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A Study Of The Various Methods For Computing Electromagnetic Field Utilizing Thin Wire Integral Equations

Tapan K. Sarkar
Department of Electrical Engineering
Rochester Institute of Technology
Rochester, New York 14623

Abstract

In this paper we analyze the numerical aspects of the various methods that have been utilized to analyze thin wire antennas. First we derive the properties of the operators for Pocklington's and Hallen's integral equations. Based on these properties we discuss the various iterative methods used to find current distribution on thin wire structures. An attempt has been made to resolve the question of numerical stability associated with various entire domain and subdomain expansion functions in Galerkin's method. It has been shown that the sequence of solutions generated by the iterative methods monotonically approaches the exact solution provided the excitations chosen for these problems are in the range of the operator. Such a statement may not hold for Galerkin's methods if the inverse operator is unbounded. Moreover if the excitation function is not in the range of the operator, then the sequence of solutions form an asymptotitc series. Examples have been presented to illustrate this point.

1. INTRODUCTION

Over the past few years several methods have been developed by many researchers [Gray (1966), Harrington (1968), King et al. (1946), Schelkunoff (1952), Siegel et al. (1934) and Wu (1969)] to analyze scattering and radiation from thin wire structures. In this presentation we investigate the properties of the integro-differential equations that arise for the various techniques developed so far. The motivation for this work is to study the causes of the numerical instabilities that sometimes arise in the solution of the current distribution on thin wire structures. The numerical instabilities may be either due to an incorrect application of the numerical techniques or due to the operator equation being actually ill-posed. In a recent note Jones [1981] has claimed that the Hallen's integral equation is a well posed problem and that the sources of numerical instabilities lie with the particular numerical technique used to solve an operator equation. As we shall presently demonstrate that the proof presented by Jones is not complete. Jones did not consider all the aspects of a well posed problem. For a problem to be well-posed, three conditions have to be met by the operator equations. According to Stakgold [1979, p. 58] the three conditions are defined as follows:

"When dealing with boundary value problems we shall still be faced with these three questions:

(1) Is there at least one solution (existence)?

(2) Is there at most one solution (uniqueness)?

(3) Does the solution depend continuously on the data?

If the answer to this trio of questions is affirmative the problems is said to be well-posed (otherwise ill-posed). Until recently it was sound dogma to require that every real physical problem be well posed. However it is now understood that ill-posed problems occur frequently in practice but that their physical interpretation and mathematical solution are somewhat more delicate."

Jones, in his paper, addresses only the first two questions. In his note he did not check whether the inverse operator in Hallen's integral equation is bounded or not! It is based on the third statement of Stakgold that the Hallen's integral equation is ill-posed. We prove later that the operator involved in Hallen's integral equation is compact, and hence its inverse is unbounded. Therefore, by definition, Hallen's integral equation is an ill-posed problem. Tikhonov and Dimitriev [1968] were the first to recognize that Hallen's integral equation is ill-posed and developed a "self-regularization" procedure to solve that integral equation. In summary, if any numerical instability is observed for Hallen's integral equation, one cannot put the blame entirely on the numerical procedure utilized to obtain a solution.

In this paper, we investigate the properties of the Pocklington's integro-differential operator and the Hallen's integral operator. We also discuss the advangates of an iterative method and the direct method (Galerkin's method) of solving the two operator equations and the numerical stabilities of the various order of the solutions given by the two techniques. Finally, we investigate the convergence properties of the approximate solutions when the excitation is not in the range of the operator.

2. PROPERTY OF THE POCKLINGTON E-FIELD OPERATOR

The Pocklington integral equation for the current on the surface of an antenna can be written by equating the total tangential electric field on the conductor surface to be zero, i.e.

$$E^{i}_{tan} + E^{s}_{tan} = 0 \tag{1}$$

where the subscript represents the tangential component of the electric field and the superscripts i and s stand for the incident and scattered fields, respectively. By assuming a time variation of the form $\exp(j\omega t)$ equation (1) can be rewritten for the tubular antenna of length L and radius a as [1]

$$k^2 \int_{-L/2}^{+L/2} dz' \, I(z') \, G(z,z') + \frac{\partial^2}{\partial z^2} \int_{-L/2}^{+L/2} dz' \, I(z') \, G(z,z') = j\omega 4\pi\varepsilon E^{i}_{tan}(z)$$

$$\text{for } -\frac{L}{2} \leq z \leq +\frac{L}{2} \tag{2}$$

where,

$$G(z,z') = \text{Green's function} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{\exp(-jkR)}{R} \, d\phi \tag{3}$$

$$R = \sqrt{(z-z')^2 + 4a^2 \sin^2 \frac{\phi}{2}} \tag{4}$$

and $k = \frac{2\pi}{\lambda}$.

In the terms of an operator equation (1) can be written as

$$PI = k^2 P_1 I + P_2 I = V \tag{5}$$

where P_1 and P_2 represents the operators in the first and second parts of the integral in (2), respectively. First, we would like to investigate the properties of the operator P in (5) as the method of solution for I(z') in (2) is dependent on whether P is bounded or unbounded.

If a constant C [independent of I(z')] exist such that the following inequality is always satisfied

$$\|P\| = \max \frac{\|PI\|}{\|I\|} = \max_{\|I\| = 1} \|PI\| \leq C \tag{6}$$

then the operator P is said to be bounded [2. p. 296]. If such a constant C exists, which is the maximum of all possible $\|PI\|$ with the constraint $\|I\| = 1$, then we say that the operator P is bounded with respect to the norm $\|\cdot\|$. The two norms that we shall be dealing with are the $\mathcal{L}^2$ norm and the Chebyshev norm. The $\mathcal{L}^2$ norm is defined as

$$\|I\|_{\mathcal{L}^2} = \left[\int_{-L/2}^{+L/2} |I(z)|^2 \, dx\right]^{1/2} \tag{7}$$

and the Chebyshev norm is defined as

$$\|I\|_T = \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} |I(z)| \tag{8}$$

When no subscripts are used then it could be either of the two norms. If we are using the $\mathcal{L}^2$ norm then we are restricting the domain of the operator P to elements which are in $\mathcal{L}^2$ (or square integrable). This does not imply that I(z') cannot be infinite within the range $-\frac{L}{2} \leq z' \leq +\frac{L}{2}$. However only those type of singularities are permitted in I(z') which are square integrable. Any function which is not square integrable is excluded from the domain of P (as they are not in $\mathcal{L}^2$). On the other hand, if we use the Chebyshev norm then the function has to be bounded. Under the Chebyshev norm any unbounded function cannot be in the domain of the operator. Thus the function log z is in $\mathcal{L}^2$ as it is square integrable [see definition (7)] but not in the domain of functions satisfying the Chebyshev norm. Physically then convergence of a sequence of functions under the $\mathcal{L}^2$ norm yields least squares convergence whereas convergence under the Chebyshev norm yields pointwise convergence.

Examination of the Green's function reveals that the kernel has a singularity. The singularity can be observed by rewriting the kernel as [Schelkunoff (1952), p. 141].

$$G(z,z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{R}\, d\phi \; - \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - e^{-jkR}}{R}\, d\phi \tag{9}$$

The first term in equation (9) can be transformed to a complete elliptic integral of the first kind. Thus we find

$$G(z,z') = \frac{p}{\pi a} F(\tfrac{\pi}{2}, p) - \frac{1 - e^{-jk|z-z'|}}{|z-z'|} + \text{terms of the order of } k^2a^2 \tag{10}$$

where

$$p = \frac{2a}{[4a^2 + (z-z')^2]^{1/2}} \tag{11}$$

As $z \to z'$, the Green's function behaves as

$$G(z,z') \to \frac{1}{\pi a} \log^4 \frac{[(z-z')^2 + 4a^2]^{1/2}}{|z-z'|} + G_2 \tag{12}$$

where G_2 contains terms which are bounded and hence square integrable. In the immediate vicinity of $z \approx z'$

$$G(z,z') \to -\frac{1}{\pi a} \log |z-z'| + G_3 \tag{13}$$

where G_3 contains terms which are bounded. So the singularity of the kernel is manifested through the log function in (13). Since log functions are square integrable, we find

$$\|P_1\|_{\mathcal{L}^2} \leq \frac{1}{\pi a}\left| \int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \{\log|z-z'| + G_3\}^2 \right|^{1/2} = C < \infty \quad (14)$$

where C is a constant. Hence P_1 is bounded under the $\mathcal{L}^2$ norm. Therefore P_1 is a Hilbert Schmidt operator as it has a square integrable kernel [2, p. 352]. It can also be shown that a Hilbert Schmidt operator is a compact operator [2, p. 353]. Under the Chebyshev norm

$$\|P\|_T \leq \max_{-\frac{L}{2} \leq z \leq \frac{L}{2}} \frac{1}{\pi a}\left| \int_{-L/2}^{+L/2} dz' \log|z-z'| \right| + \left| \int_{-L/2}^{+L/2} dz' G_3 \right|$$

$$\leq \max_{-\frac{L}{2} \leq z \leq \frac{L}{2}} \frac{1}{\pi a} \{(z+\tfrac{L}{2})\log(z+\tfrac{L}{2}) + (\tfrac{L}{2}-z)\log(\tfrac{L}{2}-z) - L\} + \text{constant}$$

$$\leq M(\text{a constant}) \text{ for all } -\frac{L}{2} \leq z \leq \frac{L}{2} \quad (15)$$

Hence the operator P_1 is also bounded under the Chebyshev norm.

Next consider the second integral in (2). We have

$$P_2 I \approx \frac{\partial}{\partial z} \int_{-L/2}^{+L/2} dz' \frac{\partial I(z')}{\partial z'} \{-\frac{1}{\pi a} \log|z-z'| + G_3\}$$

$$= \fint_{-L/2}^{L/2} dz' \cdot \frac{\partial I(z')}{\partial z'} \cdot \frac{\partial}{\partial z} \{-\frac{1}{\pi a} \log|z-z'| + G_3\} \quad (16)$$

where the bar over the second integral represents a principal value. It is clear that the operator P_2 in (16) is unbounded under the Chebyshev norm because $\frac{\partial I}{\partial z'}$ is unbounded as $z \to \pm \frac{L}{2}$. Thus there exist no constant C for all $-\frac{L}{2} \leq z \leq +\frac{L}{2}$ (because the charge $\frac{\partial I}{\partial z'} \to \infty$ at the edges)

$$||P_2||_T \leq C < \infty$$

Also the operator P_2 is unbounded under $\mathcal{L}^2$ norm as $\frac{\partial I}{\partial z'}$ and $\frac{\partial}{\partial z}$ $\{\log |z-z'|\}$ are not square integrable.

However if the antenna has no edges (and the effect of end caps is neglected) then $\frac{\partial I}{\partial z'}$ is everywhere bounded and square integrable. Even then we show that P_2 is unbounded.

Sneddon has shown through Theorem 8 [10,p. 234] that if f(z) is square integrable over $-\frac{L}{2} \leq z \leq +\frac{L}{2}$ and zero everywhere else then the formula

$$\bar{f}_H(z) = -\frac{1}{\pi}\frac{d}{dz}\int_{-\infty}^{+\infty} f(t) \log \frac{|t-z|}{|t|}\, dt \tag{17}$$

defines almost everywhere a function $\bar{f}_H(z)$ which is also square integrable and

$$||f(z)||_{\mathcal{L}^2} = ||\bar{f}_H(z)||_{\mathcal{L}^2}$$

Hence

$$||P_2|| \simeq \frac{1}{\pi a} ||\frac{\partial I}{\partial z}||$$

Even though $\frac{\partial I}{\partial z}$ is always bounded, the ratio $\frac{||\partial I/\partial z||}{||I||}$ is not bounded and hence $P = P_1 + P_2$ is an unbounded operator.

Therefore even if the antenna has no sharp edges, the operator P is unbounded under the $\mathcal{L}^2$ norm. Hence the Pocklington E-field operator,

$P = P_1 + P_2$ is unbounded both under the $\mathcal{L}^2$ and the Chebyshev norm. Since it is numerically difficult to solve an unbounded operator equation, perhaps that is why Hallen considered the potential equation. The operator for Hallen's integral equation is a bounded operator and hence easy to solve numerically. This we show next.

3. PROPERTY OF THE HALLEN OPERATOR

Hallen transformed Pocklington's equation as given by (2) into the following integral equation

$$\int_{-L/2}^{+L/2} dz'\ I(z')\ G(z,z') = D \cos kz + F \sin kz + \frac{jk}{\omega} \int_{-L/2}^{z} E^{i}_{tan}(z') \sin k(z-z')\ dz' \tag{18}$$

where D and F are obtained from the boundary conditions [i.e. $I(\pm \frac{L}{2}) = 0$]. We define the Hallen operator as

$$HI = \int_{-L/2}^{+L/2} dz'\ I(z')\ G(z,z') = P_1 I \quad [\text{from } (2)] \tag{19}$$

Hence the operator H is bounded both under the $\mathcal{L}^2$ and the Chebyshev norm. Also H is a compact operator under the $\mathcal{L}^2$ norm.

It is important to note however, that the unknown $I(z')$ in (18) is hidden in D and F. To illustrate this further, if we consider a delta gap excitation for the antenna then (18) becomes [Wu (1969), p. 325].

$$\int_{-L/2}^{+L/2} dz'\ I(z')\ G(z,z') = A \sin k\ |z| + D \cos kz$$

where A is known and D is unknown. Observe at z = 0

$$D = \int_{-L/2}^{+L/2} dz' \; I(z') \; G(0, z') .$$

If the operator H is bounded then D will be finite. If one wishes then perhaps one can transfer D to the left hand side of the equation and thus form an additional part of the operator H. But since D is a part of the operator H, whatever bound holds for H also holds for D.

Next we estimate a bound for $\|H\|$ both under the $\mathcal{L}^2$ and Chebyshev norm.

We observe

$$\|H\|_T \leq \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} \frac{1}{2\pi} \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi \; \frac{\exp(-jkR)}{R}$$

$$\lesssim \max_{-\frac{L}{2} \leq z \leq +\frac{L}{2}} \frac{1}{2\pi} \int_{-L/2}^{+L/2} dz' \int_0^{2\pi} d\phi \; \frac{1}{R} \approx \frac{L}{a} \qquad (20)$$

Schelkunoff has obtained a similar estimate for (16) but utilizing the reduced kernel. By utilizing the reduced kernel [3, p. 144] one obtains

$$\|H\|_T \lesssim \max \left[2 \log \frac{L}{a}, \; \log \frac{2L}{a}\right] + \text{ terms of the order of } a^2$$

$$\lesssim 2 \log \frac{L}{a} \quad \text{if} \quad \frac{L}{a} >> 1 \qquad (21)$$

Under the $\mathcal{L}^2$ norm we obtain

$$||H||_{\mathcal{L}^2} \leq \left[\int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \left\{ \int_0^{2\pi} d\phi \, \frac{\exp(-jkR)}{2\pi R} \right\}^2 \right]^{1/2}$$

$$\leq \left[\int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \left\{ \int_0^{2\pi} \frac{d\phi}{2\pi R} \right\}^2 \right]^{1/2} \approx \frac{L}{a} \tag{22}$$

An estimate similar to (17) can be obtained for the reduced kernel.

In our analysis, we do not consider the reduced kernel, because Schelkunoff [3, p. 149] has shown that an integral equation with the reduced kernel mathematically has no solutions (i.e. the excitation is not in the range of the operator). However if one solves such problems numerically, one does indeed get a numerical solution. In section five we discuss the convergence properties of the numerical solutions in which the excitation is not in the range of the operator.

As the Hallen integral operator H is a bounded operator, unlike the Pocklington E-field operator (which is unbounded), it may be computationally much easier to solve Hallen's integral equation than Pocklington's equation.

4. SOLUTION OF HALLEN'S INTEGRAL EQUATION

4.1. By Iterative Methods

It is well known that if H is a compact invertible operator (under $\mathcal{L}^2$ norm) on an infinite dimensional space then its inverse is often unbounded [3, p. 353]. Hence the problem of the solution of (18) in the $\mathcal{L}^2$ norm is ill-posed. If a problem is ill-posed under the $\mathcal{L}^2$ norm then it is definitely ill-posed under the Chebyshev norm. However it can be regularized in the following way. We take (18) and cast it in the form

$$HI = Q \tag{23}$$

$$I_n = [U - \tau H] I_{n-1} + \tau Q \tag{24}$$

with a starting guess of $I_o = Q$ and U is the identity operator. The sequence I_n generated by (24) converges to a solution I which satisfies HI = Q for all Q in the range of H [1, p. 196]. The sequence generated by (24) always converge to I provided

$$||[U - \tau H]|| < 1$$

or

$$|\tau| \cdot ||H|| = ||\tau H - U + U|| \leq ||U - \tau H|| + ||U|| < 2$$

or

$$\frac{1}{|\tau|} > \frac{||H||}{2} \tag{25}$$

In (25) $||H||$ could be either the $\mathcal{L}^2$ or the Chebyshev norm depending on the type of convergence desired. For all values of $\frac{1}{|\tau|} > \frac{||H||}{2}$ and Q in the range of the operator H, the iterative process defined by (24) will always converge monotonically to a solution I(z'), if it exists. This has been shown in Theorem 2 (in the appendix). By the terms of Theorems 1 and 2, the iterative process will converge for any starting value I_o if

$$\frac{1}{|\tau|_{\mathcal{L}^2}} > \frac{||H||_{\mathcal{L}^2}}{2} \quad \text{or} \quad \frac{1}{|\tau|_{\mathcal{L}^2}} \approx \frac{L}{a} \tag{26}$$

when convergence is desired in the $\mathcal{L}^2$ norm. For the convergence in the Chebyshev norm it is required that

$$\frac{1}{|\tau|_T} > \frac{||H||}{2} \quad \text{or} \quad \frac{1}{|\tau|_T} \approx \frac{L}{a} \text{ for the exact kernel} \tag{27a}$$

For the reduced kernel however, we have

$$\frac{1}{|\tau|_T} > \log \frac{L}{a} \quad \text{or} \quad \frac{1}{|\tau|_T} \approx 2 \log \frac{L}{a} \tag{27.b}$$

Hallen in his classic iterative scheme chose the value of τ as given by (27.b) [3, 4]. A detailed description on how D is solved for at each iteration is described in detail [4, p. 326]. Observe that if $\frac{1}{\tau} > \frac{||H||}{2}$ or $||U - \tau H|| < 1$ then the iterations defined by (8.149) and (8.150) of [4] would always converge for any starting I_o.

Other researchers have chosen different values of τ. For example Gray [5] chose

$$\frac{1}{\tau} = \text{Real } [2 \log \frac{L}{a} - 2\gamma - 2 \log \frac{kL}{2} - j\pi + 2\text{Ei}(+\frac{jkL}{2})] \tag{28}$$

where γ is Euler's constant and Ei is the exponential integral. King and Middleton [6] decided to make

$$\frac{1}{\tau} = \int_{-L/2}^{L/2} G(\frac{L}{2} - \frac{\lambda}{4}, z') \sin k(\frac{L}{2} - z')dz' \approx \frac{2 \log L}{a} \quad \text{(for the exact kernel)} \tag{29}$$

Whereas Siegel and Labus [7] chose

$$\frac{1}{\tau} = 2 \log \frac{L}{a} - \text{Cin } (kL) - 1 - \frac{\sin kL}{kL} \tag{30}$$

Where Cin is the special form of the cosine integral. Finally Schelkunoff

[3] after a careful analysis decided.

$$\frac{1}{\tau} = 2 \log \frac{L}{a} - \mathrm{Cin}(kL) - 1 - \frac{\sin kL}{kL} - j\,\mathrm{Si}(kL) + j\,\frac{1-\cos kL}{kL} \tag{31}$$

In general, it really does not make any difference whatsoever, what value of τ one chooses, one is guaranteed to have pointwise convergence or convergence in the mean depending on whether one chooses τ according to (27) or (26). This of course assumes that a solution to the problem exist, i.e. Q is in the range of H.

In summary, the iterative method converts $HI = Q$, a Fredholm equation of the first kind to $I_n = B\, I_{n-1} + \tau Q$, a Fredholm equation of the second kind. The advantage of the equation of the second kind is that not only is $\|[U - B]\|$ bounded but also its inverse $\|[U - B]\|$ is bounded, provided unity is not an eigenvalue of B and $\|B\|$ is bounded. Mathematically one has regularized the problem by the introduction of the parameter τ. With this regularization scheme the convergence of the sequence I_n is monotonic. The method converges as long as $\tau < \frac{a}{L}$.

Finally, we conclude by noting that as the iterative process continues the unknown D and F in (18) are determined as outlined in [4].

4.2. By Galerkin's Method

The next generation of the methods were developed primarily by Harrington [8] under the generic name of "moment methods." This very popular versatile method has been excellently documented in [8]. In Galerkin's method, the unknown function I is expressed as

$$I_N(z) = \sum_{i=1}^{N} \alpha_i \, \Psi_i(z) \tag{32}$$

where $\Psi_i(z)$ are known functions which may extend from $-\frac{L}{2} \leq z \leq +\frac{L}{2}$ or could span only a partial portion of the domain of z, i.e. $-\frac{L}{2} < \sigma_1 \leq z \leq \sigma_2 < +\frac{L}{2}$. In the former case Ψ_i 's become entire domain functions whereas in the latter Ψ_i 's are called sub domain basis functions. We solve for $I_n(z)$ by solving for the unknowns α_i in (32). We also convert the infinite dimensional problem HI = Q to a finite dimensional problem by replacing I with I_N , i.e. we solve the following equation $\sum_{i=1}^{N} \alpha_i \, H\Psi_i = Q$ in the finite dimensional space spanned by the basis functions Ψ_i, i = 1, 2, ..., N. We next find a unique solution in finite dimensional space by weighting the residual $\sum_{i=1}^{N} \alpha_i \, H\Psi_i - Q$ to zero in the following way

$$\sum_{i=1}^{N} \alpha_i \, \langle H\Psi_i, \Psi_j \rangle = \langle Q, \Psi_j \rangle \quad \text{for } j = 1, 2, \ldots, N \tag{33}$$

In a matrix form

$$[G] \, [\alpha] = [V] \tag{34}$$

where $[G] = [\langle H\Psi_i, \Psi_j \rangle]$

$[V] = [\langle Q, \Psi_j \rangle]$

and the inner product is defined as

$$\langle \phi_i, \phi_j \rangle = \int_{-L/2}^{+L/2} dz \, \phi_i(z) \, \phi_j(z)$$

The unknown α 's in (34) are obtained as

$$[\alpha] = [G]^{-1} \, [V] \tag{35}$$

The next question that normally arises in whether the sequence I_N defined in (32) approaches any limit I as $N \to \infty$. And secondly whether I satisfies the equation HI = Q. We cannot talk about convergence in the Chebyshev metric [as defined in (8)] because a Chebyshev norm cannot be derived from an inner product [2. p. 272]. In other words in an inner product space we cannot define a Chebyshev norm. Hence we shall be talking about only the $\mathcal{L}^2$ norm for Galerkin's method. So we shall be discussing about convergence in the mean. Galerkin's method guarantees the weak convergence of the residuals [from (33)], i.e.

$$\underset{N \to \infty}{\mathrm{Lt}} \quad \langle HI_N - Q, \Psi_j \rangle \to 0 \quad \text{for } j = 1, 2, \ldots, N \tag{36}$$

However if H is a bounded operator (i.e. $||H||_{\mathcal{L}^2} \leq$ a constant $< \infty$) then (36) implies strong convergence of the residuals, i.e.

$$\underset{N \to \infty}{\mathrm{Lt}} \quad ||HI_N - Q||_{\mathcal{L}^2} \to 0 \tag{37}$$

This has been proved by Mikhlin [9]. Physically, (37) implies that as $N \to \infty$, the total potential on the surface of the conductor for Hallen's method converges to zero in a least squares fashion.

Unfortunately in Galerkin's method the convergence of the residuals to zero in (37) does not imply the convergence of I_N to a solution I of HI = Q. The convergence of $I_N \to I$ in the domain of H is possible if and only if $||H^{-1}||_{\mathcal{L}^2}$ is bounded, as

$$||I_N - I||_{\mathcal{L}^2} \leq ||H^{-1}||_{\mathcal{L}^2} \cdot ||HI_N - Q||_{\mathcal{L}^2} \tag{38}$$

So if $||H^{-1}||_{\mathcal{L}^2}$ is unbounded, even though the residuals go to zero, the sequence of solutions I_N may not converge to I. This is in contrast to the iterative methods where monotonic convergence to I is guaranteed if τ and Q are chosen as prescribed.

Since $||H^{-1}||_{\mathcal{L}^2}$ is unbounded in this case, the application of Galerkin's method to HI = Q may not guarantee that $||I_N - I||_{\mathcal{L}^2} \to 0$ as $N \to \infty$. In other words, there is no quantitative way to describe the convergence of $I_N \to I$ as various expansion functions are chosen for Ψ_i. Hence we address the question: For a fixed order of approximation N, how should one choose a set of expansion functions Ψ_i such that the round-off and the truncation error in the numerical computation of α in (32) is a minimum?

Suppose the Gram matrix E is generated by the basis functions $[E_{ij} = \langle\Psi_i, \Psi_j\rangle]$ then we show in the appendix (Theorem 2) that

$$\text{cond } [G] \leq \text{cond } [\hat{H}]. \text{ cond } [E] \tag{39}$$

i.e. the condition number of the Galerkin matrix G in (34) is bounded by the condition number of the operator $\hat{H}$ in the finite N dimensional space and the Gram matrix E. Equation (39) is valid only in the finite N dimensional space spanned by Ψ_i. It is important to note that even though H may not have any eigenvalues in an infinite dimensional space, it has at least an eigenvalue on a finite dimensional space [2, p. 332]. If the homogeneous equation HI = 0 has only the trivial solution I = 0 and $||H||_{\mathcal{L}^2}$ is bounded then cond $[\hat{H}] < \infty$ and the inequality in (39) has meaning because the right hand side of (39) can never be infinity.

So (39) directly implies the following: i) Use of an orthonormal set of basis functions Ψ_i for the current implies

$$\text{cond } [G] \leq \text{cond } [\hat{H}] \tag{40}$$

i.e. the problem would not be worse conditioned than the original problem. Note that cond [E] = 1 for subdomain basis functions like pulses or entire domain orthonormal basis functions like $\sqrt{\frac{2}{\pi}}$ sin (mz) for m = 1,2,..., N. Equation (40) also implies that the solution of HI = Q by Galerkin's method in a finite dimensional space may be a better conditioned problem than the original problem posed in the finite dimensional space N.

Also from (40) there is no way to tell whether the Galerkin matrix G associated with the entire domain basis functions would be more ill-conditioned than the Galerkin matrix associated with the pulse functions.

(ii) Use of subdomain basis functions like triangles or piecewise sinusoids may deteriorate the condition number of the Galerkin matrix [G] from than that of the original problem. This is because cond [E] > 1 for these cases.

For the case when Ψ_i 's are chosen as piecewise triangles, then E is a tridiagonal matrix of the form

$$\begin{bmatrix} P & Q & 0 & . & \\ Q & P & Q & . & \\ 0 & Q & P & . & \end{bmatrix}$$

where $P = \frac{2\Delta z}{3}$ and $Q = \frac{\Delta z}{6}$ and $\Delta z = \frac{L}{N+1}$. Since the jth eigenvalue of a tridiagonal matrix is given by [1, p. 70]

$$\lambda_j = P + 2Q \cos \left(\frac{j\pi}{N+1}\right)$$

we have

$$\text{cond } [E]_{\text{triangles}} \leq \frac{|P| + 2|Q|}{|P| - 2|Q|} = 3 \tag{41}$$

Hence for all values of N the Galerkin matrix due to piecewise triangle expansion functions may have a condition number which at most can be three times as that of the original problem, i.e.

$$\text{cond } [G] \leq 3 \text{ cond } [H]$$

For the piecewise sinusoids however,

$$P = \frac{2k \Delta z - \sin 2k \Delta z}{2k \sin^2 k \Delta z} \quad \text{and } Q = \frac{\sin k \Delta z - k \Delta z \cos k \Delta z}{2k \sin^2 k \Delta z} \tag{42}$$

In this case cond [E] is bounded by

$$\text{cond } [E]_{\text{sinusoids}} \leq \frac{|2k \Delta z - \sin 2k \Delta z| + 2 |\sin k \Delta z - k \Delta z \cos k \Delta z|}{|2k \Delta z - \sin 2k \Delta z| - 2 |\sin k \Delta z - k \Delta z \cos k \Delta z|} \quad \ldots\ldots (43)$$

In the limit $\Delta z \to 0$

$$\text{cond } [E]_{\text{sinusoids}} \leq 3 \tag{44}$$

Thus (44) implies that as the dimension of the problem becomes large the Galerkin matrix due to piecewise sinusoids is no less numerically ill-conditioned than the matrix produced by piecewise triangles. It may be quite possible that for a particular value of N the Galerkin matrix due to piecewise sinusoidal functions may be better conditioned than that of the piecewise triangles or even than that of the pulse functions.

In the above analysis an attempt has been made to provide a worst case theoretical bound for the condition number of the various matrices of interest.

It is important to stress that the problem we have addressed here is not which set of basis functions would provide the best approximation for the current, but which type of expansion functions would give rise to a well conditioned Galerkin matrix G which will be easy to invert numerically. This is because truncation and round-off error associated with the solution of (34) is directly related to cond [G].

5. IS A SOLUTION POSSIBLE IF THE EXCITATION IS NOT IN THE RANGE OF THE OPERATOR?

We discuss the question of existence of a solution for the current on the antenna structure when we try to excite it with a source which is not in the range of the operator H. Clearly, if the excitation is not in the range of the operator then mathematically a solution does not exist. But numerically one could always find a solution to the integral equation. This numerical solution has some very interesting properties as outlined in Theorems 4 and 5 (in the appendix). If we try to numerically solve an integral equation $HI = Q$ with $Q \notin$ range of H, then the sequence of solutions I_N diverges even though the residuals $HI_N - Q$ associated with $HI = Q$ may approach zero monotonically. This has been proved in Theorem 4 of the appendix. In theorem 5, we develop further properties of the solution I_N. There we prove that the sequence I_N indeed form an asymptotic series. The asymptotic series has the property that it converges at first and then as more and more terms are included in the series, the series actually diverges. Even though the theorems 4 and 5 have been proved for the iterative methods, they are also valid for Galerkin's method. We now present some examples to illustrate when Q is in the range of operator and when it is not.

As an example consider the radiation problem where an antenna of length L and radius a is excited by a delta gap at the center. If we consider Pocklington's equation, then clearly the delta function excitation is not in the range of the operator. Since the delta function is not square integrable, it is not in the range of the operator under both the $\mathcal{L}^2$ norm and the Chebyshev norm. Hence if one attempts to solve Pocklington's equation for a delta function excitation, then according to Theorems 4 and 5 the sequence of solutions diverges and $\lim_{N \to \infty} ||I_N|| \to \infty$. This is indeed true, because the admittance approaches infinity as the capacitance of a delta gap is infinity.

Hallen's integral equation for a delta function excitation is given by [4, p. 321]

$$\int_{-L/2}^{+L/2} dz' \; I(z') \; G(z,z') = A \sin k|z| + D \cos kz \tag{45}$$

where A is a known constant and D is unknown. It is seen that the right hand side of (45) has a discontinuous derivative with respect to z, whereas the left hand side has a continuous derivative with respect to z. Hence the delta function excitation is not in the range of the operator. Perhaps this is the reason, why the solution yielded by the iterative methods of Hallen [3] and King-Middleton [6] seemed to diverge as the solution progressed. Whether an arbitrary excitation is in the range of the operator is difficult to verify both theoretically and numerically. When the excitation is not in the range of the operator we obtain a solution which diverges in an asymptotic sense, i.e. the solution seems to converge at first and then diverges. However this postulate may be difficult to verify numerically for certain problems. As an example consider the partial sum of the series

$$S_N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{N}$$

The partial sum S_N diverges as $N \to \infty$. This is because if we look at the following M terms of the series we find

$$\frac{1}{M+1} + \frac{1}{M+2} + \frac{1}{M+3} + \ldots + \frac{1}{2M} > \frac{1}{2M} > \frac{1}{2M} + \frac{1}{2M} + \ldots + \frac{1}{2M} = \frac{1}{2}$$

Hence

$$S_\infty > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty$$

However if we program the series on the computer and ask the computer to give us a result when the addition of the N+1 term does not change the partial sum by 10^{-10} (say) we would get a convergent result!

In conclusion, we must try to learn theoretically, as much about the problem as possible. Numerical methods may be applied as a last resort as it may be the only way to obtain a solution easily. The convergence of the numerically computed results is determined to a large extent by the theoretical analysis of the problem rather than apparent convergences in numerical computations.

6. CONCLUSIONS

In summary, we have brought out the following features.

1) The thin wire E-field Pocklington integral operator is unbounded whereas the Hallen E-field operator is bounded.

2) The inverse operator for Hallen's equation is unbounded.

3) A discussion of the various iterative methods showing how the Fredholm equation of the first kind has been converted to a Fredholm equation of the second kind is presented.

4) The conditions under which the iterative methods converge both for the $\mathcal{L}^2$ norm and the Chebyshev norm has been presented.

5) The monotonic rate of convergence of the sequence of solutions associated with iterative methods have been established for certain values of τ and for $Q \in R(H)$.

6) The numerical stability in the solution of the matrix equations for Galerkin's method for various expansion functions is examined, and

7) The sequence of solutions I_N forms an asymptotic series for both the iterative and Galerkin's method when the excitation is not in the range of the operator.

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7. REFERENCES

Gray, M. C. (1966), A Modification of Hallen's Solution of the Antenna Problem, J. of Applied Physics, Vol. 15, 61-65.

Harrington, R. F. (1968), Field Computation by Moment Methods, The Macmillan Company, New York.

Jones, D. S. (1979), Methods in Electromagnetic Wave Propagation, Clarendon Press, England.

Jones, D. S. (1981), Note on the Integral Equation for a Straight Wire Antenna, vol. 128, Pt. H, No. 2, IEE Proc.

King, R.W.P. and Middleton, D., (1946), The Cylindrical Antenna: Current and Impedance, Quarterly Appl. Math., vol. 3, pp. 302-305, corrections, ibid., vol. 6, July 1948, p. 192.

Milhlin, S. G. (1964), The Minimum of a Quadratic Functional, Holden Dey, San Francisco.

Schelkunoff, S. A. (1952), Advanced Antenna Theory, John Wiley & Sons, New York.

Siegel, L. and Labus, J. (1934), Scheinwiderstand Von Antennen," Hochfreq, U. Electroak, vol. 43, 166-172.

Sneddon, I. N. (1962), The Use of Integral Transforms, McGraw-Hill Book Co., New York.

Stakgold, I. (1979), Green's Functions and Boundary Value Problems, J. Wiley and Sons, New Jersey.

Tikhonov, A. N. and Dimitriyev, (1968), A Method for Calculating the Current Distribution in a System of Linear Bipoles and of the Radiation

Pattern in such a System, Universitet. Vychislitey'nyy Trentr. Sobrink Rabot, No. 10, 3-8. [English translation available from Defense Documentation Center, Cameron Station, Alexandria, Virginia, FSTC-HT-23-261-70, AD 709614.]

Wu, T. T. (1969), Introduction to Linear Antennas, in Antenna Theory, Pt. I, edited by R. Collin, McGraw-Hill, New York.

8. APPENDIX

<u>Theorem 1</u>: For all $Q \in$ Range of H, the sequence I_n generated by the recursion

$$I_{n+1} = [U - \tau H] I_n - \tau Q \triangleq B I_n + Q' \qquad (46)$$

where U is the identity matrix and $||\tau H||<2$) with the initial guess $I_o = Q'$ converges to I_e (the exact solution, if it exists) in the norm, i.e. $\lim_{n\to\infty} ||I_n - I_e|| \to 0$ and the convergence is strictly monotone increasing, i.e., $I_k \uparrow I_e$.

<u>Proof</u>: The iterative process (46) converges as long as the norm of B is less than one. It is clear that if $|\tau|\cdot||H|| < 2$ then $||B|| = ||U - \tau H|| < 1$. Now we have

$$I_e - I_{n+1} = I_e - BI_n - Q' = B[I_e - I_n] \qquad (47)$$

By taking the norm of both sides and simplifying

$$||I_e - I_{n+1}|| \leq ||B||\cdot||I_e - I_n|| \leq \{||B||\}^{n+1} \cdot ||I_e - I_o|| \qquad (48)$$

Since $||B|| < 1$, as $n\to\infty$ we have

$$\lim_{n\to\infty} ||I_e - I_{n+1}|| = 0 \qquad (49)$$

and thus I_{n+1} converges to the exact solution.

That $I_k \uparrow I_e$ is seen easily as

$$\varepsilon_{n+1} \triangleq I_e - I_{n+1} \qquad (50)$$

and

$$\varepsilon_n \triangleq I_e - I_n \qquad (51)$$

are related by

$$\varepsilon_{n+1} = B\, \varepsilon_n \qquad (52)$$

and so

$$||\varepsilon_{n+1}|| \leq ||B||\cdot||\varepsilon_n|| \leq ||\varepsilon_n|| \tag{53}$$

and with equality if and only if $\varepsilon_n = 0$. It follows that if n_o is the smallest integer for which $||\varepsilon_{n+1}|| = ||\varepsilon_n||$ then

$\varepsilon_n = 0$, for $n \geq n_o$ and $||\varepsilon_{n+1}|| < ||\varepsilon_n||$ for $n < n_o$, i.e.

$I_n \uparrow I_e$ and theorem 2 is proved.

<u>Theorem 2:</u> Consider the operator equation HI = Q in a finite dimensional space N. Let the unknown I be expanded in terms of the normalized basis functions ψ_i. Define $\text{cond}[\hat{H}] = \dfrac{\lambda_{max}[\hat{H}]}{\lambda_{min}[\hat{H}]}$ in the given N dimensional space.

Let cond[G] and cond[E] be the condition numbers of the Galerkin matrix $[G_{ij} = \langle\hat{H}\psi_i,\psi_j\rangle]$ and of the Gram matrix $[E_{ij} = \langle\psi_i,\psi_j\rangle]$, respectively. Then

$$\text{cond}[G] \leq \text{cond}\ [\hat{H}].\ \text{cond}\ [E] \tag{54}$$

<u>Proof:</u> Let $I = \sum_{i=1}^{N} \alpha_i\ \psi_i$, then from [2, p. 341]

$$|\langle\hat{H}I,I\rangle| = |\sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_i\ \bar{\alpha}_j\ \langle\hat{H}\psi_i,\ \psi_j\rangle| \leq ||\hat{H}||\cdot||\sum_{i=1}^{N}\alpha_i\ \ \psi_i||^2 = ||\hat{H}||\cdot\langle EI,I\rangle$$

$$\leq ||\hat{H}||\cdot\lambda_{max}\ [E]\cdot||\alpha||^2 \tag{55}$$

since $\langle I,I\rangle = ||\alpha||^2$ we have

$$\frac{|\langle\hat{H}I,\ I\rangle|}{\langle I,I\rangle} \leq ||\hat{H}||\cdot\lambda_{max}[E] \tag{56}$$

from which it follows that

$$\lambda_{max}[G] \leq ||\hat{H}|| \cdot \lambda_{max}[E] \tag{57}$$

Also since

$$|<\hat{H}I,I>| \geq \frac{||\sum_{i=1}^{N} \alpha_i \psi_i||}{||\hat{H}^{-1}||} = \frac{<EI, I>}{||\hat{H}^{-1}||} \geq \frac{\lambda_{min}[E] \cdot ||\alpha||^2}{||\hat{H}^{-1}||} \tag{58}$$

so

$$\frac{|<\hat{H}I, I>|}{<I,I>} \geq \frac{\lambda_{min}[E]}{||\hat{H}^{-1}||} \tag{59}$$

from which it follows

$$\lambda_{min}[G] \geq \frac{\lambda_{min}[E]}{||\hat{H}^{-1}||} \tag{60}$$

Hence we have

$$\frac{\lambda_{max}[G]}{\lambda_{min}[G]} \leq ||\hat{H}|| \cdot ||\hat{H}^{-1}|| \cdot \frac{\lambda_{max}[E]}{\lambda_{min}[E]} \tag{61}$$

$$\text{cond}[G] \leq \text{cond}[\hat{H}] \cdot \text{cond}\ [E].$$

<u>Theorem 3</u>: If $Q \notin R(H)$ then the sequence of approximations I_n generated by

$$I_{n+1} = BI_n + Q' \tag{62}$$

with the initialization $I_o = Q'$ yields the following relationships

i) $\lim_{n\to\infty} ||R_{n+1} - R_n|| = 0$, where $R_n = HI_n - Q$ (63)

and

ii) $\lim_{n\to\infty} ||I_n|| = \infty$ (64)

Proof: (i) We have

$$R_{n+1} - R_n = H[I_{n+1} - I_n) \qquad (65)$$

Since

$$I_o = Q'$$

$$I_1 = B\ I_o + Q' = (B + \mathfrak{U})Q'$$

$$\vdots$$

$$I_n = [B^n + B^{n-1} + \ldots + B + \mathfrak{U}]Q'$$

then

$$I_{n+1} - I_n = B^{n+1}Q$$

Since the operator H is bounded [i.e. $||H|| < M$] we have $||R_{n+1} - ||R_n|| = H\ B^{n+1}\ Q \leq M\ ||B^{n+1}\ Q'|| \leq M \cdot ||B^{n+1}|| \cdot ||Q'||$

Hence $\lim_{n\to\infty} ||R_{n+1} - R_n|| = 0$ as $\lim_{n\to\infty} \{||B^n||\} \to 0.$

ii) If $\lim_{n\to\infty} ||I_n|| = \infty$ does not hold, then we have $\lim_{n\to\infty} ||I_n|| < \infty$ (i.e a bounded sequence). Thus there is a subsequence I_n' which is bounded in norm. Now if we put the operator equation in a Hilbert space setting (now we can only talk about the $\mathcal{L}^2$ norm) and since a Hilbertspace is weakly compact [2] one can always extract from I_n' another sequence I_n'' which converges weakly to some element I of the Hilbert space, i.e. $I_n'' \overset{\omega}{\to} I$. Also we have from i) $\lim_{n\to\infty} H\ I_n \overset{S}{\to} Q$ (strong convergence in norm). However as H is a bounded operator we have

$$\lim_{n\to\infty} H\ I_n'' \overset{\omega}{\to} HI \qquad (66)$$

and also

$$\lim_{n\to\infty} H\ I_n \overset{S}{\to} Q \qquad (67)$$

Since the weak and strong limits of a sequence must concide, $H\ I_n'' = Q$. This means $Q \varepsilon R(H)$, a contradiction.

Theorem 4: The sequence I_n as derived in ii) of Theorem 3 indeed forms an asymptotic series (i.e. for obtaining a meaningful solution the series has to be truncated after a finite number of terms otherwise the results may be worse).

<u>Proof</u>: To demonstrate the source of divergence in I_n we assume

$HI_o = Q_o$ with $I_1 = Q_o + \Delta Q$

then

$$I_n = I_o - [B]^n I_o + \theta_{n-1} \text{ for } n \geq 1 \tag{68}$$

where

$$\theta_{n-1} = \sum_{i=o}^{n-1} [B]^i \Delta Q \tag{69}$$

If $Q_o + \Delta Q \notin R(H)$ then by theorem 4, $\lim_{n\to\infty} ||\theta_{n-1}|| = \infty$, since $\lim_{n\to\infty} [B]^n I_o = 0$. Observe that this holds irrespective of the size of $|\Delta Q|$. Now the error in the iterates is obtained as

$$\varepsilon_n = I_o - I_n = [B]^n I_o - \theta_{n-1} \tag{70}$$

Note that the norm of the first term is monotonically decreasing and thus it is evident that the algorithm should be terminated after a certain optimum number of steps. Unfortunately the exact number of iterations depends on the particular Q under consideration and the growth rate of $||\theta_{n-1}||$ versus the decay rate of $||[B]^n I_o||$.