Interaction Notes

Note 395

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QUASISTATIC INTERACTION BETWEEN A MAGNETIC DIPOLE AND A RESISTIVELY CAPPED CONDUCTING SPHERICAL SHELL, II: TRANSVERSE MAGNETIC DIPOLE*

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Abstract

The magnetic field inside a conducting spherical shell with a resistively loaded circular aperture is found when the structure is excited by an external magnetic dipole located above the aperture on the axis of symmetry. The dipole moment is oriented perpendicular to the symmetry axis, and a non-zero contact resistance can exist between the resistive loading and the aperture rim. Extensive numerical results are presented, and implications of these results as they pertain to the measurement of the properties of the resistive loading are discussed. Comparisons are made with the results of the axial-dipole problem considered in Part I (IN 394, 15 May 1980).

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I. INTRODUCTION

In Part I [1] we considered the quasistatic magnetic field interaction between a magnetic dipole antenna and a conducting spherical shell with resistively loaded circular aperture. The dipole was located above the aperture on the symmetry axis and was oriented parallel to this axis. In this note a similar problem is considered, with the change that the magnetic dipole is oriented perpendicular to the symmetry axis of the structure.

With the change in orientation of the source dipole, the penetration of the unloaded aperture is greatly increased [2]; furthermore, the surface current density on the spherical shell now tends to flow across the junction between the aperture loading and the spherical shell, making contact resistance effects important. It is our objective to investigate the relation between the fields at (or near) the center of the spherical shell with and without the loading and to determine how this relation is affected by the geometrical and electrical parameters of the problem.

In the next section the transverse-dipole problem is formulated in terms of dual series equations using the magnetic scalar potential. These dual series equations are reduced in Section III to an inhomogeneous Fredholm integral equation of the second kind. Exact and variational solutions are obtained in Section IV and numerical results are discussed in Section V. Section VI concludes the note with comments regarding the use of this configuration for measuring the properties of the aperture loading material.

II. FORMULATION

The geometry of the problem, which is shown in Figure 1, is identical to that considered in [1], except that the dipole moment is oriented parallel to the x-axis. As in Part I, a perfectly conducting spherical shell of radius a possessing a resistively loaded circular aperture is excited by an external magnetic dipole located above the aperture on the axis of symmetry. The object of the analysis is to determine the quasistatic magnetic field inside the spherical shell.

The magnetic field is given in terms of a magnetic scalar potential V_m by

$$\begin{aligned} \overline{H} &= -\nabla V_m \\ H_r &= -\frac{\partial V_m}{\partial r} \\ H_\theta &= -\frac{1}{r}\frac{\partial V_m}{\partial \theta} \\ H_\phi &= \frac{-1}{r\sin\theta}\frac{\partial V_m}{\partial \phi} \end{aligned} \tag{2.1}$$

where for $r \neq a$ and $(r,\theta) \neq (r_o,\pi)$, V_m satisfies Laplace's equation

$$\nabla^2 V_m = 0 \tag{2.2}$$

The scalar potential V_m is conveniently written in terms of a "primary" contribution V_m^p and an "induced" contribution V_m^i as

$$V_m = V_m^p + V_m^i \tag{2.3}$$

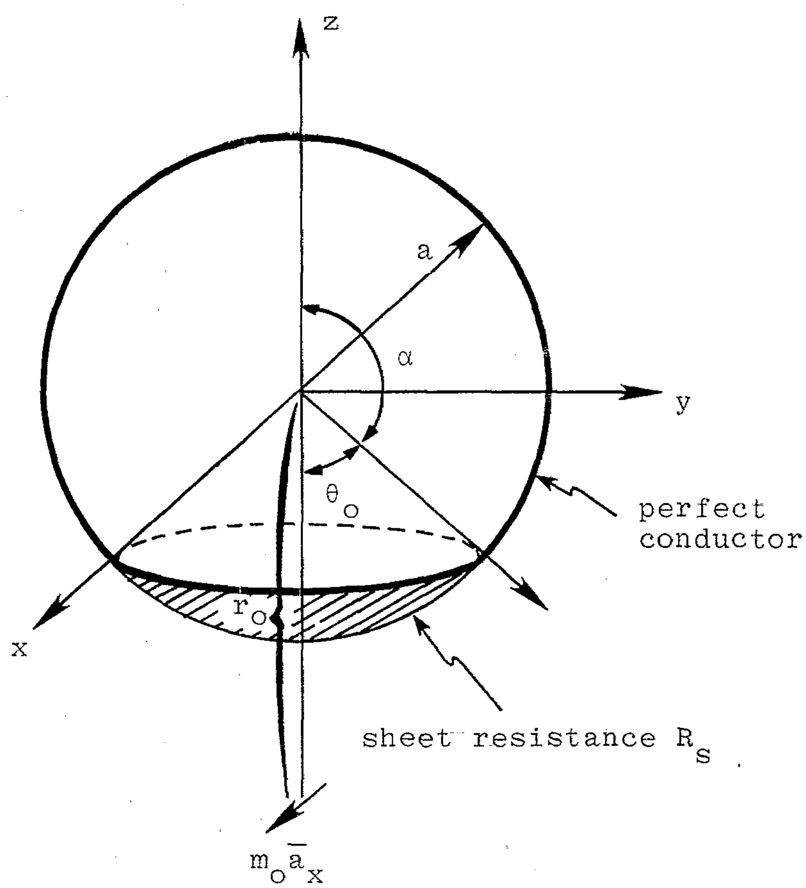

Figure 1. Geometry of the problem

in which V_m^p is the potential due to the source dipole, whose magnetic dipole moment is $m_o\bar{a}_x$, and V_m^i is due to the presence of the spherical shell:

$$
\begin{aligned}
r \le r_o: \quad V_m^p &= \frac{m_o}{4\pi r_o^2}\cos\phi \sum_{n=1}^{\infty} (-1)^n \left(\frac{r}{r_o}\right)^n P_n^1(\cos\theta) \\
r \ge r_o: \quad V_m^p &= \frac{m_o}{4\pi r_o^2}\cos\phi \sum_{n=1}^{\infty} (-1)^n \left(\frac{r}{r_o}\right)^{-n-1} P_n^1(\cos\theta) \\
r < a: \quad V_m^i &= \frac{m_o}{4\pi r_o^2}\cos\phi \sum_{n=1}^{\infty} \frac{1}{n}\, a_n \left(\frac{r}{a}\right)^n P_n^1(\cos\theta) \\
r > a: \quad V_m^i &= \frac{-m_o}{4\pi r_o^2}\cos\phi \sum_{n=1}^{\infty} \frac{a_n}{n+1} \left(\frac{r}{a}\right)^{-n-1} P_n^1(\cos\theta)
\end{aligned}
\qquad (2.4)
$$

in which P_n^1 denotes the associated Legendre function of degree n and order 1, and the coefficients a_n are to be determined. It will be noted that these representations for V_m are such that $\frac{\partial V_m^i}{\partial r}$ is continuous at $r = a$.

Over the perfectly conducting portion of the shell, the normal component of the magnetic field must vanish. Using eqs. (2.1), (2.3) and (2.4), we obtain the relation

$$\sum_{n=1}^{\infty} c_n P_n^1(\cos\theta) = 0 \qquad (0 \le \theta < \alpha) \qquad (2.5)$$

where

$$c_n \equiv a_n + (-1)^n n\left(\frac{a}{r_o}\right)^n \qquad (2.6)$$

On the resistive cap, the normal component of the magnetic field must be continuous, and [3,4]

$$\nabla_s^2(V_m^> - V_m^<) = \frac{-s\mu_o}{R_s}\frac{\partial V_m}{\partial r} \qquad (r = a,\ \alpha < \theta \leq \pi) \tag{2.7}$$

where $V_m^>$ and $V_m^<$ denote the scalar magnetic potentials at $r = a+$ and $r = a-$ respectively and R_s is the sheet resistance of the cap. On a spherical surface of radius a,

$$\nabla_s^2 F = \frac{1}{a^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial F}{\partial\theta}\right) + \frac{1}{a^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2} \tag{2.8}$$

Substituting from eqs. (2.3) and (2.4) into (2.7) and using (2.8), we obtain the relation

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{1}{\sin^2\theta}\right]\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}a_n P_n^1(\cos\theta)$$

$$= \frac{s\mu_o a}{R_s}\sum_{n=1}^{\infty} c_n P_n^1(\cos\theta) \qquad (\alpha < \theta \leq \pi) \tag{2.9}$$

Solving this differential equation yields

$$\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}a_n P_n^1(\cos\theta) = \frac{-s\mu_o a}{R_s}\sum_{n=1}^{\infty}\frac{c_n}{n(n+1)}P_n^1(\cos\theta)$$

$$+ A\tan\frac{\theta}{2} + B\cot\frac{\theta}{2} \qquad (\alpha < \theta \leq \pi) \tag{2.10}$$

in which A and B are constants to be determined.* Upon rearranging eq. (2.10) and setting $A = 0$ to remove the singularity at $\theta = \pi$, we find that

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* $\tan\frac{\theta}{2}$ and $\cot\frac{\theta}{2}$ are the solutions to the homogeneous equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) - \frac{f}{\sin^2\theta} = 0$$

$$\frac{s\mu_o a}{R_s} \sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} P_n^1(\cos\theta) + \sum_{n=1}^{\infty} \frac{(2n+1)}{n(n+1)} c_n P_n^1(\cos\theta)$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{n+1} \left(\frac{a}{r_o}\right)^n P_n^1(\cos\theta) + B \cot\frac{\theta}{2} \quad (\alpha < \theta < \pi) \tag{2.11}$$

The constant B is determined from the boundary condition at $\theta = \alpha$ of the partial differential equation (2.7). Specifically, it is not difficult to show that if there exists a net contact resistance R_c between the aperture loading and the rim, then*

$$\left(2\pi R_c \frac{\partial^2}{\partial\phi^2} + R_s \sin\theta \frac{\partial}{\partial\theta}\right)\left(V_m^{>} - V_m^{<}\right) = 0 \quad (r = a,\ \theta = \alpha+) \tag{2.12}$$

Equation (2.12) yields the condition

$$B = \frac{s\mu_o a}{Z_s} (2\pi R_c + R_s)^{-1} \tan\frac{\alpha}{2} \cdot \left(2\pi R_c - R_s \sin\theta \frac{d}{d\theta}\right)$$

$$\cdot \sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} P_n^1(\cos\theta)\Big|_{\theta=\alpha} \tag{2.13}$$

Equation (2.13) is an implicit relation from which B can be determined, since the coefficients c_n will clearly depend on B.

It will be useful in what follows to rewrite eq. (2.5) as an equivalent serio-differential equation

$$\left[\frac{1}{\sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{d}{d\theta}\right) - \frac{1}{\sin^2\theta}\right] \sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} P_n^1(\cos\theta) = 0$$

$$(0 \leq \theta < \alpha) \tag{2.14}$$

and obtain

$$\sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} P_n^1(\cos\theta) = C \tan\frac{\theta}{2} + D \cot\frac{\theta}{2} \quad (0 \leq \theta < \alpha) \tag{2.15}$$

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*See the Appendix for the derivation.

Setting D = 0 to remove the singularity at θ = 0, we have

$$\sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} P_n^1(\cos\theta) = C \tan\frac{\theta}{2} \qquad (0 \le \theta < \alpha) \tag{2.16}$$

The constant C is evaluated by noting that eq. (2.5) must be satisfied. Since as $n \to \infty$, $0 < \theta < \pi$,

$$P_n^1(\cos\theta) \sim \left(\frac{2n}{\pi\sin\theta}\right)^{\frac{1}{2}} \cos[(n + \tfrac{1}{2})\theta + \tfrac{\pi}{4}] \tag{2.17}$$

it is necessary that the coefficients c_n decrease as n^{-1} for large n. This condition implies that the function

$$F_1(\theta) = \sum_{n=1}^{\infty} \frac{c_n}{n(n+1)} P_n^1(\cos\theta) \tag{2.18}$$

is continuous and has a continuous first derivative at θ = α. Thus, using eqs. (2.13) and (2.16), we obtain a relation between the constants B and C, viz.,

$$B = C \frac{s\mu_o a}{R_s} \left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right) \tan^2 \frac{\alpha}{2} \tag{2.19}$$

Equations (2.11) and (2.16) constitute a set of dual series equations for the coefficients c_n. Specifically, defining

$$d_n = \frac{c_n}{n(n+1)} \tag{2.20}$$

and

$$G(\theta) = \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)}{n+1} \left(\frac{a}{r_o}\right)^n P_n^1(\cos\theta) \tag{2.21}$$

we have

$$\sum_{n=1}^{\infty} d_n P_n^1(\cos\theta) = C \tan\frac{\theta}{2} \qquad (0 \le \theta \le \alpha) \tag{2.22a}$$

$$\sum_{n=1}^{\infty} (2n+1) d_n P_n^1(\cos\theta) + \frac{s\mu_o a}{R_s} \sum_{n=1}^{\infty} d_n P_n^1(\cos\theta)$$

$$= G(\theta) + C \frac{s\mu_o a}{R_s} \left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s} \right) \tan^2 \frac{\alpha}{2} \cot \frac{\theta}{2} \qquad (\alpha < \theta \leq \pi) \tag{2.22b}$$

where C remains to be determined. The solution of these dual series equations is taken up in the next two sections of this note. In Section III, eqs. (2.22) are reduced to a single inhomogeneous Fredholm equation of the second kind; and in Section IV, this Fredholm equation is used to obtain both exact (numerical) and approximate solutions for the coefficients d_n.

III. REDUCTION OF DUAL SERIES EQUATIONS TO AN INTEGRAL EQUATION

We begin by defining two functions $F_1(\theta)$ and $F_2(\theta)$ over the interval $0 \leq \theta \leq \pi$ as

$$F_1(\theta) = \sum_{n=1}^{\infty} d_n P_n^1(\cos\theta) \tag{3.1a}$$

$$F_2(\theta) = \sum_{n=1}^{\infty} (2n+1) d_n P_n^1(\cos\theta) \tag{3.1b}$$

Then the dual series equations (2.22) are written

$$F_1(\theta) = C \tan\frac{\theta}{2} \qquad (0 \leq \theta \leq \alpha) \tag{3.2a}$$

$$\frac{s\mu_o a}{R_s} F_1(\theta) + F_2(\theta) = G(\theta) + C \frac{s\mu_o a}{R_s} \frac{2\pi R_c - R_s}{2\pi R_c + R_s} \tan^2\frac{\alpha}{2} \cot\frac{\theta}{2}$$
$$(\alpha < \theta \leq \pi) \tag{3.2b}$$

The coefficients d_n are expressed in terms of functions $h_1(u)$ and $h_2(u)$ as [5]

$$d_n = d_{n1} + d_{n2} \tag{3.3}$$

where

$$d_{n1} = \frac{-\sqrt{\pi}\,\Gamma(n)}{\Gamma(n+\frac{1}{2})} \int_o^{\alpha} h_1(u) \cos^3\frac{u}{2}\, P_{n-1}^{(1/2,3/2)}(\cos u)\, du \tag{3.4a}$$

$$d_{n2} = \frac{-\sqrt{\pi}\,\Gamma(n)}{\Gamma(n+\frac{1}{2})} \int_{\alpha}^{\pi} h_2(u) \cos^3\frac{u}{2}\, P_{n-1}^{(1/2,3/2)}(\cos u)\, du \tag{3.4b}$$

in which $P_{n-1}^{(\alpha,\beta)}$ denotes a Jacobi polynomial and the functions $h_1(u)$ and $h_2(u)$ are to be determined. Since d_n must decrease as n^{-3} for large n, $h_1(u)$ and $h_2(u)$ must be continuous, and in particular

$$h_1(\alpha) = h_2(\alpha) \tag{3.5}$$

Substituting eqs. (3.3) and (3.4) into eqs. (3.1), we can readily show that*

$$F_1(\theta) = \frac{1}{\sqrt{2}} \cot \frac{\theta}{2} \int_0^\theta \frac{h(u)du}{\sqrt{\cos u - \cos\theta}} \tag{3.6a}$$

$$F_2(\theta) = -\frac{1}{\sqrt{2}} \sec^2 \frac{\theta}{2} \frac{d}{d\theta} \int_\theta^\pi \frac{h(u)\cot^2 \frac{u}{2} \sin u\, du}{\sqrt{\cos\theta - \cos u}} \tag{3.6b}$$

where $h(u) = h_1(u)$ for $0 \leq u \leq \alpha$ and $h(u) = h_2(u)$ for $\alpha \leq u \leq \pi$. The first of the dual series equations thus becomes

$$\int_0^\theta \frac{h_1(u)du}{\sqrt{\cos u - \cos\theta}} = \sqrt{2}\, C \tan^2 \frac{\theta}{2} \quad (0 \leq \theta \leq \alpha) \tag{3.7}$$

whose solution is

$$h_1(u) = \frac{\sqrt{2}}{\pi} C \frac{d}{du} \int_0^u \frac{\tan^2 \frac{t}{2} \sin t\, dt}{\sqrt{\cos t - \cos u}}$$

$$= \frac{C}{\pi} \sec \frac{u}{2} \tan \frac{u}{2} (u + \sin u) \quad (0 \leq u \leq \alpha) \tag{3.8}$$

Using eq. (3.4a) and the fact that

$$P_{n-1}^{(1/2,3/2)}(\cos u) = \frac{-2\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(n+2)} \csc u \frac{d}{du}\left[\frac{\cos(n+\frac{1}{2})u}{\cos \frac{u}{2}}\right] \tag{3.9}$$

we obtain an expression for the coefficients d_{n1}, viz.

$$d_{n1} = \frac{C}{\pi n(n+1)} \left[(\alpha + \sin\alpha) \frac{\cos(n+\frac{1}{2})\alpha}{\cos \frac{\alpha}{2}} - \frac{\sin n\alpha}{n} - \frac{\sin(n+1)\alpha}{n+1}\right] \tag{3.10}$$

*See [1], pp. 9, 10 for details.

The second of the dual series equations becomes, using eqs. (3.6) and (3.2b),

$$\frac{s\mu_o a}{R_s} \cot\frac{\theta}{2} \int_o^{\alpha} \frac{h_1(u)du}{\sqrt{\cos u - \cos\theta}} + \frac{s\mu_o a}{R_s} \cot\frac{\theta}{2} \int_{\alpha}^{\theta} \frac{h_2(u)du}{\sqrt{\cos u - \cos\theta}}$$

$$- \sec^2\frac{\theta}{2} \frac{d}{d\theta} \int_{\theta}^{\pi} \frac{h_2(u) \cot^2\frac{u}{2} \sin u\, du}{\sqrt{\cos\theta - \cos u}} = \sqrt{2}\, G(\theta)$$

$$+ \sqrt{2}\, C \frac{s\mu_o a}{R_s} \left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right) \tan^2\frac{\alpha}{2} \cot\frac{\theta}{2} \quad (\alpha < \theta \leq \pi) \tag{3.11}$$

This integral equation for the unknown function $h_2(u)$ can be put into a more convenient form by using the integral equation/solution pair

$$\int_x^b \frac{f(t)dt}{\sqrt{\cos x - \cos t}} = g(x) \quad (0 \leq a < x < b \leq \pi) \tag{3.12a}$$

$$f(t) = -\frac{1}{\pi} \frac{d}{dt} \int_t^b \frac{g(u) \sin u\, du}{\sqrt{\cos t - \cos u}} \quad (a < t < b) \tag{3.12b}$$

We obtain

$$\cot\frac{x}{2} h_2(x) + \frac{s\mu_o a}{2\pi R_s} \int_{\alpha}^{\pi} \cot\frac{u}{2} h_2(u) K(x,u) du$$

$$= \frac{1}{\pi\sqrt{2}} \tan\frac{x}{2} \int_x^{\pi} \frac{G(t) \cot\frac{t}{2} \sin t\, dt}{\sqrt{\cos x - \cos t}}$$

$$+ \frac{1}{\pi\sqrt{2}} C \frac{s\mu_o a}{R_s} \left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right) \tan^2\frac{\alpha}{2} \tan\frac{x}{2} \int_x^{\pi} \frac{\cot^2\frac{t}{2} \sin t\, dt}{\sqrt{\cos x - \cos t}}$$

$$- \frac{s\mu_o a}{2\pi R_s} \tan\frac{x}{2} \int_x^{\pi} \frac{\cot^2\frac{t}{2} \sin t\, dt}{\sqrt{\cos x - \cos t}} \int_o^{\alpha} \frac{h_1(u)du}{\sqrt{\cos u - \cos t}}$$

$$(\alpha < x \leq \pi) \tag{3.13}$$

in which the kernel K(x,u) is given by

$$K(x,u) = \tan\frac{x}{2}\tan\frac{u}{2}\int_{\max(x,u)}^{\pi}\frac{\cot^2\frac{t}{2}\sin t\,dt}{\sqrt{\cos x-\cos t}\,\sqrt{\cos u-\cos t}}$$

$$(\alpha \le x,\ u \le \pi) \qquad (3.14)$$

It will be useful in what follows to extend the domain of definition of K(x,u) to $(0,\pi) \times (0,\pi)$.

Now we may obtain from eq. (3.13) an integral equation for the unknown function $h(x)\cot\frac{x}{2}$, defined by

$$h(x)\cot\frac{x}{2} = h_1(x)\cot\frac{x}{2}H(\alpha-x) + h_2(x)\cot\frac{x}{2}H(x-\alpha) \qquad (3.15)$$

We multiply eq. (3.13) through by the step function $H(x-\alpha)$, extending its domain of definition to $0 \le x \le \pi$, and add the function $h_1(x)\cot\frac{x}{2}H(\alpha-x)$ to both sides of the resulting equation.* We obtain

$$h(x)\cot\frac{x}{2} + \frac{s\mu_o a}{2\pi R_s}H(x-\alpha)\int_o^{\pi}h(u)\cot\frac{u}{2}K(x,u)du$$

$$= \frac{1}{\pi\sqrt{2}}H(x-\alpha)\tan\frac{x}{2}\int_x^{\pi}\frac{G(t)\cot\frac{t}{2}\sin t\,dt}{\sqrt{\cos x-\cos t}}$$

$$+ \frac{C}{\pi\sqrt{2}}\frac{s\mu_o a}{R_s}\left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right)\tan^2\frac{\alpha}{2}H(x-\alpha)\tan\frac{x}{2}\int_x^{\pi}\frac{\cot^2\frac{t}{2}\sin t\,dt}{\sqrt{\cos x-\cos t}}$$

$$+ \frac{C}{\pi}H(\alpha-x)\sec\frac{x}{2}(x+\sin x) \quad (0 \le x \le \pi) \qquad (3.16)$$

By forcing $h(x)\cot\frac{x}{2}$ to be continuous at $x = \alpha$ we obtain the following implicit expression for the constant C:

*See eq. (3.8).

$$C = \frac{\frac{1}{\sqrt{2}} \sin\frac{\alpha}{2} \int_{\alpha}^{\pi} \frac{G(t)\cot\frac{t}{2} \sin t\, dt}{\sqrt{\cos\alpha - \cos t}} - \frac{s\mu_o a}{2R_s} \cos\frac{\alpha}{2} \int_o^{\pi} h(u)\cot\frac{u}{2} K(\alpha,u)du}{\pi - (\pi - \alpha - \sin\alpha)\left[1 + \frac{s\mu_o a}{R_s}\left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right)\tan^2 \frac{\alpha}{2}\right]} \tag{3.17}$$

We have now obtained an integral equation for the function $h(x) \cot \frac{x}{2}$; the coefficients d_n can be evaluated via the relation

$$d_n = \frac{-\sqrt{\pi}\ \Gamma(n)}{\Gamma(n+\frac{1}{2})} \int_o^{\pi} h(u) \cot \frac{u}{2} \sin \frac{u}{2} \cos^2 \frac{u}{2} P_{n-1}^{(1/2,3/2)}(\cos u)\, du \tag{3.18}$$

We take up the problem of evaluating these coefficients in the next section.

IV. SOLUTION FOR COEFFICIENTS d_n

The relation (3.18) can be inverted by making use of the orthogonality property of the Jacobi polynomials

$$\int_0^{\pi} \sin\frac{x}{2}\cos^3\frac{x}{2}\, P_{n-1}^{(1/2,3/2)}(\cos x)\, P_{m-1}^{(1/2,3/2)}(\cos x)\sin x\, dx$$

$$= \delta_{mn}\frac{\Gamma^2(n+1/2)}{\Gamma(n)\Gamma(n+2)} \qquad (4.1)$$

in which δ_{mn} denotes the Kronecker delta-function. We obtain

$$h(u)\cot\frac{u}{2} = -\frac{1}{\sqrt{\pi}}\cos\frac{u}{2}\sin u \sum_{n=1}^{\infty}\frac{\Gamma(n+2)}{\Gamma(n+1/2)}\, d_n P_{n-1}^{(1/2,3/2)}(\cos u) \qquad (4.2)$$

Now substituting eq. (4.2) into eq. (3.16) and making use of the relation [1]

$$\int_0^{\pi} P_{n-1}^{(1/2,3/2)}(\cos u)\, K(x,u)\cos\frac{u}{2}\sin u\, du$$

$$= \frac{\pi}{n+1/2}\cos\frac{x}{2}\sin x\, P_{n-1}^{(1/2,3/2)}(\cos x) \qquad (4.3)$$

we find that

$$\cos\frac{x}{2}\sin x \sum_{n=1}^{\infty}\frac{\Gamma(n+2)}{\Gamma(n+1/2)}\, d_n P_{n-1}^{(1/2,3/2)}(\cos x)$$

$$+ \frac{s\mu_o a}{2R_s} H(x-\alpha)\cos\frac{x}{2}\sin x \sum_{n=1}^{\infty}\frac{\Gamma(n+2)}{\Gamma(n+3/2)}\, d_n P_{n-1}^{(1/2,3/2)}(\cos x)$$

$$= -\frac{1}{\sqrt{2\pi}} H(x-\alpha)\tan\frac{x}{2}\int_x^{\pi}\frac{G(t)\cot\frac{t}{2}\sin t\, dt}{\sqrt{\cos x - \cos t}}$$

$$- \frac{C}{\sqrt{\pi}}\frac{s\mu_o a}{R_s}\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\tan^2\frac{\alpha}{2}\sec\frac{x}{2} H(x-\alpha)(\pi - x - \sin x)$$

$$- \frac{C}{\sqrt{\pi}} H(\alpha - x)\sec\frac{x}{2}(x+\sin x) \qquad (0 \le x \le \pi) \qquad (4.4)$$

If we now multiply this equation through by the factor

$$\sin\frac{x}{2}\cos^2\frac{x}{2}P_{m-1}^{(1/2,3/2)}(\cos x)$$

and integrate with respect to x over the interval $0 \le x \le \pi$, making use of the orthogonality property in eq. (4.1), there results a system of linear equations in the unknown coefficients d_n, viz.,

$$\tilde{d}_m + s\tau_o \sum_{n=1}^{\infty} Q_{mn}\tilde{d}_n = R_m \quad (m \ge 1) \tag{4.5}$$

in which

$$\tilde{d}_m = -d_m\sqrt{3}\left[\frac{m(m+1)}{m+1/2}\right]^{\frac{1}{2}}\left(\frac{r_o}{a}\right) \tag{4.6}$$

$$\tau_o = \frac{\mu_o a}{3R_s} \tag{4.7}$$

$$Q_{mn} = \frac{3}{2}\,\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+1/2)\Gamma(n+1/2)}\left[\frac{m+1}{m(m+1/2)}\right]^{\frac{1}{2}}\left[\frac{n+1}{n(n+1/2)}\right]^{\frac{1}{2}}$$

$$\cdot \int_{\alpha}^{\pi} \cos^3\frac{x}{2}\sin\frac{x}{2}P_{n-1}^{(1/2,3/2)}(\cos x)\,P_{m-1}^{(1/2,3/2)}(\cos x)\sin x\,dx \tag{4.8}$$

$$R_m = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{a}{r_o}\right)^{n-1}\left[\frac{n(n+1/2)}{n+1}\right]^{\frac{1}{2}} Q_{mn}$$

$$- \sqrt{3}\,\frac{r_oC}{\pi a}\left[\frac{1}{m(m+1/2)(m+1)}\right]^{\frac{1}{2}}\Bigg\{\pi\cos(m+1/2)\,\alpha\sec\frac{\alpha}{2}$$

$$- \left[1 + \frac{s\mu_o a}{R_s}\left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right)\tan^2\frac{\alpha}{2}\right]\Bigg[(\pi-\alpha-\sin\alpha)\cos(m+1/2)\,\alpha\sec\frac{\alpha}{2}$$

$$+ \frac{\sin m\alpha}{m} + \frac{\sin(m+1)\alpha}{m+1}\Bigg]\Bigg\} \tag{4.9}$$

The coefficient $\tilde{d}_1$ is the ratio of the magnetic field H_x at the center of the sphere to that which would exist there if the sphere were absent. The time constant τ_o is that which would be characteristic of the problem if α were equal to zero, that is, if the spherical shell were entirely resistive. Explicit expressions for Q_{mn} are given in [1].

The constant C is given in terms of the coefficients $\tilde{d}_n$ by (see eq. (3.17))

$$C = \frac{\sqrt{\pi}a}{r_o}\left\{\pi - (\pi-\alpha-\sin\alpha)\left[1 + \frac{s\mu_o a}{R_s}\left(\frac{2\pi R_c - R_s}{2\pi R_c + R_s}\right)\tan^2\frac{\alpha}{2}\right]\right\}^{-1}$$

$$\cdot \sin\frac{\alpha}{2}\cos^3\frac{\alpha}{2}\sum_{n=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n+1/2)}P_{n-1}^{(1/2,3/2)}(\cos\alpha)\left\{2\left(\frac{-a}{r_o}\right)^{n-1}\right.$$

$$\left. - \frac{1}{\sqrt{3}}\left[\frac{n+1}{n(n+1/2)}\right]^{\frac{1}{2}}\frac{s\mu_o a}{R_s}\tilde{d}_n\right\} \tag{4.10}$$

If we substitute this expression for C into eq. (4.5) and rearrange the resulting system of equations, we obtain

$$\tilde{d}_m + s\tau_o\sum_{n=1}^{\infty}\hat{Q}_{mn}\tilde{d}_n = \hat{R}_m \qquad (m \geq 1) \tag{4.11}$$

in which

$$\hat{R}_m = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\left(\frac{-a}{r_o}\right)^{n-1}\left[\frac{n(n+1/2)}{n+1}\right]^{\frac{1}{2}}\hat{Q}_{mn} \tag{4.12}$$

$$\hat{Q}_{mn} = Q_{mn} - F_m G_n \tag{4.13}$$

$$F_m = [m(m+1/2)(m+1)]^{-\frac{1}{2}}\left\{\sec\frac{\alpha}{2}\cos(m+1/2)\alpha\right.$$

$$\left. + \frac{1+B'}{\pi B' - (1+B')(\alpha+\sin\alpha)}\left[\frac{\sin m\alpha}{m} + \frac{\sin(m+1)\alpha}{m+1}\right]\right\} \tag{4.14}$$

$$G_n = \frac{3}{2}\left[\frac{n(n+1)}{n+1/2}\right]^{\frac{1}{2}}\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right] \quad (4.15)$$

and

$$B' = \frac{B}{C} = 3s\tau_o \tan^2\frac{\alpha}{2}\left(\frac{2\pi R_c - R_s}{2\pi R_c - R_s}\right) \quad (4.16)$$

The system of equations (4.11) can now be solved for the coefficients $\tilde{d}_m$ on a computer.

We may also develop variational expressions for $\tilde{d}_m$ by standard means. It is not difficult to show that a variational expression for $\tilde{d}_m$, derived from the use as a trial solution of

$$\tilde{d}_{m,trial} = \hat{R}_m\Big|_{R_s\to\infty} \equiv \tilde{d}_m^{(\infty)} \quad (4.17)$$

is

$$\tilde{d}_m \simeq \frac{\hat{R}_m\tilde{d}_m^{(\infty)}}{\tilde{d}_m^{(\infty)} + s\tau_o \sum_{n=1}^{\infty} \hat{Q}_{mn}\tilde{d}_n^{(\infty)}} \quad (4.18)$$

Thus, for example,

$$\frac{\tilde{d}_1}{\tilde{d}_1^{(\infty)}} \simeq \frac{\hat{R}_1}{\tilde{d}_1^{(\infty)} + s\tau_o \sum_{n=1}^{\infty} \hat{Q}_{1n}\tilde{d}_n^{(\infty)}} \quad (4.19)$$

where

$$\tilde{d}_m^{(\infty)} = \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \left(\frac{-a}{r_o}\right)^{n-1}\left[\frac{n(n+1/2)}{n+1}\right]^{\frac{1}{2}}\hat{Q}_{mn}^{(\infty)} \quad (4.20)$$

$$\hat{Q}_{mn}^{(\infty)} = Q_{mn} - F_m^{(\infty)}G_n \quad (4.21)$$

$$F_m^{(\infty)} = [m(m+1/2)(m+1)]^{-\frac{1}{2}}\Big\{\sec\frac{\alpha}{2}\cos(m+1/2)\alpha - \frac{1}{\alpha + \sin\alpha}\left[\frac{\sin m\alpha}{m} + \frac{\sin(m+1)\alpha}{m+1}\right]\Big\} \quad (4.22)$$

In particular, when $a/r_o \to 0$,

$$\frac{\tilde{d}_1}{\tilde{d}_1^{(\infty)}} \simeq \frac{\hat{Q}_{11}}{\hat{Q}_{11}^{(\infty)} + s\tau_o \sum_{n=1}^{\infty} \hat{Q}_{1n}\hat{Q}_{n1}^{(\infty)}} \tag{4.23}$$

Unfortunately, these variational results are not sufficiently simple to be very useful. We shall return to the problem of obtaining simple approximate results for the penetrant field in the next section, wherein numerical results are presented and discussed.

V. NUMERICAL RESULTS

In this section we present numerical data for the quantity $\tilde{d}_1^{(\infty)}$ and for $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ in order to illustrate the behavior of the penetrant field under various conditions of excitation and aperture loading.

In Figure 2 are shown curves of $\tilde{d}_1^{(\infty)}$ as a function of the aperture angle θ_o ($\alpha = \pi - \theta_o$) for various values of the ratio a/r_o. It is interesting to note that, in contrast to the axial-dipole case, $\tilde{d}_1^{(\infty)}$ can exceed unity when a/r_o is sufficiently close to unity. This is easily explained by considering the limiting case $a/r_o \to 1+$, $\theta_o \to 0+$, in which half of the magnetic flux produced by the dipole source is trapped by the spherical shell, so that the field at the center of the shell is actually enhanced in comparison to that which would exist there if the shell were absent.

In order to compare the penetration into the spherical shell through an unloaded circular aperture for the axial and transverse dipole orientations, we show in Figure 3 the quantity $\tilde{d}_1^{(\infty)}$ as a function of θ_o for $a/r_o = 0$ and for both source orientations. The penetration of the transverse-dipole field is greater than that of the axial-dipole field for all angles θ_o [2].

The effect of the resistive aperture loading on the penetration of the spherical shell is shown in Figures 4 and 5, in which the magnitude of the ratio $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ is plotted as a function of normalized frequency $\omega\tau_o$ ($s = j\omega$) for various values of the aperture angle θ_o, when $a/r_o = 2\pi R_c/R_s = 0$. It is evident from

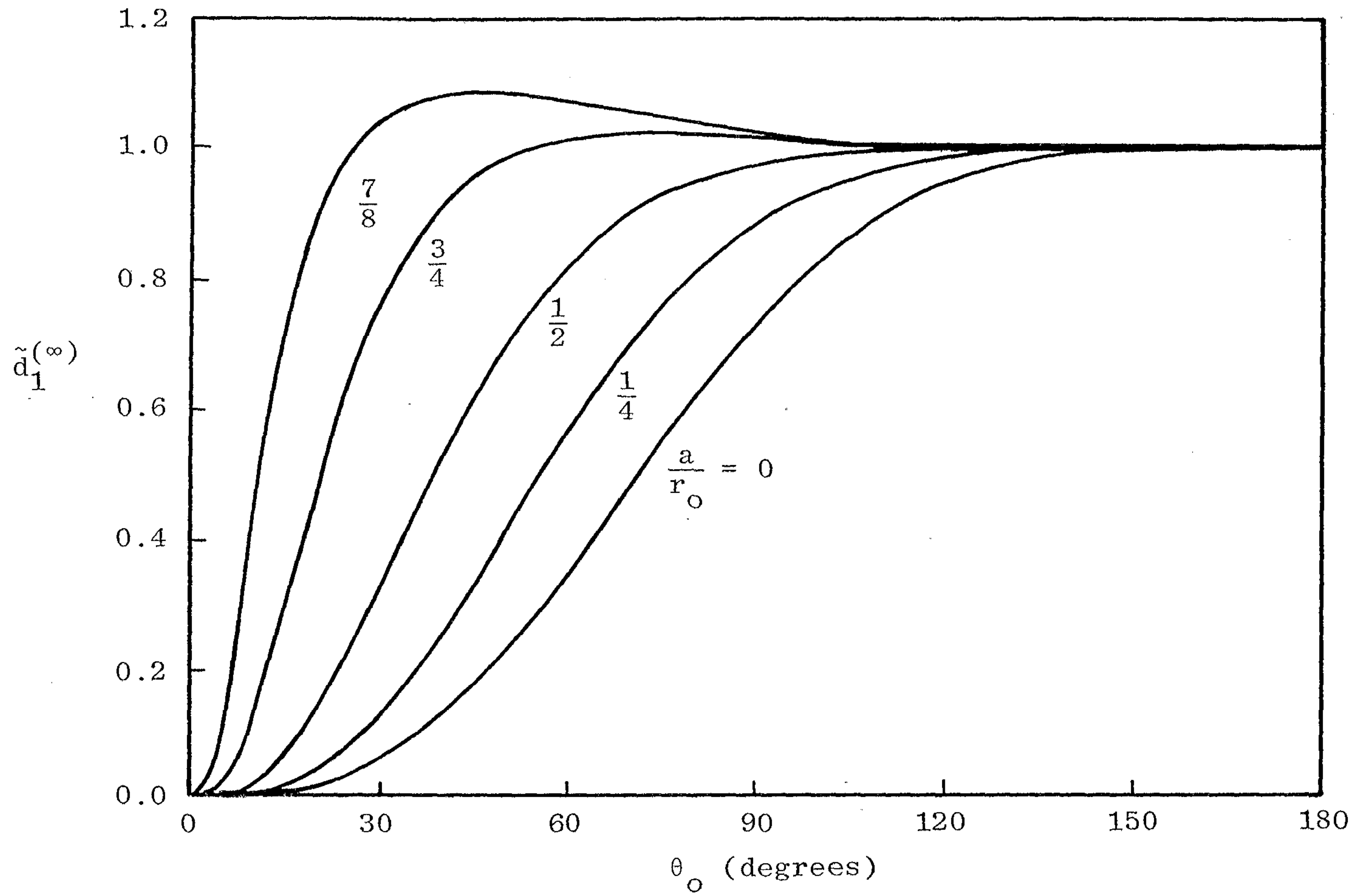

Figure 2. $\tilde{d}_1^{(\infty)}$ as a function of θ_o for various values of a/r_o

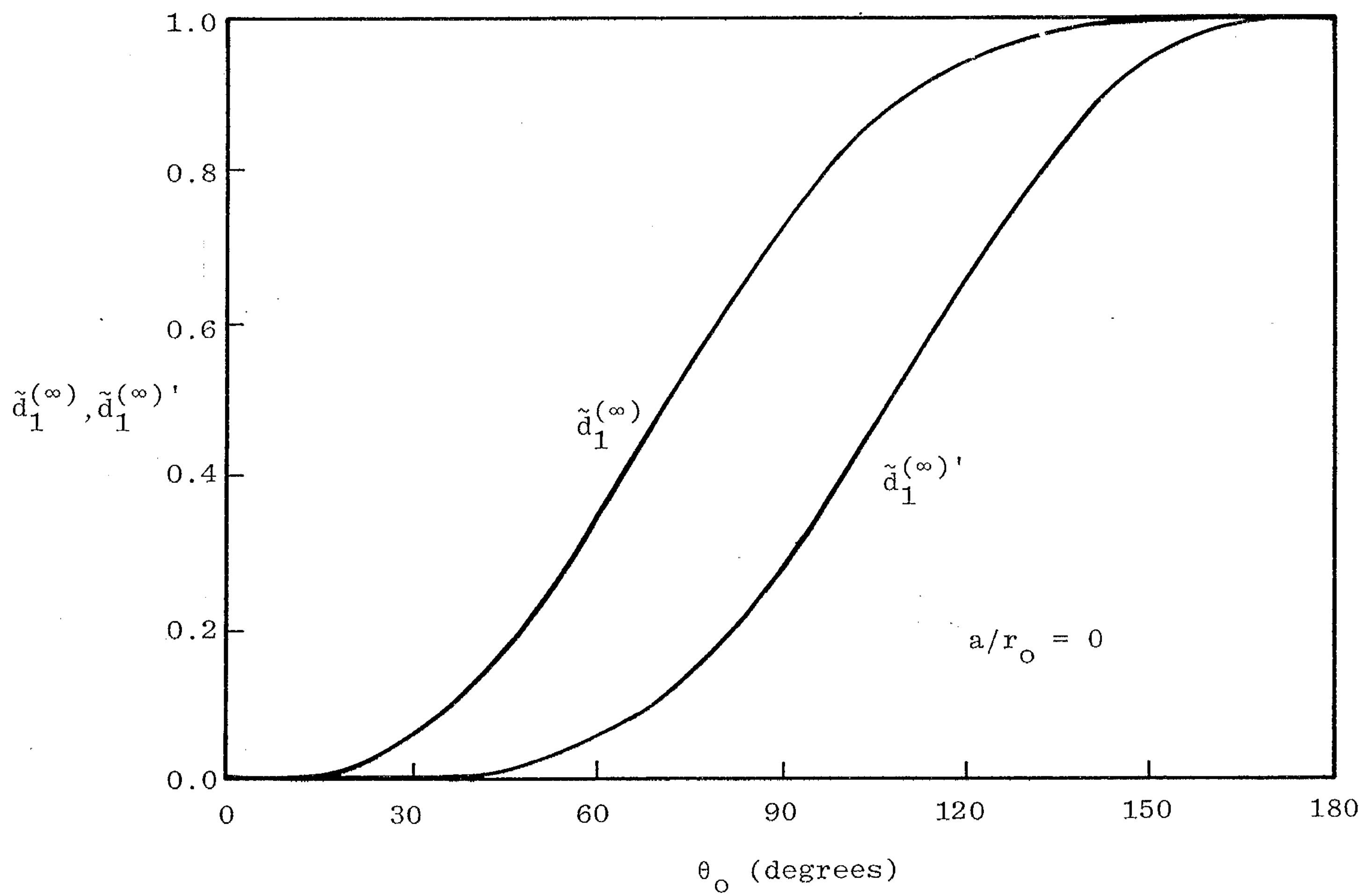

Figure 3. $\tilde{d}_1^{(\infty)}$ (transverse) and $\tilde{d}_1^{(\infty)'}$ (axial) as functions of θ_o for $a/r_o = 0$

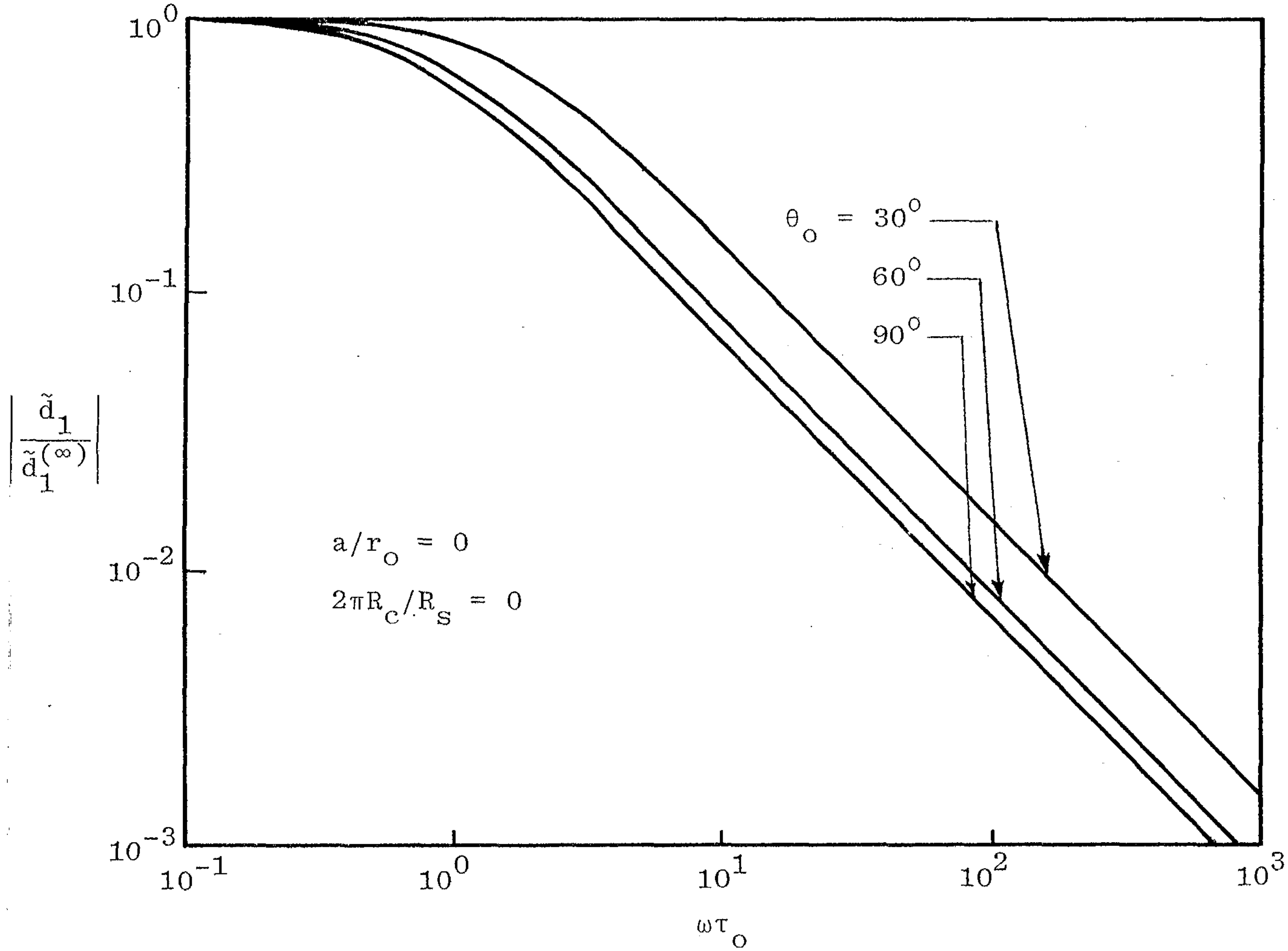

Figure 4. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$; $\theta_o = 30^o, 60^o, 90^o$

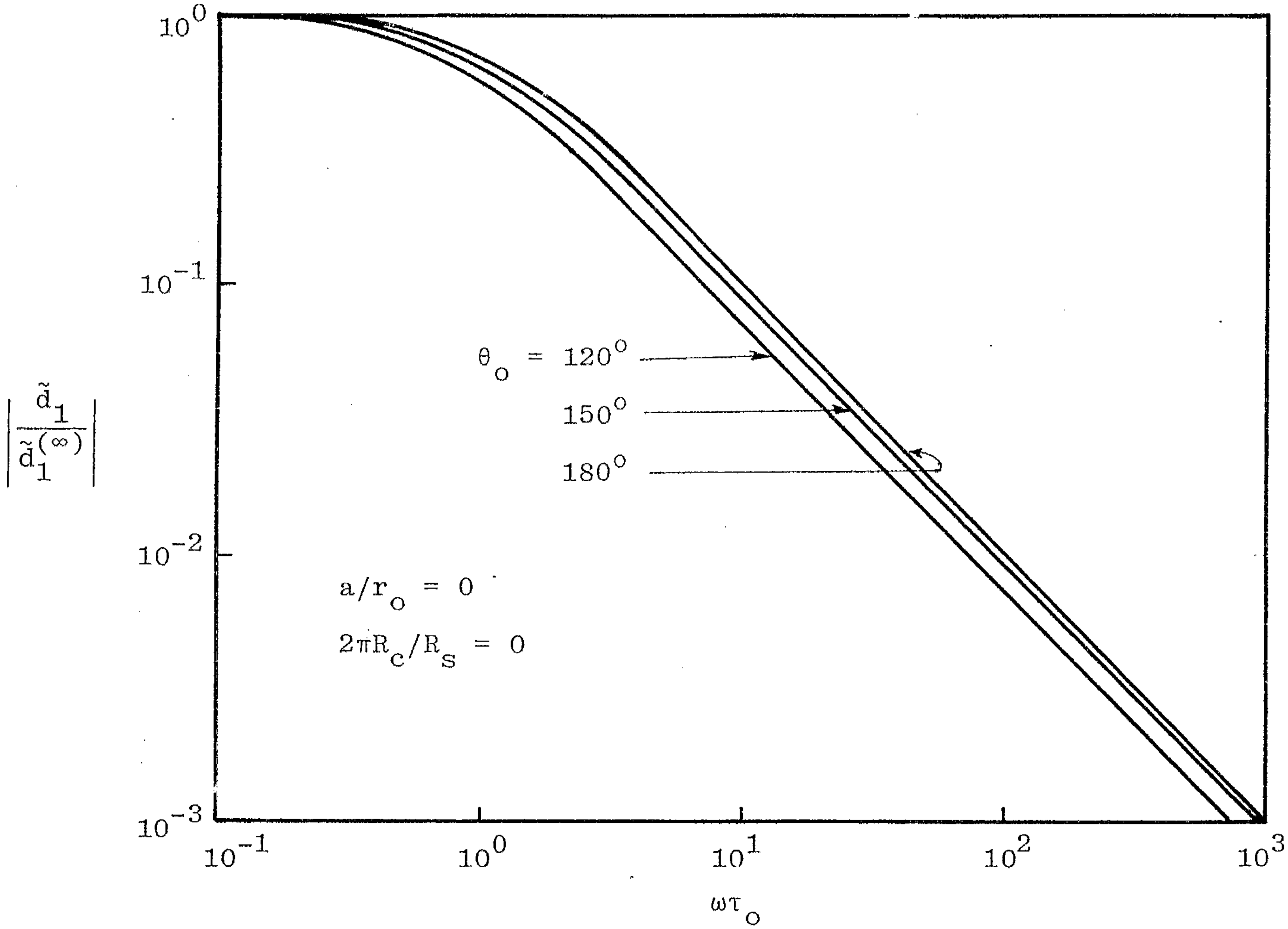

Figure 5. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$; $\theta_o = 120^o, 150^o, 180^o$

these curves that $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ can be expressed in the approximate form

$$\frac{\tilde{d}_1}{\tilde{d}_1^{(\infty)}} \simeq (1 + qs\tau_o)^{-1} \tag{5.1}$$

when $a/r_o = 0$ and $2\pi R_c/R_s = 0$. The "time constant factor" q is plotted as a function of θ_o in Figure 6. The same quantity for the axial-dipole problem is also shown on that figure. The maximum of q which occurs at $\theta_o = 90^o$ (where the spherical shell is half perfectly conducting and half resistive) can be interpreted in terms of a maximum disruption of the surface current density with respect to its distribution on a homogeneous spherical shell, and thus a maximum equivalent inductance and time constant. When $\theta_o \lesssim 30^o$ and $a/r_o = 2\pi R_c/R_s = 0$, an approximate formula for q is

$$q = \frac{4\theta_o}{3\pi} \tag{5.2}$$

Contact-resistance effects are shown in Figures 7-11, where $|\tilde{d}_1/\tilde{d}_1^{(\infty)}|$ is plotted as a function of normalized frequency $\omega\tau_o$ ($s = j\omega$) for various values of $2\pi R_c/R_s$, at fixed aperture angles θ_o and for $a/r_o = 0$. The effect becomes less pronounced as $\theta_o \to 180^o$ for obvious reasons. The increase in the penetrant field when $2\pi R_c/R_s >> 1$ is striking. Under this condition, the resistive loading in the aperture is effectively isolated from the remainder of the conducting sphere and the "inductive shielding" behavior (described by an equation of the form of eq. (5.1)) effectively disappears. It is clear from the curves in Figures 7

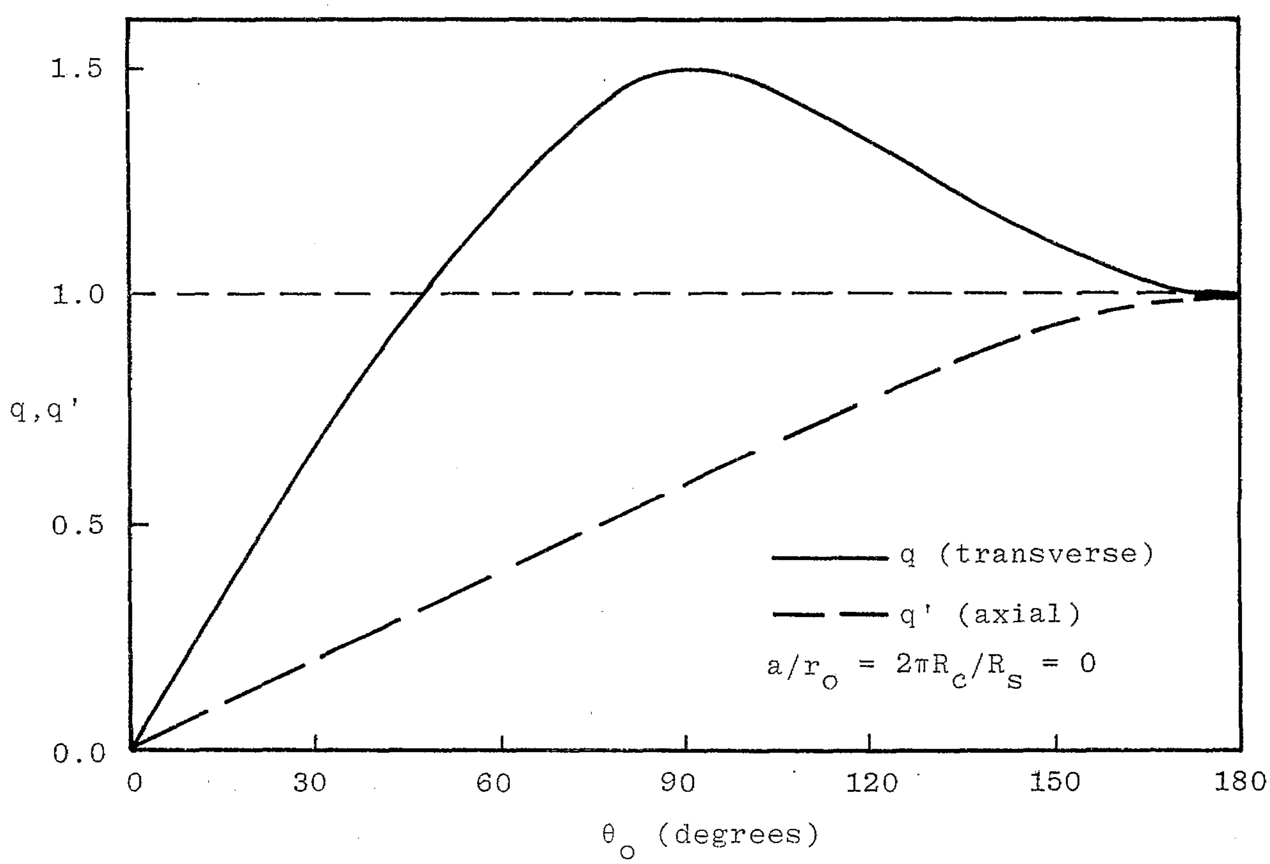

Figure 6. Time-constant factors q and q' for transverse and axial dipole orientations, $a/r_o = 0$ and $2\pi R_c/R_s = 0$

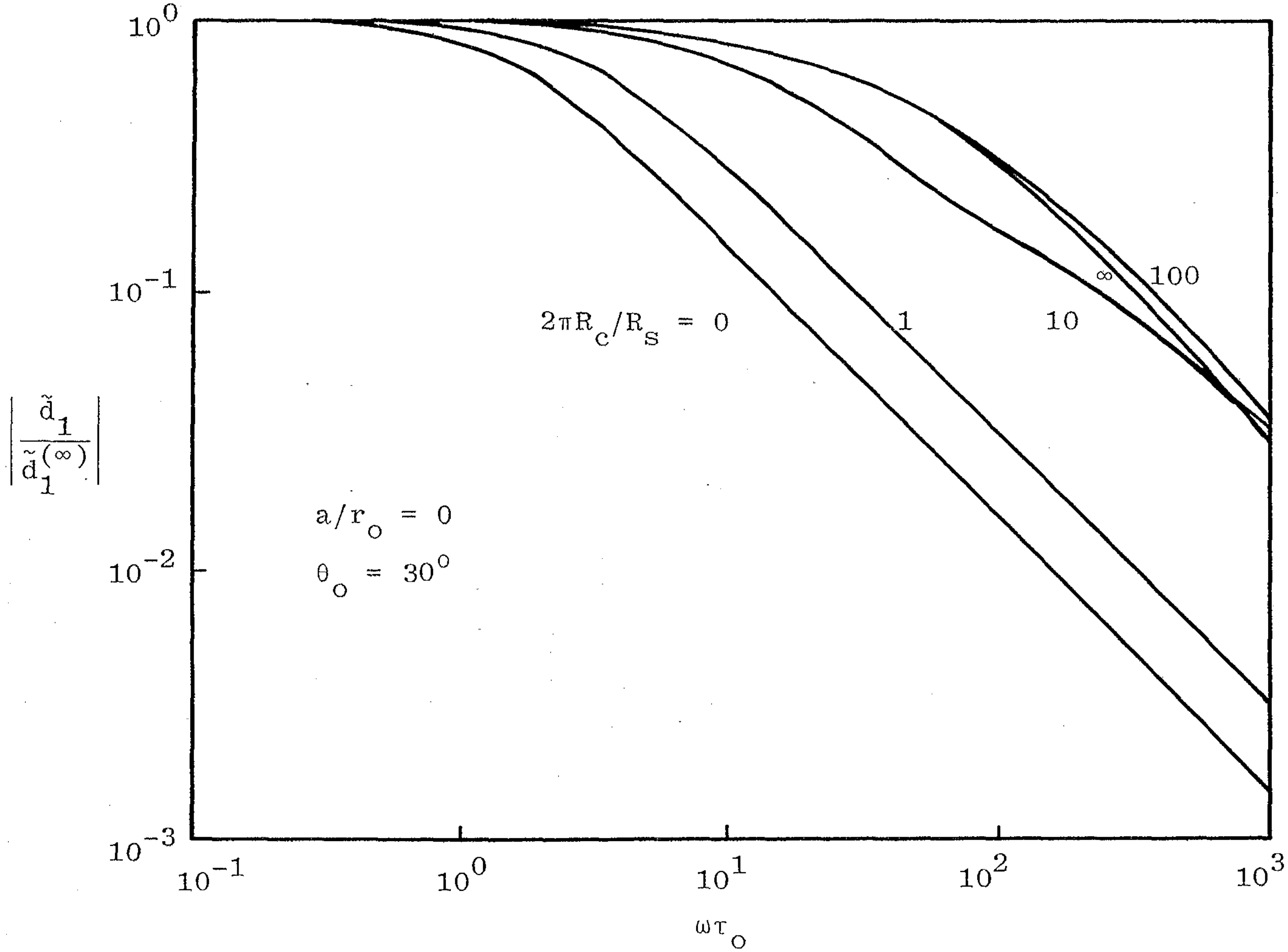

Figure 7. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$ for $a/r_o = 0$, $\theta_o = 30^o$, and various values of normalized contact resistance $2\pi R_c/R_s$

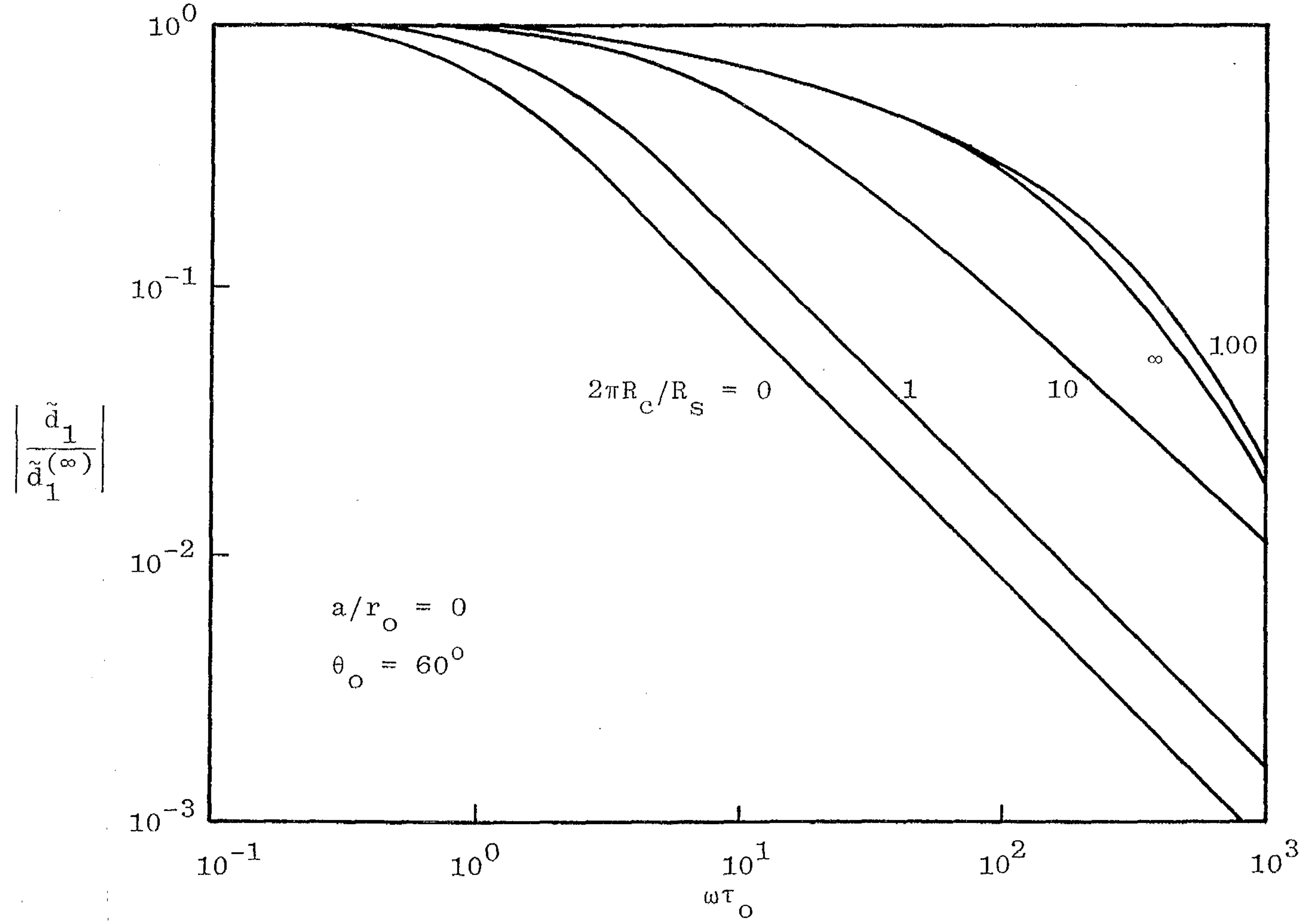

Figure 8. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$ for $a/r_o = 0$, $\theta_o = 60^o$, and various values of normalized contact resistance $2\pi R_c/R_s$

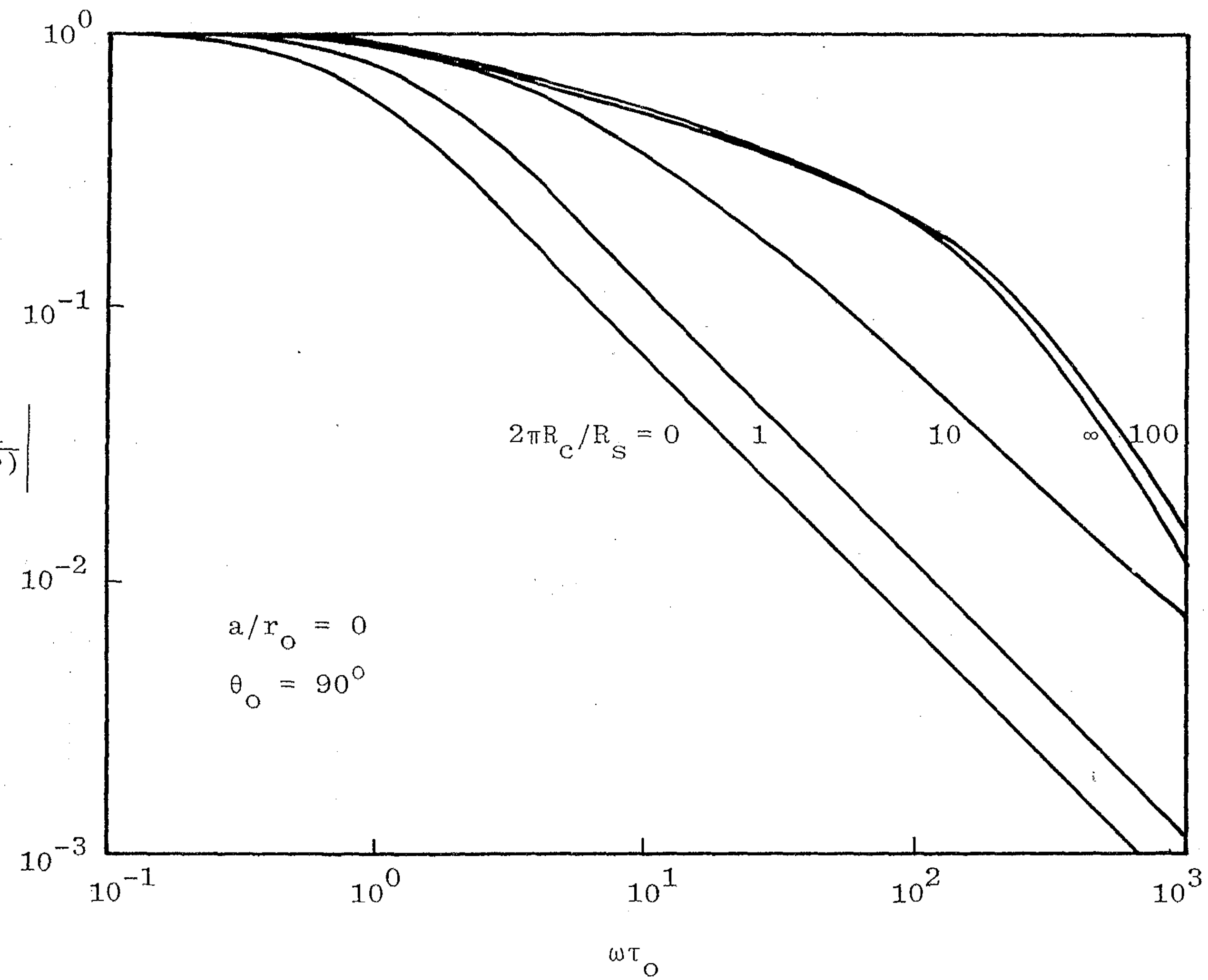

Figure 9. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$ for $a/r_o = 0$, $\theta_o = 90^0$, and various values of normalized contact resistance $2\pi R_c/R_s$

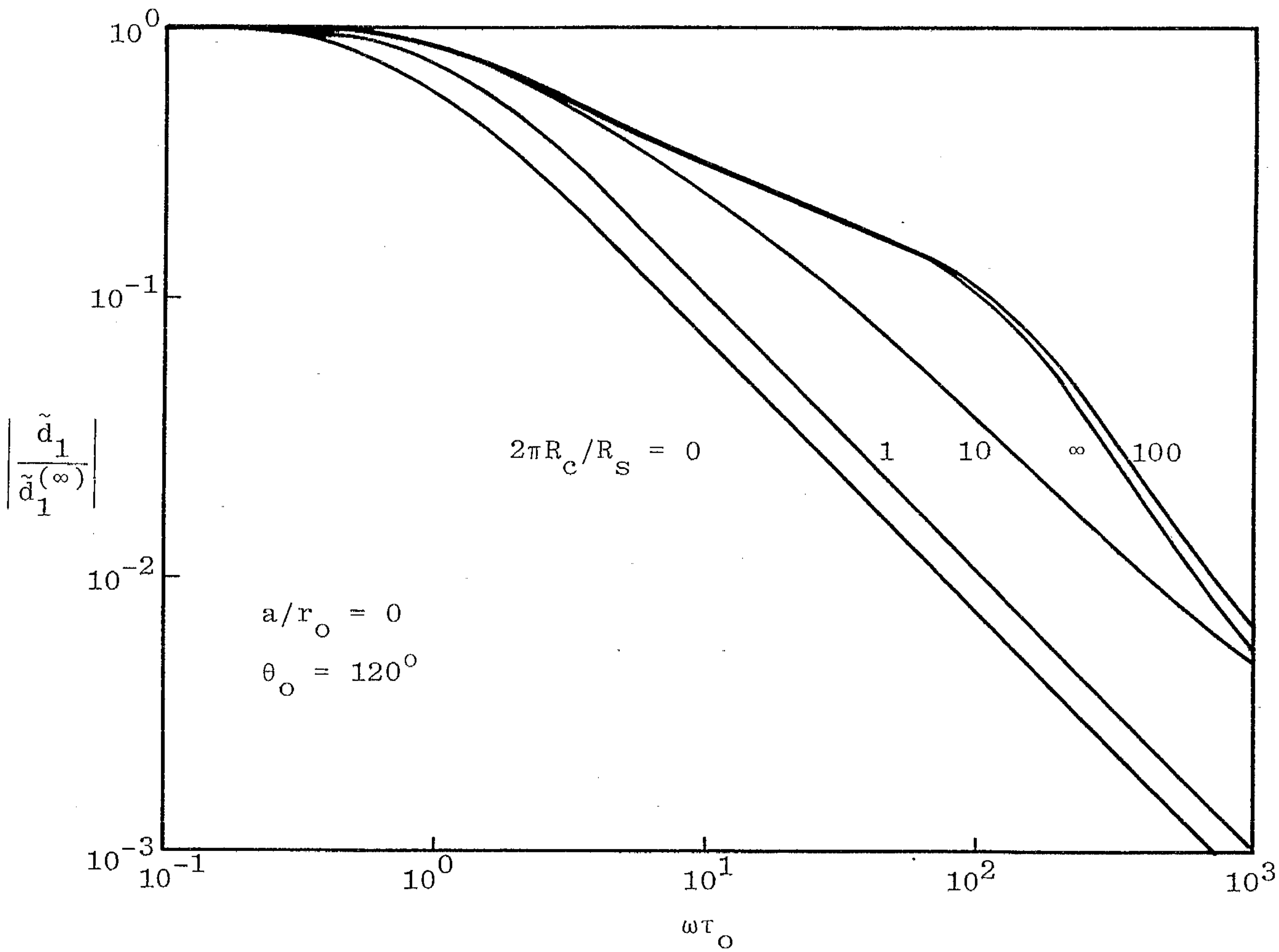

Figure 10. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$ for $a/r_o = 0$, $\theta_o = 120^o$, and various values of normalized contact resistance $2\pi R_c/R_s$

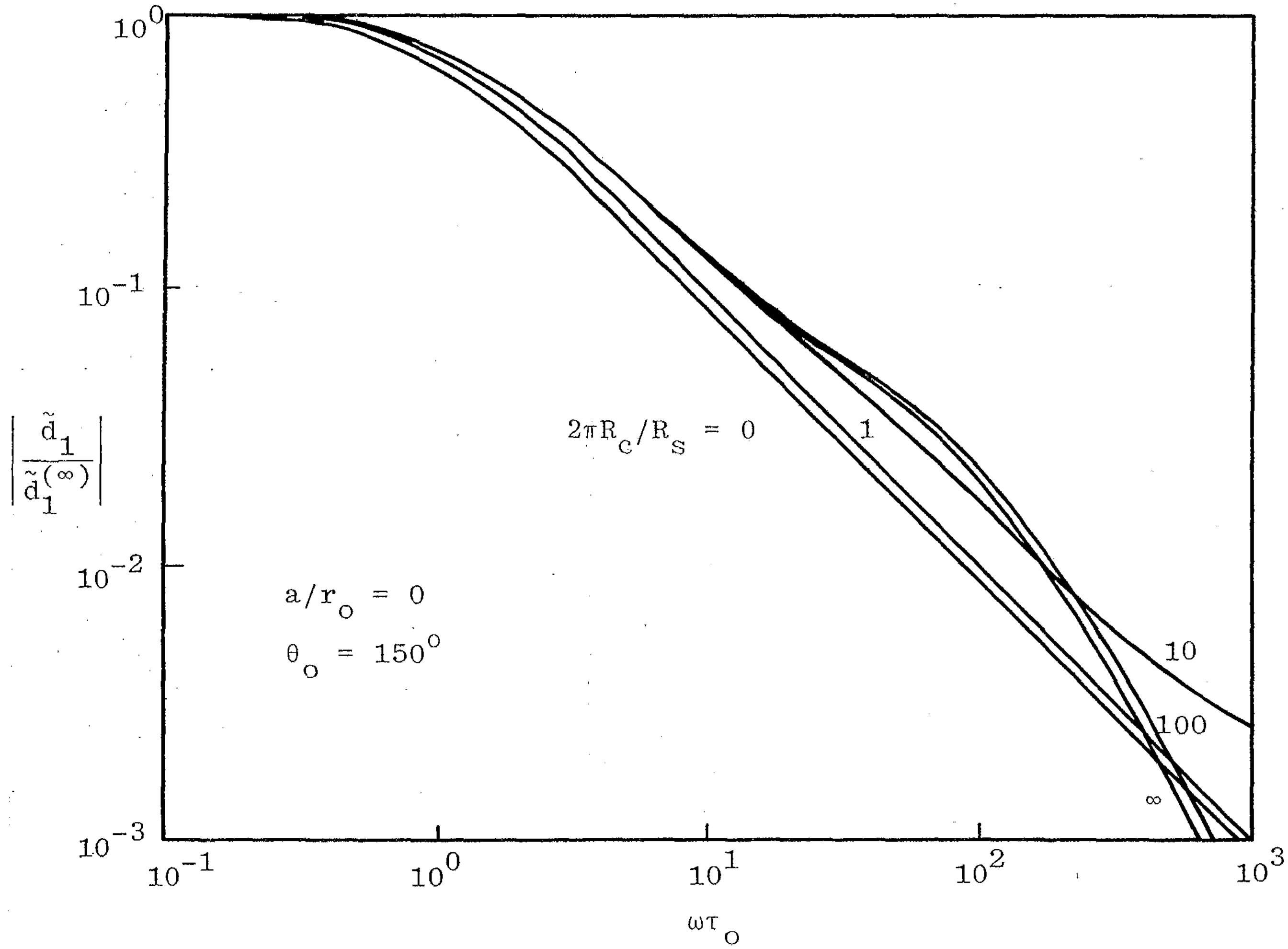

Figure 11. Magnitude of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ as a function of normalized frequency $\omega\tau_o$ for $a/r_o = 0$, $\theta_o = 150^o$, and various values of normalized contact resistance $2\pi R_c/R_s$

through 11 that when $2\pi R_c/R_s >> 1$, the frequency-domain behavior of $\tilde{d}_1/\tilde{d}_1^{(\infty)}$ is much more complicated than the $(1 + s\tau)^{-1}$ behavior characteristic of the case $R_c = 0$. It is also interesting to note that for some ranges of values of $\omega\tau_o$ and $2\pi R_c/R_s$, the penetration of the shell is lower than that for $2\pi R_c/R_s = 0$.

VI. IMPLICATIONS FOR SHIELDING AND MEASUREMENT; CONCLUDING REMARKS

When an aperture in the wall of a closed region is hardened by means of resistive loading, the effectiveness of the loading is described by a factor of the form $(1 + s\tau)^{-1}$, where τ depends on the enclosure and aperture sizes and upon the sheet resistance of the loading material, and where it is assumed that $2\pi R_c/R_s << 1$ and that the frequency is far below the first resonance of the enclosure. The presence of a contact resistance R_c which is comparable to, or greater than, $R_s/2\pi$ significantly degrades the shielding afforded by the resistive loading of the aperture. This behavior is similar to that seen in the planar-aperture problem [3]. Thus in a shielding application, the following points should be noted:

1. The contact resistance between the loading and the aperture rim should be as small as possible; specifically, $2\pi R_c/R_s << 1$.
2. The sheet resistance of the loading must be sufficiently small that the "break frequency" $1/\tau$ lies well below the lowest frequency at which the aperture loading must be effective.

If a structure which is topologically similar to that analyzed in this note is used to determine the sheet resistance R_s of the loading by means of measurements, then again the contact resistance must be as small as possible. This is necessary in order to ensure the simple and predictable $(1 + s\tau)^{-1}$ behavior of the internal field as a function of frequency. For a given geometry (say, a rectangular box with one face made of the material to be tested), one would have

$$\frac{H_{x,\text{loaded}}}{H_{x,\text{unloaded}}} = \frac{1}{1 + s\tau_1} \tag{6.1}$$

where

$$\tau_1 = \frac{L_{eq}}{R_s} = \frac{L_{eq}}{R_{so}} \frac{R_{so}}{R_s} = \tau_o \frac{R_{so}}{R_s} \tag{6.2}$$

in which $\tau_o = L_{eq}/R_{so}$ and where R_{so} is the known sheet resistance of a "calibrating" material. One would determine the time constant τ_o by measurement with the known calibrating material in place; this effectively determines the equivalent inductance L_{eq}. Then replacing the known material with the sample to be measured one would determine the time constant τ_1. Then R_s is given by

$$R_s = R_{so} \frac{\tau_o}{\tau_1} \tag{6.3}$$

Clearly, the dipole source orientation should be transverse to the axis of symmetry of the structure, in order to obtain maximum field penetration of the sample (see Figure 3). Additionally, the source dipole should probably not be placed closer to the sample than a distance corresponding to $a/r_o = 1/2$, in order that the sample be reasonably uniformly illuminated (see Figure 2).

In this note we have formulated and solved the quasistatic magnetic-field boundary value problem of a conducting spherical shell with a resistively loaded circular aperture illuminated by a transversely oriented magnetic dipole on the structure's symmetry axis. The formulation of the problem in terms of the magnetic scalar potential leads to a set of dual series equations involving a constant C whose determination follows from a requirement on the behavior for large n of the unknown coefficients d_n, or, equivalently, on the continuity of the function h(u) satisfying the Fredholm equation to which the dual series equations are reduced. We have also been able to include a non-zero contact resistance at the aperture rim in the formulation. Although a variational solution for the penetrant field can be constructed by standard means, the expression which results is not sufficiently simple to be very useful; accordingly, we have constructed an approximate solution empirically from the exact numerical results for the cases in which the contact resistance is zero. Furthermore, variational solutions can be expected to be accurate only when the contact resistance R_c is small compared to $R_s/2\pi$, since when the contact resistance is large, the loading is effectively isolated from the rest of the structure.

In applying the results of this analysis to other topologically equivalent geometries, one assumes that the shielding behavior, when the contact resistance is zero, is described by a factor of the form $(1 + s\tau)^{-1}$, where the time constant $\tau = L_{eq}/R_s$. This "inductive shielding" behavior of the enclosure and its loaded aperture can be used in an experimental configuration to measure the sheet resistance of a given material, provided that the structure is calibrated (i.e., L_{eq} is determined) through the use of a material with known sheet resistance.

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APPENDIX

THE APERTURE-RIM CONDITION

Consider the line integral of the electric field around the contour C shown in Figure 12. The surface $r = a$, $0 \leq \theta < \alpha$ is perfectly conducting; for $r = a$, $\alpha \leq \theta < \alpha + \Delta\theta$, the sheet resistance is R_{sc}; and for $r = a$, $\alpha + \Delta\theta \leq \theta \leq \pi$, the sheet resistance is R_s. The angle $\Delta\theta$ will be allowed to approach zero in the following, so that

$$\lim_{\Delta\theta \to 0} \oint_C \overline{E} \cdot \overline{d\ell} = 0 \tag{A.1}$$

Furthermore, R_{sc} will be allowed to approach infinity in such a way that

$$\lim_{\substack{\Delta\theta \to 0 \\ R_{sc} \to \infty}} \Delta\theta R_{sc} = 2\pi R_c \sin\alpha \tag{A.2}$$

where R_c denotes the net contact resistance across the junction between the aperture loading and the rim.

It is easy to see that

$$\oint_C \overline{E} \cdot \overline{d\ell} = a \sin\alpha \, \Delta\phi \, E_\phi(a, \alpha+\Delta\theta, \phi)$$

$$-a\Delta\theta\Delta\phi \frac{1}{\Delta\phi} [E_\theta(a, \alpha+\Delta\theta/2, \phi+\Delta\phi/2) - E_\theta(a, \alpha+\Delta\theta/2, \phi-\Delta\phi/2)] \tag{A.3}$$

or, as $\Delta\phi \to 0$

$$\sin\alpha \, E_\phi(a, \alpha+\Delta\theta, \phi) = \Delta\theta \frac{\partial}{\partial\phi} E_\theta(a, \alpha+\Delta\theta/2, \phi) \tag{A.4}$$

Now

$$E_\phi(a, \alpha+\Delta\theta, \phi) = R_s J_{s\phi}(\alpha+\Delta\theta, \phi) \tag{A.5}$$

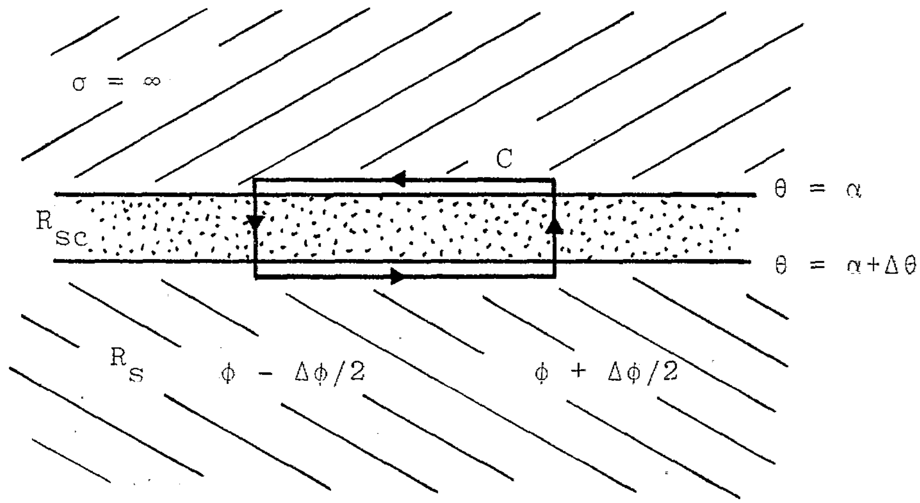

Figure 12. Integration contour for aperture-rim boundary condition. The radius of the spherical surface is <u>a</u>

$$E_\theta(a,\alpha+\Delta\theta/2,\phi) = R_{sc} J_{s\theta}(\alpha+\Delta\theta/2,\phi) \qquad (A.6)$$

so that, substituting Equations A.5 and A.6 into Equation A.4 and using Equation A.2, we obtain upon taking the limit $\Delta\theta \to 0$

$$R_s J_{s\phi}(\alpha+,\phi) = 2\pi R_c \frac{\partial}{\partial\phi} J_{s\theta}(\alpha,\phi) \qquad (A.7)$$

since $J_{s\theta}$ is continuous at $\theta = \alpha$.

The current-density components $J_{s\theta}$ and $J_{s\phi}$ are expressed in terms of the discontinuity in the magnetic scalar potential $V_m^> - V_m^<$ as

$$J_{s\theta} = \frac{1}{a\ \sin\theta} \frac{\partial}{\partial\phi} (V_m^> - V_m^<) \qquad (A.8)$$

$$J_{s\phi} = -\frac{1}{a} \frac{\partial}{\partial\theta} (V_m^> - V_m^<) \qquad (A.9)$$

so that Equation A.7 becomes

$$2\pi R_c \frac{\partial^2}{\partial\phi^2} (V_m^> - V_m^<) + R_s \sin\theta \frac{\partial}{\partial\theta} (V_m^> - V_m^<) = 0 \qquad (A.10)$$

at $\theta = \alpha+$. This is the required aperture-rim boundary condition on $V_m^> - V_m^<$.