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Stochastic Behavior Of Random Lay Cables

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ABSTRACT

The analysis of EMP effects depends on parameters which in effect may be random variables. One such example is the physical orientation of a cable within a structure. Further there are situations where a parameter may be deterministic, e.g., terminal impedance, but to investigate system behavior over a broad range of possible terminations one is forced to treat terminal impedance as a random variable. In this report we develop a model for a random lay cable with random termination and derive the stochastic properties of the associated electromagnetic matrices of interest.

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1. INTRODUCTION

Given a particular visualization of a cable and its terminations such parameters of interest as the scattering matrix can be calculated and the system's response to EMP determined for that particular visualization. However, there are other visualizations that might be encountered. For example, one may find pin #2 in the uppermost position of a connector rather than pin #1 as was assumed. Will the EMP response of this second visualization differ significantly from that of the first? This question can in principle be answered by a recalculation of all parameters for this second visualization. But the number of possible visualizations that need to be calculated can get rapidly out of hand even for relatively simple systems. This direct calculation approach fails even more rapidly when one extends his area of inquiry beyond a specific cable to a generic type of cable, e.g., aircraft communications cables in general.

The approach we will take in this effort is to treat the parameters of the cable and its terminations as random variables. We will develop a stochastic model for a random lay cable. Based on this model, the question: "Will the EMP response of this second visualization differ significantly from that of the first?" will be answered probabilistically.

2. OBJECTIVES OF THE RESEARCH EFFORT

The first goal of this effort is to develop a stochastic model that describes a random lay cable. Clearly there are as many random lay cable models as there are possible definitions for what constitutes a random lay. In this study we adopt the approach that the parameters of the cable elements are fixed and that it is their spatial orientation and terminal impedances that are random.

Having defined our model, our next goal is to develop the stochastic properties of the model. Finally, we derive the distribution theory for scattered current in a random lay cable.

3. PERMUTATION MATRICES

In constructing our model of the random lay cable we will make extensive use of the concept of a permutation matrix. It is therefore appropriate at this point to review the properties of such matrices.

A. ALGEBRAIC PROPERTIES

Property 1:

Let Π_{ij} be an identity matrix with its ith and jth rows interchanged. Then $\Pi_{ij}M$ is the matrix M with its ith and jth rows interchanged.

Property 2:

$M\Pi_{ij}$ is the matrix M with its ith and jth columns interchanged.

Property 3:

$\Pi_{ij}M\Pi_{ij}$ is the matrix M with its ith and jth rows and columns interchanged.

Property 4:

The Π_{ij}'s are symmetric, orthogonal matrices; $\Pi^{-1} = \Pi^T$.

Property 5:

Define a permutation matrix P:

$$P = \Pi_{ij} \cdot \Pi_{kl} \cdot \cdots \Pi_{yz} .$$

Then P is an orthogonal matrix; $P^{-1} = P^T$.

Property 6:

$P^T MP$ is the matrix M with a sequence of row and column interchanges.

Property 7:

P is of the form that each column (row) contains one 1. The rest of the elements in the column (row) are 0.

Property 8:

P is of full rank.

Property 9:

$$[P^T MP]^{-1} = P^T M^{-1} P$$

Property 10:

$$(P^T MP)(P^T NP) = P^T MNP$$

Property 11:

The effect of $P^T MP$ is to relocate diagonal elements of M onto the diagonal of $P^T MP$ and to relocate off diagonal elements of M onto off diagonal elements of $P^T MP$.

B. STOCHASTIC PROPERTIES

Stochastic Property 1:

Assuming that all permutations are equally likely, then the expected value of the permutation matrix P is

$$E[P] = \frac{1}{N} J ,$$

where J is an $N \times N$ matrix all of whose elements are 1 .

Stochastic Property 2:

The expected value of the matrix $P^T MP$ is

$$E[P^T MP] = \begin{cases} \overline{m}_d & \textit{on diagonal} \\ \overline{m}_o & \textit{off diagonal} \end{cases}$$

where $\overline{m}_d$ is the average of the diagonal element of M , $\overline{m}_o$ is the average of the off diagonal elements of M and M is an arbitrary square matrix.

Stochastic Property 3

The expected value of the matrix $(P^T MP)^2 = P^T M^2 P$ is

$$E[(P^T MP)^2] = \begin{cases} \frac{1}{N} \sum_{ik} m_{ki} m_{ki} & \textit{on diagonal} \\ \frac{1}{N^2 - N} \sum_{ijk} m_{ki} m_{kj} & \textit{off diagonal} \end{cases}$$

Stochastic Property 4

The variance of $P^T MP$ is

$$Var[P^T MP] = \begin{cases} \frac{1}{N} \sum_{ik} m_{ki}^2 - (\overline{m}_d)^2 & \textit{on diagonal} \\ \frac{1}{N^2 - N} \sum_{ijk} m_{ki} m_{kj} - (\overline{m}_o)^2 & \textit{off diagonal} \end{cases}$$

4. THE RANDOM LAY CABLE MODEL

Consider a cable consisting of N parallel conductors and an infinite ground plane.

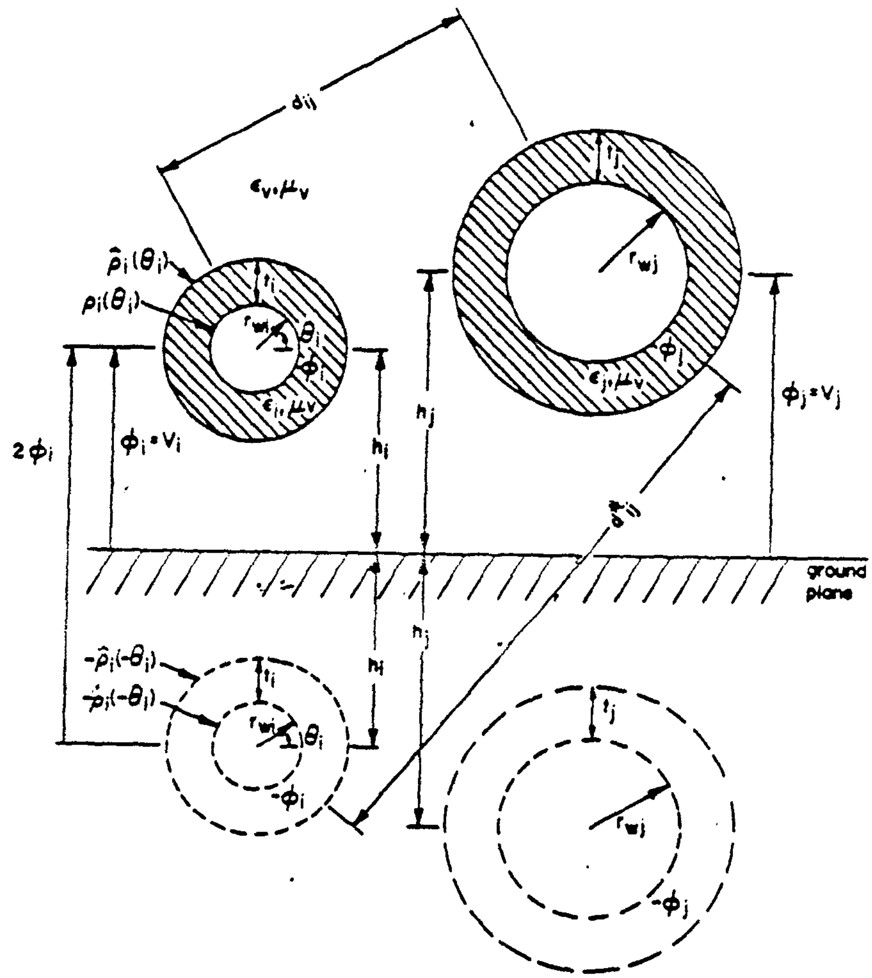

Fig. 1. A system of n wires above a ground plane [1].

Figure 1

The ith conductor is characterized by

$r_{wi} \equiv$ the radius of the wire

$t_i \equiv$ the thickness of the insulation

$d_{ij} \equiv$ the distance form the center of the ith wire to the center of the jth wire

$h_i \equiv$ the distance of the center of the ith wire above the ground plane.

Following Paul and Feather's [1] development, let the per unit length bound charge distribution at the dielectric surface be expanded as

$$\hat{\rho}_i(\theta_i) = \hat{a}_{i0} + \sum_{m=1}^{\hat{A}_i} \hat{a}_{im}\cos(m\theta_i) + \sum_{m=1}^{\hat{B}_i} \hat{b}_{im}\sin(m\theta_i)$$

and the per unit length total charge distribution (bound plus free) at the conductor surface as

$$\rho(\theta_i) = a_{i0} + \sum_{m=1}^{A_i} a_{im}\cos(m\theta_i) + \sum_{n=1}^{B_i} b_{im}\sin(m\theta_i)$$

where θ_i is an arbitrary reference angle for the ith conductor. Boundary conditions are enforced by requiring the potential of the ith conductor due to all charge distributions in the system at a set of matched points on this surface to be ϕ_i , and the normal component of the displacement vector due to all charge distributions to be continuous at a set of match points on each dielectric surface. The boundary conditions yield a set of simultaneous equations

$$D\begin{bmatrix}\underline{\rho}\\ \hat{\underline{\rho}}\end{bmatrix}=\begin{bmatrix}\underline{\phi}\\ \underline{0}\end{bmatrix}.$$

Paul and Feather show that the per unit length capacitance matrix is of the form

$$c_{ij}' = k_1(ij)2\pi r_{wi} + k_2(ij)2\pi(r_{wi}+t_i) \tag{1}$$

$k_1(ij)$ is the sum of all elements in D^{-1} which are in the row associated with a_{i0} and the columns associated with ϕ_j . $k_2(ij)$ is the sum of all elements in D^{-1} which are in the row associated with $\hat{a}_{i0}$ and the columns associated with $\hat{\phi}_j$. They show further that when the wires are sufficiently separated from each other and the ground plane, an approximate per unit length inductance matrix is of the form

$$l_{ii}' = \frac{\mu_\nu}{2\Pi}\ \ln\left(\frac{2h_i}{r_{wi}}\right) \tag{2}$$

$$l_{ij}' = \frac{\mu_\nu}{2\pi}\ \ln\left(\frac{\sqrt{d_{ij}^2+4h_ih_j}}{d_{ij}}\right) \tag{3}$$

From our point of view, the importance of these results is that the ij elements of these matrices are direct functionals of parameters of the ith and jth wires only.

Let L' and C' be respectively the per unit length inductance and capacitance matrices associated with some reference cable orientation. Then from (1), (2), (3) and Property 3 it follows that ${}_kL'$ and ${}_kC'$, for the kth permutation $({}_kP)$, can be written as

$${}_kC' = ({}_kP^T)(C')({}_kP)$$

(4)

$$ {}_kL' = ({}_kP^T(L')({}_kP) $$

Having modeled the capacitance and inductance matrix associated with an arbitrary cable lay in terms of a permutation matrix, we now apply our knowledge of the properties of such matrices to derive the stochastic properties of the random lay cable.

A. EXPECTED VALUE OF THE PER UNIT LENGTH INDUCTANCE, THE PER UNIT LENGTH CAPACITANCE AND CHARACTERISTIC IMPEDANCE MATRICES FOR A RANDOM LAY CABLE:

From (4), the kth permutation of the cable will result in capacitance and inductance matrices of the form

$$ ({}_kC') = ({}_kP^T)(C')({}_kP) $$

$$ ({}_kL') = ({}_kP^T)(L')({}_kP) . $$

Since every permutation is considered equally likely, the expected value of the L and C matrices of a random lay cable comes directly from Stochastic Property 2.

$$ E[C'] = \begin{cases} \frac{1}{N} \sum c_{ii} & \textit{on diagonal} \\ \frac{1}{N^2-N} \sum c_{ij} & \textit{off diagonal} \end{cases} $$

$$ E[L'] = \begin{cases} \frac{1}{N} \sum l_{ii} & \textit{on diagonal} \\ \frac{1}{N^2-N} \sum l_{ij} & \textit{off diagonal} \end{cases} $$

where N is the number of conductors in the cable and c_{ij} and l_{ij} are the elements of the per unit length capacitance and inductance matrices as defined in (1), (2) and (3) for the arbitrarily selected reference orientation of the cable.

The expected value of the characteristic impedance and admittance matrices follow in a similar manner. Defining the characteristic impedance matrix Z_c as [2]

$$Z_c = C'^{-1} T[\lambda]^{1/2} T^{-1}$$

where $[\lambda]$ is the diagonal matrix of eigenvalues and T the matrix of eigenvectors associated with the decomposition

$$C' L' = T[\lambda] T^{-1}$$

then the characteristic impedance matrix for the kth permutation is

$$_kZ_c = {}_kC'^{-1} T[\lambda_k]^{1/2} T^{-1}$$

By Properties 3 and 10

$$[\lambda_k] = T^{-1}(P^T C' L' P) T$$

Since $C' L' = T^{-1}[\lambda] T$

$$[\lambda_k] = T^{-1} P^T T^{-1} [\lambda] T P T$$

$$_kZ_c = [P^T C' P]^{-1} T \cdot T^{-1} P^T T^{-1} [\lambda^{1/2}] T P T T^{-1}$$

By Property 9

$$_kZ_c = P^T C'^{-1} P P^T T^{-1} [\lambda^{1/2}] T P$$

$$k_{Z_c} = P^T C'^{-1} T^{-1} [\lambda^{1/2}] T P$$

$$_kZ_c = P^T Z_c P$$

Further

$$
\begin{aligned}
{}_kY_c = {}_kZ_c^{-1} &= [P^T Z_c P]^{-1} \\
{}_kY_c &= P^T Z_c^{-1} P \\
{}_kY_c &= P^T Y_c P
\end{aligned}
$$

By Stochastic Property 2 the expected value of the impedance matrix is

$$
E[Z] = \begin{cases} \dfrac{1}{N} \sum z_{ii} & \textit{on diagonal} \\[2ex] \dfrac{1}{N^2-N} \sum z_{ij} & \textit{off diagonal} \end{cases}
$$

and the expected value of the admittance matrix is

$$
E[Y] = \begin{cases} \dfrac{1}{N} \sum y_{ii} & \textit{on diagonal} \\[2ex] \dfrac{1}{N^2-N} \sum y_{ij} & \textit{off diagonal} \end{cases} .
$$

B. THE EXPECTED VALUE OF THE SCATTERING MATRIX

Let the terminal admittance of the random lay cable be Y_T . Assume that the elements of Y_T are fixed. Then [3]

$$
{}_kS = [I - ({}_kZ)({}_kY_T)][I + ({}_kZ({}_kY_T)]^{-1}
$$

$$
{}_kS = [P^T(I - Z_{ref}\, Y_{T_{ref}})P][P^T(I + Z_{ref}\, Y_{T_{ref}})P]^{-1}
$$

$$
{}_kS = [P^T(I - Z_{ref}\, Y_{T_{ref}})P]P^T(I + Z_{ref}\, Y_{T_{ref}})^{-1}P
$$

Since $P^TP = I$ (Property 5)

$$ {}_kS = P^T(I - Z_{ref}\, Y_{T_{ref}})(I + Z_{ref}\, Y_{T_{ref}})^{-1}P $$

$$ {}_kS = P^T S P $$

Using Stochastic Property 2, the expected value of the scattering matrix is

$$ E[S] = \begin{cases} \frac{1}{N} \sum s_{ii} = \overline{s}_{ii} & \textit{on diagonal} \\ \frac{1}{N^2 - N} \sum s_{ij} = \overline{s}_{ij} & \textit{off diagonal} \end{cases} $$

C. VARIABILITY OF EM MATRICES:

So far we have looked at the expected value of some matrices of interest under our random lay cable model. We have found that they all have essentially the same form of expected value. While the values of the elements of the matrix will tend to have the expected value, any one visualization of a random cable is likely to produce different results. The question to be addressed now is the extent to which the elements of an EM matrix are likely to vary from their expected value.

Worst case variability is of interest. Clearly the maximum possible variation in any of the electromagnetic matrices considered is simply the difference between the largest and smallest on (off) diagonal elements of the reference matrix for diagonal (off diagonal) terms. In addition we will find a measure of the average expected variation of matrix elements.

A common measure of variability is the Coefficient of Variation defined as

$$ \nu = \frac{\textit{standard deviation of } y}{\textit{expected value of } y} . \tag{5} $$

Clearly $\nu = 0$ implies no variability what so ever. Hence the expected value of y is a perfect description of y. While on the other hand, $\nu = \infty$ implies that the random variable y

is so variable that its expected value is of little use in describing the variable. In our treatment we will use a form equivalent to (5)

$$\nu = \left[\frac{E[Y^2]}{(E[Y])^2} - 1 \right]^{1/2} \tag{6}$$

The electromagnetic matrices of interest have all been representable in the form ${}_kM = P^T M P$ and their expected values have had the form

$$E[M] = \begin{bmatrix} a & b & \cdots & b \\ b & a & & \\ \vdots & & & \\ b & & & a \end{bmatrix}.$$

We will find the same true for variation. Therefore, we will develop the variation only for the scattering matrix since these results are immediately transferable to any other matrix of interest.

From Stochastic Property 3 we have the expected value of the ijth element of the matrix $S^2 = S^T S$ as

$$E[S^2] = \begin{cases} \dfrac{1}{N} \sum_i \sum_k s_{ki}^2 & \textit{diag.} \\ \dfrac{1}{N^2-N} \sum_i \sum_{\substack{k \\ i \neq j}} \sum_j s_{ki_k j} & \textit{off diag.} \end{cases}.$$

Then by (4) and (5) the coefficient of variation for on diagonal elements is

$$\nu = \left[\frac{\frac{1}{N} \sum_{ki} s_{ki}^2}{(\frac{1}{N} \sum_{i} s_{ii})^2} - 1 \right]^{1/2}$$

For the off diagonal elements the coefficient of variation is from (5) and (6)

$$\nu = \left[\frac{\frac{1}{N^2-N} \sum_{i,n} s_{ki} s_{kj}}{\left(\frac{1}{N} \sum_{i} s_{ii}\right)^2} - 1 \right]^{1/2} \tag{7}$$

5. PROPERTIES OF A RANDOM LAY CABLE WITH STOCHASTIC RESISTIVE TERMINATION

Consider a random lay cable of N conductors with a ground plane. Assume that its characteristic impedance is of the form

$$Z_c = \begin{bmatrix} a & b & \cdots & b \\ b & a & & \\ \vdots & & & \vdots \\ b & & & a \end{bmatrix} . \tag{8}$$

Let the ith conductor be terminated with a resistance r_i which is an independent random variable taking on values zero to infinity with a probability density $f(r_i)$. Then

$$Z_T = \begin{bmatrix} r_1 & & \\ & r_2 & \\ & & \\ & & r_N \end{bmatrix} . \tag{9}$$

Consider the Scattering Matrix in the form [3]

$$S = [Z_T - Z_c] \cdot [Z_T + Z_c]^{-1}$$

An invigorating exercise in matrix algebra using (8) and (9) yields for the ith row and jth column of the scattering matrix $\underset{\sim}{S}$

$$\frac{-kb(r_i - a)}{(r_i + a - b)(r_i + a - b)} - \frac{(r_j + a - b - bk)b}{(r_j + a - b)^2} + W(i,j) \quad i \neq j$$

$$\frac{(r_i + a - b - bk)(r_i - a)}{(r_i + a - b)^2} + W(i,j) \quad i = j \tag{10}$$

where

$$k = 1 - b \sum_i \frac{1}{(r_i + a - b)}$$

$$W(i,j) = \sum_{\substack{l \\ l \neq i,j}} \frac{kb^2}{(r_i + a - b)(r_l + a - b)}$$

Example Calculation 1

We first consider a single conductor cable and ground plane with a resistive termination. Let $N = 1 \quad Z_T = R \quad Z_c = 1$. In this case the scattering matrix reduces to the scalar

$$S = \frac{R-1}{R+1} \, . \tag{11}$$

We wish to find the distribution function of R , $f(R)$, such that S will be distributed uniform on the interval -1 to 1 . From (11) the Jacobian of the transformation is

$$\frac{\partial S}{\partial R} = 2(R+1)^{-2}$$

and by hypothesis $f(S) = 1/2$. Thus the distribution function of R is

$$f(R) = [R+1]^{-2} \qquad 0 \leq R < \infty . \tag{12}$$

Example Calculation 2

We now wish to determine the probability that a differential mode current is larger than the common mode current. Let I be the current vector at the termination. Then from [4]

$$I = \frac{1}{2} Y_c [V_+ - V_-]$$

where Y_c is the characteristic admittance of the cable, V_+ is the voltage associated with the outgoing wave and V_- is the voltage associated with the incoming wave. Take

$$V_+ = \left(2 \frac{I_0}{N} \right) S \cdot Z_c \cdot \mathbf{1}$$

$$V_- = \left(2 \frac{I_0}{N} \right) Z_c \cdot \mathbf{1}$$

where $\mathbf{1}$ is a column of 1's . It follows that

$$I = \left[\frac{I_0}{N} \right] \left[\mathbf{1} - Z_c^{-1} \cdot S \cdot Z_c \cdot \mathbf{1} \right]$$

For the jth element of the current vector I to be larger than the bulk current I_0 it is necessary that

$$\{ \mathbf{1} - Z_c^{-1} \cdot S \cdot Z_c \cdot \mathbf{1} \}_j > N . \tag{13}$$

For the $N = 2$ case, take Z_c as in (8) and from (10)

$$\mathcal{S} = \begin{bmatrix} \frac{(r_1-a)(r_2+a) + b^2}{(r_1+a)(r_2+a) - b^2} & \frac{-2b(r_1-a)}{(r_1+a)(r_2+a) - b^2} \\ \frac{-2b(r_2-a)}{(r_1+a)(r_2+a) - b^2} & \frac{(r_2-a)(r_1+a) + b^2}{(r_1+a)(r_2+a) - b^2} \end{bmatrix} \tag{14}$$

Simplifying notation

$$\mathcal{S} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}$$

Then condition (13) is equivalent to

$$-(a-b) > a(s_{11}+s_{12}) - b(s_{21}+s_{22}) \tag{15}$$

where it has been assumed that $a - b > 0$. If not, reverse the inequality.

Assume that r_1 and r_2 are independent random variables with probability density functions

$$f(r_1) = [r_1+1]^{-2} \qquad 0 \le r_1 \le \infty$$

$$f(r_2) = [r_2+1]^{-2} \qquad 0 \le r_2 \le \infty .$$

Then

$$Prob\{-(a-b) > a(s_{11}+s_{12}) - b(s_{21}+s_{22})\}$$

$$= a[b+1][1-k]^{-2}\ln(k) - [k-ab-a-2(a^2+b^2)]k^{-1} \tag{16}$$

where

$$k = \frac{2a^3-4a^2b-3ab^2+b^3+ab+a}{2(a^2+b^2)}$$

Note that for (15) to be possible k must be non negative and (16) must yield a result between zero and one.

6. RECOMMENDATIONS

I believe that there are two areas for further research and at least one immediate application for these results.

1. Intuitively it appears that the random lay cable results will be applicable far beyond the simplistic assumptions of the model. Indeed any equally probable random perturbations which can be expressed as ij interactions should produce Stochastic Property 2. This assertion needs proof.

2. The treatment of random resistive termination can be extended to random impedance in a straightforward manner. However, in general, the resulting probability integrals will require numerical techniques for solution. This is fine for a specific calculation but hinders development of the distribution theory.

In approaching the general distribution theory one can perhaps identify assumptions about the distributions of the Z_T and Z_C matrices that lead to tractable probability integrals. The preferred approach, however, is to identify probability distributions for Z_T and Z_C that in some sense agree with the real world and then do battle with $\mathfrak{S}$. This will almost certainly produce an intractable result. But, since the result must be a probability distribution function it can almost certainly be approximated with a tractable expression. Such a result would not only further development of the general distribution theory but knowledge of the behavior of the functions is likely to yield physical insight into the random lay cable problem.

Finally, even in this rudimentary from the random lay cable model has an immediate practical application to EMP experiments. The model can at least give an indication whether in a particular experiment "random lay" will be an important factor that might alter results.

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