Interaction Notes

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Toward an Increased Understanding of the Singularity Expansion Method

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Abstract

Four separate efforts related to the singularity expansion method (SEM) are described in this report. Two of the efforts deal with acoustic scattering and two deal with electromagnetic scattering. The rationale for treating acoustic scattering is that useful inferences can be drawn from the results obtained to enhance our understanding of electromagnetic scattering.

The first acoustic scattering effort was to establish that important theoretical results valid for electromagnetic SEM, are also valid for scalar (acoustic) SEM. In the process of accomplishing this task, an interesting relationship was established between the interior Dirichlet and exterior Neumann problem. This relationship is viewed as the scalar analogue of the pseudosymmetric argument developed for electromagnetic SEM. The remaining scalar scattering effort consisted of an additional theoretical development that established the relationship between scalar SEM theory and a prominent scattering theory. A byproduct of the second effort is a formal proof that SEM poles are simple.

One of the two electromagnetic SEM subjects treated in this report is a continued study of an old question concerning the class of coupling coefficient issue that exists in electromagnetic SEM. The remaining subject is a numerical study that demonstrates the dependence of the numerically determined electromagnetic SEM pole locations on aspects of the procedure employed to find them.

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I. INTRODUCTION

The singularity expansion method (SEM) has received considerable attention by workers in the EMP community. This report does not address the applications of the method to EMP system effects, rather we elaborate on the contribution of SEM to an increased understanding of basic scattering theory.

One of the major objectives of this report is to demonstrate how SEM is connected to general scattering concepts developed by workers in other communities. This objective evolved as a result of our having demonstrated that important theoretical SEM results developed for electromagnetic scattering were obtainable for scalar (acoustic) scattering. Our purpose in establishing scalar SEM was to have the benefit of dealing with scalar scattering to address some of the open questions that exist for electromagnetic SEM. Some of the scalar results that we obtained do not appear in the scalar related literature we utilized, and may be new.

In addition to our scalar SEM results as well as the connection to related theories, this report presents results from two separate and distinct SEM studies, both dealing specifically with electromagnetic SEM. One study was an examination of the relationship between class 1 and class 2 coupling coefficients, and the other study was a demonstration of the sensitivity of SEM pole location values as a function of the numerical procedure used to obtain them.

In a previous work (Ref. 1) we showed that class 2 coupling coefficients yielded results that were in conflict with the known sphere solution. Despite this, it seemed class 2 coupling coefficients might still be of some use if they were viewed as wrong in principle but possibly a good approximation in practice. With this as the motivation for further study, we obtained the following results. First, we showed that after a time which is the sum of the transit time across an obstacle of an incident plane wave pulse, plus the time corresponding to its pulse width, class 1 and class 2 coupling coefficients yield the same result. Second, for the case where the incident field was a step function, and consequently had an infinite pulse width, we showed that class 1 coupling coefficients yield the correct late time magnetostatic solution while class 2 coupling coefficients do not. These coupling coefficient efforts are presented in Appendix D and Appendix E.

We now return to our scalar SEM investigation and describe it in more detail. It is our view that scalar SEM theory would be comparably established relative to electromagnetic SEM if the scalar SEM equivalent of the following electromagnetic SEM results could be obtained: (1) a demonstration that the basic concept works for a special case; the work of Baum (Ref. 2), which initiated electromagnetic SEM accomplished this for the perfectly conducting sphere scattering problem; (2) the establishment of the meromorphic nature of the scattering solution; (3) the establishment of the relationship between SEM and eigenmode expansion method (EEM) solutions in order to obtain explicit relations for coupling coefficients; (4) a demonstration that the EEM-derived solution works for a particular case; and (5) the establishment that cavity SEM pole locations do not cause nonphysical results for the external scattering problem. Results (2), (3), and (5) appear in the work of Marin and Latham (Ref. 3). Later Baum elaborated on the relationship between EEM and SEM in Reference 4 and obtained expressions that had a wider use. In Reference 1, we established result (4) and in addition we further clarified the relationship of cavity SEM pole locations with external scattering results. In that work we also presented a more useful form of EEM solutions than had previously existed. Much of the remaining literature concerning electromagnetic SEM deals with numerical calculations for special cases.

In this report we emphasize the scalar scattering problem in which Neumann boundary conditions are imposed on the surface of the scatterer. We choose to deal with this scattering problem because it directly leads to a scalar integral equation that is the analogue of the equation that provided the basis for the theoretical investigations contained in Reference 3, i.e., the magnetic field integral equation (MFIE). As will be explained shortly, even though we emphasize the Neumann problem, the structure of scalar SEM will force us also to treat the scalar Dirichlet scattering problem in some detail.

Utilizing both the Neumann and Dirichlet scalar integral equations, we establish the scalar equivalent of results (1), (3), (4), and (5) within the body of the text. Concerning result (2), the meromorphic character of the scalar scattering solution, the literature dealing with scalar scattering treats this feature of the solution as a foregone conclusion. A procedure to obtain this result is to use a theorem historically described in the very informative review article by Dolph and Scott (Ref. 5) and ultimately attributed to Steinberg. It is well known that the scalar integral equation we derive conforms to the Steinberg conditions which in turn guarantees the desired meromorphic properties.

Before leaving this topic, it is appropriate to describe some of the work in more detail. To provide a demonstration that scalar SEM exists for a particular problem, we considered a plane wave incident on a spheroid which then included the sphere as a special case. For the reasons previously discussed, we consider the case where Neumann boundary conditions are satisfied on the surface of the spheroid. We show that the solution obtained by the EEM is the same as the standard solution one obtains by separation of variables. We then specialize this solution for the case where the spheroid becomes a sphere. We rewrite this scalar sphere solution in a manner which exhibits all of the SEM properties that Baum showed for the electromagnetic sphere problem in Reference 2. Having done this, we were immediately in a position to increase our knowledge as a result of treating the scalar problem. The only analytic solution for scattering from a finite object that it is possible to examine in the electromagnetic case is the solution for the sphere. For the sphere, we find that the eigenmodes do not depend on frequency. The scalar sphere EEM solution also has this property; however, the scalar spheroid eigenmodes do depend on frequency.

The spheroid EEM solution provided information in another related area. Conditions are given by Ramm in his very informative review article (Ref. 6) that can readily be interpreted to be a set of sufficient conditions for the EEM solution to our scalar integral equation to be meaningful. Ramm presents enough detail in that article for us to conclude that we would not meet the described sufficient conditions unless our scalar integral operator is normal. We are able to show that this is the case when the scatterer is the sphere, but we were unable to show this when the object was the spheroid. For the spheroid we show that the integral operator is complex symmetric and, despite this, the EEM solution is shown to be the standard separation of variables solution.

An aspect of the scalar EEM solution worth noting is the scalar pseudosymmetric analogue of the electromagnetic pseudosymmetric theory presented in Reference 1. We show that the set of eigenvalues for the Neumann integral operator for the external scattering problem is the same set of eigenvalues one obtains for the Dirichlet integral operator for the interior scattering problem. This set equality implies the existence of purely imaginary zeros of the exterior Neumann eigenvalues. Just as in electromagnetic SEM, these extraneous zeros must be shown not to contribute to the resulting scalar SEM expansion and we provide this demonstration.

We now present a more detailed description of how we relate scalar SEM to general scattering concepts that workers in other communities have developed. In particular, we will connect SEM to a theory of scattering developed by Lax and Phillips (Refs. 7 and 8). This theory deals with solutions valid over a volume as opposed to solutions on a surface and it still has the satisfying quality that it leads one to view the internal scattering (cavity) problem and the external scattering problem in a manner that exhibits considerable uniformity. A connection to the Lax-Phillips theory acts as a connection to a chain of theories as effort on their part was made to connect to other prominent scattering theories and effort on the part of workers outside the EMP community was made to connect to the Lax-Phillips theory.

Our introduction to the Lax-Phillips theory came from reading the review article by Dolph and Scott (Ref. 5). This informative article not only summarized important features of the Lax-Phillips theory but connected that theory to other theories; however, it only minimally mentioned SEM. We were motivated to pursue the connection because we had been speculating on the insights into SEM one could obtain if a theory such as that of Lax-Phillips existed. Lax and Phillips as well as Dolph and Scott, made special effort to relate to an approach that has been much studied in quantum mechanical scattering. This approach utilizes the concept of a scattering operator and the S-matrix. Another work that makes a connection between the S-matrix approach and the Lax-Phillips theory is the book by Nussenzveig (Ref. 9). This book contains detailed material concerning applications of the S-matrix approach and presents a general review of other efforts where it has been utilized.

This report demonstrates that scalar SEM theory, based on the EEM expansion, is related to the Lax-Phillips theory. In particular, we prove that the set of complex eigenvalues that play a central role in the Lax-Phillips theory is exactly the same set as the one consisting of the (nonextraneous) zeros of the eigenvalues of the surface integral equation with the latter set being the SEM pole locations. At this point, one of the noteworthy connections between Lax-Phillips theory and S-matrix theory is that the complex eigenvalues have been shown to be in one-to-one correspondence with the poles of the S-matrix. We have now established that SEM pole locations are in one-to-one correspondence with these S-matrix poles. Another important feature of the Lax-Phillips theory is an asymptotic expansion (valid for late times) that has as its only time dependence, exponential functions having an argument that is the product of the complex eigenvalues and the time variable. The set equality we proved concerning the complex eigenvalues and SEM pole locations, as well as the form of the SEM expansion, allow us to conclude a term-by-term equality of the two expansions. This conclusion is based on the argument that the common time functions employed in both expansions are linearly independent functions.

The connection between the Lax-Phillips theory and scalar SEM based on the EEM approach benefits both efforts. The Lax-Phillips approach has a more advanced theoretical foundation. As an example of this, some conditions have been established on the shape of scattering surfaces to which this theory can be applied. This information had not yet been determined by the SEM/EEM approach. In addition, quantative and qualitative estimating techniques have been developed for the complex eigenvalues. Finally, as has already been discussed, the Lax-Phillips theory has been connected to other scattering theories. The SEM/EEM approach contributes to the scattering problem by providing explicit expressions for the expansion coefficients and these expansion coefficients, as well as SEM pole locations, have been numerically determined by workers in the EMP community. More generally, workers in the EMP community have developed the capability to numerically obtain SEM solutions for scattering shapes that are beyond analytic treatment.

A generalization of the arguments presented for the scalar case may readily be possible for the electromagnetic case since the Lax-Phillips theory has been extended to electromagnetic scattering (Ref. 10). We would be inclined to continue focusing attention on scalar SEM because of the untapped information contained in the explicit analytic scalar solutions, the analogue of which does not exist for electromagnetic scattering. We certainly have not exhausted the information in the spheroid solution presented in this report.

Finally, we describe a result that we were able to obtain due to the fact that we have made the described connection between scalar SEM and Lax-Phillips theory. We were able to obtain a formal proof that the SEM poles are simple, and this has long been identified as an open question by workers in the EMP community. This formal proof is totally constructive in nature and such a proof can alternately be thought of as providing a sufficient set of conditions for the desired conclusion. The sufficient condition nature of the proof is implicit in that certain formal manipulations are assumed valid and that the sufficient conditions are those implied to make the manipulations valid. For the case of the sphere, we have verified that these manipulations are valid.

II. A SCALAR THEORY OF THE SINGULARITY EXPANSION METHOD

The presentation of the material in this section is facilitated by referring to Figure 1.

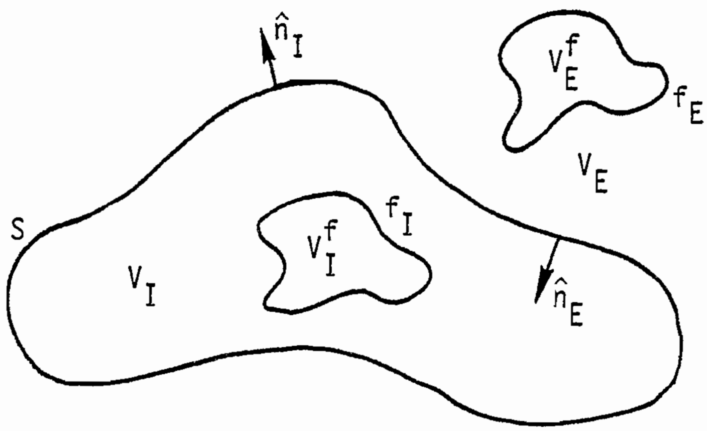

Figure 1. Separation of space into an interior and exterior region.

In this figure we introduce a surface, S, that separates all of space into an interior region, V_I, and an exterior region, V_E. At this point S is just a mathematically constructed surface; however, as this presentation proceeds, S will correspond to a physical surface on which boundary conditions are satisfied and it will also have shape requirements placed on it. Also in Figure 1 are sources denoted f_E and f_I, which are nonzero on finite volumes V_E^f and V_I^f contained within V_E and V_I. We are interested in finding solutions to the scalar wave equation in each region, which is given by

$$\nabla^2 \phi_\alpha(\underline{r},t) - \frac{1}{c^2} \frac{\partial^2 \phi_\alpha(\underline{r},t)}{\partial t^2} = f_\alpha(\underline{r},t) \qquad (1)$$

where

$$\alpha = E, I \text{ for } \underline{r} \varepsilon V_E \text{ or } \underline{r} \varepsilon V_I$$

Employing the Laplace transform with the convention

$$\tilde{f}(s) = \int_{-\infty}^{\infty} f(t) e^{-st} \, dt \tag{2a}$$

$$f(t) = \frac{1}{2\pi i} \int_{Br} \tilde{f}(s) e^{st} \, ds \tag{2b}$$

where Br indicates an appropriate Bromwich path, we obtain

$$(\nabla^2 - \gamma^2) \; \tilde{\phi}_\alpha(\underline{r}, \gamma) = \tilde{f}_\alpha(\underline{r}, \gamma) \tag{3}$$

where $\gamma = s/c$. In the remaining portion of this section we will omit the ~ notation. We now introduce the Green's function satisfying the equation

$$(\nabla^2 - \gamma^2) \; G(\underline{r}, \underline{r}', \gamma) = -\delta(\underline{r} - \underline{r}') \tag{4}$$

which is given by

$$G(\underline{r}, \underline{r}', \gamma) = \frac{e^{-\gamma|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|} \tag{5}$$

This Green's function is used in conjunction with the identity

$$\nabla \cdot [G\nabla\phi_\alpha - \phi_\alpha \nabla G] = G\nabla^2\phi_\alpha - \phi_\alpha \nabla^2 G \tag{6}$$

Substituting Equations (3) and (4) into Equation (6) and integrating over V_α we have

$$\int_{V_\alpha} \nabla \cdot [G\nabla\phi_\alpha - \phi_\alpha \nabla G]\, dV = \phi_\alpha(\underline{r}') + Q_\alpha(\underline{r}') \tag{7}$$

where

$$Q_\alpha(\underline{r}') = \int_{V_\alpha^f} f_\alpha(\underline{r})\, G(\underline{r},\underline{r}',\gamma)\, dV \tag{8}$$

At this point we make a distinction between the cases $\alpha = E,I$. In particular, we wish to apply the divergence theorem to the left-hand side of Equation (7). In order to do this, it is necessary to identify the surfaces bounding the volume over which the integration occurs. For the integration over V_I, the bounding surface is S and we have

$$\int_{V_I} \nabla \cdot [G\nabla\phi_I - \phi_I \nabla G]\, dV = \int_S \hat{n}_I \cdot [G\nabla\phi_I - \phi_I \nabla G]\, dS \tag{9}$$

For the integration over V_E, we identify the bounding surface as being S as well as a spherical surface that has a radius approaching infinity. We now have

$$\int_{V_E} \nabla \cdot [G\nabla\phi_E - \phi_E \nabla G]\, dV = \int_S \hat{n}_E \cdot [G\nabla\phi_E - \phi_E \nabla G]\, dS + I_L \tag{10}$$

where

$$I_L = \lim_{r\to\infty} \int \hat{a}_r \cdot [G\nabla\phi_E - \phi_E \nabla G]\, r^2 \sin\theta\, d\theta\, d\phi \tag{11}$$

with θ, ϕ, and r being spherical coordinates defined in a coordinate system having its origin at some finite distance from S. At this point we will consider that the surface S is the physical surface corresponding to a hard acoustic scatterer so that ϕ_I and ϕ_E satisfy the Neumann boundary conditions on S. That is,

$$\hat{n} \cdot \nabla\phi_\alpha = \frac{\partial\phi_\alpha}{\partial n} = 0 \tag{12}$$

In addition, $\phi_E(\underline{r})$, in the terminology of Reference 7, must satisfy the "γ-outgoing" condition, which is expressed as

$$\phi_E(\underline{r}) \sim K(\theta,\phi)r^{-1}e^{-\gamma r} \tag{13}$$

for large r. For a γ-outgoing solution for ϕ_E one can show that

$$\hat{a}_r \cdot [G\nabla\phi_E - \phi_E\nabla G] \sim F(\underline{r},\underline{r}',\gamma)e^{-2\gamma r} \tag{14}$$

for large r and further that F is such that

$$\lim_{r\to\infty} r^2 F(\underline{r},\underline{r}',\gamma)e^{-2\gamma r} = 0 \tag{15}$$

for

$$\mathrm{Re}\gamma \geq 0 \tag{16}$$

The significance of Equation (16) is that it defines the region for a Bromwich path that in turn allows us to conclude that

$$I_L = 0 \tag{17}$$

It is worth noting that the specification of a region for the Bromwich path has always been inherent in the more usual Fourier treatment of the wave equation. The Sommerfeld radiation condition is satisfied if Equation (16) is satisfied, as can be seen by making the substitution

$$\gamma = -ik \tag{18}$$

This substitution is consistent with the Fourier transform convention

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}\, dt \tag{19a}$$

$$f(t) = \frac{1}{2\pi} \int_{Br} \tilde{f}(\omega)e^{-i\omega t}\, d\omega \tag{19b}$$

Both Equations (14) and (15) are valid under this substitution and, accordingly, the region for the Bromwich path which allows I_L to vanish is

$$\ell mk \geq 0 \tag{20}$$

For γ or k restricted to the identified regions so that $I_L = 0$, utilizing the Neumann boundary conditions expressed in Equation (12), and changing the notation so that the roles of $\underline{r}$ and $\underline{r}'$ are interchanged, we combine Equations (7), (9), (10), and (17) to obtain

$$\phi_\alpha(\underline{r}) + \int_S [\hat{n}_\alpha(\underline{r}') \cdot \nabla' G(\underline{r},\underline{r}')]\phi_\alpha(\underline{r}')dS' = -Q_\alpha(\underline{r}) \qquad (21)$$

Equations (8) and (21) are sufficient to give $Q_\alpha(\underline{r})$ meaning without any explicit evaluation of integrals. From Equation (8) we see that $Q_\alpha(\underline{r})$ is independent of the presence of the surface S, so by letting the surface integral in Equation (20) vanish, we conclude that

$$-Q_\alpha(\underline{r}) = \phi_\alpha^{inc}(\underline{r}) \qquad (22)$$

where $\phi_\alpha^{inc}(\underline{r})$ is the field radiated by the sources without the surface being present. Substituting Equation (22) into Equation (21) yields

$$\phi_\alpha(\underline{r}_V) + \int_S [\hat{n}_\alpha(\underline{r}') \cdot \nabla' G(\underline{r}_V,\underline{r}')]\, \phi_\alpha(\underline{r}')dS' = \phi_\alpha^{inc}(\underline{r}_V) \qquad (23)$$

and we have changed the notation, temporarily replacing $\underline{r}$ by $\underline{r}_V$, in order to emphasize that Equation (23) was derived with $\underline{r}_V$ ranging over the volume V_α. We will reserve the notion $\underline{r}$ to represent an observation point on the surface S. Next we consider the limit as $\underline{r}_V$ approaches the surface, i.e., r, in both the left- and right-hand sides of Equation (23). Both $\phi_\alpha(\underline{r}_V)$ and $\phi_\alpha^{inc}(\underline{r}_V)$ are continuous functions as this limit is taken for values of $\underline{r}_V$ in the appropriate V_α so that we need only consider

$$I_\alpha = \lim_{\underline{r}_V \to \underline{r}} \int_S [\hat{n}_\alpha(\underline{r}') \cdot \nabla' G(\underline{r}_V,\underline{r}')]\, \phi_\alpha(\underline{r}')dS' \qquad (24)$$

The limit described in Equation (24) is a frequently treated limit and our major focus in this report is to present a simple bookkeeping scheme to identify critical plus and minus signs without addressing the subtleties of this limit. Referring to Figure 1 we see that the sign of the dot product of the vectors $\underline{r}_V - \underline{r}'$ and $\hat{n}_\alpha(\underline{r}')$ is the same when $\underline{r}_V \epsilon V_\alpha$, whether $\alpha = E$ or $\alpha = I$. It is the sign of this dot product that determines the sign multiplying the term $\frac{1}{2}\phi_\alpha(\underline{r})$ which is part of the expression that will be used to describe the limit. Since the sign of the dot product is independent of the choice for α, the sign multiplying the $\frac{1}{2}\phi_\alpha(\underline{r})$ term in the resulting expression is also independent of the choice for α. Using this information we have

$$I_\alpha = -\frac{1}{2}\phi_\alpha(\underline{r}) + \int_S [\hat{n}_\alpha(\underline{r}') \cdot \nabla' G(\underline{r},\underline{r}')]\phi_\alpha(\underline{r}')dS' \qquad (25)$$

and we see that the common sign is a negative sign. We also note that the factor $\frac{1}{2}$ is appropriate only if $\underline{r}$ does not correspond to a pathological point on the surface, e.g., a tip or an edge. We now introduce the notation

$$\hat{n}(\underline{r}') = \hat{n}_I(\underline{r}') = -\hat{n}_E(\underline{r}') \qquad (26)$$

$$\frac{\partial G}{\partial n'} = \hat{n}(\underline{r}') \cdot \nabla' G(\underline{r},\underline{r}') \qquad (27)$$

$$K\,\phi(\underline{r}) = \int_S \frac{\partial G}{\partial n'}\,\phi(\underline{r}')dS' \qquad (28)$$

Combining Equations (23) through (28), we have

$$L_s^E \phi^E = \phi_{inc}^E \tag{29}$$

$$L_s^I \phi^I = \phi_{inc}^I \tag{30}$$

where

$$L_s^E = \frac{1}{2} - K \tag{31}$$

$$L_s^I = 1 - L_S^E \tag{32}$$

In Equations (29) through (32) we have introduced the subscript "s" to the operator L_s in order to indicate that we are treating the scalar scattering problem. The reason for this additional notation is that we are now in a position to make an important distinction between the scalar scattering problem and the vector scattering problem.

From Reference 1, we see that the vector perfect conductor scattering problem treated by the MFIE has a generic representation that is equivalent to Equations (29) and (30). It is given by

$$L_V^E \underline{J}_E = \underline{J}_E^{inc} \tag{33}$$

and

$$L_V^I \underline{J}_I = \underline{J}_I^{inc} \tag{34}$$

$$L_V^I = 1 - L_V^E \tag{35}$$

The point we wish to make is independent of the specific representation of L_V^α and we refer the reader to Reference 1 if he is interested. In Equations (33) and (34) the quantities $\underline{J}_E$, $\underline{J}_I$, $\underline{J}_E^{inc}$, and $\underline{J}_I^{inc}$ have analogous meanings to ϕ_E, ϕ_I, ϕ_E^{inc}, and ϕ_I^{inc}. If we consider that the same surface depicted in Figure 1 was perfectly conducting and f_E and f_I were replaced by electromagnetic rigid sources, $\underline{f}_E$ and $\underline{f}_I$, then we can describe the electromagnetic quantities as follows. The quantity $\underline{J}_E$ is the total current density induced on the exterior of the surface that had as its prime excitation $\underline{f}_E$, and $\underline{J}_E^{inc}$ is a current density that would be excited by $\underline{f}_E$ at the same mathematical points describing the scattering surface; however, the physical surface is considered to be removed. The quantity $\underline{J}_I$ is the total current density induced on the interior of the surface by $\underline{J}_I^{inc}$, which again is the current density excited by $\underline{f}_I$ on the mathematical surface S, but the physical surface is considered to be removed. We will shortly present essential details of an eigenmode solution to Equation (31) as we present eigenmode solutions to the scalar scattering problem.

We now consider the scalar eigenvalue equation

$$L_s^\alpha \phi_n^\alpha = \lambda_n^{\alpha,s} \phi_n^\alpha \qquad \alpha = E,I \tag{36}$$

and in Appendix A we define an inner product between two functions f and g and introduce the standard notation to represent this inner product, i.e., (f,g). This inner product is used in the determination of the adjoint operator $L_s^{E\dagger}$ which is also presented in Appendix A. The eigenmode expansion method (EEM) solution to Equation (36) is expressed in terms of the eigenmodes ϕ_n^α and eigenvalues $\lambda_n^{\alpha,s}$ satisfying Equation (36), as well as the eigenmodes satisfying the adjoint equation

$$L_s^{\alpha\dagger}\phi_n^{\alpha\dagger} = \lambda_n^{\alpha,s*}\phi_n^{\alpha\dagger} \qquad \alpha = E,I \tag{37}$$

The corresponding EEM solutions are

$$\phi^E = \sum_{n=1}^{\infty} \frac{\left(\phi_n^{E\dagger},\phi_{inc}^E\right)}{\lambda_n^{E,s}}\phi_n^E \tag{38}$$

$$\phi^I = \sum_{n=1}^{\infty} \frac{\left(\phi_n^{I\dagger},\phi_{inc}^I\right)}{\lambda_n^{I,s}}\phi_n^I \tag{39}$$

Using results from Reference 1, we see that the EEM solution to Equation (31) is

$$\underline{J}^E = \sum_{n=1}^{\infty} \frac{\left\{\underline{\tilde{J}}_n^E,\underline{J}_{inc}\right\}}{\lambda_n^{E,v}}\underline{J}_n^E \tag{40}$$

where the quantity $\left\{\underline{\tilde{J}}_n^E,\underline{J}_{inc}\right\}$ is a pseudo-inner product which is defined in Reference 1 and its explicit representation is not required for us to make our desired point. The eigenmode $\underline{J}_n^E$ and eigenvalue $\lambda_n^{E,v}$ satisfy the equation

$$L_v^E\underline{J}_n^E = \lambda_n^{E,v}\underline{J}_n^E \tag{41}$$

where L_v^E is defined in Reference 1. A feature that makes the connection between scalar and vector EEM as well as SEM more dramatic, is the scalar analogue of the vector pseudo-symmetric theory, presented in Reference 1. In order to make the analogue, we retrieve essential results from Reference 1. First we introduce the vector interior equation given by

$$L_V^I \underline{J}_n^I = \lambda_n^{I,v} \underline{J}_n^I \tag{42}$$

To make the desired point we present the equation satisfied by the $\underline{\tilde{J}}_n^E$ which appeared in the pseudo-inner product contained in the EEM solution given by Equation (40). The equation satisfied by this quantity was obtained from adjointness considerations that are presented in detail in Reference 1. Retrieving the desired equation from that reference, we have

$$L_V^I \underline{\tilde{J}}_n^E = \lambda_n^{E,v} \underline{\tilde{J}}_n^E \tag{43}$$

Comparing Equations (42) and (43), we conclude the set equality

$$\left\{\lambda_n^{E,v}\right\} = \left\{\lambda_n^{I,v}\right\} \tag{44}$$

For the scalar case, there is also a connection between the interior and exterior eigenvalues that appears to be fundamentally different. However, viewing the SEM consequences of the relationship, shows that it is, in fact, fundamentally the same. In Appendix B, we present the details to support the following statements. First, we cite the surface eigenvalue equation for the interior Dirichlet problem. It is

$$L_s^{E^{\dagger *}} \sigma_n^I = \lambda_n^{I,s,D} \sigma_n^I \tag{45}$$

where σ_n is defined in Appendix B, the superscript D is affixed to the eigenvalue to indicate Dirichlet boundary conditions, and the crucial point is that the $L_s^{E^\dagger}$ turns out to be the adjoint of the Neumann exterior integral operator. Comparing the complex conjugate of this equation with Equation (37), we conclude the set equality

$$\left\{\lambda_n^{I,s,D}\right\} = \left\{\lambda_n^{E,s}\right\} \tag{46}$$

which states that the set of exterior Neumann surface eigenvalues is the same set as the set of interior Dirichlet surface eigenvalues. In addition, we conclude that the set of exterior adjoint Neumann eigenfunctions are the complex conjugate of the set of interior Dirichlet eigenfunctions. That is,

$$\left\{\phi_n^{E\dagger *}\right\} = \left\{\sigma_n^I\right\} \tag{47}$$

Using this result we can write Equation (38) in a manner analogous to Equation (40) as follows:

$$\phi^E = \sum_{n=1}^{\infty} \frac{\left\{\sigma_n^I, \phi_{inc}^E\right\}}{\lambda_n^{E,s}} \phi_n^E \tag{48}$$

where

$$\left\{\sigma_n^I, \phi_{inc}^E\right\} = \left(\sigma_n^{I*}, \phi_{inc}^E\right) = \left(\phi_n^{E\dagger}, \phi_{inc}^E\right) \tag{49}$$

The SEM consequences of the analogous relations, Equations (40) and (48), will now be discussed. First we express the following relations in a form that exhibits a desired amount of uniformity. Equations (38), (39), (40), and (48) can be written as

$$Y_\beta^\alpha(s) = \sum_{n=1}^{\infty} \frac{N_\beta^\alpha(s)}{\lambda_n^{\alpha,\beta}} Y_{\beta n}^\alpha(s) \qquad \begin{array}{l} \beta = s,\ \alpha = E,I \\ \beta = v,\ \alpha = E \end{array} \tag{50}$$

$$N_s^{\alpha}(s) = \left(\phi_n^{\alpha\dagger}(s), \phi_{inc}^{\alpha}(s)\right) \quad (51a)$$

$$N_v^{E}(s) = \left\{\tilde{\underline{J}}_n^{E}(s), \underline{J}_{inc}(s)\right\} \quad (51b)$$

$$N_s^{E}(s) = \left\{\sigma_n^{I}(s), \phi_{inc}^{E}(s)\right\} \quad (51c)$$

$$y_{sn}^{\alpha}(s) = \phi_n^{\alpha}(s) \quad (51d)$$

$$y_s^{\alpha}(s) = \phi^{\alpha}(s) \quad (51e)$$

$$y_{vn}^{E}(s) = \underline{J}_n^{E}(s) \quad (51f)$$

$$y_v^{E}(s) = \underline{J}^{E}(s) \quad (51g)$$

The argument s is the Laplace transform variable while the subscript s indicates scalar. We arrive at the desired SEM related residue series by taking the inverse of that equation using the Laplace transform convention presented in Equation (2). From independent analyses as discussed in the Introduction, we know that $y_{\beta}^{\alpha}(s)$ is a meromorphic function of s and from additional arguments we can confine the poles of $y_{\beta}^{\alpha}(s)$ to the left-half s plane with the presence or absence of poles lying on the imaginary axis requiring special attention. The argument that there are no right-half plane poles is related to causality. It is argued that the Bromwich path can be closed by increasingly large right-half plane semicircles and further that the integral of $y_{\beta}^{\alpha}(s)e^{st}$ over these semicircles vanishes for all t less than the time when the scattering object is initially struck by an incident field.

This allows us to close the Bromwich path to the right for these values of time. Because of causality, we do not wish to enclose any poles since they would yield residue contributions to the inverse transform for nonphysically interpretable values of time. There is a connection between causality and the radiation condition in that the radiation condition arguments led to the specification that the Bromwich path lie to the right of the singularities, thus making the zero residue response possible.

We now employ analytic continuation and close the Bromwich path to the left for a range of positive times. The results for this case are formal in that we will simply make any mathematical assumption required to obtain the results desired. If we proceed along this line, a question arises as to the utility of our results. In a subsequent section we will make some of these results rigorous, and in addition we will explain how these results are consistent with those obtained by analyses which did not make these assumptions. These results will also be shown to be in agreement with a known sphere solution. To obtain the desired form, we look for the totality of the zeros of the eigenvalues $\lambda_n^{\alpha,\beta}(s)$ and we denote an arbitrary zero as $s_{nn'}^{\alpha,\beta,T}$ where

$$\lambda_n^{\alpha,\beta}(s_{nn'}^{\alpha,\beta,T}) = 0 \qquad n' = 1,2,\ldots, N_T(n) \tag{52}$$

and the superscript T is used to indicate totality. As a result of Equation (44) for the vector case and Equation (46) for the scalar case, we expect a decomposition of the total set of zeros as follows

$$\left\{s_{nn'}^{\alpha,\beta,T}\right\} = \left\{s_{nn'}^{\alpha,\beta}\right\} \cup \left\{s_{nn'}^{\alpha,\beta,EX}\right\} \tag{53}$$

The zeros $s_{nn'}^{\alpha,\beta}$ are the significant zeros and the zeros having the superscript EX attached are the extraneous zeros. For the exterior problem, the significant zeros are the ones having a negative imaginary part and the extraneous ones are purely imaginary. For the interior problem, the significant ones are purely imaginary and the extraneous ones are in the left-half plane. Equations (44) and (46) demonstrate the existence of the extraneous zeros for the exterior problem because we know that the interior surface eigenvalue contained in each of these equations has purely imaginary zeros. The value of the pseudosymmetric argument presented in Reference 1 and in Appendix B is to eliminate nonphysical residue results due to the extraneous zeros for the exterior problem. More specifically, in Reference 1 it is shown that

$$\left\{N_v^E(s_{nn'}^{E,v,EX})\right\} = 0 \tag{54}$$

and in Appendix B, we show that

$$\left\{N_s^E(s_{nn'}^{E,s,EX})\right\} = 0 \tag{55}$$

Assuming that the significant zero is simple, we write the residue arising in the inverse Laplace transform corresponding to the significant zero $s_{nn'}^{\alpha,\beta}$ as $a_{nn'}^{\alpha,\beta} Y_{\beta nn'}^{\alpha}(r) e^{s_{nn'}^{\alpha,\beta} t}$ where

$$a_{nn'}^{\alpha,\beta} = \frac{N_\beta^\alpha(s_{nn'}^{\alpha,\beta})}{\lambda'^{\alpha,\beta}_n(s_{nn'}^{\alpha,\beta})} \tag{56}$$

$$y^{\alpha}_{\beta nn'}(\underline{r}) = y^{\alpha}_{\beta n}(s^{\alpha,\beta}_{nn'},\underline{r}) \qquad (57)$$

$$\lambda'^{\alpha,\beta}_{n}(s^{\alpha,\beta}_{nn'}) = \left.\frac{d\lambda^{\alpha,\beta}_{n}(s)}{ds}\right|_{s = s^{\alpha,\beta}_{nn'}} \qquad (58)$$

Using these residues, we write the response as

$$y^{\alpha}_{\beta}(\underline{r},t) = S^{\alpha}_{\beta}(\underline{r},t) + R^{\alpha}_{\beta}(\underline{r},t) \qquad (59)$$

where

$$S^{\alpha}_{\beta}(\underline{r},t) = \sum_{n=1}^{\infty}\sum_{n'=1}^{N(n)} a^{\alpha,\beta}_{nn'} y^{\alpha}_{\beta nn'}(\underline{r}) e^{s^{\alpha,\beta}_{nn'} t} \qquad (60)$$

and $R^{\alpha}_{\beta}(\underline{r},t)$ is just the remaining function needed to augment the sum $S^{\alpha}_{\beta}(\underline{r},t)$ in order to represent $y^{\alpha}_{\beta}(\underline{r},t)$. The real assumption is that $S^{\alpha}_{\beta}(\underline{r},t)$ converges in some meaningful sense and then it also dominates $R^{\alpha}_{\beta}(\underline{r},t)$. By comparing the surface approach used in the EMP community to the volume approach used by workers outisde of that community (Ref. 8), we can infer that $S^{\alpha}_{\beta}(\underline{r},t)$ does have the desired property. Finally, we note that an alternate form for the scalar expansion, one that bears a closer resemblance to the more prominent electromagnetic SEM expansion, is presented in Appendix C.

III. BACKGROUND MATERIAL FOR THE CONNECTION TO THE LAX-PHILLIPS THEORY

To facilitate the desired connection between the theory just presented and the Lax-Phillips theory which is a volume approach, we introduce the volume eigenvalue equations for both the interior and exterior scalar scattering problems

$$(\nabla^2 - \gamma_{\alpha n}^2)\phi_{\alpha n} = 0 \qquad \alpha = E, I \tag{61}$$

For the interior problem, $\alpha = I$, the boundary condition

$$\frac{\partial \phi_{\alpha n}}{\partial n} = 0 \tag{62}$$

on the surface as well as certain volume behavior requirements, e.g., requirements which force us to reject the explicit solutions obtainable for separable coordinates that become unbounded, are known to lead to denumerable sets of eigenfunctions $\{\phi_{In}\}$ and $\{\gamma_{In}\}$. Furthermore, for the eigenfunction equation convention used in Equation (61), it is also known that the γ_{In}'s are purely imaginary and occur in complex conjugate pairs. To clarify this and to provide a point of reference for a subsequent discussion, we note that the more common form of the interior eigenvalue equation is

$$(\nabla^2 + k_{In}^2)\phi_{In} = 0 \tag{63}$$

which, when the same boundary conditions and volume behavior requirements are imposed, it is known to lead to a discrete set of eigenfunctions and eigenvalues with the further result that the k_{In}'s lie on the real axis.

Returning to Equation (61), we consider the exterior scattering problem. The fact that, subject to appropriate boundary conditions, there exists only a denumerable set of eigenfunctions and eigenvalues is not as well known to be the case for the exterior problem as it is for the interior problem. The boundary conditions that lead to these denumerable sets are $\frac{\partial \phi_{En}}{\partial n} = 0$ on the surface the the γ-outgoing condition expressed as

$$\phi_{En} \sim K_n(\theta,\phi) r^{-1} e^{-\gamma_{En} r} \qquad (64)$$

for large r. As discussed in Reference 8, we also have the additional information that

$$\mathrm{Re}\gamma_n^E < 0 \qquad (65)$$

The requirement on the shape of the scatterer is an issue that has received attention. Many of the cited properties have been proved when the object is star shaped, i.e., a point within the object can be found from which a straight line can be drawn that connects this point to any other point within the volume bounded by the surface of scatterer. Star-shaped surfaces include convex surfaces. There has been some work and some conjecture concerning the applicability of the work to surfaces described as confining or nonconfining (Ref. 8). It should be noted that the predominant situation of an imperfectly sealed enclosure, i.e., a finitely thick-walled enclosure containing an aperture, is not a star-shaped surface.

We now restrict our attention to surfaces for which we have the desired discrete spectrum for the exterior scattering situation and utilize the identity

$$\nabla \cdot [\phi_{\alpha n} \nabla G - G \nabla \phi_{\alpha n}] = \phi_{\alpha n} \nabla^2 G - G \nabla^2 \phi_{\alpha n} \tag{66}$$

Substituting Equations (4) and (61) into Equation (66), we have

$$\nabla \cdot [\phi_{\alpha n} \nabla G - G \nabla \phi_{\alpha n}] = (\gamma^2 - \gamma_{\alpha n}^2) \phi_{\alpha n} G - \delta(\underline{r} - \underline{r}') \phi_{\alpha n} \tag{67}$$

First we consider this equation for the interior problem and integrate both sides over the interior volume and use the divergence theorem as well as the boundary condition given by Equation (62) to obtain

$$\phi_{In}(\underline{r}') + \int_S \hat{n}_I(\underline{r}) \cdot \nabla G(\underline{r}, \underline{r}') \phi_{In}(\underline{r}) dS = (\gamma^2 - \gamma_{In}^2) \int_{V_I} \phi_{In} G dV \tag{68}$$

Interchanging the notation $\underline{r}$ and $\underline{r}'$ and taking the limit as $\underline{r}$ approaches the surface, we have

$$L_s^I \phi_{In} = (\gamma_{In}^2 - \gamma^2) \int_{V_I} \phi_{In} G dV' \tag{69}$$

where L_s^I is defined by Equations (27), (28), (31), and (32). We now rewrite Equations (36) and (69) emphasizing the γ dependence. Equation (36), after changing the dummy index to m, is

$$L_s^\alpha(\gamma) \phi_m^\alpha(\gamma, \underline{r}) = \lambda_m^{\alpha, s}(\gamma) \phi_m^\alpha(\gamma, \underline{r}) \tag{70}$$

and Equation (69) is

$$L_s^I(\gamma) \phi_{In}(\underline{r}) = (\gamma_{In}^2 - \gamma^2) I_I(\gamma, \gamma_{In}, \underline{r}) \tag{71}$$

where

$$I_I(\gamma,\gamma_{In},\underline{r}) = \int_{V_I} \phi_{In}(\underline{r}')G(\gamma,|\underline{r}-\underline{r}'|)dV' \tag{72}$$

Rather than discuss the connection between SEM and the volume approach at this stage, i.e., having only provided material for the connection to the interior problem, we will obtain an equation that has the same significance for the exterior scattering problem as does Equation (71) for the interior problem.

To obtain the desired equation, we consider two mathematical surfaces to be introduced in the exterior domain. One is the smallest sphere that can circumscribe the scattering object, and the other is a sphere whose radius we will allow to approach infinity. Initially, we will not make use of the smaller sphere and will integrate both sides of Equation (67) over the volume, V_E , which is bounded by the scattering object and the large sphere, denoted S_r , and having a radius r. Performing this integration we obtain

$$\phi_{En}(\underline{r}') + \int_S \hat{n}_E(\underline{r}) \cdot \nabla G(\underline{r},\underline{r}')\phi_{En}(\underline{r})dS$$

$$= (\gamma^2 - \gamma_{En}^2) \int_{V_E} \phi_{En}GdV - I_r \tag{73}$$

where

$$I_r = \lim_{r\to\infty} \int_{S_r} \hat{a}_r \cdot [G\nabla\phi_{En} - \phi_{En}\nabla G]dS \tag{74}$$

Now the "γ-outgoing" condition leads to conclusions analogous to those expressed by Equations (14-16). That is, for large r

$$\hat{a}_r \cdot [G\nabla\phi_{En} - \phi_{En}\nabla G] \sim F_n(\underline{r},\underline{r}',\gamma,\gamma_{En})e^{-(\gamma+\gamma_{En})r} \tag{75}$$

where F_n is such that

$$\lim_{r\to\infty} r^2 F_n(\underline{r},\underline{r}',\gamma,\gamma_{En})e^{-(\gamma+\gamma_{En})r} = 0 \tag{76}$$

for

$$\text{Re}(\gamma+\gamma_{En}) \geq 0 \tag{77a}$$

Recognizing the $\text{Re}\gamma_{En} = -|\text{Re}\gamma_{En}|$, this condition becomes

$$\text{Re}\gamma \geq |\text{Re}\gamma_{En}| \tag{77b}$$

Since $|\text{Re}\gamma_{En}|$ is a nondecreasing function of n, we see that the specification of the Bromwich path moves to the right as n increases. Despite this, we can make the choice of the path independent of n, at least for some time, by employing the following argument. We can analytically continue any function of γ that is of interest back to the line $\text{Re}\gamma = -|\text{Re}\gamma_{E1}|$ where $|\text{Re}\gamma_{E1}|$ is the smallest $|\text{Re}\gamma_{En}|$ for all n. We can perform this analytic continuation in the finite γ plane without encountering any singularities since all the γ_{En} have negative real parts. We now see that with the Bromwich path initially chosen according to Equation (77), it follows that

$$I_r = 0 \tag{78}$$

We now turn our attention to the volume integral appearing in Equation (73) and denote this integral as I_E^n. We write this integral as the sum of three terms and in order to do this, we utilize the previously defined smallest circumscribing sphere. That is, we express

$$I_E^n = \int_{V_E} \phi_{En}(\underline{r})\, G(\gamma, |\underline{r}-\underline{r}'|)\, dV \qquad (79)$$

alternately as the sum of the terms

$$I_E^n = I_a^n + I_{bd}^n + I_{bf}^n \qquad (80)$$

where

$$I_a^n = \int_{V_a} \phi_{En} G dV \qquad (81)$$

$$I_{bd}^n = \lim_{r \to \infty} \int_{V_b} (\phi_{En} G - f_n) dV \qquad (82)$$

$$I_{bf}^n = \lim_{r \to \infty} \int_{V_b} f_n dV \qquad (83)$$

To complete the definitions expressed by Equations (80), (81), and (82), it is necessary to define V_a , V_b , and f_n . The quantity V_a is the volume between the surface of the scatterer and the circumscribing sphere whose radius we denote as a. The quantity V_b is the volume between the circumscribing sphere and the increasingly large sphere having the radius r. The quantity f_n is a function that is constructed as the product of the functions describing the large r asymptotic behavior of ϕ_{En} and G. For large r, the asymptotic behavior of ϕ_{En} is given in Equation (64) and we just note that the K_n given in that equation is a function of angular variables which will now be defined. Let us consider a spherical coordinate system defined as having its origin at the center of the circumscribing sphere and let us denote the usual angular variables as θ and ϕ. Then we can write

$$I^n_{bf} = \lim_{r\to\infty} \int_a^r dr'' \int_o^{2\pi} d\phi \int_o^{\pi} d\theta \; r''^2 \sin\theta \; f_n \tag{84}$$

For our purposes it is sufficient to represent f_n as

$$f_n = \psi_n(\underline{r}', \theta, \phi, \gamma, \gamma_{En}) \frac{1}{r''^2} e^{-(\gamma_{En} + \gamma) r''} \tag{85}$$

and we shall see shortly that the exact form of ψ_n is not important. Substituting Equation (85) into Equation (84), we have

$$I^n_{bf} = A_n(\underline{r}', \gamma, \gamma_{En}) I^{sing}_n \tag{86}$$

where

$$A_n = \int_o^{2\pi} d\phi \int_o^{\pi} d\theta \; \psi_n \sin\theta \tag{87}$$

and

$$I_n^{sing} = \lim_{r\to\infty} \int_a^r e^{-(\gamma+\gamma_{En})r''} dr''$$

$$= \lim_{r\to\infty} \frac{1}{\gamma + \gamma_{En}} \left[e^{-(\gamma+\gamma_{En})a} - e^{-(\gamma+\gamma_{En})r} \right] \tag{88}$$

If Equation (77) is satisfied, then

$$I_n^{sing} = (\gamma+\gamma_{En})^{-1} e^{-(\gamma+\gamma_{En})a} \tag{89}$$

Equation (73) can now be written as

$$\phi_{En}(\underline{r}') + \int_S \hat{n}_E(r) \cdot \nabla G(\underline{r},\underline{r}')\phi_{En}(r')dS$$

$$= (\gamma^2 - \gamma_{En}^2) I_c^n(\gamma,\gamma_{En},\underline{r}') + (\gamma - \gamma_{En}) I_D^n(\gamma,\gamma_{En},\underline{r}') \tag{90}$$

where

$$I_c^n = I_a^n(\underline{r}',\gamma,\gamma_{En}) + I_{bd}^n(\underline{r}',\gamma,\gamma_{En}) \tag{91}$$

and

$$I_D^n = A_n(\underline{r}',\gamma,\gamma_{En}) e^{-(\gamma+\gamma_{En})a} \tag{92}$$

The significance of the terms contained in Equations (91) and (92) is that they are not singular when $\gamma = \gamma_{En}$, and this is still the case when $\underline{r}'$ approaches the surface of the scatterer. We are now concerned with taking this limit in Equation (90) and we introduce the same notation change that was used in taking the same limit of Equation (68). That is, we interchange $\underline{r}$ and $\underline{r}'$ notation, use Equation (26), as well as the limiting arguments preceding that equation to obtain

$$L_s^E(\gamma)\phi_{En}(\underline{r}) = (\gamma^2 - \gamma_{En}^2) I_c^n(\gamma, \gamma_{En}, \underline{r}) + (\gamma - \gamma_{En}) I_D^n(\gamma, \gamma_{En}, \underline{r}) \tag{93}$$

where L_s^E is defined in Equation (31).

IV. CONCLUDING ARGUMENTS CONCERNING THE CONNECTION BETWEEN THE TWO THEORIES

At this point we will retrieve appropriate equations from the preceding section and rewrite them in a more convenient form. We begin by presenting Equation (70), without modification, to make it convenient for reference.

$$L_S^{\alpha}(\gamma)\phi_m^{\alpha}(\gamma,\underline{r}) = \lambda_m^{\alpha,s}(\gamma)\phi_m^{\alpha}(\gamma,\underline{r}) \qquad \alpha = E,I \qquad (70)$$

We rewrite Equations (71) and (93) as

$$L_S^{\alpha}(\gamma)\phi_{\alpha n}(\underline{r}) = (\gamma-\gamma_{\alpha n})F_{\alpha n}(\gamma,\underline{r}) \qquad (94a)$$

and

$$L_S^{I}(\gamma)\phi_{In}(\underline{r}) = (\gamma+\gamma_{In})F_{In}^{c}(\gamma,\underline{r}) \qquad (94b)$$

where

$$F_{In} = -(\gamma+\gamma_{In})I_I(\gamma,\gamma_{In},\underline{r}) \qquad (95a)$$

$$F_{In}^{c} = (\gamma_{In}-\gamma)I_I(\gamma,\gamma_{In},\underline{r}) \qquad (95b)$$

$$F_{En} = (\gamma+\gamma_{En})I_c^n(\gamma,\gamma_{En},\underline{r}) + I_D^n(\gamma,\gamma_{En},\underline{r}) \qquad (96)$$

A major conclusion that we wish to draw concerns the following sets. One set is described following Equation (53) in conjunction with the definition $\gamma_{nn'}^{\alpha,s} = \frac{s_{nn'}^{\alpha,s}}{c}$, and we write this set as

$$\left\{\gamma_{nn'}^{\alpha,s}\right\} = \left\{\gamma_{nn'}^{\alpha,s} \;\middle|\; \lambda_n^{\alpha,s}(\gamma_{nn'}^{\alpha,s}) = 0 \;,\; R_e\gamma_{nn'}^{I,s} = 0 \;,\; R_e\gamma_{nn'}^{E,s} < 0\right\} \qquad (97)$$

The other set is

$$\left\{\gamma_{\alpha m}\right\} = \left\{\gamma_{\alpha m} \;\middle|\; (\nabla^2-\gamma_{\alpha m}^{\;2})\phi_{\alpha m} = 0 + \begin{matrix}\text{boundary}\\ \text{conditions}\end{matrix}\right\} \tag{98}$$

The first conclusion that we wish to draw is the set equality

$$\left\{\gamma_{nn'}^{\alpha,s}\right\} = \left\{\gamma_{\alpha m}\right\} \tag{99}$$

To prove that

$$\left\{\gamma_{\alpha m}\right\} \subset \left\{\gamma_{nn'}^{\alpha,s}\right\} \tag{100}$$

we consider the interior problem first. We do this to emphasize a major difference between the interior and exterior problem. This difference manifested itself in the need to present Equation (94b) for the interior problem. Consider a typical γ_{Im} and let $\gamma = \gamma_{Im}$ in Equation (94a) to obtain

$$L_S^I(\gamma_{Im})\phi_{Im}(\underline{r}) = 0 \tag{101}$$

Letting $\gamma = -\gamma_{Im}$ in Equation (94b) yields

$$L_S^I(-\gamma_{Im})\phi_{Im}(\underline{r}) = 0 \tag{102}$$

Equation (101) implies that there exists some $\lambda_\ell^{I,s}(\gamma)$ such that $\lambda_\ell^{I,s}(\gamma_{Im}) = 0$, while Equation (102) implies that there exists some $\lambda_j^{I,s}(\gamma)$ such that $\lambda_j^{I,s}(-\gamma_{Im}) = 0$. This not only proves Equation (100) for $\alpha = I$, but it also proves that if

$$\gamma_{Im} \varepsilon \left\{ \gamma_{nn'}^{I,s} \right\} \quad (103a)$$

then

$$-\gamma_{Im} \varepsilon \left\{ \gamma_{n,n'}^{I,s} \right\} \quad (103b)$$

The conclusion expressed by Equation (103) is acceptable for $\alpha = I$ but would not be acceptable for $\alpha = E$. The reason for making this observation is to emphasize the need for the bounding arguments presented in the previous section that led to Equation (94a) for $\alpha = E$, and at the same time eliminated the need for an equation corresponding to Equation (94b) for the exterior problem. Substituting $\gamma = \gamma_{Em}$ into Equation (94a) leads to

$$L_S^E(\gamma_{Em})\phi_{Em}(\underline{r}) = 0 \quad (104)$$

which (by exactly the same arguments applied to Equation (101)) leads to the result

$$\gamma_{Em} \varepsilon \left\{ \gamma_{nn'}^{I,s} \right\} \; ; \quad (105)$$

however, there is no result corresponding to Equation (103b). This is necessary in that

$$\mathrm{Re}\gamma_{Em} < 0 \quad (106)$$

while, as is well known for the interior problem,

$$\mathrm{Re}\gamma_{Im} = 0 \quad (107)$$

Equation (107), together with the fact expressed in Equations (103a) and (103b), simply states that γ_{Im} occur in complex conjugate pairs. If an equation corresponding to Equation (103b) were true for the exterior problem, then because of Equation (106), we would be led to contradictions concerning the existence of right half-plane values of γ_{Em}. The bounding arguments presented in the previous section precluded this undesirable conclusion. Finally, we note that the more familiar form of Equation (107) is obtained by using the complex plane rotation described by Equation (18), which was presented in conjunction with the complex Fourier transform. The statement equivalent to Equation (107), after making the indicated substitution $k_{Im} = i\gamma_{Im}$ is

$$\mathrm{Im} k_{Im} = 0 \tag{108}$$

Equation (108) is the more familiar result corresponding to internal resonances.

Equations (103) and (105) prove Equation (100). To complete the proof of Equation (99), we will prove that

$$\left\{\gamma_{nn'}^{\alpha,s}\right\} \subset \left\{\gamma_{\alpha m}\right\} \tag{109}$$

To accomplish this, we form the following function

$$\psi_n^{\alpha}(\underline{r}_v) = \int_S G(\underline{r}_v,\underline{r}',\gamma_{nn'}^{\alpha,s})\,\phi_n^{\alpha\dagger^*}(\gamma_{nn'}^{\alpha,s},\underline{r}')\,dS' \tag{110}$$

where * indicates complex conjugate and $\phi_n^{\alpha\dagger}(\gamma,\underline{r}')$ satisfies the adjoint eigenvalue equation. The adjoint operator is given in Appendix A, and the adjoint eigenvalue equation is given as

$$L_s^{\alpha\dagger}(\gamma)\phi_n^{\alpha\dagger}(\gamma,\underline{r}) = \lambda_n^{\alpha,s*}(\gamma)\phi_n^{\alpha\dagger}(\gamma,\underline{r}) \qquad (111)$$

As in the previous section, subscript v is attached to the $\underline{r}$ to indicate that $\underline{r}_v$ varies over the appropriate three-dimensional volumes V_I or V_E. Direct substitution of $\psi_n^{\alpha}(\underline{r}_v)$ given by Equation (110) into

$$(\nabla_v^2 - \gamma_{nn'}^{\alpha,s^2})\psi_n^{\alpha}(\underline{r}_v) = 0 \qquad (112)$$

shows that this equation is satisfied for $\underline{r}_v$ in the appropriate open volumes V_E or V_I. In addition, $\psi_n^{E}(\underline{r}_v)$ can readily be seen to satisfy the γ-outgoing condition. For Equation (105) to be true, it remains to show that the Neumann boundary conditions on S are satisfied. To show this we take the gradient of both sides of Equation (110) to obtain

$$\nabla_v\psi_n^{\alpha}(\underline{r}_v) = \int_S \nabla_v G(\underline{r}_v,\underline{r}',\gamma_{nn'}^{\alpha,s})\phi_n^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s},\underline{r}')dS' \qquad (113)$$

Identifying $\underline{r}$ as a point on S which will be approached by $\underline{r}_v$, and defining $\hat{n}_\alpha(\hat{\underline{r}})$ as the appropriate normal defined at the point $\underline{r}$, consistent with the convention depicted in Figure 1, we have

$$\lim_{\underline{r}_v\to\underline{r}} \hat{n}_\alpha(\hat{r})\cdot\nabla_v\psi_n^{\alpha}(\underline{r}_v) = \lim_{\underline{r}_v\to\underline{r}} \int_S \hat{n}_\alpha(\hat{r})\cdot\nabla_v G(\underline{r}_v,\underline{r}',\gamma_{nn'}^{\alpha,s})\phi_n^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s},\underline{r}')dS' \qquad (114)$$

The appropriate 1/2 factors come from the limit expressed by the right-hand side of Equation (114) so that the following equation is true:

$$\hat{n}_\alpha(\underline{r}) \cdot \nabla\psi_n^\alpha(\underline{r}) = L_s^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s})\,\phi_n^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s},\underline{r}) \tag{115}$$

First we recognize that the left-hand side of Equation (115) is the desired normal derivative. Next we substitute $\gamma = \gamma_{nn'}^{\alpha,s}$ into Equation (111) and then take the complex conjugate of the resulting equation to obtain

$$L_s^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s})\,\phi_n^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s},\underline{r}) = \lambda_n^{\alpha,s}(\gamma_{nn'}^{\alpha,s})\,\phi_n^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s},\underline{r}) \tag{116}$$

Combining Equations (115) and (116), we obtain

$$-\frac{\partial\psi_n^\alpha(\underline{r})}{\partial n} = \lambda_n^{\alpha,s}(\gamma_{nn'}^{\alpha,s})\,\phi_n^{\alpha\dagger *}(\gamma_{nn'}^{\alpha,s},\underline{r}) \tag{117}$$

where we have used the convention that

$$\frac{\partial\psi_n^\alpha(\underline{r})}{\partial n} = -\hat{n}_\alpha(\underline{r}) \cdot \nabla\psi_n^\alpha(\underline{r}) \tag{118}$$

Referring to the definition of the $\gamma_{nn'}^{\alpha,s}$ given in Equation (97), we see from Equation (117) that

$$\frac{\partial\psi_n^\alpha(\underline{r})}{\partial n} = 0 \tag{119}$$

Equations (112) and (119) yield the desired result

$$\gamma_{nn'}^{\alpha,s} \in \left\{\gamma_{\alpha m}\right\} \tag{120}$$

which in turn implies that Equation (109) is true. Equations (100) and (109) together prove the desired result expressed by Equation (99).

Shortly, we will interpret the significance of Equation (99). However, we will first point out the intrinsic consistency of the equations that led to this result. There were two intermediate results that occurred during the course of this study that appeared as detriments to drawing desired conclusions. One was the fact that the integral I_E^n given by Equation (79) became unbounded. The other initially appeared as a nuisance factor when we determined that the operators L_s^α turned out not to be self-adjoint as described in Appendix A. The analysis presented in this section showed that both of these apparently undesirable results turned out to be very desirable. The unboundedness of I_n^E is the underlying reason that precluded the consideration of $-\gamma_{Em}$ as belonging to $\{\gamma_{nn'}^{E,s}\}$. The nonself-adjointness of L_s^α greatly facilitated the proof of Equation (109).

Returning to the interpretation of Equation (99), we first prove that the elements of this single set which we now denote as $\{\gamma_\ell^\alpha\}$ for notational convenience, i.e.,

$$\left\{\gamma_\ell^\alpha\right\} = \left\{\gamma_{\alpha m}\right\} = \left\{\gamma_{nn'}^{\alpha,s}\right\} \tag{121}$$

contain the pole locations of the ϕ^α that satisfy Equations (29) and (30), which we rewrite as

$$L_s^\alpha(\gamma)\phi^\alpha(\gamma,\underline{r}) = \phi_{inc}^\alpha(\gamma,\underline{r}) \tag{122}$$

From arguments already discussed, we know that $\phi^{\alpha}(\gamma,\underline{r})$ is a memomorphic function of γ having all of its poles in the closed left-half γ plane. First we will consider the possibility that a pole at the location $p_{n\alpha}$ has multiplicity $M_{n\alpha}$.

Next we multiply both sides of Equation (122) by $(\gamma-p_{n\alpha})^{M_{n\alpha}}$ and then set $\gamma= p_{n\alpha}$. Assuming we have not chosen an incident field having the same pole location, the resulting equation is

$$L_s^{\alpha}(p_{n\alpha})R_n^{\alpha}(\underline{r}) = 0 \tag{123}$$

where

$$R_n^{\alpha}(\underline{r}) = (\gamma-p_{n\alpha})^{M_{n\alpha}} \phi^{\alpha}(\gamma,\underline{r})\Big|_{\gamma=p_{n\alpha}} \tag{124}$$

Equation (123) implies

$$p_{n\alpha} \in \left\{\gamma_{nn'}^{\alpha,s}\right\} \tag{125}$$

which, according to Equation (121), implies

$$p_{n\alpha} \in \left\{\gamma_n^{\alpha}\right\} \tag{126}$$

At this stage we have not presented an argument that all of the members of $\{\gamma_n^{\alpha}\}$ are pole locations.

A major goal of this work is to facilitate the use of complementary efforts related to SEM that have been generated by different communities. In Reference 8, the following asymptotic expansion, expressed in the notation of this work, is given as

$$\phi_E(\underline{r},t) \sim S_E(\underline{r},t) \tag{127}$$

where

$$S_E(\underline{r},t) = \sum_{m=1}^{\infty} c_m \, e^{\gamma_{Em}\tau} \, \phi_{Em}(\underline{r}) \tag{128}$$

and $\tau = ct$. $\phi_E(\underline{r},t)$ is said to behave asymptotically like $S_E(\underline{r},t)$ for large t.

After performing only a cursory search, we found no explicit representations for the c_m's nor did we find estimates on errors introduced by terminating the sum as a function of either $\underline{r}$ or t. Despite our inability to obtain this information, the existence of Equation (127) together with our result, Equation (99), benefits our state of knowledge. A formal comparison of $S_s^E(\underline{r},t)$, given by Equation (60), after the following substitution,

$$e^{s_{nn'}^{E,s}t} = e^{\gamma_{nn'}^{E,s}\tau} = e^{\gamma_{Em}\tau} \tag{129}$$

with $S_E(\underline{r},t)$ given by Equation (128) yields the following formal conclusions. The sum of all of the terms multiplying a particular $e^{\gamma_{Em}\tau}$, i.e., allowing for degeneracy of the complex eigenvalue γ_{Em}, in the representation $S_E(\underline{r},t)$ given by Equation (128), is equal to the sum of the terms multiplying the same $e^{\gamma_{Em}\tau}$ in the representation of $S_s^E(\underline{r},t)$ given by Equation (60). This formal equality has mutual benefits. The terms contained in the representation $S_s^E(\underline{r},t)$ have explicit representations in terms of surface quantities and furthermore they are numerically calculable.

The benefit in going the other direction is that we now have additional reasons to believe that the remainder term $R_s^E(\underline{r},t)$ becomes unimportant for large t. This result has frequently been assumed to be true by workers in the EMP community, based on the argument that $R_V^E(\underline{r},t)$ was the inverse transform of an entire function. This argument suffers in that a very simple entire function can be identified whose inverse transform would dominate $S_\beta^\alpha(\underline{r},t)$ in a controlled manner. This function for the scalar case is simply

$$f_E(\gamma) = Ae^{-\gamma T} \tag{130}$$

If $f_E(\gamma)$ were the additive entire function for the case where the time dependence of the incident field had a Dirac delta time dependence, then the inverse transform of $f_E(\gamma)$ convolved with the time dependence of a general incident field could be chosen to yield $R_\beta^\alpha(\underline{r},t)$ which is larger than the $S_\beta^\alpha(\underline{r},t)$. All that would be required would be to choose the constants A and T so that this occurred. The asymptotic nature of $S_E(\underline{r},t)$ precludes this possibility for the scalar scattering problem and suggests there is no hidden surprise in the $R_V^E(\underline{r},t)$ for the vector scattering problem.

Before leaving the formal connection between $S_E(\underline{r},t)$ and $S_s^E(\underline{r},t)$, two additional comments are in order. The first is just to note that the $S_E(\underline{r},t)$, which results from the volume approach, contains the features discussed in the Introduction. The second is that the form of S_E is very indicative that $\phi_E(\underline{r},\gamma)$ has only simple poles at the location γ_{Em}. This follows from taking a term-by-term Laplace transform of the sum $S_E(\underline{r},t)$. We will shortly present a separate structured argument that this is actually the case.

Our argument that $\phi_E(\underline{r},\gamma)$ has only simple poles requires a number of clearly identified mathematical statements to be true. A crucial mathematical requirement is that Equation (29) have the eigenmode solution expressed by Equation (38). Changing the notation of Equation (25), we ask whether a solution $\psi(\gamma,\underline{r})$ exists to the

$$L_s^E(\gamma)\psi(\gamma,\underline{r}) = F(\gamma,\underline{r}) \qquad (131)$$

for a given $F(\gamma,\underline{r})$ and furthermore we ask that the solution can be expressed as

$$\psi(\gamma,\underline{r}) = \sum_{n=1}^{\infty} \frac{\left(\phi_n^{E\dagger}(\gamma,\underline{r}),F(\gamma,\underline{r})\right)}{\lambda_n^{E,s}(\gamma)} \phi_n^E(\gamma,\underline{r}) \qquad (132)$$

The conditions that ψ exist for a specified F can be found in discussions of Fredholm operators since $L_S^E(\gamma)$ is such an operator. The conditions for the validity of the EEM expressed by Equation (132) require further discussion. A recently distributed report by Ramm (Ref. 6) presented conditions that an operator very closely related to $L_S^E(\gamma)$ could be inverted according to Equation (132). In particular, he treated the operator $K^\dagger$ presented in Appendix A. He concluded that, for a suitable class of scattering surface shapes, the root system of $K^\dagger$ formed an appropriate expansion basis. He then emphasized the limitations of this conclusion in that the root system consists of both eigenvectors and root vectors. In order to utilize a standard EEM expansion, it is necessary that there be only eigenvectors and no root vectors in the root system. His conclusions concerning the root system for $K^\dagger$ can be extended to apply to the operators L_s^E and L_S^I and we now focus on L_s^E. In order that Equation (132) be valid, it is necessary that the root system of L_s^E contain only eigenvectors and no root vectors. Ramm (Ref. 6) gives a set of sufficient conditions for this to be the case, and shows that all of these are readily satisfied with the exception of the condition that L_s^E is normal. Ramm points out that whether this sufficient condition is satisfied depends on the shape of the scatterer. He cites some of his earlier work which demonstrated for the analogous Dirichlet problem that a spherical scatterer yielded a normal operator. Later in this work, we examine whether L_s^E is normal for both the sphere and spheroid. For the sphere, we conclude that it is normal while for the spheroid we could only establish that it is a complex symmetric operator. Despite this, we conclude that the EEM solution for the spheroid yields the correct solution that is obtained by the standard separation of variables procedure.

We now consider Equation (132) as applied to a special case of Equation (131). This special case is Equation (94a) with α = E and the dummy index changed to i. For this situation Equation (132) becomes

$$\phi_{Ei}(\underline{r}) = \sum_{n=1}^{\infty} \frac{\gamma-\gamma_{Ei}}{\lambda_n^{E,s}(\gamma)} \left(\phi_n^{E\dagger}(\gamma,\underline{r}),F_{Ei}(\gamma,\underline{r})\right)\phi_n^E(\gamma,\underline{r}) \quad (133)$$

We have already shown that for some n, say N,

$$\lambda_N^{E,s}(\gamma_{Ei}) = 0 \quad (134)$$

In order that the right-hand side remain finite and nonzero at $\gamma = \gamma_{Ei}$, we conclude that $\lambda_N^{E,s}(\gamma)$ has only a simple zero at γ_{Ei}.

Referring to Equation (132) for a source $F_D(\gamma,\underline{r})$ corresponding to an incident field have a Dirac delta time dependence, we make the following observation concerning the meromorphic function $\psi_D(\gamma,\underline{r})$. The poles of $\psi_D(\gamma,\underline{r})$ corresponding to the zeros of the eigenvalues are simple poles. It is conjectured that for many surfaces these are the only poles. For the case where the incident field has a more general time dependence, we simply convolve the inverse transform of the Dirac solution $\psi_D(\gamma,\underline{r})$ with the specified time dependence.

We conclude this section by making an important distinction between the vector and scalar scattering problems. We have just presented sufficient conditions for the eigenvalues $\lambda_n^{E,s}(\gamma)$ to have simple zeros. Equation (60), which represents a standard SEM expansion for all cases of interest, vector and scalar, interior and exterior, is based on the assumption that all of the eigenvalues $\lambda_n^{\alpha,\beta}(\gamma)$ have simple zeros. In this section we have presented sufficient conditions for this to be the case for $\lambda_n^{E,s}(\gamma)$. Identically, the same arguments could be applied to conclude that $\lambda_n^{I,s}(\gamma)$ had only simple poles. All that would be required would be to use Equation (94a) and (94b) for $\alpha = I$ to obtain an equation analogous to Equation (133) for the interior problem. It now remains to discuss the vector problem. As mentioned in the Introduction, there has been work directed at extending the scalar volume approach to the vector (electromagnetic) case. We have not pursued that vector extension to determine whether the vector analogue of Equation (61) has been established for the exterior vector problem. If it has, the arguments presented in this work would require minimum modification to draw analogous conclusions to the ones drawn for the scalar case. The primary modifications would be to utilize vector Green's theorems and the free space dyadic Green's functions. This might be readily accomplished; however, it is our view that further insight into SEM can be obtained by continuing the study of the scalar problem. We have two underlying reasons that led us to this view. One is that only sufficient conditions have been identified that allow desired SEM conclusions to be drawn. The other is that there are more finite shapes for which analytic solutions exist for the scalar scattering problem, the spheroid is only one of them. In this report, we have gained some information by examining the scalar spheroid scattering solution; however, we have not exhausted the information contained in that solution.

V. PROLATE SPHEROID

In this section we consider an acoustically hard prolate spheroid and determine the eigenfunctions and eigenvalues of the surface integral operator. These eigenfunctions have an explicit dependence on the Fourier complex variable k (or γ if Laplace transforms are employed) whereas, as in the vector case, the eigenfunctions for the special case of a sphere are independent of k.

The scalar integral equation for an acoustically hard body is

$$\frac{1}{2}\,\phi(\underline{r}) - \int_S \phi(\underline{r}')\,\frac{\partial}{\partial n'}\,G(\underline{r}';\underline{r})\,dS' = \phi^{inc}(\underline{r}),\quad \underline{r}\varepsilon S \tag{135}$$

where

$$G = \frac{e^{ikR}}{4\pi R}\ ,\quad R = |\underline{r}-\underline{r}'|$$

$$\frac{\partial G}{\partial n'} = \hat{n}(\underline{r}')\cdot\nabla' G$$

and $\hat{n}(\underline{r}')$ is the outward unit normal at $\underline{r}'$. The corresponding eigenvalue problem has the form

$$L\Phi_n = \lambda_n \Phi_n \tag{136}$$

where

$$L\Phi_n = \left(\frac{I}{2} - K\right)\,\Phi_n$$

$$K\Phi_n = \int_S \Phi_n(\underline{r}')\,\frac{\partial G}{\partial n'}\,dS \tag{137}$$

The prolate spheroidal curvilinear coordinates ξ, η, ϕ are related to the Cartesian coordinates x, y, z by the transformation

$$x = \frac{1}{2}\left[\left(\xi^2-1\right)\left(1-\eta^2\right)\right]^{1/2} \cos\phi$$

$$y = \frac{1}{2}\left[\left(\xi^2-1\right)\left(1-\eta^2\right)\right]^{1/2} \sin\phi$$

$$z = \frac{1}{2} d\xi\eta$$

where ξ is the "radial" coordinate ($1\leq\xi<\infty$), $\eta(-1\leq\eta\leq 1)$ and $\phi(0\leq\phi\leq 2\pi)$ are the angular coordinates and d is the interfocal distance. The surface of the spheroid is defined by $\xi = \xi_1$. The correspondence with the spherical coordinates r, θ is

$$\frac{1}{2}\xi d \to r, \quad \eta \to \cos\theta \quad \text{as} \quad \xi \to \infty$$

i.e., we let the interfocal distance go to zero and keep ξd finite. We also present the formulas for the metric coefficients h_ξ, h_η, h_ϕ because we will use them shortly:

$$h_\xi = \frac{d}{2}\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)^{1/2}$$

$$h_\eta = \frac{d}{2}\left(\frac{\xi^2-\eta^2}{1-\eta^2}\right)^{1/2}$$

$$h_\phi = \frac{d}{2}\left[\left(1-\eta^2\right)\left(\xi^2-1\right)\right]^{1/2}$$

The scalar Hermholtz equation

$$\left(\nabla^2+k^2\right) \Phi = 0$$

has the following solution for outgoing waves

$$\Phi = S_{mn}(c,\eta) \; R_{mn}^{(3)}(c,\xi) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix}$$

where

$$c = \frac{1}{2} kd$$

and $S_{mn}(c,\eta)$, $R_{mn}^{(3)}(c,\xi)$ are the angular and radial functions respectively. The equations they satisfy are given in the monograph by C. Flammer (Ref. 11). In the limit of a sphere, $S_{mn}(c,\eta)$ reduces to the associated Legendre Polynominal $P_n^m(\cos\theta)$ and $R_{mn}^{(3)}(\xi)$ to the spherical Bessel Function $h_n^{(1)}(r)$. The angular functions $S_{mn}(c,\eta)$ satisfy the following orthogonality relationship

$$\int_{-1}^{+1} S_{mn}(c,\eta) \; S_{mn'}(c,\eta) \; d\eta = \delta_{nn'} \; N_{mn} \tag{138}$$

where N_{mn} is the normalization factor given in Reference 11. In terms of the spheroidal functions the Green's function G has the following expansion

$$\frac{e^{ikR}}{4\pi R} = \sum_{p=0}^{\infty} \sum_{\ell=p}^{\infty} \frac{ik\left(2-\delta_{op}\right)}{2\pi N_{p\ell}} \; S_{p\ell}(c,\eta) \; S_{p\ell}(c,\eta') \cos p(\phi-\phi')$$

$$R_{p\ell}^{(1)}(c,\xi') \; R_{p\ell}^{(3)}(c,\xi) \qquad \xi > \xi' \tag{139}$$

where $R_{p\ell}^{(1)}(c,\xi)$ corresponds to $j_\ell(kr)$, $R_{p\ell}^{(3)}(c,\xi)$ to $h_\ell^{(1)}(kr)$ and $N_{p\ell}$ is the same normalization factor as in Equation (138).

We are now in a position to show that the eigenfunctions of L defined by Equation (137) are:

$$\Phi_{mn} = S_{mn}(c,\eta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix} \tag{140}$$

Equation (140) shows the explicit dependence of Φ_{mn} on k or $\gamma(c = (1/2)kd = (i/2)\gamma d)$. We begin by noting that the integral operator involves the normal derivative of the Green's function G. In order to be able to employ the representation given by Equation (139) and operate on $R_{p\ell}^{(1)}(c,\xi')$ inside the sum, we recall that

$$\frac{1}{2}\,\Phi_{mn}(\underline{r}) - \int_S \Phi_{mn}(\underline{r}')\,\frac{\partial G}{\partial n'}\,dS'$$

$$= \Phi_{mn}(\underline{r}) - \lim_{\xi\to\xi_1}\int_S \phi_{mn}(\underline{r}')\,\frac{\partial G}{\partial n'}\,dS' \tag{141}$$

where $G(\xi_1,\eta',\phi';\xi,\eta,\phi)$ in the second line has $\xi \neq \xi_1$; i.e., its unprimed radius vector corresponds to a point off the surface. In the first line the limiting process has been completed and both $\underline{r}$ and $\underline{r}'$ correspond to points on the surface $\xi = \xi_1$. In view of Equation (139) we can write

$$\frac{\partial G}{\partial n'} = \frac{1}{h_{\xi'}} \frac{\partial G}{\partial \xi'}\Bigg|_{\xi' = \xi_1}$$

$$= \sum_{p=0}^{\infty} \sum_{\ell=p}^{\infty} \frac{ik\left(2-\delta_{op}\right)}{2\pi N_{p\ell}} S_{p\ell}(c,\eta)\, S_{p\ell}(c,\eta')\, \cos p(\phi-\phi')$$

$$\frac{1}{h_{\xi_1}} \frac{d}{d\xi_1} R_{p\ell}^{(1)}\left(c,\xi_1\right) R_{p\ell}^{(3)}(c,\xi) \qquad \xi > \xi_1 \tag{142}$$

If we now notice that

$$dS' = h_{\eta'} d\eta' h_{\phi'} d\phi'$$

$$\frac{h_{\eta'} h_{\phi'}}{h_{\xi_1}} = \xi_1^2 - 1$$

and take into account the orthogonality relationships of S_{mn} and of the trigonometric functions, we find that

$$\lim_{\xi \to \xi_1} \int_S \Phi_{mn}(\underline{r}') \frac{\partial G}{\partial n'} dS'$$

$$= \frac{ikd}{2} \left(\xi_1^2-1\right) R_{mn}^{(3)}\left(c,\xi_1\right) \frac{d}{d\xi_1} R_{mn}^{(1)}\left(c,\xi_1\right) \Phi_{mn}(\underline{r}) \tag{143}$$

where Φ_{mn} is given by Equation (140). Combining Equations (136), (137), (141), and (143) we obtain

$$\lambda_{mn} = 1 - ic\left(\xi_1^2-1\right) R_{mn}^{(3)}\left(c,\xi_1\right) \frac{d}{d\xi_1} R_{mn}^{(1)}\left(c,\xi_1\right) \tag{144}$$

where the relationship $c = (1/2)kd$ has been used. In order to simplify Equation (144) we employ the equations satisfied by the radial functions $R_{mn}^{(3)}$, $R_{mn}^{(1)}$ (Ref. 11) along with their asymptotic expressions for large ξ (also given in Reference 11) to obtain the Wronskian relationship

$$R_{mn}^{(3)} \frac{dR_{mn}^{(1)}}{d\xi} - R_{mn}^{(1)} \frac{dR_{mn}^{(3)}}{d\xi} = - \frac{i}{c\left(\xi^2-1\right)} \tag{145}$$

Combining Equations (144) and (145) we obtain

$$\lambda_{mn} = ic\left(\xi_1^2-1\right) R_{mn}^{(1)}\left(c,\xi_1\right) \frac{d}{d\xi_1} R_{mn}^{(3)}\left(c,\xi_1\right) \tag{146}$$

If the Laplace transform variable γ is used we simply replace c by $i\gamma d/2$ wherever c appears.

Now that we have determined the eigenfunctions and eigenvalues for Equation (136) by utilizing Equation (142), we can go back and give an explicit expansion of $\partial G/\partial n'$ when both $\underline{r}$ and $\underline{r}'$ are on the surface of the spheroid. The answer is

$$\frac{\partial G}{\partial n'} = \sum_{\ell=0}^{\infty} \sum_{p=0}^{\ell} \sum_{\sigma} \frac{1-2\lambda_{p\ell}}{2N_{p\ell}} \Phi_{p\ell\sigma}(\underline{r}) \Phi_{p\ell\sigma}(\underline{r}') \tag{147}$$

where σ stands for odd, even. In Appendix A we show that for both the spheroid and the sphere the adjoint operator $L^{\dagger}$ is equal to L^{*} where the asterisk signifies complex conjugation. For the sphere we also show that L is normal. In general, in order to solve Equation (135) we need the adjoint eigenfunctions. A formula for the adjoint operator is derived in Appendix A.

In the limit of a sphere we easily obtain

$$\Phi_{mn} = P_n^m(\cos\theta) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \tag{148}$$

$$\lambda_n = - i(ka)^2 \, j_n(ka) \, h_n^{(1)'}(ka) .$$

$$= - (\gamma a)^2 \, i_n(\gamma a) \, k_n'(\gamma a) \tag{149}$$

where $f'(x) \equiv df/dx$. The eigenfunctions given by Equations (140) and (148) form complete orthogonal sets in their respective domains ($-1 \leq \eta \leq 1$, $0 \leq \phi \leq 2\pi$ for the surface of the prolate spheroid and $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ for the sphere) and, consequently, they represent the complete solutions to the eigenvalue problems. Along with the eigenvalues they can be used to solve the integral

$$\Phi(\underline{r}) = \sum_{\ell} \frac{\left(\Phi_\ell, \Phi^{inc}\right)}{N_\ell \lambda_\ell} \Phi_\ell \tag{150}$$

where

$$\left(\Phi_\ell, \Phi^{inc}\right) = \int_S \Phi_\ell(\underline{r}) \, \Phi^{inc}(\underline{r}) \, d\eta d\phi$$

and Φ_ℓ, λ_ℓ, N_ℓ are the eigenfunctions, eigenvalues and normalization factors, respectively. Employing Equations (140) and (146), we can show that the scalar field Φ evaluated at the surface of a hard spheroid excited by an incident plane wave e^{ikz} is given by the well-known formula

$$\Phi = \frac{2}{c(\xi_1^2 - 1)} \sum_{n=0}^{\infty} \frac{i^{n-1}}{N_{on}} \left[\frac{d}{d\xi_1} R_{on}^{(3)}(c,\xi_1) \right]^{-1} S_{on}(c,1)\, S_{on}(c,\eta) \tag{151}$$

by noting that on the surface of the spheroid

$$e^{ikz} = \sum_n \frac{2i^n}{N_{on}} S_{on}(c,1)\, S_{on}(c,\eta)\, R_{on}^{(1)}(c,\xi_1)$$

and that

$$\int_{-1}^{1} e^{ikz} S_{on}\, d\eta = 2i^n S_{on}(c,1)\, R_{on}^{(1)}(c,\xi_1)$$

For the sphere it is easy to derive the SEM expansion by first employing Equations (148) and (149) and assuming an incident plane wave $e^{-\gamma z}$

$$\Phi(\underline{r},\gamma) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\left(P_n^m(\cos\theta), e^{-\gamma z} \begin{matrix}\cos\\ \sin\end{matrix} m\phi \right)}{-(\gamma a)^2\, i_n(\gamma a)\, k_n'(\gamma a)\, N_{mn}} P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} m\phi .$$

Following the methods employed in References 1 and 2, we arrive at the following SEM expansion

$$\Phi(\underline{r},\gamma) = e^{-\gamma c t_o} \sum_n \sum_{n'} \frac{(2n+1)(-1)^{n+1}}{F'(\gamma_{nn'} a)} \frac{1}{(\gamma - \gamma_{nn'})^a} P_n(\cos\theta) \tag{152}$$

where

$$F(x) \equiv e^x x^2 k_n'(x)\,, \quad F'(x) \equiv \frac{dF}{dx}$$

$\gamma_{nn'}$ are the roots of $k_n'(\gamma a) = 0$ and $t_o (= -a/c)$ is the instant at which the wavefront first hits the sphere. Equation (152) corresponds to class 1 coupling coefficients as expected from the vector case.

VI. NUMERICAL RESULTS

In this section we present numerical results for the zero locations corresponding to the vector sphere problem. These zeros were found by employing the method described in Reference 1. Figure 2 shows the division of the surface of the sphere into zones, where N_{LONG} is the number of slices in the azimuthal direction and N_{LAT} is the number of slices in the polar direction.

Table 1 presents the numerical results. Notice that the search routine (Ref. 1) locates clusters of zeros corresponding to a true single zero. For example, for $n = 1$ and $N_{LONG} = 4$, $N_{LAT} = 2$, we obtain the three zeros $-1.0202+i0.0862$, $-1.1012 -i0.0522$, $-0.9408-i0.0365$, whereas the real zero is $s_{110} = -1+i0$. The reason for the clustering is that zoning transforms the surface of a sphere into a different surface and the three-fold degeneracy ($n=1$, $m=-1,0,1$) is resolved. As $N_{LONG} \to \infty$, $N_{LAT} \to \infty$ all three zeros will coalesce into the one zero s_{110} if we ignore numerical roundoff error. Also notice that as we increase the number of zones a) more zeros belonging to a cluster are found and b) the norm of the error of the average cluster value for a zero decreases.

The rectangular area we employed is defined by:

Coordinates of the lower right corner: (-1.55, -0.30)

Coordinates of the upper right corner: (0.00, 1.30)

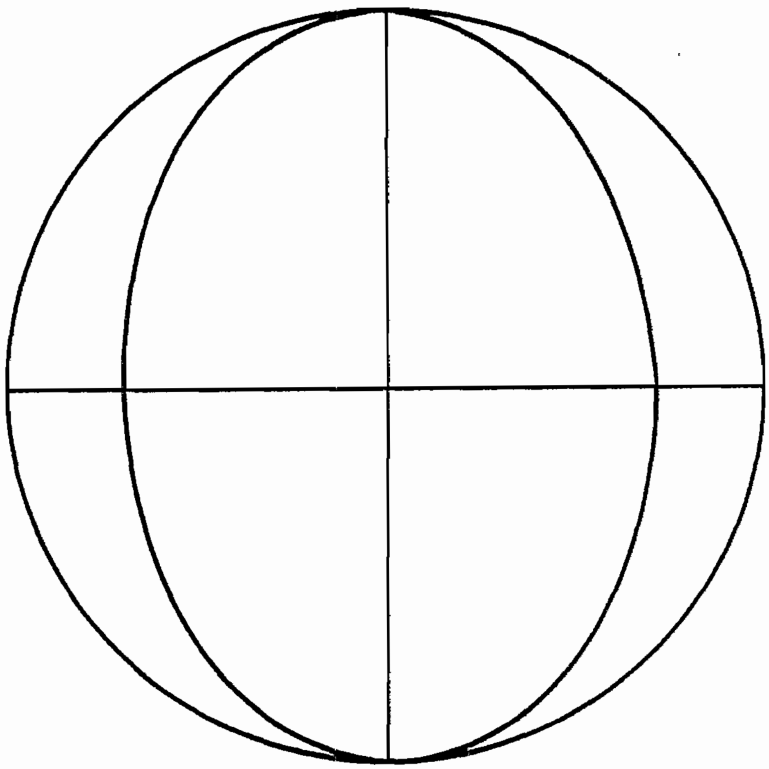

Figure 2. Zoning scheme. For the figure $N_{LONG} = 8(\Delta\phi=45^{\circ})$, $N_{LAT} = 2(\Delta\theta=90^{\circ})$.

TABLE 1. POLE LOCATIONS AS A FUNCTION OF ZONING

Number of zones		Zeros			
N_{LONG}	N_{LAT}	No. of zeros	Designation of zeros	Real part ×(-1)	Imaginary part
1	1	2	s_{110}	0.8787	0
			s_{211}	0.5554	0.8763
2	1	4	s_{110}	0.9763	0
			s_{110}	0.9306	0
			s_{211}	0.5701	0.8663
			s_{211}	0.4525	0.8250
2	2	7	s_{220}	1.3252	0
			s_{110}	1.0013	0
			s_{110}	0.9620	0
			s_{121}	1.4398	0.8410
			s_{121}	1.1146	1.0637
			s_{211}	0.5374	0.8897
			s_{211}	0.5128	0.8523
4	2	8	s_{110}	1.0202	0.0862
			s_{110}	1.1012	-0.0522
			s_{110}	0.9408	-0.0365
			s_{121}	1.4621	0.8491
			s_{121}	1.4610	0.8483
			s_{211}	0.56912	0.912328
			s_{211}	0.56913	0.912322
			s_{211}	0.4892	0.8586
4	4	9	s_{220}	1.5479	0.0033
			s_{110}	0.9813	0.0023
			s_{110}	1.0396	0.0103
			s_{110}	1.0269	-0.0125
			s_{121}	1.4957	0.8339
			s_{121}	1.4588	0.8742
			s_{211}	0.5670	0.8737

TABLE 1. CONCLUDED

Number of zones		Zeros			
N_{LONG}	N_{LAT}	No. of zeros	Designation of zeros	Real part ×(-1)	Imaginary part
8	4	8	s_{211}	0.5016	0.8648
			s_{211}	0.5229	0.9242
			s_{110}	1.0055	0.01258
			s_{110}	0.9921	-0.0059
			s_{110}	1.0179	-0.0067
			s_{121}	1.5063	0.9291
			s_{121}	1.4548	0.8388
			s_{211}	0.5600	0.9043
			s_{211}	0.5071	0.8196
			s_{211}	0.4651	0.8979

$s_{110} = 1$, $s_{211} = -0.5000 + i0.8660$, $s_{121} = -1.5000 + i0.8660$
$s_{220} = -1.6961 + i0.0000$

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APPENDIX A. ADJOINTNESS RELATIONSHIPS

In this appendix we derive an expression for the adjoint to the scalar integral operator corresponding to an acoustically hard body and we specialize it for the case of the spheroid and sphere.

We start with the exterior equation

$$\frac{1}{2}\,\Phi(\underline{r}) - \int_S \Phi(\underline{r}')\,\frac{\partial G(\underline{r}';\underline{r})}{\partial n'}\,dS' \equiv \left(\frac{I}{2} - K\right)\Phi \equiv L\Phi \qquad \text{(A1)}$$

where I is the identity operator.

The adjoint is defined through the relationship

$$(\psi, L\Phi) \equiv (L^{\dagger}\psi, \Phi) \qquad \text{(A2)}$$

where we have used the inner product definition

$$(f,g) \equiv \int_S f^*(\underline{r})\; g(\underline{r})\,dS$$

i.e., we can define

$$L^{\dagger} \equiv \frac{I}{2} - K^{\dagger} \qquad \text{(A3)}$$

To find $K^{\dagger}$ we write

$$(\psi, K\Phi) = \int_S \psi^*(\underline{r})\; K\Phi(\underline{r})\,dS$$

$$= \int_S\int_S \psi^*(\underline{r})\;\Phi(\underline{r}')\;\frac{\partial G(\underline{r}';\underline{r})}{\partial n'}\,dS'\,dS$$

$$= \int_S \int_S \psi^* (\underline{r}') \; \Phi(\underline{r}) \; \frac{\partial G(\underline{r};\underline{r}')}{\partial n} \; dS dS'$$

$$= \int_S \Phi(\underline{r}) \left[\int_S \psi(\underline{r}') \; \frac{\partial G^*(\underline{r};\underline{r}')}{\partial n} \; dS' \right]^* dS$$

where the penultimate step involved an interchange between $\underline{r}$ and $\underline{r}'$.

Thus, according to Equation (A2)

$$K^{\dagger}\psi = \int_{S'} \psi(\underline{r}') \; \frac{\partial G^*(\underline{r};\underline{r}')}{\partial n} \; dS' \qquad \text{(A4)}$$

where

$$\frac{\partial G(\underline{r};\underline{r}')}{\partial n} = \frac{\partial G(\underline{r}';\underline{r})}{\partial n} \equiv \hat{n}(\underline{r}) \cdot \nabla G$$

$$= -\left(\frac{1}{R} + \gamma\right) \frac{e^{-\gamma R}}{4\pi R} \; \underline{R} \cdot \hat{n}(\underline{r})$$

and

$$\frac{\partial G(\underline{r}';\underline{r})}{\partial n'} \equiv \hat{n}(\underline{r}') \cdot \nabla' G$$

$$= \left(\frac{1}{R} + \gamma\right) \frac{e^{-\gamma R}}{4\pi R} \; \underline{R} \cdot \hat{n}(\underline{r}')$$

The interior integral equation is

$$\frac{1}{2} \Phi(\underline{r}) + \int \Phi(\underline{r}') \; \frac{\partial G(\underline{r}';\underline{r})}{\partial n'} \; dS' \equiv \left(\frac{I}{2} + K\right) \Phi \equiv L^I \Phi$$

and thus the adjoint operator is $(I/2) + K^{\dagger}$ where $K^{\dagger}$ is given by Equation (A4).

For a spheroid one can use Equation (147) to cast the operator in the form

$$L\Phi = \frac{1}{2}\,\Phi - \int \Phi(\underline{r}')\sum_{m} \frac{1-2\lambda_m}{2N_m}\;\Phi_m(\underline{r})\;\Phi_m(\underline{r}')\;d\underline{r}'$$

and employ the procedure outlined in this appendix to show that

$$L^{\dagger} = L^{*}$$

The adjoint eigenfunctions of $L^{\dagger}$ are then ϕ_m^{*}. For a sphere, in particular, one can choose real eigenfunctions ϕ_m and consequently the adjoint eigenfunctions are identical to the ϕ_m's; i.e., the operator L is normal.

APPENDIX B. SCALAR ANALOGUE OF THE ELECTROMAGNETIC PSEUDOSYMMETRIC THEORY

We begin this appendix by viewing certain properties of the eigenvalue for the exterior spheroid scattering problem, which was derived in Section V. This eigenvalue is reproduced here for convenient reference.

$$\lambda_{mn}(\gamma) = ic(\xi_1^2-1)R_{mn}^{(1)}(c,\xi_1)\,\frac{d}{d\xi_1}\,R_{mn}^{(3)}(c,\xi_1) \qquad \text{(B1)}$$

where

$$c = \frac{i}{2}\,\gamma d \qquad \text{(B2)}$$

and the associated functions and parameters have been defined in Section V. Our effort in this appendix is motivated by examining the two ways in which the zeros of $\lambda_{mn}(\gamma)$ are obtained. They are

$$\frac{d}{d\xi_1}\,R_{mn}^{(3)}(c,\xi_1) = 0 \qquad \text{(B3)}$$

and

$$R_{mn}^{(1)}(c,\xi_1) = 0 \qquad \text{(B4)}$$

An examination of whether λ_{mn} had an as yet unexplained zero at $\gamma = 0$ was conducted. We found that the multiplicative γ that appears in this eigenvalue is cancelled by an appropriate term that is contained within the $dR_{mn}^{(3)}(c,\xi_1)/d\xi_1$ factor. Equation (B3) is an explicit example of the connection we proved between the significant zeros of the eigenvalue and the Lax-Phillips complex eigenvalues. Equation (B4) is the clue to pursue the results developed in this appendix. That is, it yields the interior volume eigenvalues associated with the Dirichlet problem.

We now begin our investigation of this issue by deriving the interior Dirichlet integral equation. First we combine Equations (7), (9), and (22), employ the change of variables previously described in which $\underline{r}'$ becomes the integration variable and $\underline{r}_V$ an observation variable in V_I, and employ the Dirichlet condition $\phi_I^D = 0$ on S, to obtain

$$\phi_I^D(\underline{r}_V) + \int_S G(\underline{r}_V,\underline{r}')\sigma^I(\underline{r}')dS' = \phi_I^{inc}(\underline{r}_V) \tag{B5}$$

where we have affixed the superscript D to indicate Dirichlet boundary conditions are employed and we introduce the definition

$$\sigma^I(\underline{r}') = -\hat{n}_I(\underline{r}')\nabla'\phi_I^D(\underline{r}') \tag{B6}$$

We now take the gradient of both sides of Equation (B5) to obtain

$$\nabla_V\phi_I^D(\underline{r}_V) + \int_S \nabla_V G(\underline{r}_V,\underline{r}')\sigma^I(\underline{r}')dS' = \nabla_V\phi_I^{inc}(\underline{r}_V) \tag{B7}$$

Identifying a point $\underline{r}$ on the surface as the point approached by $\underline{r}_V$, and defining $\hat{n}_I(\underline{r})$ at the point $\underline{r}$, we first take $-\hat{n}_I(\underline{r})$ dotted into the preceding equation and then take the limit as $\underline{r}_V$ approaches $\underline{r}$ to obtain

$$\frac{1}{2}\,\sigma^{I}(\underline{r}) - \int_{S} \frac{\partial G(\underline{r},\underline{r}')}{\partial n}\,\sigma^{I}(\underline{r}') = F^{I}_{\sigma}(\underline{r}) \qquad \text{(B8)}$$

where we have used Equation (26) and the definitions

$$\hat{n}(\underline{r}) \cdot \nabla_V G(\underline{r}_V,\underline{r}')\Big|_{\underline{r}_V=\underline{r}} = \frac{\partial G(\underline{r},\underline{r}')}{\partial n} \qquad \text{(B9)}$$

and

$$-\hat{n}(\underline{r}) \cdot \nabla_V \phi^{inc}_{I}(\underline{r}_V)\Big|_{\underline{r}_V=\underline{r}} = F^{I}_{\sigma}(\underline{r}) \qquad \text{(B10)}$$

The sign of the $\frac{1}{2}\,\sigma$ factor in the described limit is determined in the same manner that was described in treating Equation (24). We will now write Equation (B9) as

$$L_{ID}\sigma^{I} = F^{I}_{\sigma} \qquad \text{(B11)}$$

and this equation serves as the definition of L_{ID}. We now compare L_{ID} with $L^{E\dagger}_{s}$ given in Appendix A and find that

$$L_{ID} = L^{E\dagger *}_{s} \qquad \text{(B12)}$$

We now consider the interior Dirichlet eigenvalue equation

$$L_{ID}\sigma^{I}_{n} = \lambda^{I,s,D}_{n}\sigma^{I}_{n} \qquad \text{(B13)}$$

Next, we take the complex conjugate of Equation (B13), substitute Equation (B12) into the result to find

$$L_s^{E\dagger}\sigma_n^{I*} = \lambda_n^{I,s,D*}\sigma_n^{I*} \tag{B14}$$

Comparing Equation (B14) and Equation (37), we conclude that

$$\left\{\lambda_n^{E,s}\right\} = \left\{\lambda_n^{I,s,D}\right\} \tag{B15}$$

and

$$\left\{\phi_n^{E\dagger *}\right\} = \left\{\sigma_n^{I}\right\} \tag{B16}$$

Equation (B15) is consistent with the spheroid eigenvalue result expressed in Equation (B4) and in addition, it is very significant by itself, as discussed in Section II. In particular, it implies the existence of the extraneous zeros of the external eigenvalue, which we will now argue do not contribute to the desired SEM expansions.

Specifically, for the external Neumann surface problem, we want to show that

$$N_s^E(s_{nn'}^{E,s,EX}) = 0 \tag{B17}$$

where $s_{nn'}^{E,s,EX}$ is an extraneous zero of $\lambda_n^{E,s}$ corresponding to the zero of the interior Dirichlet problem. To accomplish this, we first employ the identity

$$\nabla \cdot \left[\phi_{inc}^{E}\nabla\phi_n^{ID} - \phi_n^{ID}\nabla\phi_{inc}^{E}\right] = \phi_{inc}^{E}\nabla^2\phi_n^{ID} - \phi_n^{ID}\nabla^2\phi_{inc}^{E} \tag{B18}$$

This identity is true for general functions ϕ_{inc}^{E} and ϕ_n^{ID}, but we are interested in the case where ϕ_{inc}^{E} is the incident field excited by the source in the exterior region. That is,

$$(\nabla^2 - \gamma^2)\, \phi^E_{inc} = f_E \qquad \text{(B19)}$$

We are interested in the case where ϕ^{ID}_n are interior volume Dirichlet eigenfunctions satisfying the equation

$$(\nabla^2 - \gamma^{D^2}_{In})\, \phi^{ID}_n = 0 \qquad \text{(B20)}$$

and the Dirichlet condition on the surface

$$\phi^{ID}_n = 0 \qquad \text{(B21)}$$

Substituting Equation (B19) and (B20) into Equation (B18), integrating over the interior volume, applying the divergence theorem, and the Dirichlet boundary condition, yields

$$\int_S \phi^E_{inc} \hat{n}_I \cdot \nabla\phi^{ID}_n \, dS = \left(\gamma^{D^2}_{In} - \gamma^2\right) \int_{V_I} \phi^E_{inc} \phi^{ID}_n \, dV - \int_{V_I} \phi^{ID}_n f^E \, dV \qquad \text{(B22)}$$

Recalling that f^E is nonzero only in V_E , we rewrite Equation (B22) as

$$\int_S \left(- \hat{n}_I \cdot \nabla\phi^{ID}_n\right) \phi^E_{inc} \, dS = \left(\gamma^2 - \gamma^{D^2}_{In}\right) \int_{V_I} \phi^E_{inc} \phi^{ID}_n \, dV \qquad \text{(B23)}$$

In order to draw our desired conclusions from (B23) without leaving any gaps, we would have to reproduce the results connecting the surface and volume solutions for Neumann boundary conditions contained in Section IV all over again for the Dirichlet boundary conditions. At this point we will not provide these details and we will thus have the indicated gap; however, we see no difficulty in eliminating this gap. Assuming the Dirichlet connection yields the same results as the Neumann connection, it then follows that the volume eigenvalue γ_{In}^{D} is a zero of the surface eigenvalue $\lambda_{n}^{I,s,D}$. According to Equation (B15), it follows that γ_{In}^{D} is an extraneous zero of $\lambda_{n}^{E,s}$ which we can formally state as

$$s_{nn'}^{E,s,EX} = c\gamma_{Im}^{D} \tag{B24}$$

where c is the acoustic propagation velocity as opposed to the quantity defined by Equation (B2), and m can represent a multiple counting index. To proceed with the argument it is necessary to relate $-\hat{n}_I \cdot \nabla\phi_n^{ID}$ and the σ_n^I which satisfy (B13). A Dirichlet extension of the Neumann arguments presented in Section IV would be necessary to conclude that $\sigma_n^I(\gamma_{In}^D) = A_n \hat{n}_I \cdot \nabla\phi_n^{ID}$ where A_n is just a proportionality constant. Assuming the described Dirichlet equivalents of the Neumann arguments have been established, it follows from (B23) that

$$\int_S \sigma_n^I\left(s_{nn'}^{E,s,EX}\right)\phi_{inc}^E\left(s_{nn'}^{E,s,EX}\right)dS = 0 \tag{B25}$$

This can be written as

$$\left(\sigma_n^{I*}(s_{nn'}^{E,s,EX}),\ \phi_{inc}^E(s_{nn'}^{E,s,EX})\right) = 0 \tag{B26}$$

and using Equation (B16), we write this as

$$\left(\phi_n^{E+}(s_{nn'}^{E,s,EX}),\ \phi_{inc}^E(s_{nn'}^{E,s,EX})\right) = 0 \tag{B27}$$

The left-hand side of Equation (B27) is the original definition $N_s^E(s_{nn'}^{E,s,EX})$ and we now have our desired result.

Having obtained the desired result, we could end this appendix. Instead, we will present a number of results that have some aspects in common with the results that have been presented. To do this, we summarize the essential forms of the scalar operators that have already been defined. For the external Neumann problem, we have

$$L_s^E = \frac{1}{2} - K \quad , \tag{B28}$$

for the internal Neumann problem we have

$$L_s^I = \frac{1}{2} + K \quad , \tag{B29}$$

for the internal Dirichlet problem we have

$$L_{ID} = \frac{1}{2} - K^{\dagger *} , \tag{B30}$$

and for the external Dirichlet problem, it can be shown that

$$L_{ED} = \frac{1}{2} + K^{\dagger *} \tag{B31}$$

From these forms the set equalities follow

$$\left\{\phi_n^E\right\} = \left\{\phi_n^I\right\} \tag{B32}$$

$$\left\{\sigma_n^E\right\} = \left\{\sigma_n^I\right\} \tag{B33}$$

$$\left\{\phi_n^{E\dagger}\right\} = \left\{\phi_n^{I\dagger}\right\} \tag{B34}$$

$$\left\{\sigma_n^{E\dagger}\right\} = \left\{\sigma_n^{I\dagger}\right\} \tag{B35}$$

$$\left\{\sigma_n^{E\dagger *}\right\} = \left\{\phi_n^{I}\right\} \tag{B36}$$

$$\left\{\sigma_n^{I\dagger *}\right\} = \left\{\phi_n^{E}\right\} \tag{B37}$$

$$\left\{\phi_n^{I\dagger *}\right\} = \left\{\sigma_n^{E}\right\} \tag{B38}$$

$$\left\{\lambda_n^{I,s}\right\} = \left\{\lambda_n^{E,s,D}\right\} \tag{B39}$$

as well as Equations (B15) and (B16), which played a central role in this appendix. In addition, the following eigenvalue equalities hold.

$$\lambda_n^{E,s} + \lambda_n^{I,s} = 1 \tag{B40}$$

$$\lambda_n^{E,s,D} + \lambda_n^{I,s,D} = 1 \tag{B41}$$

$$\lambda_n^{E,s} + \lambda_n^{E,s,D} = 1 \tag{B42}$$

$$\lambda_n^{I,s} + \lambda_n^{I,s,D} = 1 \tag{B43}$$

Finally, for the spheroid, which includes the sphere as a special case, we have

$$K^{\dagger *} = K \tag{B44}$$

and we have the following additional set equalities:

$$\left\{\phi_n^E\right\} = \left\{\phi_n^{E\dagger *}\right\} \tag{B45}$$

$$\left\{\phi_n^I\right\} = \left\{\phi_n^{I\dagger *}\right\} \tag{B46}$$

$$\left\{\sigma_n^E\right\} = \left\{\sigma_n^{E\dagger *}\right\} \tag{B47}$$

$$\left\{\sigma_n^I\right\} = \left\{\sigma_n^{I\dagger *}\right\} \tag{B48}$$

For the sphere, the eigenfunctions can be chosen to be real, thus eliminating the need for the conjugate. This aspect of the sphere solution is sufficient to show that K and $K^\dagger$ commute, thus making K normal.

APPENDIX C. A USEFUL SCALAR SEM RELATIONSHIP

In this appendix we will present an alternate representation for $a_{nn'}^{\alpha,s}$ which plays a prominent role in the final SEM expansion given by Equation (60). There are several alternate forms already possible for this quantity, based on material already presented in this report, and the form we now choose is

$$a_{nn'}^{\alpha,s} = \left. \frac{\left(\phi_n^{\alpha\dagger}(s),\ \phi_{inc}^{\alpha}(s)\right)}{\lambda_n'^{\alpha,s}} \right|_{s=s_{nn'}^{\alpha,s}} \tag{C1}$$

Implied by this form is a normalization of the adjoint eigenfunction that appears in the expression as well as a normalization of eigenfunction ϕ_n^{α}. The normalization that has been implied so far in the text is

$$\left(\phi_n^{\alpha\dagger}(s),\ \phi_n^{\alpha}(s)\right) = 1 \tag{C2}$$

Such a normalization was always possible because we could have chosen the unnormalized functions $\phi_{nQ}^{\alpha\dagger}$ and ϕ_{nQ}^{α} and formed the inner product

$$(\phi_{nQ}^{\alpha\dagger}, \phi_{nQ}^{\alpha}) = Q_n^{\alpha} \tag{C3}$$

The normalized eigenfunction and its adjoint can then be obtained by dividing each by $(Q_n^{\alpha})^{1/2}$. We are now in a position to express Equation (60) in terms of the unnormalized eigenfunctions as

$$S_s^{\alpha}(\underline{r},t) = \sum_{n=1}^{\infty} \sum_{n=1}^{N(n)} a_{Qnn'}^{\alpha,s} \phi_{Qnn'}^{\alpha}(\underline{r}) e^{s_{nn'}^{\alpha,s} t} \qquad \text{(C4)}$$

where

$$\phi_{Qnn'}^{\alpha}(\underline{r}) = \phi_{nQ}^{\alpha}(s_{nn'}^{\alpha,s},\underline{r}) \qquad \text{(C5)}$$

and

$$a_{Qnn'}^{\alpha,s} = \left.\frac{\left(\phi_{nQ}^{\alpha\dagger},\phi_{inc}^{\alpha}\right)}{Q_n^{\alpha}\,\lambda'^{\,\alpha,s}_{n}}\right|_{s=s_{nn'}^{\alpha,s}} \qquad \text{(C6)}$$

It is the denominator of the last expression that can be written in an alternate form. To accomplish this, we write the original eigenvalue equation

$$L_s^{\alpha}(s)\phi_{nQ}^{\alpha}(s,\underline{r}) = \lambda_n^{\alpha,s}(s)\phi_{nQ}^{\alpha}(s,\underline{r})$$

and take the derivative of both sides with respect to s to obtain

$$\frac{\partial L_s^{\alpha}}{\partial s}\phi_{nQ}^{\alpha} + L_s^{\alpha}\frac{\partial \phi_{nQ}^{\alpha}}{\partial s} = \frac{d\lambda_n^{\alpha,s}}{ds}\phi_{nQ}^{\alpha} + \lambda_n^{\alpha,s}\frac{\partial \phi_{nQ}^{\alpha}}{\partial s} \qquad \text{(C7)}$$

Next we take the inner product of all terms with $\phi_{nQ}^{\alpha\dagger}$ to obtain

$$\left(\phi_{nQ}^{\alpha\dagger}, \frac{\partial L_s^{\alpha}}{\partial s}\phi_{nQ}^{\alpha}\right) + \left(\phi_{nQ}^{\alpha\dagger}, L_s^{\alpha}\frac{\partial \phi_{nQ}^{\alpha}}{\partial s}\right) = \frac{d\lambda_n^{\alpha,s}}{ds} Q_n^{\alpha} + \lambda_n^{\alpha,s}\left(\phi_{nQ}^{\alpha,\dagger}, \frac{\partial \phi_{nQ}^{\alpha}}{\partial s}\right)$$

(C8)

The second term can be written as

$$\left(\phi_{nQ}^{\alpha\dagger}\ ,\ L_s^{\alpha}\ \frac{\partial\phi_{nQ}^{\alpha}}{\partial s}\right) = \left(L_s^{\alpha\dagger}\ \phi_{nQ}^{\alpha\dagger}\ ,\ \frac{\partial\phi_{nQ}^{\alpha}}{\partial s}\right) = \lambda_n^{\alpha,s}\left(\phi_{nQ}^{\alpha\dagger}\ ,\ \frac{\partial\phi_n^{\alpha}}{\partial s}\right) \quad \text{(C9)}$$

Substituting Equation (C9) into Equation (C8) and canceling terms yields

$$\frac{d\lambda_n^{\alpha,s}}{ds}\ Q_n^{\alpha} = \left(\phi_{nQ}^{\alpha\dagger}\ ,\ \frac{\partial L_s}{\partial s}\ \phi_{nQ}^{\alpha}\right) \quad \text{(C10)}$$

The desired expression is obtained by substituting (C10) into (C6) and it is

$$a_{Qnn'}^{\alpha,s} = \left.\frac{\left(\phi_{nQ}^{\alpha\dagger}\ ,\ \phi_{inc}^{\alpha}\right)}{\left(\phi_{nQ}^{\alpha\dagger}\ ,\ \frac{\partial L_s^{\alpha}}{\partial s}\ \phi_{nQ}^{\alpha}\right)}\right|_{s=s_{nn'}^{\alpha,s}} \quad \text{(C11)}$$

We now have a choice of dealing with an explicit normalization procedure implied by Equation (C6) or the normalization insensitive procedure implied by Equation (C11). It should be noted that the electromagnetic SEM equivalent of the material presented in this appendix has long been established using virtually the same procedure.

APPENDIX D. A TIME FOR THE EQUIVALENCE OF CLASS 1 AND CLASS 2 COUPLING COEFFICIENTS

In this appendix we show that when a perfectly conducting body is illuminated by an electromagnetic pulse of finite width T the SEM responses with class 1 and class 2 coupling coefficients are identical for times $t \geq T + (L/c) + t_0$, where L is the maximum body dimension in the direction of propagation and $t=t_0$ corresponds to the instant at which the pulse wavefront first hits the body.

First we show that this is true for a delta function incident pulse, i.e., for T=0. The coupling coefficients are defined as follows

$$\eta_\alpha^{(1)} \equiv Ae^{(\gamma_\alpha-\gamma)ct_0}\int_S \underline{J}^{inc}(\underline{r},\gamma_\alpha)\cdot\underline{\Phi}_\alpha(\underline{r})\,dS \tag{D1}$$

$$\eta_\alpha^{(2)} \equiv A\int_S \underline{J}^{inc}(\underline{r},\gamma)\cdot\underline{\Phi}_\alpha(\underline{r})\,dS \tag{D2}$$

where A is a common factor and $\underline{\Phi}_\alpha(\underline{r})$ is a coupling vector. For a delta function pulse

$$\underline{J}^{inc}(\underline{r},\gamma) = \hat{p}(\underline{r})e^{-\gamma z} \tag{D3}$$

where, for simplicity, the direction of propagation has been chosen along the z axis, with z=0 at the "center" of the body, $\hat{p} = \hat{n} \times \hat{h}$, $\hat{n}$ is the unit normal to the surface of the body and $\hat{h}$ is the polarization vector for the incident magnetic field. In view of Equation (D3), Equation (D2) can be rewritten as

$$\eta_\alpha^{(2)}/A = \int_S e^{-\gamma z} \psi(\underline{r}) \, dS$$

where

$$\psi(\underline{r}) \equiv \hat{p}(\underline{r}) \cdot \underline{\Phi}_\alpha(\underline{r})$$

If we Laplace invert the SEM simple-pole expansion term is

$$T^{(2)} \equiv L^{-1}\left[\frac{\eta_\alpha^{(2)}}{A\left(\gamma-\gamma_\alpha\right)}\right] = \frac{1}{2\pi i}\int_S\int_{C_B} \frac{e^{\gamma(ct-z)}}{\gamma-\gamma_\alpha} \psi(\underline{r}) \, dS \qquad \text{(D4)}$$

where C_B is the Bromwich path. For times $t>(z/c)$

$$T^{(2)} = \int_S \psi(\underline{r}) \; u(t-z/c) \; e^{\gamma_\alpha(ct-z)} \, dS$$

The pulse wavefront has just passed the body at $t=(L/c) + t_o$ and

$$T^{(2)} = e^{\gamma_\alpha ct}\int_S \psi(\underline{r}) \; e^{-\gamma_\alpha z} \, dS \qquad t \geq (L/c) + t_o \qquad \text{(D5)}$$

The corresponding term for class 1 is

$$T^{(1)} = e^{\gamma_\alpha ct_o} \frac{1}{2\pi i} \int_S \int_{C_B} \frac{e^{-\gamma ct_o - \gamma_\alpha z + \gamma ct}}{\gamma - \gamma_\alpha} \psi(\underline{r}) \, d\gamma dS$$

$$= e^{\gamma_\alpha ct} \int_S \psi(\underline{r}) \, e^{-\gamma_\alpha z} \, dS \qquad t > t_o$$

and $T^{(1)}$ is identical to $T^{(2)}$ in Equation (D5).

For a general incident pulse

$$\underline{J}^{inc} = f(\gamma) \, \hat{p}(\underline{r}) \, e^{-\gamma z}$$

and

$$T^{(2)} = \frac{1}{2\pi i} \int_S \int_{C_B} \frac{f(\gamma) e^{\gamma(ct-z)}}{\gamma - \gamma_\alpha} \psi(\underline{r}) \, d\gamma dS$$

$$T^{(1)} = e^{\gamma_\alpha ct_o} \frac{1}{2\pi i} \int_S \int_{C_B} \frac{f(\gamma) e^{\gamma(t-t_o)c}}{\gamma - \gamma_\alpha} e^{-\gamma_\alpha z} \psi(\underline{r}) \, d\gamma dS$$

We see that the above expressions involve the product of $f(\gamma)$ with another function of γ and, consequently, the convolution theorem can be employed. If we rewrite $T^{(2)}$ as

$$T^{(2)} = \frac{1}{2\pi i} \int_{C_B} f(\gamma) \, p(\gamma) \, e^{\gamma ct} \, d\gamma$$

$$p(\gamma) \equiv \int_S \frac{e^{-\gamma z}}{\gamma-\gamma_\alpha}\, \psi(\underline{r})\; dS$$

then

$$T^{(2)} = \int_{-\infty}^{t} F(t-\tau)\; P(\tau)\; d\tau \qquad \text{(D6)}$$

where F(t) and P(t) are the inverse Laplace transforms of $f(\gamma)$ and $p(\gamma)$ respectively. Similarly, for $T^{(1)}$ we can write

$$T^{(1)} = \frac{1}{2\pi i} \int_{C_B} f(\gamma)\; q(\gamma)\; e^{\gamma ct}\; d\gamma$$

$$q(\gamma) \equiv e^{\gamma_\alpha c t_o} \int_S \frac{e^{-\gamma t_o}}{\gamma-\gamma_\alpha}\, \psi(\underline{r})\; dS$$

$$T^{(1)} = \int_{-\infty}^{t} F(t-\tau)\; Q(\tau)\; d\tau \qquad \text{(D7)}$$

The calculation for the delta function pulse showed that P(t) is equal to Q(t) for $t \geq (L/c) + t_o$. We will now show that $T^{(2)}$ given by Equation (6) is equal to $T^{(1)}$ given by Equation (D7) for $t \geq (L/c) + T + t_o$ where T is the pulse width (Fig. D1). This is done simply by plotting the products $F(t-\tau)\ P(\tau)$, $F(t-\tau)\ Q(\tau)$, and observing with the aid of Figure D1 that for $t \geq (L/c) + T + t_o$, $T^{(2)}$ and $T^{(1)}$ are indeed equal. Notice that as $T \to \infty$ the SEM responses with class 1 and class 2 coupling coefficients are different for all times as noted in Reference 1.

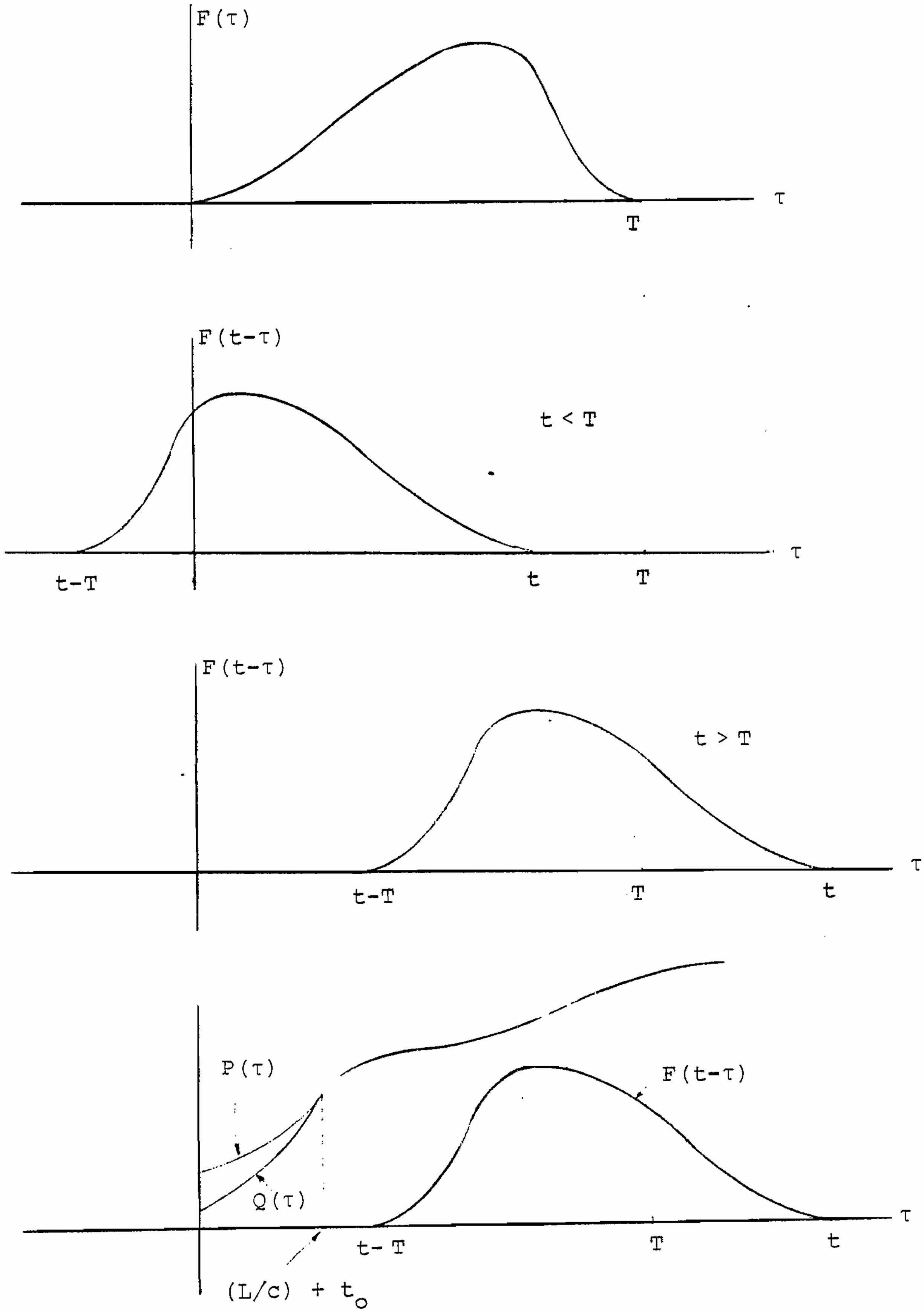

Figure D1. The four graphs show how to plot the products $F(t-\tau)\ P(\tau)$ and $F(t-\tau)\ Q(\tau)$.

APPENDIX E. MAGNETOSTATIC IMPLICATIONS OF COUPLING COEFFICIENTS

In this appendix we calculate the static current density induced on the surface of a perfectly conducting sphere immersed in a homogeneous magnetic field by calculating the class 2 SEM current density for a step function incident pulse as $t \to \infty$. This static response has the correct θ and ϕ dependence, as it should, but an incorrect numerical coefficient.

To facilitate the analysis we assume that the direction of propagation is in the z direction and that $\underline{H}^{inc}$ is in the y direction. This situation corresponds to a polarization index p=2 with $\hat{e}_1=\hat{e}_z$, $\hat{e}_2=\hat{e}_x$, $\hat{e}_z=\hat{e}_y$ ($\theta_1=0$, $\phi_1=\pi$) (Ref. 12). Thus, following Baum's notation and the results in Reference 12

$$C_{1,n,n',1,o,2}(0,\pi) = (-1)^{n+1} \frac{2n+1}{n(n+1)} D_{1,n,n'}$$

$$C_{1,n,n',1,e,2}(0,\pi) = 0$$

$$C_{2,n,n',1,o,2}(0,\pi) = 0$$

$$C_{2,n,n',1,e,2}(0,\pi) = (-1)^{n+1} \frac{2n+1}{n(n+1)} D_{2,n,n'}$$

where o,e stand for odd, even and only the m=1 terms are non-zero. The class 2 coupling coefficients have been evaluated in Reference 1.

$$\eta^R = \frac{1}{c} e^{\gamma^R_{n,n'} a} \; C_{1,n,n',m,\sigma,p} \frac{\left[\gamma a i_n(\gamma a)\right]' / \gamma a}{\left\{\left[\gamma a i_n(\gamma a)\right]' / \gamma a\right\}_{\gamma=\gamma^R_{n,n'}}}$$

$$\eta^{Q} = \frac{1}{c} e^{\gamma^{Q}_{n,n'}a} C_{2,n,n',m,\sigma,p} \frac{i_n(\gamma a)}{i_n\left(\gamma^{Q}_{n,n'}a\right)}$$

Thus the current density induced on the surface of the sphere is

$$\underline{J}(\underline{r},\gamma) = f(\gamma) \sum_{n=1}^{\infty} \left(\sum_{n'} \eta^{R}_{n,n',m,o,2} \frac{1}{\gamma-\gamma^{R}_{n,n'}} \underline{R}_{n,1,o} + \sum_{n'} \eta^{Q}_{n,n',m,e,2} \frac{1}{\gamma-\gamma^{Q}_{n,n'}} \underline{Q}_{n,1,e} \right)$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} \left\{ \left\{ \sum_{n'} e^{\gamma^{R}_{n,n'}a} \frac{1}{\left(\gamma-\gamma^{R}_{n,n'}\right)a} \times \frac{(a/c)\, D_{1,n,n'}}{\left\{\left[\gamma a i_n(\gamma a)\right]'/\gamma a\right\}_{\gamma=\gamma^{R}_{n,n'}}} \right\} \frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a} \underline{R}_{n,1,o} \right.$$

$$\left. + \left\{ \sum_{n'} e^{\gamma^{Q}_{n,n'}a} \frac{1}{\left(\gamma-\gamma^{Q}_{n,n'}\right)a} \frac{(a/c)\, D_{2,n,n'}}{i_n\left(\gamma^{Q}_{n,n'}a\right)} \right\} \times i_n(\gamma a)\, \underline{Q}_{n,1,e} \right\} \tag{E1}$$

where $f(\gamma)$ is the Laplace transform of $u(t)$, i.e., $\frac{1}{\gamma c}$.

To evaluate $\underline{J}(\underline{r},t)$ as $t\to\infty$ we employ the limiting process

$$\lim_{s\to o} s\tilde{f}(s) = \lim_{t\to\infty} f(t)$$

where $\tilde{f}(s)$ is the Laplace transform of $f(t)$. Thus we should evaluate $c\gamma \underline{J}(\underline{r},\gamma)$ as $\gamma\to o$. We have

$$\frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a} = i_n'(\gamma a) + \frac{i_n(\gamma a)}{\gamma a} = i_{n+1}(\gamma a) + \frac{n+1}{\gamma a}\, i_n(\gamma a)$$

and

$$\lim_{\gamma\to o} i_n(\gamma a) = 0 \qquad n \geq 1$$

$$\lim_{\gamma\to o} \frac{n+1}{\gamma a}\, i_n(\gamma a) \begin{cases} 0 & n > 1 \\ \frac{2}{3} & n = 1 \end{cases}$$

Thus

$$\lim_{t\to\infty} \underline{J}(\underline{r},t) = \lim_{\gamma\to o} c\gamma \underline{J}(\underline{r},\gamma)$$

$$= \frac{-(a/c)\ D_{1,1,o}}{\gamma^R_{1,o}a} \frac{e^{\gamma^R_{1,o}a}}{i_2\left(\gamma^R_{1,o}a\right) + \left(2/\gamma^R_{1,o}a\right)\, i_1\left(\gamma^R_{1,o}a\right)}\ \underline{R}_{1,1,o}$$

We have

$$(a/c)\ D_{1,1,o} = 1 \qquad \gamma^R_{1,o}a = -1$$

$$i_1(-1) = -i_1(1) = -.36788, \quad i_2(-1) = i_2(1) = .07156$$

$$\underline{R}_{1,1,o} = -\cos\phi\ \hat{e}_\theta + \cos\theta\ \sin\phi\ \hat{e}_\phi$$

and

$$\lim_{t\to\infty} \underline{J}(\underline{r},t) = .45568(-\cos\phi\,\hat{e}_\theta + \cos\theta\,\sin\phi\,\hat{e}_\phi)$$

The correct response has a numerical coefficient equal to 1.5 instead of 0.45568. Since the SEM solution which employs class 1 coupling coefficients is equivalent to the Mie solution, the use of class 1 coupling coefficients must yield the correct magnetostatic solution.