A consider a pair of measurements at C and D with vector orientation parallel to S and each a mirror of the other (in the electric sense), i.e., define unit vectors at these two locations parallel to S such that

$$\vec{1}_C = \overset{\leftrightarrow}{R} \cdot \vec{1}_D \quad , \quad \vec{1}_D = \overset{\leftrightarrow}{R} \cdot \vec{1}_C \tag{3.10}$$

These unit vectors denote the sense in which the surface current density is to be measured at each of these locations.

As discussed in [7] the signals from the sensors can be added and subtracted as in (3.2) and (3.3) to give in scalar form

$$\begin{aligned} \vec{1}_C \cdot \vec{J}_{s_{sy}}(\vec{r}_C,t) &= \frac{1}{2}\{\vec{1}_C \cdot \vec{J}_s(\vec{r}_C,t) + \vec{1}_D \cdot \vec{J}_s(\vec{r}_D,t)\} = \vec{1}_D \cdot \vec{J}_{s_{sy}}(\vec{r}_D,t) \\ \vec{1}_D \cdot \vec{J}_{s_{as}}(\vec{r}_C,t) &= \frac{1}{2}\{\vec{1}_C \cdot \vec{J}_s(\vec{r}_C,t) - \vec{1}_D \cdot \vec{J}_s(\vec{r}_D,t)\} = -\vec{1}_D \cdot \vec{J}_{s_{as}}(\vec{r}_D,t) \end{aligned} \tag{3.11}$$

In this concept the sum responds to the symmetric part and the difference responds to the antisymmetric part.

Heretofore we have discussed the surface-current-density natural modes. There are other (related) parameters on S that one can measure. One scalar measure is the surface charge density given by

$$\nabla_s \cdot \vec{J}_s(\vec{r},t) = -\frac{\partial}{\partial t}\rho_s(\vec{r},t) \tag{3.12}$$

This can be used to define corresponding natural modes via

$$\rho_{s_\alpha}(\vec{r}) = \ell\, \nabla_s \cdot \vec{j}_{s_\alpha}(\vec{r}) \tag{3.13}$$

where ℓ is some characteristic dimension of the object of interest. In turn the surface-charge-density modes are also symmetric and antisymmetric with properties as in (3.2) and (3.3). The normalization in (3.13) is somewhat arbitrary, but preserves dimensions.

In addition, there is the equivalent magnetic charge density on S discussed in [5]. This second scalar measure on S is given by

$$k(\vec{r},t) = \nabla_s \cdot \vec{B}_t^{(t)}(\vec{r},t) = \mu_0 \vec{1}_s \cdot [\nabla_s \times \vec{J}_s(\vec{r},t)] \tag{3.14}$$

where the subscript t indicates the components of the magnetic field parallel to S (on the exterior where applicable). Again natural modes can be defined for the equivalent magnetic charge density as

$$k_\alpha(\vec{r}) = \ell \, \vec{1}_s \cdot [\nabla_s \times \vec{J}_{s_\alpha}(\vec{r})] \tag{3.15}$$

These modes are also symmetric and antisymmetric (in the magnetic sense) as in (3.2) and (3.3). Again the normalization in (3.15) is somewhat arbitrary, but preserves dimensions.

IV. Modal Filtering According to E and H Modes

Another property of the natural modes which we can use to aid in separating them for separate identification in the data is their curl/ divergence property. As in (3.12) and (3.13) one can relate the surface charge density to the surface divergence of the surface current density and define surface-charge-density natural modes. Correspondingly in (3.14) and (3.15) one can relate the equivalent magnetic charge density to the normal component of the surface curl of the surface current density and define equivalent-magnetic-charge-density natural modes.

An interesting question is whether each natural mode has zero normal surface curl or zero surface divergence (but not both), and whether some natural modes have zero normal surface curl while some others have zero surface divergence. Let us denote modes symbolically with

$$\begin{aligned} E &\Rightarrow \text{zero normal surface curl } (k \equiv 0), \\ &\quad\ \text{non-zero surface divergence } (\rho_s \not\equiv 0) \\ H &\Rightarrow \text{zero surface divergence } (\rho_s \equiv 0), \\ &\quad\ \text{non-zero normal surface curl } (k \not\equiv 0) \end{aligned} \tag{4.1}$$

So the question is whether some or all of the natural modes can be classified this way. If they can, then separate measurements of ρ_s and k can filter the modes by separating them and their corresponding natural frequencies in the data according to the above property.

Let us generalize this question of E and H modes to the case of eigenmodes. As discussed in [11,13] one can define eigenmodes of the scattering integral-equation operators. Using for convenience the symmetric E-field or impedance integral equation (3.6) we have

$$\left\langle \tilde{\tilde{Z}}(\vec{r}_s, \vec{r}_s\,';s) \,;\, \tilde{\vec{J}}_{s_\beta}(\vec{r}_s\,',s) \right\rangle = Z_\beta(s)\tilde{\vec{J}}_{s_\beta}(\vec{r}_s,s) \tag{4.2}$$

Analogous to (2.1) an eigenmode expansion method (EEM) representation of the surface current density on S can be written in complex frequency domain as

$$\tilde{\vec{J}}_s(\vec{r}_s,s) = \sum_\beta \frac{1}{\tilde{Z}_\beta(s)} \frac{\left\langle \tilde{\vec{J}}_{s_\beta}(\vec{r}_s\,',s) \; ; \; \tilde{\vec{E}}^{(inc)}(\vec{r}_s\,',s) \right\rangle}{\left\langle \tilde{\vec{J}}_{s_\beta}(\vec{r}_s\,',s) \; ; \; \tilde{\vec{J}}_{s_\beta}(\vec{r}_s\,',s) \right\rangle} \tilde{\vec{J}}_{s_\beta}(\vec{r}_s,s) \tag{4.3}$$

If desired one can set the denominator symmetric product to 1 giving orthonormalized eigenmodes. Alternatively the above form can be used to allow us to normalize the eigenmodes to some peak or typical value, as is often done with natural modes.

Consider the E-mode and H-mode properties of such eigenmodes. As a canonical problem consider the perfectly conducting sphere of radius a. In this case the eigenmodes are well known [8,10]. They are the vector spherical harmonics $\vec{Q}$ and $\vec{R}$ given by [8]

$$Y_{n,m,\sigma}(\theta,\phi) \equiv P_n^{(m)}(\cos(\theta)) \left\{ \begin{matrix} \cos(m\phi) \\ \sin(m\phi) \end{matrix} \right\}$$

$$m = 1,2,\ldots,n \quad \text{and } \sigma = 1 \text{ (upper)}, 2 \text{ (lower)}$$

$$\begin{aligned}
\vec{P}_{n,m,\sigma}(\theta,\phi) &\equiv Y_{n,m,\sigma}(\theta,\phi)\, \vec{1}_r \\
\vec{Q}_{n,m,\sigma}(\theta,\phi) &\equiv a\nabla_s\, Y_{n,m,\sigma}(\theta,\phi) \\
&= \vec{1}_\theta \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi) + \vec{1}_\phi \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} Y_{n,m,\sigma}(\theta,\phi) \\
\vec{R}_{n,m,\sigma}(\theta,\phi) &\equiv a\nabla_s \times [\vec{1}_r Y_{n,m,\sigma}(\theta,\phi)] \\
&= \vec{1}_\theta \frac{1}{\sin(\theta)} \frac{\partial}{\partial\phi} Y_{n,m,\sigma}(\theta,\phi) - \vec{1}_\phi \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi)
\end{aligned} \tag{4.4}$$

These are related by

$$\begin{aligned}
\vec{Q}_{n,m,\sigma}(\theta,\phi) &= \vec{1}_r \times \vec{R}_{n,m,\sigma}(\theta,\phi) \\
\vec{R}_{n,m,\sigma} &= -\vec{1}_r \times \vec{Q}_{n,m,\sigma}(\theta,\phi)
\end{aligned} \tag{4.5}$$

Summarizing some results for surface curl and divergence [5,20] we have

$$\nabla_S \times \vec{1}_S = \vec{0} \quad , \quad \nabla_S \times [g\vec{1}_S] = [\nabla_S g] \times \vec{1}_S$$

$$\nabla_S \cdot [\vec{F} \times \vec{1}_S] = \vec{1}_S \cdot [\nabla_S \times \vec{F}]$$

$$\nabla_S \cdot [\nabla_S \times \vec{F}] = 0 \quad \text{if } \vec{F} \text{ is normal to } S$$

$$\vec{1}_S \cdot [\nabla_S \times [\nabla_S g]] = 0 \tag{4.6}$$

$$\vec{1}_S \cdot [\nabla_S \times [\vec{1}_S \cdot [\nabla_S g]]] = \nabla_S^2 g$$

$$\nabla_S \cdot [\nabla_S g] = \nabla_S^2 g$$

There are other formulas in [20] involving surface operators. Note that the above nowhere explicitly include the principal radii of curvature of S.

Applying these formulas to the eigenmodes of a sphere we have

$$\vec{1}_S \cdot [\nabla_S \times \tilde{\vec{J}}^{(E)}_{S_{\beta_1}}(\vec{r}_S,s)] = \vec{1}_r \cdot [\nabla_S \times \vec{Q}_{n,m,\sigma}(\theta,\phi)]$$

$$= a\vec{1}_r \cdot [\nabla_S \times [\nabla_S Y_{n,m,\sigma}(\theta,\phi)]] = 0 \tag{4.7}$$

$$\nabla_S \cdot \tilde{\vec{J}}^{(H)}_{S_{\beta_2}}(\vec{r}_S,s) = -\nabla_S \cdot \vec{R}_{n,m,\sigma}(\theta,\phi)$$

$$= -a\nabla_S \cdot [\nabla_S \times [\vec{1}_r Y_{n,m,\sigma}]] = 0$$

with a minus sign in the second equation for later convenience. This allows us to identify for the sphere

$$\tilde{\vec{J}}^{(E)}_{S_{\beta_1}}(\vec{r}_S,s) \equiv \vec{Q}_{n,m,\sigma}(\theta,\phi) \equiv \text{E (or TM) eigenmodes for surface current density}$$

$$\tilde{\vec{J}}^{(H)}_{S_{\beta_2}}(\vec{r}_S,s) \equiv -\vec{R}_{n,m,\sigma}(\theta,\phi) \equiv \text{H (or TE) for eigenmodes for surface current density} \tag{4.8}$$

Note that in the case of the sphere these eigenmodes can be taken as purely real valued. They are also frequency independent and are hence in this case also the natural modes.

Now compute the surface-charge-density and equivalent-magnetic-charge-density eigenmodes for the perfectly conducting sphere (with ℓ in (3.13) and (3.15) chosen as a) as

$$\begin{aligned}
\tilde{\rho}_{s_{\beta_1}}(\vec{r}_s,s) &= a\nabla_s \cdot \tilde{\vec{J}}^{(E)}_{s_{\beta_1}}(\vec{r}_s,s) = a\nabla_s \cdot \vec{Q}_{n,m,\sigma}(\theta,\phi) \\
&= a^2\nabla_s^2 Y_{n,m,\sigma}(\theta,\phi) \\
&= \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left[\sin(\theta)\frac{\partial}{\partial\theta}Y_{n,m,\sigma}(\theta,\phi)\right] + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}Y_{n,m,\sigma}(\theta,\phi)
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\tilde{K}_{s_{\beta_2}}(\vec{r}_s,s) &= a\vec{1}_r \cdot [\nabla_s \times \tilde{\vec{J}}^{(H)}_{s_{\beta_2}}(\vec{r}_s,s)] = -a\vec{1}_r \cdot [\nabla_s \times \vec{R}_{n,m,\sigma}(\theta,\phi)] \\
&= -a^2\vec{1}_r \cdot [\nabla_s \times [\nabla_s \times [\vec{1}_r Y_{n,m,\sigma}(\theta,\phi)]]] \\
&= -a^2\vec{1}_r \cdot [\nabla_s \times [[\nabla_s Y_{n,m,\sigma}(\theta,\phi)] \times \vec{1}_r]] \\
&= a^2\nabla_s^2\, Y_{n,m,\sigma}(\theta,\phi)
\end{aligned}$$

Here we have chosen β_1 and β_2 as the eigenmode indices for these two kinds of modes above. Note that β_1 and β_2 have to be related to the index set $\{n,m,\sigma\}$ of the spherical harmonics as well as the E and H mode indices. What we have here is the result that, at least for the perfectly conducting sphere, it is possible to define an eigenmode index β such that

$$\tilde{\rho}_{s_\beta}(\vec{r}_s,s) = \tilde{K}_{s_\beta}(\vec{r}_s,s) \tag{4.10}$$

indicating that the eigenmodes can be paired with each E mode corresponding to an H mode and conversely. This result can also be seen in terms of the surface-current-density eigenmodes in (4.5) which can be written as

$$\tilde{\vec{J}}^{(H)}_{s_\beta}(\vec{r}_s,s) = \vec{1}_r \times \tilde{\vec{J}}^{(E)}_{s_\beta}(\vec{r}_s,s) \tag{4.11}$$

$$\tilde{\vec{J}}_{s_\beta}^{(E)}(\vec{r}_s,s) = -\vec{1}_r \times \tilde{\vec{J}}_{s_\beta}^{(H)}(\vec{r}_s,s)$$

These results for the perfectly conducting sphere in (4.10) and (4.11) show that one need only have determined one kind of eigenmode (E or H) and the other kind is also immediately determined.

Consider now the surface Laplacian as in (4.9). In spherical coordinates on a surface $r = a$ we have

$$a^2\nabla_s^2 Y_{n,m,\sigma}(\theta,\phi) = \frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta} [\sin(\theta) \frac{\partial}{\partial\theta} Y_{n,m,\sigma}(\theta,\phi)] + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial\phi^2} Y_{n,m,\sigma}(\theta,\phi) \tag{4.12}$$

Now we have from (4.4)

$$\frac{\partial^2}{\partial\phi^2} Y_{n,m,\sigma}(\theta,\phi) = -m^2 Y_{n,m,\sigma}(\theta,\phi) \tag{4.13}$$

giving

$$a^2\nabla_s^2 Y_{n,m,\sigma}(\theta,\phi) = \{\frac{1}{\sin(\theta)} \frac{d}{d\theta}[\sin(\theta) \frac{d}{d\theta} P_n^{(m)}(\cos(\theta))] - \frac{m^2}{\sin^2(\theta)} P_n^{(m)}(\cos(\theta))\} \begin{Bmatrix}\cos(m\phi)\\ \sin(m\phi)\end{Bmatrix} \tag{4.14}$$

Now the Legendre functions satisfy the differential equation [19 (Chap. 7)]

$$\frac{1}{\sin(\theta)} \frac{d}{d\theta} [\sin(\theta) \frac{d}{d\theta} P_n^{(m)}(\cos(\theta))] + [n(n + 1) - \frac{m^2}{\sin^2(\theta)}] P_n^{(m)}(\cos(\theta)) = 0 \tag{4.15}$$

Substituting this into (4.14) we have

$$a^2\nabla_s^2 Y_{n,m,\sigma}(\theta,\phi) = -n(n + 1) Y_{n,m,\sigma}(\theta,\phi) \tag{4.16}$$

Stated in another way

$$[\nabla_s^2 + \frac{n(n + 1)}{a^2}] Y_{n,m,\sigma}(\theta,\phi) = 0 \tag{4.17}$$

which means that the $Y_{n,m,\sigma}$ are the solutions of a scalar Helmholz equation on the surface of the sphere.

Let us then define some scalar potential function $\tilde{\Phi}_\beta$ which, at least in the case of the perfectly conducting sphere, can be used to characterize the eigenmodes for the surface current density, the surface charge density, and the equivalent magnetic charge density. These can be reconstructed as

$$
\begin{aligned}
\tilde{\vec{J}}^{(E)}_{s_\beta}(\vec{r}_s,s) &= \ell\nabla_s\tilde{\Phi}_\beta(\vec{r}_s,s) = -\vec{1}_S \times \tilde{\vec{J}}^{(H)}_{s_\beta}(\vec{r}_s,s) \\
\tilde{\vec{J}}^{(H)}_{s_\beta}(\vec{r}_s,s) &= -\ell\nabla_s \times [\vec{1}_S\tilde{\Phi}_\beta(\vec{r}_s,s)] = \vec{1}_S \times \tilde{\vec{J}}^{(E)}_{s_\beta}(\vec{r}_s,s) \\
\tilde{\rho}_{s_\beta}(\vec{r}_s,s) &= \ell^2\nabla_s^2\tilde{\Phi}_\beta(\vec{r}_s,s) \\
&= -(\text{function } \beta,\ \ell,\ \text{and } s)\ \tilde{\Phi}_\beta(\vec{r}_s,s) \\
\tilde{K}_\beta(\vec{r}_s,s) &= \ell^2\nabla_s^2\tilde{\Phi}_\beta(\vec{r}_s,s) \\
&= -(\text{function of } \beta,\ \ell,\ \text{and } s)\ \tilde{\Phi}_\beta(\vec{r}_s,s)
\end{aligned}
\tag{4.18}
$$

where $\tilde{\Phi}_\beta$ satisfies a Helmholz equation on S as

$$
[\nabla_s^2 + (\text{function } \beta,\ \ell,\ \text{and } s)]\ \tilde{\Phi}_\beta(\vec{r}_s,s) = 0 \tag{4.19}
$$

Note that we have included a possible dependence on s in the Helmholz equation since we expect the $\tilde{\vec{J}}^{(E,H)}_{s_\beta}$ and the $\tilde{\Phi}_\beta$ to depend somewhat on frequency in the general case.

At least for the sphere the eigenmodes have some important properties which could be very useful. An important question concerns the degree to which these properties apply to arbitrary perfectly conducting bodies of finite linear dimensions in free space. A very interesting paper [14] has considered the magnetic-field integral equation, giving a "pseudosymmetric" eigenmode expansion with a pairing of the eigenmodes like (4.11). Here we would like to conjecture that this pairing applies in general to the electric-field integral equation and that the modes can be considered as E and H modes as in (4.11) and (4.18).

This problem of separating the eigenmodes into E and H modes is related to the problem of separating vector fields into parts with zero divergence and zero curl. That this can be done is known as the Helmholz theorem. Explicit formulas have been developed for a three-dimensional current density field in [2]. An interesting paper [6] has discussed the application of this type of decomposition to surface curl and surface divergence such as we are considering. The two parts also are related as in (4.11) which is consistent with our conjecture.

It is known [11,13] that the eigenmodes and natural modes are related. Referring to the integral equation (4.2) we have

$$\left\langle \tilde{\tilde{Z}}(\vec{r}_s,\vec{r}_s\,';s_\alpha)\ ;\ \tilde{\vec{J}}_{s_\beta}(\vec{r}_s,s_\alpha)\right\rangle = \vec{0} \tag{4.20}$$

so that comparing to (3.7)

$$\begin{gathered} \tilde{Z}_\beta(s_\alpha) = 0 \\ \tilde{\vec{J}}_{s_\beta}(\vec{r}_s,s_\alpha) = \tilde{\vec{J}}_{s_\alpha}(\vec{r}_s) \text{ (times an arbitrary constant)} \end{gathered} \tag{4.21}$$

The eigenmodes are natural modes at certain natural frequencies s_α for which the corresponding eigenvalues (eigenimpedances) are zero. The natural mode index α can then be partitioned as

$$\alpha = (\beta,\beta') \tag{4.22}$$

where β' indicates the β'th zero of the βth eigenvalue. As indicated previously, β can also be partitioned according to E and H modes, and symmetric and antisymmetric modes (or partitioning according to higher order symmetries). In the present context one might set

$$\begin{gathered} \beta = \left(\begin{smallmatrix}E\\H\end{smallmatrix}, \begin{smallmatrix}sy\\as\end{smallmatrix}, n\right) \\ \alpha = \left(\begin{smallmatrix}E\\H\end{smallmatrix}, \begin{smallmatrix}sy\\as\end{smallmatrix}, n, n'\right) \end{gathered} \tag{4.23}$$

with n' denoting the n'th zero of the nth eigenvalue after separating out (E,H) and (sy,as).

Now one difference between the modes of an arbitrary perfectly conducting surface S and the perfectly conducting sphere concerns the real-valued nature of the modes. For the perfectly conducting sphere the eigenmodes for the surface current density are the vector spherical harmonics in (4.4). These are purely real-valued vector functions. However, in [10] some of the natural modes are computed for the prolate sphere, a limiting case of which is the sphere. For the prolate sphere the natural modes are not purely real but include an imaginary part. Fortunately, the imaginary part is small compared to the peak magnitude of the mode, at least for the first few (lowest order axial) natural modes considered. Furthermore, it is known that in the limit of a thin wire the natural modes can also be represented as purely real [12,23]. Since the natural modes are special cases of the eigenmodes the above observations can also be applied to the eigenmodes. Thus the property of pure realness of the eigenmodes does not generalize from the sphere to more general shapes. Furthermore, since the natural frequencies occur in complex conjugate pairs and the corresponding natural modes are also complex conjugate to each other, then any eigenmode with not all purely real natural modes must also vary as a function of s.

Now as to measurements, one can measure the various natural modes. By measuring ρ_s and k as functions of time or frequency one can determine the natural frequencies and corresponding natural modes. To the extent that the natural modes separate (or even approximately separate) into E and H modes, such a separation can be used to improve the determination of the natural frequencies by reducing the number of significant natural frequencies in each measurement. If, as indicated in the case of the sphere, the $\tilde{\rho}_{s_\beta}$ and $\tilde{k}_\beta$ are the same, even though they have different natural frequencies in general, this should simplify matters somewhat.

V. Modal Filtering According to Modal Quasi-Orthogonality

An important property of the eigenmodes is their orthogonality, i.e.,

$$\left\langle \tilde{\vec{J}}_{s_{\beta_1}}(\vec{r}_s,s) \; ; \; \tilde{\vec{J}}_{s_{\beta_2}}(\vec{r}_s,s) \right\rangle \begin{cases} = 0 \text{ for } \beta_1 \neq \beta_2 \\ \neq 0 \text{ for } \beta_1 = \beta_2 \end{cases} \tag{5.1}$$

where we exclude possible peculiarities associated with modal degeneracy from our present discussion. Let us use this orthogonality property as a modal filter.

Expanding the surface current density in an eigenmode expansion as in (4.3), let us take the symmetric product of the surface current density with one of the eigenmodes giving

$$\left\langle \tilde{\vec{J}}_{s_{\beta}}(\vec{r}_s,s) \; ; \; \tilde{\vec{J}}_{s}(\vec{r}_s,s) \right\rangle = \frac{1}{\tilde{\mathcal{Z}}_{\beta}(s)} \left\langle \tilde{\vec{J}}_{s_{\beta}}(\vec{r}_s,s) \; ; \; \tilde{\vec{E}}^{(inc)}(\vec{r}_s,s) \right\rangle \tag{5.2}$$

This indicates that one might use an eigenmode (if one had such a thing, even approximately) to filter the data in a way which produced something which was proportional to $\tilde{\mathcal{Z}}_{\beta}^{-1}$, and which contained only those object poles which were the zeros of $\tilde{\mathcal{Z}}_{\beta}$, i.e., the $s_{\beta,\beta'}$. The poles associated with other eigenmodes would be suppressed, thereby increasing the effective signal-to-noise ratio in determining the particular subset of the poles, the $s_{\beta,\beta'}$.

Let us perform a gedankenexperiment. Suppose that as in section 2 we measure the surface current density at N positions on S designed $\vec{r}_{s_n}$ for n = 1,2,...,N. At each $\vec{r}_{s_n}$ there are two components of the surface current density. Designate one of the component directions as $\vec{1}_n$ and the other as $\vec{1}_{n+N}$ with

$$\begin{aligned} \vec{1}_n \cdot \vec{1}_{n+N} &= 0 \quad \text{for } n = 1,2,\ldots,N \\ \vec{r}_{s_{n+N}} &= \vec{r}_{s_n} \quad \text{for } n = 1,2,\ldots,N \end{aligned} \tag{5.3}$$

Then by letting n vary from 1 to 2N all patches are covered twice, one for each of the two orthogonal components of the surface current density.

The integral in (5.1) can then be approximated by a sum giving

$$\sum_{n=1}^{2N} [\vec{1}_n \cdot \tilde{\vec{J}}_{s_{\beta_1}}(\vec{r}_{s_n},s)][\vec{1}_n \cdot \tilde{\vec{J}}_{s_{\beta_2}}(\vec{r}_{s_n},s)]\, A_n$$

$$\simeq \left\langle \tilde{\vec{J}}_{s_{\beta_1}}(\vec{r}_s,s)\; ; \;\tilde{\vec{J}}_{s_{\beta_2}}(\vec{r}_s,s) \right\rangle \begin{cases} = 0 \text{ for } \beta_1 \neq \beta_2 \\ \neq 0 \text{ for } \beta_1 = \beta_2 \end{cases} \tag{5.4}$$

$S_n \equiv$ surface patch with "center" $\vec{r}_{s_n}$

$A_n \equiv$ area of S_n

Let us apply this approximate orthogonality to the measured data.

Similarly approximating the integrals in (5.2) we have

$$\sum_{n=1}^{2N} [\vec{1}_n \cdot \tilde{\vec{J}}_{s_\beta}(\vec{r}_{s_n},s)][\vec{1}_n \cdot \tilde{\vec{J}}_s(\vec{r}_{s_n},s)]\, A_n$$

$$\simeq \vec{Z}_\beta^{-1}(s) \sum_{n=1}^{2N} [\vec{1}_n \cdot \tilde{\vec{J}}_{s_\beta}(\vec{r}_{s_n},s)][\vec{1}_n \cdot \tilde{\vec{E}}^{(inc)}(\vec{r}_{s_n},s)]\, A_n$$

$$\equiv \tilde{F}_\beta(s) \tag{5.5}$$

This $\vec{F}_\beta$ has the property then that

$$\vec{F}_\beta^{-1}(s_{\beta,\beta'}) \simeq 0 \tag{5.6}$$

so if we can construct the $\tilde{F}_\beta$ from the data we have a means of determining the $s_{\beta,\beta'}$.

Well now we need one or more eigenmodes to make this work. Eventually there may be various ways that are known to be able to do this. Basically we need a set of coefficients approximating $\vec{1}_n \cdot \tilde{\vec{J}}_\beta(\vec{r}_{s_n})$ to weight our measurements of the surface current density so as to bring out the poles $s_{\beta,\beta'}$ associated with this eigenmode. One approach would be to select these coefficients so as to minimize the residues of poles associated with some $s_{\beta_1,\beta'}$ or set of these for which $\beta_1 \neq \beta$.

Another approach is to take some estimate of the βth eigenmode obtained by other means. Say as in section 2 we have already determined some approximation of one or more natural modes. Then as in (4.21) note that natural modes are associated with eigenmodes. Let us assume that

$$\tilde{\vec{j}}_{s_\beta}(\vec{r}_s,s) \simeq \vec{j}_{s_{\beta,\beta'}}(\vec{r}_s) \tag{5.7}$$

i.e., that at least some eigenmodes of interest can be approximated by natural modes, the natural frequencies of which are zeros of the corresponding eigenvalues.

Now there is some evidence for the above. The case of the perfectly conducting sphere in (4.4) has the eigenmodes independent of frequency. In this case the eigenmodes are all natural modes as well. Furthermore, note that the eigenmodes in this case can be represented as purely real-valued vector functions tangential to S. Another case of interest is the thin wire in which case the natural modes are also purely real functions (as the wire thickness tends to zero) [12,23]. A numerical study of the thin wire shows the almost realness of the principal natural modes [9]. A very interesting study considers the prolate spheriod, exhibiting the natural modes for various ratios of minor to major radii [10]. Varying this ratio from the case of the sphere to that of the "thin wire" shows that the lowest order natural modes are almost real over the range of this parameter.

Let us then take some natural mode $\vec{j}_{s_{\beta,\beta'}}$ determined by some other means (e.g., section 2). Note that we also have immediately $\vec{j}_{s_{\beta,-\beta'}}$ since the natural modes occur in complex conjugate pairs, i.e.,

$$\vec{j}_{s_{\beta,-\beta'}}(\vec{r}_s) = \vec{j}^{\,*}_{s_{\beta,\beta'}}(\vec{r}_s)$$

$$* \equiv \text{complex conjugate} \tag{5.8}$$

$$s_{\beta,-\beta'} = s^{*}_{\beta,\beta'}$$

Here $-\beta'$ is used in the indexing system to give the natural frequencies in the third quadrant of the s plane, positive values of β' applying to the second quadrant. From (5.8) one may also define an averge natural mode as

$$\vec{J}^{(avg)}_{s_{\beta,\beta'}}(\vec{r}_s) = \frac{1}{2}\,[\vec{J}_{s_{\beta,\beta'}}(\vec{r}_s) + \vec{J}_{s_{\beta,-\beta'}}(\vec{r}_s)]$$

$$= \mathrm{Re}[\vec{J}_{s_{\beta,\beta'}}(\vec{r}_s)] \tag{5.9}$$

This is also an approximation to the βth eigenmode. In the case of a real eigenmode the average natural mode is the same as the actual eigenmode, at least for the particular natural frequency under consideration.

Heretofore the problem in this section has been formulated in frequency domain (complex). However, to the extent that eigenmodes can be characterized as frequency independent the problem can be characterized in time domain. In terms of natural modes, frequency-independent eigenmodes give the orthogonality condition

$$\left\langle \vec{J}_{s_{\beta_1,\beta'}}(\vec{r}_s)\ ;\ \vec{J}_{s_{\beta_2,\beta''}}(\vec{r}_s) \right\rangle \begin{cases} \simeq 0 \text{ for } \beta_1 \neq \beta_2 \\ \neq 0 \text{ for } \beta_1 = \beta_2 \end{cases} \tag{5.10}$$

Note that this is only approximate since some frequency variation of the eigenmodes, and hence some variation of the natural modes over β' (for fixed β) is in general expected.

Applying this result to the SEM representation in (2.1) we have

$$\left\langle \vec{J}_{s_{\beta,\beta'}}(\vec{r}_s)\ ;\ \vec{J}_s(\vec{r}_s,t) \right\rangle$$

$$\simeq E_0 \sum_{\beta''} \tilde{f}_p(s_{\beta,\beta'})\ n^{(max)}_{\beta,\beta''} n^{(1)}_{\beta,\beta''}(\vec{1}_1,\vec{1}_p) \left\langle \vec{J}_{s_{\beta,\beta'}}(\vec{r}_s)\ ;\ \vec{J}_{s_{\beta,\beta''}}(\vec{r}_s) \right\rangle$$

$$e^{s_{\beta,\beta''}t}\ u(t - t_0)$$

$$+ \text{ other SEM terms } + \text{ noise}$$

$$= \sum_{\beta''} R_{\beta,\beta',\beta''}\ e^{s_{\beta,\beta''}t}\ u(t - t_0)$$

$$+ \text{ other SEM terms } + \text{ noise} \tag{5.11}$$

Here the $R_{\beta,\beta'\beta''}$ are the residues of the unsuppressed poles. Applying (5.7) and (5.9) we can regard

$$\tilde{\vec{J}}_{s_\beta}(\vec{r}_s, s) \simeq \vec{J}_{s_\beta}(\vec{r}_s) \simeq \vec{J}^{(avg)}_{s_\beta}(\vec{r}_s) = \text{purely real natural mode} \tag{5.12}$$

giving

$$\left\langle \vec{J}^{(avg)}_{s_\beta}(\vec{r}_s) \; ; \; \vec{J}_s(\vec{r}_s, t) \right\rangle \simeq \sum_{\beta'} R_{\beta,\beta'} \, e^{s_{\beta,\beta'} t} \, u(t - t_0)$$
$$+ \text{ other SEM terms } + \text{ noise} \tag{5.13}$$

Now approximate the integral over S by a sum as before giving

$$\sum_{n=1}^{2N} [\vec{1}_n \cdot \vec{J}^{(avg)}_{s_\beta}(\vec{r}_{s_n})][\vec{1}_n \cdot \vec{J}_s(\vec{r}_{s_n}, t)] \, A_n \equiv G_\beta(t)$$
$$\simeq \sum_{\beta'} R_{\beta,\beta'} \, e^{s_{\beta,\beta'} t} \, u(t - t_0)$$
$$+ \text{ other SEM terms } + \text{ noise} \tag{5.14}$$

Here we have constructed one or more $G_\beta(t)$ which are dominated by the $s_{\beta,\beta'}$ natural frequencies. The other natural frequencies are hopefully significantly suppressed. Note that this is a time-domain waveform. Our gedanken-experiment in this case has taken 2N waveform measurements, preliminary estimates of some important natural modes, and combination of the waveform measurements by a set of weights which enhance the set of natural frequencies associated with the particular βth eigenmode of interest.

Note that while (5.14) is written in time domain a Laplace transform gives $\tilde{G}_\beta(s)$ and the exponentials becomes poles. Setting $s = j\omega$ then this technique for finding poles applies in frequency domain as well.

Another result of the development in this section is the implication of the quasi-orthogonality formulas (5.5) and (5.14). Not only do they have application for finding natural frequencies, they also group the natural frequencies according to the eigenvalues. Note that this grouping can be done experimentally by these formulas which imply processing the experimental data in certain ways. From (5.5) one can even in principle reconstruct the eigenvalues from the experimental data. Of course, the techniques discussed are only approximate for this partitioning of the $s_{\beta,\beta'}$ according to the $\tilde{Z}_\beta(s)$ since the natural modes only approximate the eigenmodes and their orthogonality property.

VI. Summary

Well, this paper has covered quite a lot of ground. Perhaps it will stimulate some readers to explore some of the points contained and to implement some versions of the experimental procedures discussed. There are various properties of the various modes which are known for special cases. Are they exactly or approximately the same for more general cases? Some evidence exists; more is needed.

The ideas explored here are grouped into four major categories. Section 2 discusses the implications of the factoring of the SEM pole terms and the experimental determination of these parameters. Section 3 considers the use of symmetry, in particular a symmetry plane, for separating the natural frequencies, modes, etc., into distinct sets present in distinct sets of experimental data. Section 4 discusses the use of surface-charge-density and equivalent-magnetic-charge-density measurements to separate the modes and associated natural frequencies into two distinct sets of data. Section 5 considers the application of eigenmode orthogonality in an approximate way to the natural modes to separate the natural frequencies into sets associated with each eigenvalue.

As a last point it should be emphasized that theory and experiment should not be thought of as totally separate. Each should shed light on the other. In particular, all possible a priori knowledge concerning the character of the electromagnetic response should be applied to the problem of the design of optimum experiments, including what is to be measured and how the data is to be analyzed. On the other hand, well designed experiments can give results which give insight into the theoretical description of the electromagnetic response.

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