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Some Bounds Concerning the Response of Linear Systems with a Nonlinear Element

Carl E. Baum
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Abstract

This paper addresses the problem of nonlinear elements present in otherwise linear electromagnetic systems. What effect does such an element have on a linear analysis of such systems? Here we obtain some bounds on such effects. At a minimum these bounds can give some criteria as to under what conditions some nonlinearities can be neglected in the analysis of the system electromagnetic response.

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Contents

I. Introduction

Much analysis of the properties of electromagnetic scattering (or interaction) problems, as well as the properties of electrical networks, relies on the assumption of linearity and time invariance. This permits the ready introduction of the Laplace (or Fourier) transform, replacing convolution (or equivalently integro-differential operators) with respect to time by multiplication in complex frequency domain. In one sense this assumption is in principle rather restrictive, representing a subset of electromagnetic systems. On the other hand, systems are often constructed to be at least approximately linear precisely so that they can be more simply analyzed (or synthesized) using the properties implied by linearity.

To say that a system is nonlinear does not say much in that nonlinearity represents an infinity of possibilities. One needs to specify what kinds of nonlinearities he has in mind so as to proceed with the analysis. In this context linearity is only one of a large number of possible assumptions concerning the system, i.e., is only a special case.

In this paper we address the problem of what happens when there is an identifiable nonlinear element in the system which can be isolated from the system by definition of a single two-terminal port so that the nonlinearity can be thought of as being removed from the system (the remainder being now linear). Examples of nonlinear elements meeting such conditions are simple elements such as resistors or capacitors (perhaps under breakdown conditions), diodes, arcs between two nearby conductors in the system, etc.

Using the linearity of the remaining system, linear time-invariant passive loads (impedances) are maneuvered into and out of the system via the ports to form what we refer to as the modified system. This modified system is defined in a form that simplifies the wave transport via the associated scattering matrix. Treating the nonlinear element we assume that it is passive. This allows us to get a bound on the 2-norm of the signal scattered from the nonlinear element, the 2-norm being related to energy. This bound is transported through the system via the scattering matrix to obtain bounds of the effect produced by the nonlinearity at other ports including a source driving the system and an arbitrarily chosen impedance somewhere in the system.

II. General Considerations

Let us first consider the simpler problem of a 2-port linear time-invariant system (or network) as indicated in fig. 2.1. We assume that the system is driven by some source at port 1 with a Thevenin equivalent voltage source $V_s(t)$ with a Laplace transform (two sided) $\tilde{V}_s(s)$ and an impedance $\tilde{Z}_s(s)$ with

$$\tilde{f}(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st}\, ds \tag{2.1}$$

Br ≡ Bromwich contour parallel to $j\omega$ axis in strip of convergence (assumed to exist)

$s \equiv \Omega + j\omega$ = Lapace-transform variable or complex frequency

Note that the Laplace transform applies to vector and matrix functions as well.

Now consider a general N-port which is a linear time-invariant system (or network) as illustrated in fig. 2.1. This is characterized by a scattering matrix $(\tilde{S}_{n,m}(s))$ or an admittance matrix $(\tilde{Y}_{n,m}(s))$ with [1,3]

$$(\tilde{Y}_{n,m}(s)) = (\tilde{Z}_{n,m}(s))^{-1}$$

$$\begin{aligned}
(\tilde{S}_{n,m}(s)) &= [(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}^{(ref)}_{n,m}(s)) + (1_{n,m})]^{-1} \\
&\quad \cdot [(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}^{(ref)}_{n,m}(s)) - (1_{n,m})] \\
&= [(1_{n,m}) + (\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))]^{-1} \\
&\quad \cdot [(1_{n,m}) - (\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))] \\
&= [(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}^{(ref)}_{n,m}(s)) - (1_{n,m})] \\
&\quad \cdot [(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}^{(ref)}_{n,m}(s)) + (1_{n,m})]^{-1}
\end{aligned}$$

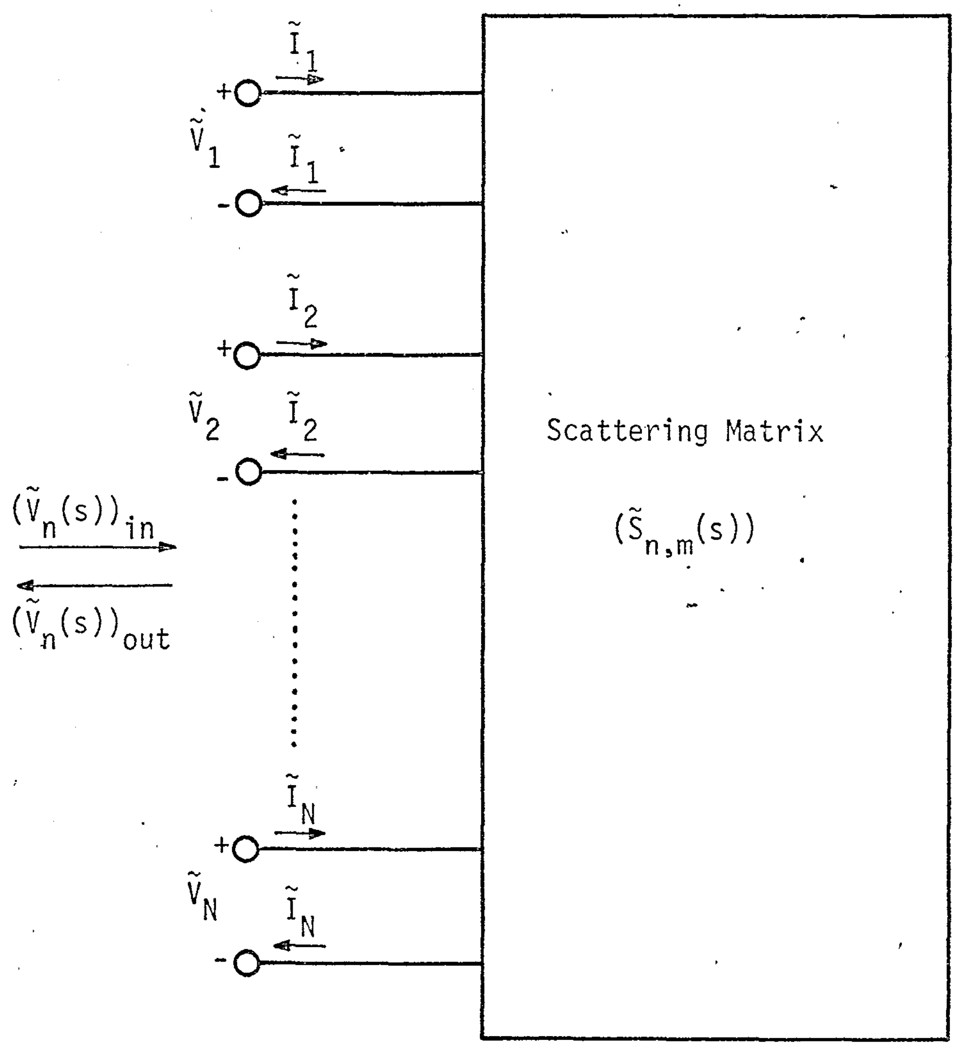


Fig. 2.1. N-Port Linear Time-Invariant System (Network)

$$= [(1_{n,m}) - (\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))] \qquad (2.2)$$

$$\cdot [(1_{n,m}) + (\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s))]^{-1}$$

$$(\tilde{Y}^{(ref)}_{n,m}(s)) = (\tilde{Z}^{(ref)}_{n,m}(s))^{-1}$$

The normalizing or reference admittance matrix $(\tilde{Y}^{(ref)}_{n,m}(s))$ and impedance matrix $(\tilde{Z}^{(ref)}_{n,m}(s))$ are used to define the wave variables (vectors)

$$(\tilde{V}_n(s))_{in} \equiv (\tilde{V}_n(s)) + (\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{I}_n(s))$$

$$(\tilde{V}_n(s))_{out} \equiv (\tilde{V}_n(s)) - (\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{I}_n(s)) \qquad (2.3)$$

where the voltage and current vectors are related by

$$(\tilde{V}_n(s)) = (\tilde{Z}_{n,m}(s)) \cdot (\tilde{I}_n(s))$$

$$(\tilde{I}_n(s)) = (\tilde{Y}_{n,m}(s)) \cdot (\tilde{V}_n(s)) \qquad (2.4)$$

with voltage and current conventions as indicated in fig. 2.1. The scattering matrix is used to relate the wave variables as

$$(\tilde{V}_n(s))_{out} = (\tilde{S}_{n,m}(s)) \cdot (\tilde{V}_n(s))_{in} \qquad (2.5)$$

Given a scattering matrix the impedance and admittance matrices can be reconstructed via [3]

$$(\tilde{Z}^{(ref)}_{n,m}(s)) \cdot (\tilde{Y}_{n,m}(s)) = [(1_{n,m}) + (\tilde{S}_{n,m}(s))] \cdot [(1_{n,m}) - (\tilde{S}_{n,m}(s))]^{-1}$$

$$= [(1_{n,m}) - (\tilde{S}_{n,m}(s))]^{-1} \cdot [(1_{n,m}) + (\tilde{S}_{n,m}(s))]$$

$$(\tilde{Z}_{n,m}(s)) \cdot (\tilde{Y}^{(ref)}_{n,m}(s)) = [(1_{n,m}) - (\tilde{S}_{n,m}(s))] \cdot [(1_{n,m}) + (\tilde{S}_{n,m}(s))]^{-1} \qquad (2.6)$$

$$= [(1_{n,m}) + (\tilde{S}_{n,m}(s))]^{-1} \cdot [(1_{n,m}) - (\tilde{S}_{n,m}(s))]$$

In a previous paper [3] constraints were established on the allowable form for this reference admittance or impedance matrix. In particular, requiring an energy (or 2 norm) relation for the wave variables and bounding the 2 norm of the scattering matrix by unity (on the $j\omega$ axis of the s plane) as

$$0 \leqslant \|(S_{n,m}(j\omega))\|_2 \leqslant 1 \tag{2.7}$$

gave

$$(\tilde{Y}^{(ref)}_{n,m}(j\omega)) = \tilde{\lambda}(j\omega)(1_{n,m})$$

$$\tilde{\lambda}(j\omega) > 0 \tag{2.8}$$

Hence on the $j\omega$ axis, the N x N normalizing admittance and impedance matrices must be diagonal with all elements the same and positive. In this form we have on the $j\omega$ axis

$$\|(\tilde{V}_n(j\omega))_{in}\|_2 \geqslant \|(\tilde{V}_n(j\omega))_{out}\|_2 \geqslant 0 \tag{2.9}$$

which is discussed in Appendix C. Note that for a lossless network we also have

$$\|(\tilde{S}_{n,m}(j\omega))\|_2 \Big|_{lossless} = 1 \tag{2.10}$$

and that for a perfect terminating network, i.e.,

$$\text{if } (\tilde{Z}_{n,m}(s)) = (\tilde{Z}^{(ref)}_{n,m}(s)), \text{ then } (\tilde{S}_{n,m}(s)) = (0_{n,m}) \tag{2.11}$$

In Appendix D of this paper it is also shown that a passive realizable normalizing admittance (or impedance) matrix makes

$$\tilde{\lambda}(s) = \lambda_0 > 0 \quad \text{(a positive constant)} \tag{2.12}$$

the only possible choice. Hence we set

$$\left(\hat{Z}_{n,m}^{(ref)}(s)\right) \equiv R(1_{n,m})$$

$$\left(\hat{Y}_{n,m}^{(ref)}(s)\right) \equiv G(1_{n,m}) \qquad (2.13)$$

$$R = \frac{1}{G}\ , \quad R > 0, \quad G > 0$$

III. Modified Scattering Matrices

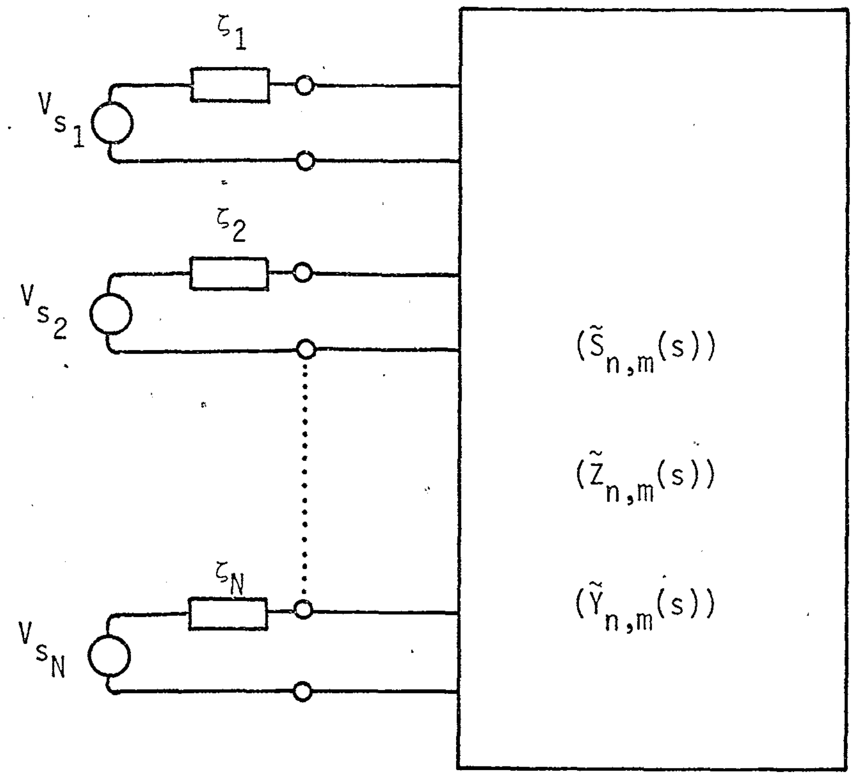
Now the scattering matrix $(\tilde{S}_{n,m}(s))$ represents the system in some configuration as one may find it, the matrix elements perhaps being determined by experiment. In order to simplify the analysis, in this paper let us combine the system (as in fig. 2.1) with other impedance (or admittance) elements as indicated in fig. 3.1. Here the intent, as will be made specific later, is to control the reflection at some of the ports back into the system, as least in its formal representation.

As indicated in fig. 3.1 we start with some "original" form of the system as in fig. 3.1A. (For later, note that nonlinear elements, one in our present analysis, have been pulled out and included in the elements attached to the ports.) In this form we assume that there are Thevenin equivalent sources at each port including voltage sources V_{s_n} and series "impedances." These may be impedances in the strict sense or may be nonlinear elements as designated in a generic way by ζ_n. Let us move some or all of these elements, in whole or part, into the system representation and move other elements out of the system into the loads at the ports, at our convenience, to give what we refer to as the modified system in fig. 3.1B with elements (linear or nonlinear) at the ports designated by $\zeta_n^{(1)}$. Of course, no nonlinear elements are to be moved into the system representation so that this representation will remain linear. While our current procedure involves Thevenin equivalent voltage sources V_{s_n} or $V_{s_n}^{(1)}$ and associated "impedance-like" elements ζ_n or $\zeta_n^{(1)}$, one can also approach this problem from a set of Norton equivalent current sources and associated "admittance-like" elements.

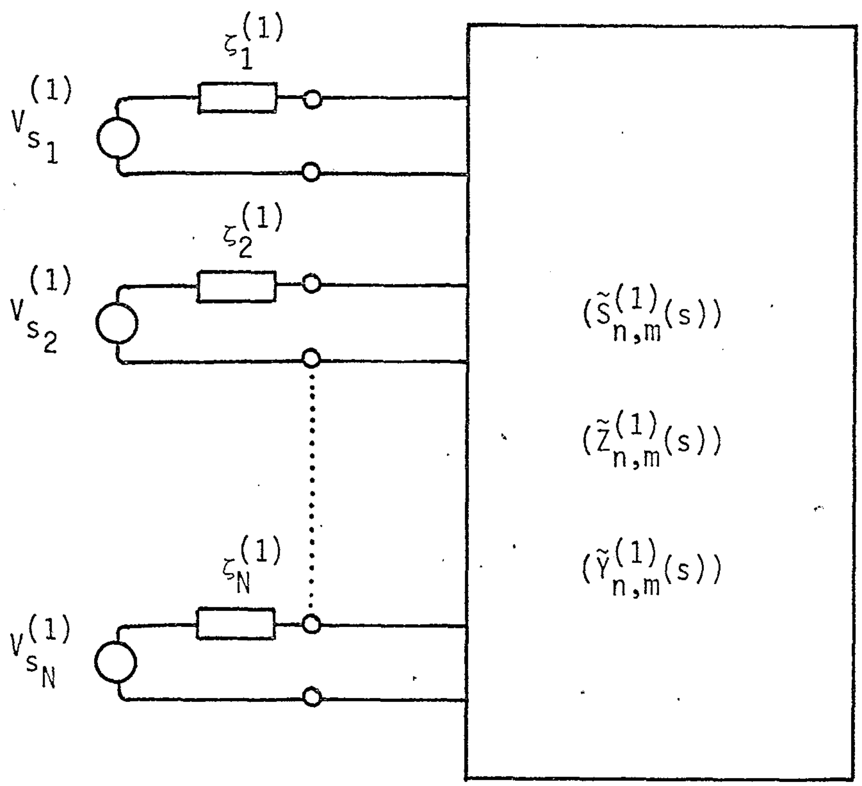
Following up on our Thevenin-equivalent approach let us modify the impedance representation of the system as

$$(\tilde{Z}_{n,m}^{(1)}(s)) = (\tilde{Z}_{n,m}(s)) + (\tilde{Z}_n^{(+)} 1_{n,m}) - (\tilde{Z}_n^{(-)} 1_{n,m}) \qquad (3.1)$$

where the diagonal matrix $(\tilde{Z}_n^{(+)} 1_{n,m})$ represents the augmentation impedance matrix associated with moving certain series impedances from the ports into the system, and the diagonal matrix $(\tilde{Z}_n^{(-)} 1_{n,m})$ represents a diminution matrix from the system into its ports.



A. System



B. Modified System

Fig. 3.1. Modification of N-Port System by Addition of and/or Removal of Impedances at Ports

Now the modified impedance matrix $(\tilde{Z}^{(1)}_{n,m}(s))$ will be restricted to passive, or positive real, in the parlance of circuit theory. This restricts the allowable choices of $(\tilde{Z}^{(+)}_n 1_{n,m}) - (\tilde{Z}^{(-)}_n 1_{n,m})$ depending on the specific character of the unmodified impedance matrix $(\tilde{Z}_{n,m}(s))$ which is also assumed to be passive.

Note that at least in a formal sense we have

$$
\begin{aligned}
(\tilde{Z}^{(+)}_n 1_{n,m}) - (\tilde{Z}^{(-)}_n 1_{n,m}) &= (\tilde{Z}^{(1)}_{n,m}(s)) - (\tilde{Z}_{n,m}(s)) \\
&= (\zeta_n 1_{n,m}) - (\zeta^{(1)}_n 1_{n,m}) \qquad (3.2)
\end{aligned}
$$

Now we can define the modified scattering matrix $(\tilde{S}^{(1)}_{n,m}(s))$ for the system by using (2.2) with the same $(\tilde{Z}^{(ref)}_{n,m}(s))$ and $(\tilde{Y}^{(ref)}_{n,m}(s))$ as before, but with $(\tilde{Z}_{n,m}(s))$ and $(\tilde{Y}_{n,m}(s))$ replaced respectively by $(\tilde{Z}^{(1)}_{n,m}(s))$ and $(\tilde{Y}^{(1)}_{n,m}(s))$. With the aforegoing restrictions then the modified scattering matrix $(\tilde{S}^{(1)}_{n,m}(s))$ is linear and passive as desired. In terms of the modified scattering matrix we have

$$
(\tilde{V}^{(1)}_n(s))_{out} = (\tilde{S}^{(1)}_{n,m}(s)) \cdot (\tilde{V}^{(1)}_n(s))_{in} \qquad (3.3)
$$

for the modified system.

IV. 2-Port Case: Linear Source and Nonlinear Element

Now consider a 2-port network or system ($\tilde{N} = 2$) as illustrated in fig. 4.1. Let there be some (in general) nonlinear load ζ attached to port 2. This nonlinear load is chosen to represent some nonlinearity in a system of interest which has been identified and "removed" so that the linear remainder of the system is represented by $(\tilde{Y}_{n,m}(s))$, $(\tilde{Z}_{n,m}(s))$ and/or $(\tilde{S}_{n,m}(s))$. This nonlinearity might be an arc between 2 wires, a single-port element (like a resistor, capacitor, etc.), or whatever, provided it can be represented by a single port (i.e., a single voltage-current pair). Port 1 has a source $\tilde{V}_s(s)$ and a source impedance (passive) $\tilde{Z}_s(s)$. Summarizing symbolically

$$(\tilde{V}_{s_n}(s)) = \begin{pmatrix} \tilde{V}_s(s) \\ 0 \end{pmatrix} \quad , \quad (\zeta_n 1_{n,m}) = \begin{pmatrix} \tilde{Z}_s(s) & 0 \\ 0 & \zeta \end{pmatrix} \tag{4.1}$$

Figure 4.2 shows the modified system and port networks. The port networks are now described by

$$(\tilde{V}^{(1)}_{s_n}(s)) = \begin{pmatrix} \tilde{V}_s(S) \\ 0 \end{pmatrix} \quad , \quad (\zeta^{(1)}_n 1_{n,m}) = \begin{pmatrix} R & 0 \\ 0 & \zeta \end{pmatrix} \tag{4.2}$$

The sources are unchanged but the loads are in general changed. At port 2 no change is made; the same nonlinear load, ζ, is present. At port 1, however, $\tilde{Z}_s(s)$ has been absorbed into the system, and a frequency-independent resistance, R, has been pulled out. Not coincidentally, and consistent with section 2, the resistance at port 1 is made to equal the normalizing constant resistance used in the normalizing impedance (2.13) to defined the wave variables which are used to define in turn the (now modified) scattering matrix. Of course, we now require that the modified impedance matrix from (3.1) be

$$(\tilde{Z}^{(1)}_{n,m}(s)) = (\tilde{Z}_{n,m}(s)) + \begin{pmatrix} \tilde{Z}_s(s) - R & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

noting that ζ is not included here so that $(\tilde{Z}^{(1)}_{n,m}(s))$ is still an impedance in the strict sense (in particular, no nonlinearities). Let us further assume

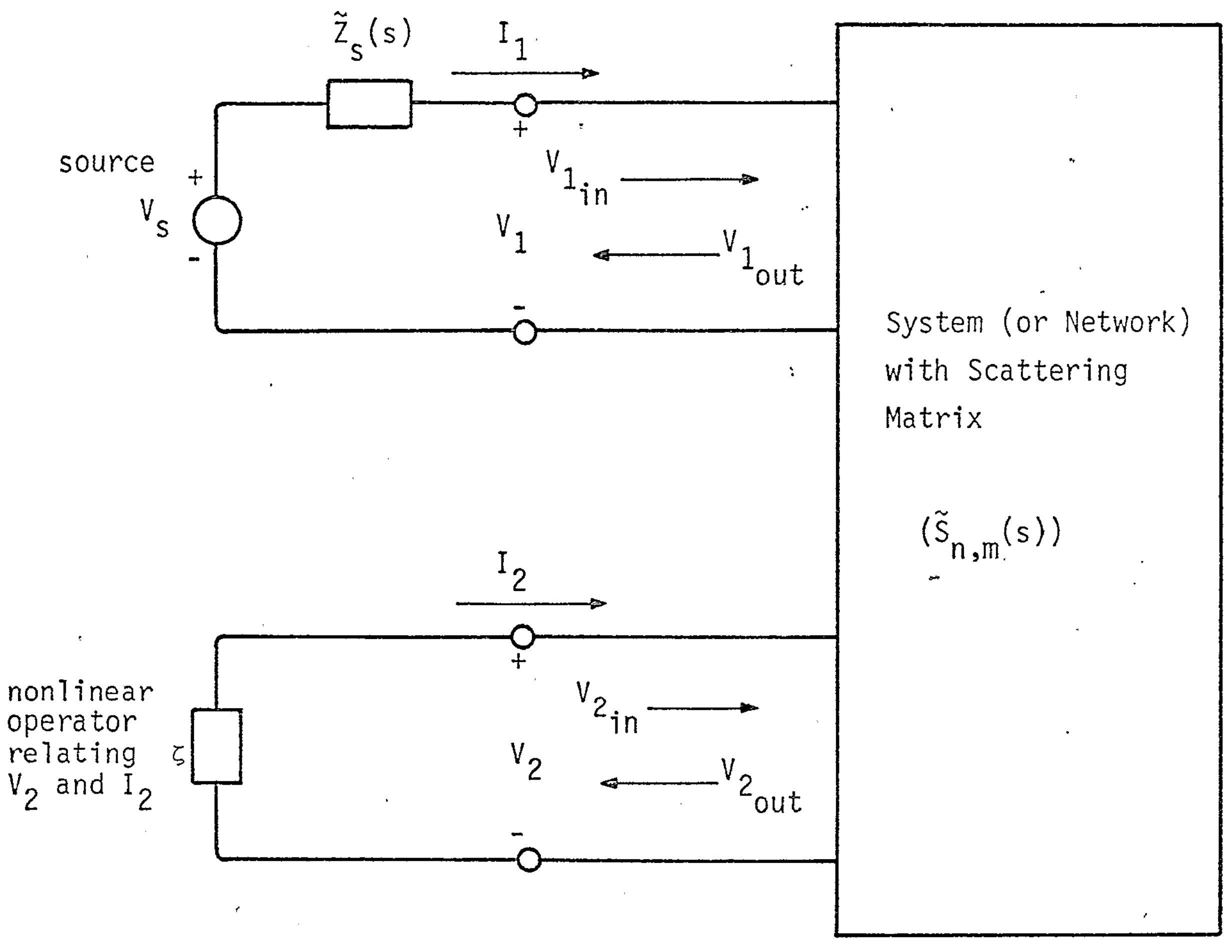


Fig. 4.1. Representation of Linear Source and Nonlinear Element Connected to a 2-Port System (or Network)

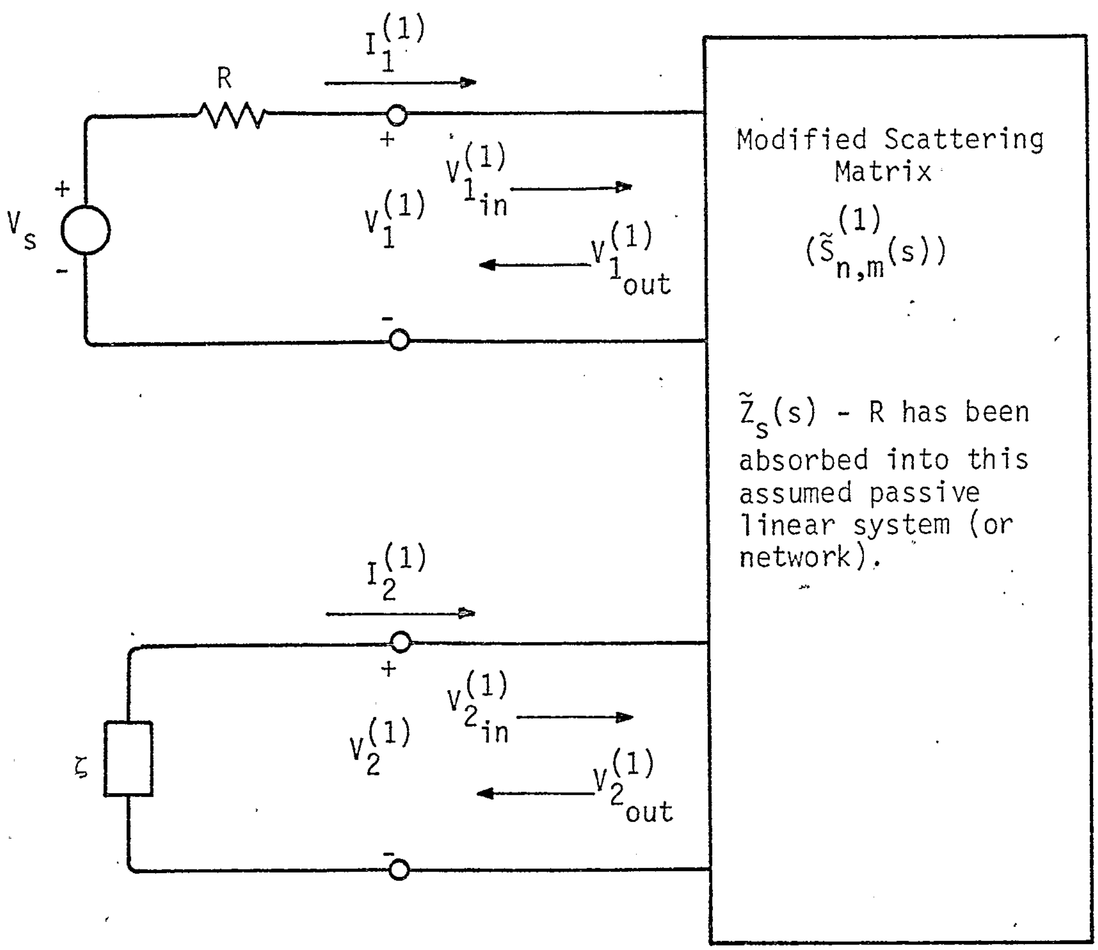


Fig. 4.2. Modified Scattering Matrix for 2-Port

that $(\tilde{Z}^{(1)}_{n,m}(s))$ and $\tilde{Z}_s(s)$ are passive (restricting values for R) so that $(\tilde{S}^{(1)}_{n,m}(s))$ is passive.

We are now in a position to identify R with the diagonal reference-impedance matrix in (2.13). This is done so that the wave leaving port 1 is terminated (i.e., not rescattered back into the system). The wave variables are defined in (2.3). There are two waves at each port, designated by subscripts "in" and "out" related by the modified scattering matrix as in (3.3). Writing out this relation in component form we have

$$\tilde{V}^{(1)}_{1_{out}}(s) = \tilde{S}^{(1)}_{1,1}(s)\ \tilde{V}^{(1)}_{1_{in}}(s) + \tilde{S}^{(1)}_{1,2}\ \tilde{V}^{(1)}_{2_{in}}(s)$$

$$\tilde{V}^{(1)}_{2_{out}}(s) = \tilde{S}^{(1)}_{2,1}(s)\ \tilde{V}^{(1)}_{1_{in}}(s) + \tilde{S}^{(1)}_{2,2}(s)\ \tilde{V}^{(1)}_{2_{in}}(s) \qquad (4.4)$$

Next consider the relation of $\tilde{V}^{(1)}_{1_{in}}(s)$ and $\tilde{V}^{(1)}_{1_{out}}(s)$ at port 1. Here we have

$$\tilde{V}_s(s) - R\ \tilde{I}^{(1)}_1(s) = \tilde{V}^{(1)}_1(s) = \frac{1}{2}\quad \tilde{V}^{(1)}_{1_{in}}(s) + \tilde{V}^{(1)}_{1_{out}}(s)$$

$$V_s(s) = \tilde{V}^{(1)}_1(s) + R\ I^{(1)}_1(s) = \tilde{V}^{(1)}_{1_{in}}(s) \qquad (4.5)$$

Then, (4.4) becomes

$$\tilde{V}^{(1)}_{1_{out}}(s) = \tilde{S}^{(1)}_{1,1}(s)\ \tilde{V}_s(s) + \tilde{S}^{(1)}_{1,2}(s)\ \tilde{V}^{(1)}_{2_{in}}(s)$$

$$\tilde{V}^{(1)}_{2_{out}}(s) = \tilde{S}^{(1)}_{2,1}(s)\ \tilde{V}_s(s) + \tilde{S}^{(1)}_{2,2}(s)\ \tilde{V}^{(1)}_{2_{in}}(s) \qquad (4.6)$$

Now we need to relate $\tilde{V}^{(1)}_{2_{in}}(s)$ and $\tilde{V}^{(1)}_{2_{out}}(s)$ at port 2 through the non-linear load ζ. From Appendix C and using the assumption that ζ is passive we have in time and frequency domains

$$\|V_{2_{out}}^{(1)}(t)\|_{2,t} \geq \|V_{2_{in}}^{(1)}(t)\|_{2,t} \geq 0$$
$$\|\tilde{V}_{2_{out}}^{(1)}(j\omega)\|_{2,\omega} \geq \|\tilde{V}_{2_{in}}^{(1)}(j\omega)\|_{2,\omega} \geq 0 \tag{4.7}$$

noting the interchange of the roles "in" and "out" in going from a reference to the system to a reference to the load at the port.

Applying the 2-norm with respect to frequency to (4.6) gives

$$\|\tilde{V}_{1_{out}}^{(1)}(j\omega)\|_{2,\omega} \leq \|\tilde{S}_{1,1}^{(1)}(j\omega)\ \tilde{V}_s(j\omega)\|_{2,\omega} + \|\tilde{S}_{1,2}(j\omega)\ \tilde{V}_{2_{in}}(j\omega)\|$$
$$\|\tilde{V}_{2_{out}}^{(1)}(j\omega)\|_{2,\omega} \leq \|\tilde{S}_{2,1}^{(1)}(j\omega)\ \tilde{V}_s(j\omega)\|_{2,\omega} + \|\tilde{S}_{2,2}^{(1)}(j\omega)\ \tilde{V}_{2_{in}}(j\omega)\|_{2,\omega} \tag{4.8}$$

Define

$$|\tilde{S}_{n,m}^{(1)}|_{sup} \equiv \sup_{\omega} |\tilde{S}_{n,m}^{(1)}(j\omega)| \tag{4.9}$$

where the supremum is over all ω (real) or at least the range of ω of interest. Then using (A.5) the inequalities (4.8) can be extended to

$$\|\tilde{V}_{1_{out}}^{(1)}(j\omega)\|_{2,\omega} \leq |\tilde{S}_{1,1}^{(1)}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} + |\tilde{S}_{1,2}^{(1)}|_{sup}\ \|\tilde{V}_{2_{in}}(j\omega)\|_{2,\omega}$$
$$\|\tilde{V}_{2_{out}}^{(1)}(j\omega)\|_{2,\omega} \leq |\tilde{S}_{2,1}^{(1)}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} + |\tilde{S}_{2,2}^{(1)}|_{sup}\ \|\tilde{V}_{2_{in}}(j\omega)\|_{2,\omega} \tag{4.10}$$

For a port-2 result substitute the second of (4.7) into the second of (4.10) to give

$$\|\tilde{V}_{2_{out}}^{(1)}(j\omega)\|_{2,\omega} \leq |\tilde{S}_{2,1}^{(1)}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} + |\tilde{S}_{2,2}^{(1)}|_{sup}\ \|\tilde{V}_{2_{out}}^{(1)}(j\omega)\|_{2,\omega} \tag{4.11}$$

which can be rearranged as

$$\|\tilde{V}^{(1)}_{2_{out}}(j\omega)\|_{2,\omega} \leq [1 - |\tilde{S}^{(1)}_{2,2}|_{sup}]^{-1} |\tilde{S}^{(1)}_{2,1}|_{sup} \|\tilde{V}_s(j\omega)\|_{2,\omega} \tag{4.12}$$

and carried directly into time domain as

$$\|V^{(1)}_{2_{out}}(t)\|_{2,t} \leq [1 - |\tilde{S}^{(1)}_{2,2}|_{sup}]^{-1} |\tilde{S}^{(1)}_{2,1}|_{sup} \|V_s(t)\|_{2,t} \tag{4.13}$$

Combined with (4.7) we thus have bounds for the 2-norm in time and frequency domains for both outgoing and incoming waves at port 2, even though the load at port 2 is by hypothesis in general nonlinear. Note the direct (first power) dependence on the supremum of the magnitude of the scattering from port 1 to port 2.

Turning to port 1 the first of (4.10) combined with (4.12) and the second of (4.7) gives

$$\|\tilde{V}^{(1)}_{1_{out}}(j\omega)\|_{2,\omega} \leq \left\{ |\tilde{S}^{(1)}_{1,1}|_{sup} + [1 - |\tilde{S}^{(1)}_{2,2}|_{sup}]^{-1} |\tilde{S}^{(1)}_{1,2}|_{sup} \, |\tilde{S}^{(1)}_{2,1}|_{sup} \right\} \|\tilde{V}_s(j\omega)\|_{2,\omega} \tag{4.14}$$

However, if we note that the $\tilde{S}^{(1)}_{1,1}$ term represents the scattering back into the source from port 1 with no reflection from port 2 with the nonlinear load, then we can obtain another, perhaps more interesting, bound. Rewriting the first of (4.6) and taking 2-norms with respect to ω (and using (4.7)) gives

$$\begin{aligned}
&\|\tilde{V}^{(1)}_{1_{out}}(j\omega) - \tilde{S}_{1,1}(j\omega) \, \tilde{V}_s(j\omega)\|_{2,\omega} \\
&\quad = \|\tilde{S}^{(1)}_{1,2}(j\omega) \, \tilde{V}_{2_{in}}(j\omega)\|_{2,\omega} \\
&\quad \leq |\tilde{S}^{(1)}_{1,2}|_{sup} \|\tilde{V}_{2_{in}}(j\omega)\|_{2,\omega} \\
&\quad \leq [1 - |\tilde{S}^{(1)}_{2,2}|_{sup}]^{-1} |\tilde{S}^{(1)}_{1,2}|_{sup} \, |\tilde{S}^{(1)}_{2,1}|_{sup} \|\tilde{V}_s(j\omega)\|_{2,\omega}
\end{aligned} \tag{4.15}$$

The last expression then represents a bound on the wave (in 2-norm sense) scattered from port 2 back to port 1, and hence some bound of the effect of the nonlinearity at port 2 back on port 1. Note the first power dependence on the scattering elements from port 1 to port 2 and from port 2 to port 1. This is effectively a second power dependence on the transfer function through the system between ports 1 and 2.

Note, however, the dependence of the results in (4.12) - (4.15) on the factor $[1 - |\tilde{S}^{(1)}_{2,2}|_{sup}]^{-1}$. By our choice of $(\tilde{Z}^{(ref)}_{n,m}(s))$ as a positive constant times the identity matrix as in (2.13) we have insured that if our modified system is passive then [3]

$$\|(\tilde{S}^{(1)}_{n,m}(j\omega))\|_2 \leq 1 \tag{4.16}$$

Now $\tilde{S}^{(1)}_{2,2}(j\omega)$ represents the scattering of a wave incident on port 2 towards the nonlinear load. Energy must also be conserved here and hence for all elements on the diagonal

$$|\tilde{S}^{(1)}_{n,m}(j\omega)| \leq 1 \quad \text{for } n = 1,2, \ldots ,N \tag{4.17}$$

if the system is passive. However, the result is "less than or equal to 1." If equality applies for some ω then, for that n, $|\tilde{S}_{n,n}|_{sup} = 1$ and the factor $[1 - |\tilde{S}^{(1)}_{2,2}|]^{-1}$ (in this case) blows up and the bounds are not applicable. This points to the necessity for loss in the modified system, precisely to avoid this problem.

Note that R has not yet been chosen. It must be small enough (and still positive) that $(\tilde{Z}^{(1)}_{n,m}(s))$ in (4.3) is passive. Aside from this restriction one might choose R so as to minimize the bounds in (4.12) - (4.15), noting the importance of moving $|\tilde{S}_{2,2}|_{sup}$ away from 1.

Noting that the 2-norm of a wave over frequency or time is proportional to energy, but not exactly the same, let us consider the energy delivered out of port 2. In the time domain the power delivered to the load (in general nonlinear) at port 2 is

$$P_\zeta(t) = -V_2^{(1)}(t)I_2^{(1)}(t) = -\frac{1}{2}\,[V_{2_{in}}^{(1)}(t) + V_{2_{out}}^{(1)}(t)]\,\frac{1}{2R}\,[V_{2_{in}}^{(1)}(t) - V_{2_{out}}^{(1)}(t)]$$

$$= \frac{1}{4R}\,[V_{2_{out}}^{(1)^2}(t) - V_{2_{in}}^{(1)^2}(t)] \tag{4.18}$$

The energy received at the nonlinear load at port 2 is

$$U_\zeta = \int_{-\infty}^{\infty} P_\zeta(t)\,dt = \frac{1}{4R}\,[\|V_{2_{out}}^{(1)}(t)\|_{2,t}^2 - \|V_{2_{in}}^{(1)}(t)\|_{2,t}^2]$$

$$= \frac{1}{8\pi R}\,[\|\tilde{V}_{2_{out}}^{(1)}(j\omega)\|_{2,\omega}^2 - \|\tilde{V}_{2_{in}}^{(1)}(j\omega)\|_{2,\omega}^2] \tag{4.19}$$

For a given R this energy has an upper bound from setting

$$\|\tilde{V}_{2_{in}}^{(1)}(j\omega)\|_{2,\omega} = 0 \quad , \quad \|\tilde{V}_{2_{in}}^{(1)}(t)\|_{2,t} = 0 \tag{4.20}$$

which is achieved by setting

$$\tilde{V}_{2_{in}}^{(1)}(j\omega) = 0 \quad , \quad V_{2_{in}}^{(1)}(t) = 0 \tag{4.21}$$

except for discrete points, which when integrated over ω (in a square sense) give 0. This case is achieved, for example, by zero scattering from ζ, or formally, a particular linear case

$$\zeta = R \tag{4.22}$$

which is "perfect" termination. Hence

$$U_\zeta \le \frac{1}{4R}\,\|V_{2_{out}}^{(1)}(t)\|_{2,t}^2 = \frac{1}{8\pi R}\,\|\tilde{V}_2^{(1)}(j\omega)\|_{2,\omega}^2 \tag{4.23}$$

Combining this with (4.12) or (4.13) gives an upper bound on U_ζ in terms of $V_s(t)$ or $\tilde{V}_s(j\omega)$. The new factor introduced is 1/R, so that if one wishes to choose R so as to minimize this bound on V_ζ, then one needs to minimize the coefficients in this bound

$$U_\zeta \leq \frac{1}{4R}\left[1 - |\hat{S}^{(1)}_{2,2}|_{sup}\right]^{-2} |\hat{S}^{(1)}_{2,1}|^2_{sup} \| \hat{V}_s(t) \|^2_{2,t} \qquad (4.24)$$

V. 3-Port Case: Linear Source, Linear Victim, and Nonlinear Element

Now consider a 3-port network or system ($N = 3$) as illustrated in fig. 5.1. As in section 4 we have some linear source $\tilde{V}_s(s)$ and source impedance (passive) $\tilde{Z}_s(s)$ attached to port 1. There is the nonlinear load ζ, now attached to port 3. Consider some element $\tilde{Z}_v(s)$ which is attached to port 2. This element is pulled out of the system for special attention and initially defines port 2. This element is further assumed linear, passive, and time invariant. Our concern is what effect ζ will have on the signals reaching $\tilde{Z}_v(s)$, and whether one need be concerned about damage or upset there, at least in a bound sense. In this sense $\tilde{Z}_v(s)$ represents an arbitrarily chosen element from the system which we regard as a potential victim to the source $\tilde{V}_s(s)$ as influenced by the nonlinearity ζ. Summarizing symbolically the sources and elements at the ports are

$$(\tilde{V}_{s_n}(s)) = \begin{pmatrix} \tilde{V}_s(s) \\ 0 \\ 0 \end{pmatrix} \quad , \quad (\zeta_n 1_{n,m}) = \begin{pmatrix} \tilde{Z}_s(s) & 0 & 0 \\ 0 & \tilde{Z}_v(s) & 0 \\ 0 & 0 & \zeta \end{pmatrix} \tag{5.1}$$

Figure 5.2 shows the modified system and port networks. The port networks are now described by

$$(\tilde{V}^{(1)}_{2_n}(s)) = \begin{pmatrix} \tilde{V}_s(s) \\ 0 \\ 0 \end{pmatrix} \quad , \quad (\zeta^{(1)}_n 1_{n,m}) = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & \zeta \end{pmatrix} \tag{5.2}$$

The sources are unchanged, but the loads are in general changed. Port 3 still has the nonlinear load ζ. However, both ports 1 and 2 have the loads replaced by R. The modified impedance matrix from (3.1) is now

$$(\tilde{Z}^{(1)}_{n,m}(s)) = (\tilde{Z}_{n,m}(s)) + \begin{pmatrix} \tilde{Z}_s(s)-R & 0 & 0 \\ 0 & \tilde{Z}_v(s)-R & 0 \\ 0 & 0 & \zeta \end{pmatrix} \tag{5.3}$$

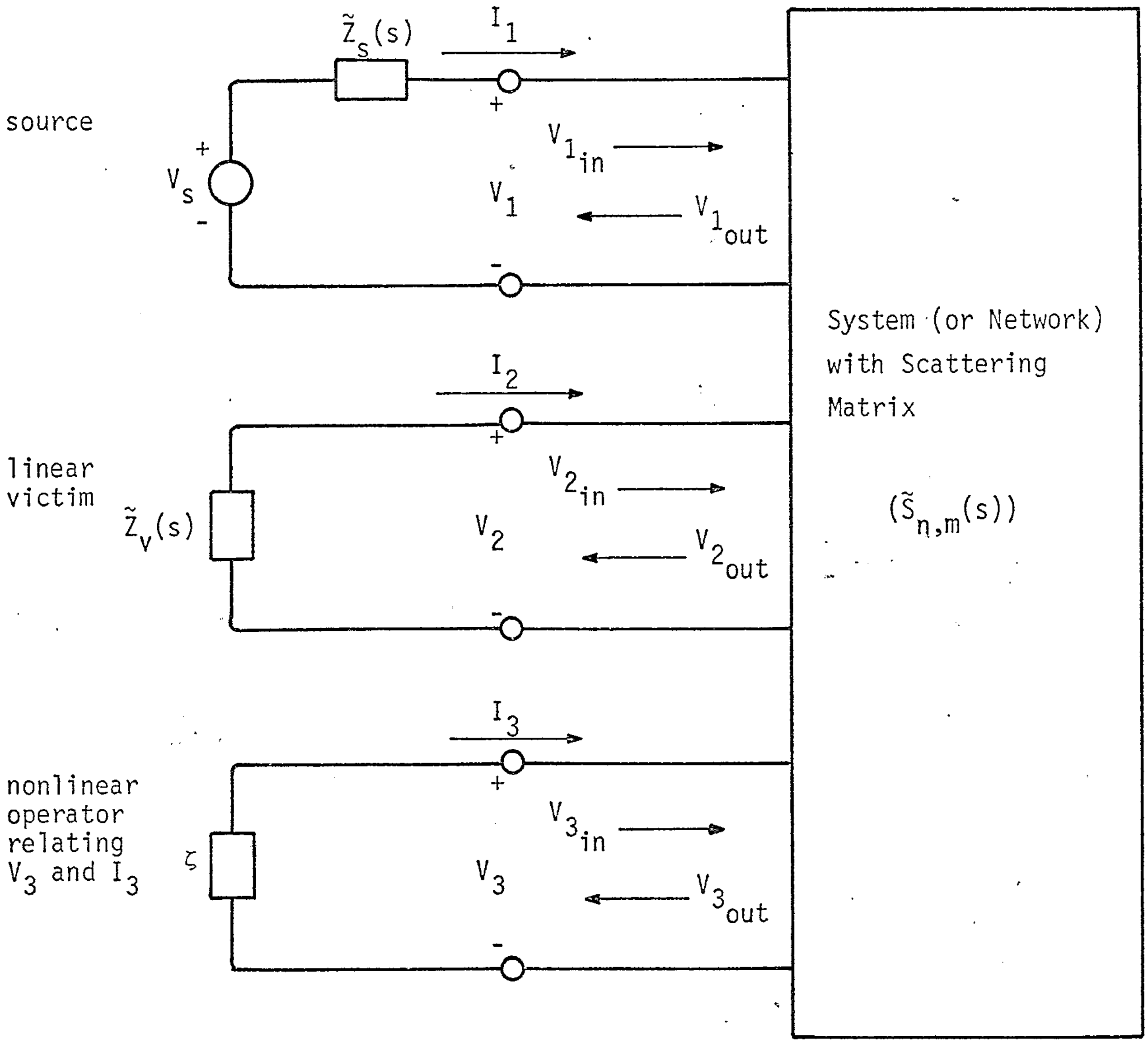


Fig. 5.1. Representation of Linear Source, Linear Victim, and Nonlinear Element Connected to a 3-Port System (or Network)

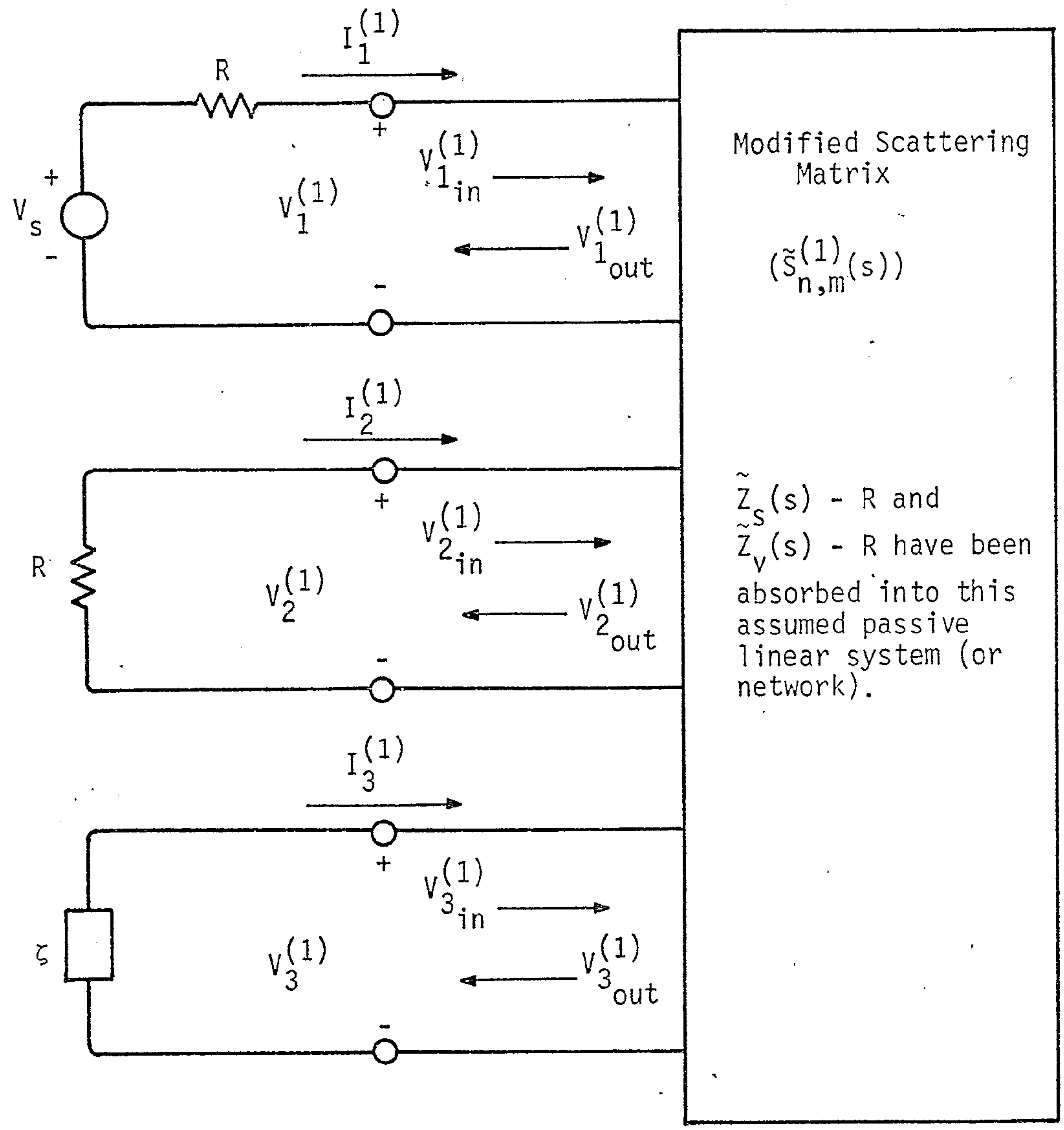


Fig. 5.2. Modified Scattering Matrix for 3-Port

In this configuration any waves scattered through the system into ports 1 and 2 are terminated (i.e., not rescattered back into the system). As indicated in fig. 5.2 we immediately have

$$\tilde{V}^{(1)}_{2_{in}}(s) = 0 \tag{5.4}$$

simplifying the problem somewhat. The effect of having a resistance R as the loads on ports 1 and 2 is summarized in fig. 5.3 where, as one can see, only six scattering elements are needed to characterize the scattering of the various waves. Three scattering elements, representing the scattering at port 2, are not needed since there is no source there and all waves incident on the load, R, are terminated. Again, as in section 2, the inclusion of $\tilde{Z}_s(s)$ and $\tilde{Z}_v(s)$ which are passive impedances into the modified system keep the system passive and linear, but one must be cautious about the removal of the resistance R out of the system via ports 1 and 2 since it is desirable that the modified system remain passive; this limits the allowable range of R (positive).

Identifying R with the diagonal reference-impedance matrix in (2.13), let us write out the scattering equations for the modified system as

$$\begin{aligned}
\tilde{V}^{(1)}_{1_{out}}(s) &= \tilde{S}^{(1)}_{1,1}(s)\ \tilde{V}^{(1)}_{1_{in}}(s) + \tilde{S}^{(1)}_{1,2}(s)\ \tilde{V}^{(1)}_{2_{in}}(s) + \tilde{S}^{(1)}_{1,3}(s)\ \tilde{V}^{(1)}_{3_{in}}(s) \\
\tilde{V}^{(1)}_{2_{out}}(s) &= \tilde{S}^{(1)}_{2,1}(s)\ \tilde{V}^{(1)}_{1_{in}}(s) + \tilde{S}^{(1)}_{2,2}(s)\ \tilde{V}^{(1)}_{2_{in}}(s) + \tilde{S}^{(1)}_{2,3}(s)\ \tilde{V}^{(1)}_{3_{in}}(s) \\
\tilde{V}^{(1)}_{3_{out}}(S) &= \tilde{S}^{(1)}_{3,1}(s)\ \tilde{V}^{(1)}_{1_{in}}(s) + \tilde{S}^{(1)}_{3,2}(s)\ \tilde{V}^{(1)}_{2_{in}}(s) + \tilde{S}^{(1)}_{3,3}(s)\ \tilde{V}^{(1)}_{3_{in}}(s)
\end{aligned} \tag{5.5}$$

This is further simplified by noting as in (5.4) that the incoming wave at port 2 is zero.

At port 1 we again have

$$\begin{aligned}
\tilde{V}_s(s) - R\ \tilde{I}_1(s) &= \tilde{V}^{(1)}_1(s) = \frac{1}{2}\left\{\tilde{V}^{(1)}_{1_{in}}(s) + \tilde{V}^{(1)}_{1_{out}}(s)\right\} \\
\tilde{V}_s(s) &= \tilde{V}^{(1)}_1(s) + R\ \tilde{I}^{(1)}_1(s) = \tilde{V}_{1_{in}}(s)
\end{aligned} \tag{5.6}$$

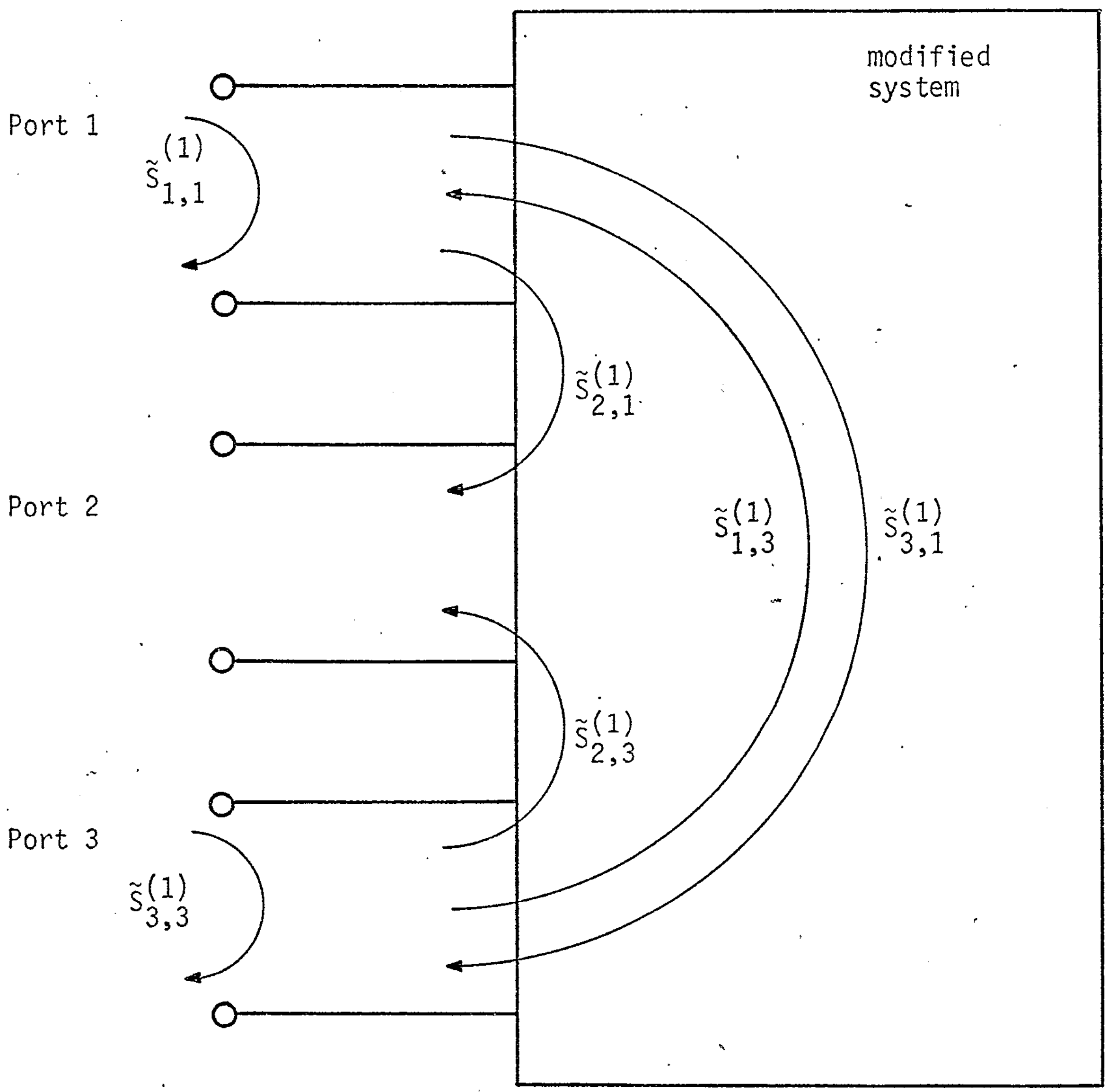


Fig. 5.3. Diagram of Wave Transport Through Modified System

Then (5.5) becomes

$$\hat{V}^{(1)}_{1_{out}}(s) = \hat{S}^{(1)}_{1,1}(s)\ \hat{V}_s(s) + \hat{S}^{(1)}_{1,3}(s)\ \hat{V}^{(1)}_{3_{in}}(s)$$

$$\hat{V}^{(1)}_{2_{out}}(s) = \hat{S}^{(1)}_{2,1}(s)\ \hat{V}_s(s) + \hat{S}^{(1)}_{2,3}(s)\ \hat{V}^{(1)}_{3_{in}}(s) \qquad (5.7)$$

$$\hat{V}^{(1)}_{3_{out}}(s) = \hat{S}^{(1)}_{3,1}(s)\ \hat{V}_s(s) + \hat{S}^{(1)}_{3,3}(s)\ \hat{V}^{(1)}_{3_{in}}(s)$$

Now $\hat{V}^{(1)}_{3_{in}}(s)$ and $\hat{V}^{(1)}_{3_{out}}(s)$ at port 3 can be related through the assumed passive nonlinear load ζ giving the results

$$\|\hat{V}^{(1)}_{3_{out}}(t)\|_{2,t} \geq \|\hat{V}^{(1)}_{3_{in}}(t)\|_{2,t} \geq 0$$

$$\|\hat{V}^{(1)}_{3_{out}}(j\omega)\|_{2,\omega} \geq \|\hat{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega} \geq 0 \qquad (5.8)$$

which are of the same form as (4.7).

Applying the 2-norm with respect to frequency to (5.7) gives

$$\|\hat{V}^{(1)}_{1_{out}}(j\omega)\|_{2,\omega} \leq \|\hat{S}^{(1)}_{1,1}(j\omega)\ \hat{V}_s(j\omega)\|_{2,\omega} + \|\hat{S}^{(1)}_{1,3}(j\omega)\ \hat{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

$$\|\hat{V}^{(1)}_{2_{out}}(j\omega)\|_{2,\omega} \leq \|\hat{S}^{(1)}_{2,1}(j\omega)\ \hat{V}_s(j\omega)\|_{2,\omega} + \|\hat{S}^{(1)}_{2,3}\ \hat{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega} \qquad (5.9)$$

$$\|\hat{V}^{(1)}_{3_{out}}(j\omega)\|_{2,\omega} \leq \|\hat{S}^{(1)}_{3,1}(j\omega)\ \hat{V}_s(j\omega)\|_{2,\omega} + \|\hat{S}^{(1)}_{3,3}(j\omega)\ \hat{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

Defining the supremum over frequency of the scattering elements as in (4.9) and using (A.5) the inequalities become

$$\|\hat{V}^{(1)}_{1_{out}}(j\omega)\|_{2,\omega} \leq |\hat{S}^{(1)}_{1,1}|_{sup}\ \|\hat{V}_s(j\omega)\|_{2,\omega} + |\hat{S}^{(1)}_{1,3}|_{sup}\ \|\hat{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

$$\|\tilde{V}^{(1)}_{2_{out}}(j\omega)\|_{2,\omega} \leq |\tilde{S}^{(1)}_{2,1}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} + |\tilde{S}^{(1)}_{2,3}|_{sup}\ \|\tilde{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega} \tag{5.10}$$

$$\|\tilde{V}^{(1)}_{3_{out}}(j\omega)\|_{2,\omega} \leq |\tilde{S}^{(1)}_{3,1}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} + |\tilde{S}^{(1)}_{3,3}|_{sup}\ \|\tilde{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

For a port-3 result substitute the second of (5.8) into the third of (5.10) to give

$$\|\tilde{V}^{(1)}_{3_{out}}(j\omega)\|_{2,\omega} \leq |\tilde{S}_{3,1}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} + |\tilde{S}^{(1)}_{3,3}|_{sup}\ \|\tilde{V}^{(1)}_{3_{out}}(j\omega)\|_{2,\omega} \tag{5.11}$$

which can be rearranged as

$$\|\tilde{V}^{(1)}_{3_{out}}(j\omega)\|_{2,\omega} \leq [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1}\ |\tilde{S}^{(1)}_{3,1}|_{sup}\ \|\tilde{V}_s(j\omega)\|_{2,\omega} \tag{5.12}$$

and expressed also in time domain as

$$\|V^{(1)}_{3_{out}}(t)\|_{2,t} \leq [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1}\ |\tilde{S}^{(1)}_{3,1}|_{sup}\ \|V_s(t)\|_{2,t} \tag{5.13}$$

These results for the port with the nonlinear load ζ directly parallel those in section 2. Note again the importance of having $|\tilde{S}^{(1)}_{3,3}|_{sup}$ less than one, while a general condition of passivity will only require that it be less than or equal to one as in (4.17). Again R needs to be optimally chosen to minimize the bound in (5.12) and (5.13). By including 1/(4R) in (5.13) this bound can be put on an energy basis as in (4.24) for minimization with respect to R.

For a port-1 result the first of (5.10) can be combined with (5.12) and the second of (5.8) to give

$$\|\tilde{V}^{(1)}_{1_{out}}(j\omega)\|_{2,\omega} \leq \left\{ |\tilde{S}^{(1)}_{1,1}|_{sup} + [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1}\ |\tilde{S}^{(1)}_{1,3}|_{sup}\ |\tilde{S}^{(1)}_{3,1}|_{sup} \right\} \|\tilde{V}_s(j\omega)\|_{2,\omega} \tag{5.14}$$

which is like the result in (4.14). Another bound is obtained by rearranging the first of (5.7) and taking 2-norms with respect to ω (and using (5.8) and (5.12)) to give

$$\|\tilde{V}^{(1)}_{1_{out}}(j\omega) - \tilde{S}^{(1)}_{1,1}(j\omega)\ \tilde{V}^{(1)}_{s}(j\omega)\|_{2,\omega}$$

$$= \|\tilde{S}^{(1)}_{1,3}(j\omega)\ \tilde{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

$$< |\tilde{S}^{(1)}_{1,3}|_{sup}\ \|\tilde{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

$$< [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1}\ |\tilde{S}^{(1)}_{1,3}|_{sup}\ |\tilde{S}^{(1)}_{3,1}|_{sup}\ \|\tilde{V}_{s}(j\omega)\|_{2,\omega} \qquad (5.15)$$

This is a 2-norm bound on the wave scattered from port 3 back to port 1, and hence some bound of the effect of the nonlinearity at port 3 on port 1. Note that this effectively is a second power dependence on the transfer function through the system between ports 1 and 3.

Now consider port 2, the victim port. The second of (5.7) gives, with 2-norm over ω (and using (5.8)),

$$\|\tilde{V}^{(1)}_{2_{out}}(j\omega)\|_{2,\omega} < \|\tilde{S}^{(1)}_{2,1}(j\omega)\ \tilde{V}_{s}(j\omega)\|_{2,\omega} + \|\tilde{S}^{(1)}_{2,3}(j\omega)\ \tilde{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

$$< |\tilde{S}^{(1)}_{2,1}|_{sup}\ \|\tilde{V}_{s}(j\omega)\|_{2,\omega} + |\tilde{S}^{(1)}_{2,3}|_{sup}\ \|\tilde{V}^{(1)}_{3_{in}}(j\omega)\|_{2,\omega}$$

$$< \left\{ |\tilde{S}^{(1)}_{2,1}|_{sup} + [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1}\ |\tilde{S}^{(1)}_{2,3}|_{sup}\ |\tilde{S}^{(1)}_{3,1}|_{sup} \right\}$$

$$\|\tilde{V}_{s}(j\omega)\|_{2,\omega} \qquad (5.16)$$

In this last form we have terms which represent bounds on the norms of signals arriving at port 2 from ports 1 and 3. Now the nonlinear load ζ can be considered to have negligible effect on the victim at port 2 provided generally that

$$|\tilde{S}^{(1)}_{2,1}|_{sup} \gg [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1} |\tilde{S}^{(1)}_{2,3}|_{sup}\ |\tilde{S}^{(1)}_{3,1}|_{sup} \qquad (5.17)$$

This gives some measure of the significance of the nonlinear load when considering some other element (victim) elsewhere in the system. Note that (5.16) can also be considered directly in time domain as

$$\|\tilde{V}^{(1)}_{2_{out}}(t)\|_{2,t} < \left\{ |\tilde{S}^{(1)}_{2,1}|_{sup} + [1 - |\tilde{S}^{(1)}_{3,3}|_{sup}]^{-1} |\tilde{S}^{(1)}_{2,3}|_{sup} |\tilde{S}^{(1)}_{3,1}|_{sup} \right\} \|V_s(t)\|_{2,t} \tag{5.18}$$

Now a certain problem presents itself. We have some bounds in terms of the 2-norm of the wave incident on the load at port 2 of the modified system, but the modified system has R at the port instead of $\tilde{Z}_v(s)$ in the case of he original or unmodified system. In order to relate the aforegoing results to the situation of the original (unmodified) system, note that the difference between the two involves a change from the original waves $(V_n(t))_{\substack{in\\out}}$ to the modified waves $(V^{(1)}_n(t))_{\substack{in\\out}}$. In the process the currents at the ports remain the same while the voltages are changed due to the fact that the changes involve series (not parallel) movement of impedance elements, i.e.,

$$(I^{(1)}_n(t)) = (I_n(t)) \tag{5.19}$$

Since the currents are unchanged then, in particular at port 2, we have the same current through R as through $\tilde{Z}_v(s)$. At port 2 we have

$$\begin{aligned} V^{(1)}_{2_{in}}(t) &= 0 = V^{(1)}_2(t) + R\, I^{(1)}_2(t) \\ V^{(1)}_{2_{out}}(t) &= V^{(1)}_2(t) - R\, I^{(1)}_2(t) \end{aligned} \tag{5.20}$$

From which we have

$$\begin{aligned} V_2(t) &= \frac{1}{2} V^{(1)}_{2_{out}}(t) \\ I_2(t) &= -\frac{1}{2R} V^{(1)}_{2_{out}}(t) \end{aligned} \tag{5.21}$$

Hence we have

$$\| I_2^{(1)}(t) \|_{2,t} = \| I_2(t) \|_{2,t}$$

$$< \frac{1}{2R} \left\{ |S_{2,1}^{(1)}|_{sup} + [1 - |\tilde{S}_{3,3}^{(1)}|_{sup}]^{-1} |\tilde{S}_{2,3}^{(1)}|_{sup} |\tilde{S}_{3,1}^{(1)}|_{sup} \right\}$$

$$\| \tilde{V}_s(t) \|_{2,t} \quad (5.22)$$

Consider now the energy deposited in $\tilde{Z}_v(s)$, the impedance at port 2 in the original (or unmodified) system. Noting that the port currents are not changed by the system modification we have energy delivered out of port 2 as

$$U_v = \int_{-\infty}^{\infty} P_v(t)\, dt = - \int_{-\infty}^{\infty} V_2(t)\, I_2(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}_2(j\omega)\, \tilde{I}_2^*(j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}_2^*(j\omega)\, \tilde{I}_2(j\omega)\, d\omega \quad (5.23)$$

using (C.3). Using $\tilde{Z}_v(s)$ we can eliminate $\tilde{V}_2(s)$ giving

$$U_v = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}_2(j\omega)\, \tilde{Z}_v(j\omega)\, \tilde{I}_2^*(j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{I}_2^*(j\omega)\, \tilde{Z}_v^*(j\omega)\, \tilde{I}_2(j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}[\tilde{Z}_v(j\omega)]\, |\tilde{I}_2(j\omega)|^2\, d\omega$$

$$\leq \frac{1}{2\pi} |\mathrm{Re}[\tilde{Z}_v]|_{sup} \| \tilde{I}_2(j\omega) \|_{2,\omega}^2$$

$$= |\mathrm{Re}[\tilde{Z}_v]|_{sup} \| I_2(t) \|_{2,t}^2 \quad (5.24)$$

using (A.5). Then from (5.21)

$$U_v < \frac{1}{4R^2} \left|\mathrm{Re}[\hat{Z}_v]\right|_{sup} \| \hat{V}^{(1)}_{2_{out}}(t) \|^2 \qquad (5.25)$$

which combines with (5.18) to give

$$U_v < \frac{1}{4R^2} \left|\mathrm{Re}[\hat{Z}_v]\right|_{sup} \left\{ |\hat{S}^{(1)}_{2,1}|_{sup} + [1 - |\hat{S}^{(1)}_{3,3}|_{sup}]^{-1} |\hat{S}^{(1)}_{2,3}|_{sup} |\hat{S}^{(1)}_{3,1}|_{sup} \right\}^2$$
$$\| V_s(t) \|^2_{2,t} \qquad (5.26)$$

This result contains the same two terms from ports 1 and 3 as in (5.16), but they are now squared after summation because this result is in the form of energy. Note the inclusion of $\left|\mathrm{Re}[\hat{Z}_v]\right|_{sup}$ from the original (unmodified) system in this result. By judicious choice of R one may minimize the bound on U_v, or on the parts of the bound associated with ports 1 or 3, but this depends on presently unspecified characteristics of the system or network of interest.

VI. Summary

In this paper we have obtained bounds on the effect of a certain class of passive nonlinear elements on linear elements elsewhere in a general linear electromagnetic system. Note, however, these bounds are in general a function of a resistance (R > 0) used in defining the reference impedance matrix which in turn defines the wave variables and the modified system scattering matrix. Perhaps one can vary R within some acceptable limits and find a minimum value for the appropriate bound, but this may be very system specific. Note, in particular, that these bound formulas depend on bounding certain scattering-matrix elements by values less than unity. This in turn requires that these elements contain loss, and in general that the modified system contain loss (i.e., absorbs energy). Hence, these bounds do not apply to every conceivable case, some of which may require other approaches.

The bounds in this paper show that for at least some cases a nonlinearity need not be understood in detail because its effect can be negligibly small and hence neglected. Note that the requisite scattering matrix elements can be measured on a real system and transformed to the modified scattering matrix to obtain <u>empirical</u> bounds to determine when such a nonlinearity can be neglected.

Perhaps future papers can obtain additonal bounds concerning this and other similar nonlinear problems.

Appendix A. Norm Conventions

Fundamental to our present considerations is the concept of a norm. Various texts discuss norms of vectors and matrices and the continuous case of functions and operators. The case of vectors and matrices has been discussed in a previous note [4]. Norms of functions have been used in another previous note [2]. In this note we distinguish these two kinds of norms.

First we have the vector norm in its common manifestation, the p norm, as

$$\|(x_n)\|_p = \left\{ \sum_{n=1}^{N} |x_n|^p \right\}^{1/p} \tag{A.1}$$

Now (x_n) may be a function of some variable such as time, t, or complex frequency (or Laplace transform variable), s ($= \Omega + j\omega$), which can be specialized as the usual radian frequency variable, ω. One could consider a norm as in (A.1) as a function of t or ω or whatever. Let us define vector norms which also consider such variables as

$$\|(x_n(y))\|_{p,y} \equiv \left\{ \int_{-\infty}^{\infty} \sum_{n=1}^{N} |x_n(y)|^p \, dy \right\}^{1/p}$$

$$y = \text{a real variable} \tag{A.2}$$

where, of course, this norm is defined only for vector functions for which the integral exists. As a special case $(x_n(y))$ can be a scalar $x(y)$ for which $N = 1$ and the above definition (A.2) still applies.

Observe that with the integration over the variable, y, there are mathematical peculiarities in the norm depending on the continuity properties of the $x_n(y)$. Specifically isolated points of the $x_n(y)$ which are not continuous on both sides of the function for, say, $y = y_0$, do not contribute to the integration. In terms of the commonly used ∞ norm

$$\|(x_n(y))\|_\infty = \sup_{\substack{-\infty<y<\infty \\ 1\leq n\leq N}} |x_n(y)| \tag{A.3}$$

such isolated points of discontinuity must be excluded from consideration in the supremum since they contribute nothing to the integral. Note that (A.3) is a special case which is not necessarily normalized in the same way as (A.2), and might better have a special symbol, such as sup.

A useful result for these functional norms is

$$
\begin{aligned}
\|f(y)(x_n(y))\|_{p,y} &= \left\{ \int_{-\infty}^{\infty} \sum_{n=1}^{N} |f(y)x_n(y)|^p \, dy \right\}^{1/p} \\
&= \left\{ \int_{-\infty}^{\infty} |f(y)|^p \sum_{n=1}^{N} |x_n(y)|^p \, dy \right\}^{1/p} \\
&\leq \left\{ \int_{-\infty}^{\infty} |f|_{\sup}^p \sum_{n=1}^{N} |x_n(y)|^p \, dy \right\}^{1/p} \\
&= \left\{ |f|_{\sup}^p \int_{-\infty}^{\infty} \sum_{n=1}^{N} |x_n(y)|^p \, dy \right\}^{1/p} \\
&= |f|_{\sup} \left\{ \int_{-\infty}^{\infty} \sum_{n=1}^{N} |x_n(y)^p \, dy \right\}^{1/p} \qquad \text{(A.4)}
\end{aligned}
$$

Summarizing then

$$
\begin{aligned}
&\|f(y)(x_n(y))\|_{p,y} \leq |f|_{\sup} \|(x_n(y))\|_{p,y} \\
&|f|_{\sup} \equiv \sup_{y \text{ real}} |f(y)|
\end{aligned}
\qquad \text{(A.5)}
$$

assuming that $|f|_{\sup}$ and $\|(x_n(y))\|_{p,y}$ exist.

Appendix B. Parseval's Theorem and The 2-Norm

In this paper our concern is primarily with the 2-norm, particularly because of its association with energy considerations which are directly connected with the concept of passivity, including passive nonlinear elements. Central to our considerations is the relation between time and frequency domains in such energy considerations, as can be found in Parseval's theorem.

We have the two-sided Laplace transform

$$\tilde{f}(s) \equiv \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi j} \int_{Br} \tilde{f}(s)\, e^{st}\, ds \tag{B.1}$$

Br ≡ Bromwich contour in strip of convergence parallel to $j\omega$ axis

$s = \Omega + j\omega$ ≡ Laplace transform variable or complex frequency

We have the common convolution theorems

$$\int_{-\infty}^{\infty} f_1(t) f_2(t)\, e^{-st}\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s')\tilde{f}_2(s-s')\, ds'$$

$$\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{t} f_1(t') f_2(t-t')\, dt' \right\} e^{-st}\, dt = \tilde{f}_1(s)\tilde{f}_2(s) \tag{B.2}$$

which expresses the well-known result that multiplication of two functions in one domain corresponds to convolution of the two functions in the other domain. Of course, certain restrictions must be placed on the functions so that the transforms exist. Without getting into detail one can note that a sufficient condition is the square integrability of the functions [5].

Specializing the above result (the first of (B.2) to the case of $s = 0$) we have with appropriate change of variables,

$$\int_{-\infty}^{\infty} f_1(t) f_2(t)\, dt = \frac{1}{2\pi j} \int_{Br} \tilde{f}_1(s)\tilde{f}_2(-s)\, ds \tag{B.3}$$

If we further restrict $f_1(t)$ and $f_2(t)$ to be real valued (for real t) then $f_1(s)$ and $f_2(s)$ are conjugate symmetric. Specializing the Bromwich contour to the $j\omega$ axis then gives

$$\int_{-\infty}^{\infty} f_1(t)f_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}_1(j\omega)\ \tilde{f}_2(-j\omega)\ d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}_1(j\omega)\ \tilde{f}_2^{*}(j\omega)\ d\omega \qquad \text{(B.4)}$$

A more limited form is obtained if we take $f_1(t)$ and $f_2(t)$ as the same function giving

$$\int_{-\infty}^{\infty} f^2(t)\ dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\tilde{f}(j\omega)\right|^2 d\omega \qquad \text{(B.5)}$$

which is the usual form of the Parseval theorem. One can think of this as a statement relating energy in time domain to energy in frequency domain.

Looking at the norms discussed in Appendix A one can readily see that (B.5) takes the form of a relation between 2 norms as

$$\|f(t)\|_{2,t} = \frac{1}{\sqrt{2\pi}}\ \|\tilde{f}(j\omega)\|_{2,\omega} \qquad \text{(B.6)}$$

which is a compact statement of the Parseval theorem.

Appendix C. Passivity and The 2-Norm.

Consider an N-port network (or more general distributed elctromagnetic system). Let the port variables be

$$\begin{aligned} V_n(t) &\equiv \text{voltage at nth port} \\ I_n(t) &\equiv \text{current at nth port} \end{aligned} \tag{C.1}$$

with the convention that positive $V_n(t)I_n(t)$ represents power flow <u>into</u> the port.

Now this network may be linear or nonlinear. However, we assume that the network is <u>passive</u>, i.e., that

$$\int_{-\infty}^{t} \left(V_n(t')\right) \cdot \left(I_n(t')\right) dt \geqslant 0 \tag{C.2}$$

with zero initial stored energy in the network. Referring back to (B.4) we then have

$$\begin{aligned} \int_{-\infty}^{\infty} \left(V_n(t)\right) \cdot \left(I_n(t)\right) dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{V}_n(j\omega)\right) \cdot \left(\tilde{I}_n(-j\omega)\right) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\tilde{V}_n(j\omega)\right) \cdot \left(\tilde{I}_n(j\omega)\right)^* d\omega \end{aligned} \tag{C.3}$$

This result is remarkable in that it does not imply that the power into the network is positive for any given ω, but that the integral of such power over all ω is still positive. If the network is linear then one can say that the power into the network is non-negative for each ω.

Now a previous paper [3] has shown that a similar power relationship applies to wave and scattering variables if the normalizing admittance (or impedance) is appropriately chosen. Defining wave variables

$$\left(\tilde{V}_n(s)\right)_{in} = \left(\tilde{V}_n(s)\right) + \left(\tilde{Z}_{n,m}^{(ref)}(s)\right) \cdot \left(\tilde{I}_n(s)\right)$$

$\equiv$ vector wave into N-port

$$(\tilde{V}_n(s))_{out} = (\tilde{V}_n(s)) - (\tilde{Z}_{n,m}^{(ref)}(s)) \cdot (\tilde{I}_n(s))$$

$\equiv$ vector wave out of (scattered from) N-port (C.4)

$(\tilde{Y}_{n,m}^{(ref)}(s)) = (\tilde{Z}_{n,m}^{(ref)}(s))^{-1} \equiv$ normalizing admittance matrix for wave vectors

Provided [3]

$$(\tilde{Y}_{n,m}^{(ref)}(j\omega)) = \tilde{\lambda}(j\omega)(1_{n,m})$$

$$\tilde{\lambda}(j\omega) > 0 \text{ (and thereby real)} \tag{C.5}$$

then in frequency domain the wave vectors do represent power flow, and for a linear, passive, time invariant network we have

$$\frac{1}{2}\left[(\tilde{V}_n(j\omega))^* \cdot (\tilde{I}_n(j\omega)) + (\tilde{V}_n(j\omega)) \cdot (\tilde{I}_n(j\omega))^*\right] \geq 0 \tag{C.6}$$

which in terms of wave variables becomes [3]

$$\begin{aligned} &(\tilde{V}_n(j\omega))_{in} \cdot (\tilde{V}_n(j\omega))_{in}^* \\ &\quad \geq (\tilde{V}_n(j\omega))_{out} \cdot (\tilde{V}_n(j\omega))_{out}^* \\ &\quad \geq 0 \end{aligned} \tag{C.7}$$

Construct now in time domain a wave variable of the form

$$\begin{aligned} (V_n(t))_{in} &= (V_n(t)) + R(I_n(t)) \\ (V_n(t))_{out} &= (V_n(t)) - R(I_n(t)) \\ R &= \frac{1}{G} > 0 \end{aligned} \tag{C.8}$$

where R and G are now positive constants representing a simple resistance and conductance, respectively. Having chosen the normalizing impedance as a positive constant times the identity matrix we have avoided the presence of a convolution in (C.8).

Since now power into the network is proportional to

$$(V_n(t))_{in} \cdot (V_n(t))_{in} - (V_n(t))_{out} \cdot (V_n(t))_{out}$$
$$= 4R(V_n(t)) \cdot (I_n(t)) \tag{C.9}$$

with

$$(V_n(t))_{in} \cdot (V_n(t))_{in} > 0$$
$$(V_n(t))_{out} \cdot (V_n(t))_{out} \geq 0 \tag{C.10}$$

Using (C.2) assuming that we are dealing with a passive, but perhaps non-linear, network with zero initial energy, we have

$$\int_{-\infty}^{t} (V_n(t'))_{in} \cdot (V_n(t'))_{in}\, dt'$$
$$\geq \int_{-\infty}^{t} (V_n(t'))_{out} \cdot (V_n(t'))_{out}\, dt'$$
$$\geq 0 \tag{C.11}$$

Extending the upper limit $t \to \infty$ gives

$$\|(V_n(t))_{in}\|_{2,t} > \|(V_n(t))_{out}\|_{2,t} \geq 0 \tag{C.12}$$

Applying the Parseval theorem gives

$$\|(\tilde{V}_n(j\omega))_{in}\|_{2,\omega} > \|(\tilde{V}_n(j\omega))_{out}\|_{2,\omega} > 0 \tag{C.13}$$

Again note that for nonlinear networks the inequality does not apply to individual frequencies in a pulse, but to an integral over all frequencies in the 2-norm sense.

Appendix D. Passive Realizability Condition for Normalizing Impedance or Admittance Matrix for Wave Variables

In a previous paper [3] the desirability of having wave variables have a power relationship (like the usual product of voltage and current) led to the result that the normalizing admittance for defining the wave variables takes the form

$$\left(\tilde{Y}_{n,m}^{(ref)}(j\omega)\right) = \tilde{\lambda}(j\omega)(1_{n,m})$$

$$\tilde{\lambda}(j\omega) > 0 \tag{D.1}$$

Here let us extend the argument further.

Let us constrain that $\tilde{\lambda}(s)$ be passive so that it may be physically realizable in a convenient form. Since $\tilde{\lambda}(s)$ is a self or driving-point admittance, then passivity requires that $\tilde{\lambda}(s)$ (as well as $\tilde{\lambda}(s)^{-1}$) be a p.r. (positive real) function in the usual sense [7].

Furthermore, let us constrain that $\tilde{\lambda}(s)$ have no singularities or zeros on the $j\omega$ axis. This is required by our requirement for $s = j\omega$ that the normalizing admittance actually represent a linear combination of voltage and current variables into wave variables; $\tilde{\lambda}(j\omega) = 0$ would eliminate voltage from the combination and $\tilde{\lambda}(j\omega) = \infty$ would eliminate current from the combination. Hence we require

$$0 < \varepsilon_1 \leq \tilde{\lambda}(j\omega) \leq \varepsilon_2 < \infty$$

$$\text{for } -\infty \leq \omega \leq \infty \tag{D.2}$$

One can also consider the question of zeros and singularities of $\tilde{\lambda}(s)$ on the $j\omega$ axis by noting that the phase is $\arg[\tilde{\lambda}(j\omega)] = 0$ since $\mathrm{Im}[\tilde{\lambda}(j\omega)] = 0$ and $\mathrm{Re}[\tilde{\lambda}(s)] \geq 0$ for $\mathrm{Re}[s] \geq 0$. The continuity of the phase staying approximately zero in the vicinity of the $j\omega$ axis ($\mathrm{Re}[s] \geq 0$) removes the possibility of such zeros or singularities.

With this restriction on $\tilde{\lambda}(j\omega)$ let us look at the relation between the real and the imaginary parts of a p.r. function, $\tilde{\lambda}(s)$, as discussed in various texts [7], i.e.,

$$\mathrm{Re}[\hat{\lambda}(j\omega)] = \mathrm{Re}[\hat{\lambda}(j\infty)] - \frac{2}{\pi}\int_0^\infty \frac{\omega' \,\mathrm{Im}[\hat{\lambda}(j\omega')]\, d\omega'}{\omega'^2 - \omega^2}$$

$$\mathrm{Im}[\hat{\lambda}(j\omega)] = \frac{2\omega}{\pi}\int_0^\infty \frac{\mathrm{Re}[\hat{\lambda}(j\omega)]}{\omega'^2 - \omega^2}\, d\omega' \tag{D.3}$$

Now applying our result that

$$\mathrm{Im}[\hat{\lambda}(j\omega)] = 0 \tag{D.3}$$

gives

$$\begin{aligned} \mathrm{Re}[\hat{\lambda}(j\omega)] &= \mathrm{Re}[\hat{\lambda}(j\infty)] \\ &\equiv \lambda_o \quad \text{(a constant)} \end{aligned} \tag{D.4}$$

and hence

$$\hat{\lambda}(j\omega) = \lambda_o$$

$$0 < \varepsilon_1 \leqslant \lambda_o \leqslant \varepsilon_2 < \infty \tag{D.5}$$

Since $\hat{\lambda}(s) = \lambda_o$ on the $j\omega$ axis then let us apply the concept of analytic continuation into the right half of the s plane. Clearly

$$\hat{\lambda}(s) = \lambda_o \quad \text{for } \mathrm{Re}[s] \geqslant 0 \tag{D.6}$$

is a solution and the uniqueness of the analytic continuation [6] makes this solution the only one. Note that the above solution is easily analytically continued into the left half plane as well.

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