C.1 N-C

Interaction Notes

Note 394

15 May 1980

QUASISTATIC INTERACTION BETWEEN A MAGNETIC DIPOLE AND A RESISTIVELY CAPPED CONDUCTING SPHERICAL SHELL, I: AXIAL MAGNETIC DIPOLE*

K. F. Casey
The Dikewood Corporation
1009 Bradbury Drive, S.E.
University Research Park
Albuquerque, New Mexico 87106

Abstract

The magnetic field inside a conducting spherical shell with a resistively loaded circular aperture is found when the structure is excited by an external magnetic dipole located above the aperture on the axis of symmetry. The dipole moment is oriented parallel to the symmetry axis. Both exact (numerical) and approximate (variational) results are obtained; it is found that the agreement between the approximate and the exact results is very good. Implications of the results as they pertain to the measurement of the properties of the resistive loading are discussed.

*The research reported in this Note was performed under Subcontract No. SC-0082-79-0005 with Mission Research Corporation, under Contract No. F29601-78-C-0082 with the Air Force Weapons Laboratory.

I. INTRODUCTION

The problems to be discussed in this note and in the companion Part II deal with the interaction between a point source and a conducting body possessing a resistively loaded aperture. The frequency range is such that the body is electrically small, so that a quasistatic analysis is appropriate, and the source is taken to be a magnetic dipole. This choice is made because the magnetic-field penetration is the dominant effect in the low-frequency limit. In both Parts I and II of this note, the conducting body is taken to be a spherical shell with a circular aperture. The source is placed above the aperture on the symmetry axis of the structure. In Part I, the dipole moment is taken to be oriented parallel to this axis; in Part II, it is perpendicular to the axis.

The motivations for this problem are two: first, we wish to consider the effect of surface curvature on penetration of a loaded aperture. Penetration of an unloaded circular aperture in a spherical shell has been discussed in [1] and penetration of a loaded circular aperture in an infinite ground plane was considered in [2]. In these notes the combined problem of a loaded circular aperture in a curved surface is addressed.

The second motivation has to do with assessing a measurement technique for certain advanced composite materials [3-6]. It is our intent to investigate, by means of the canonical problems mentioned above, the relation between the fields at (or near) the center of the spherical shell with and without the resistive loading, and to determine how this relation is affected by such quantities as the opening angle of the loaded aperture, the position of the source dipole, and the contact resistance between the resistive loading and the conducting spherical shell.*

In the next section, the axial-dipole problem is formulated in terms of dual series equations, using the magnetic vector potential. These dual series equations are reduced in Section III to an inhomogeneous Fredholm integral equation of the second kind. Numerical and approximate variational solutions are obtained in Section IV and numerical data are discussed in Section V. Section VI concludes the note.

*The contact resistance is important only for the transverse dipole orientation and is considered in Part II.

II. FORMULATION

The geometry of the problem is shown in Figure 2.1. A perfectly conducting spherical shell of negligible thickness is located at $r = a$, $0 \leq \theta < \alpha$ in the spherical coordinates (r,θ,ϕ). The aperture in the shell is covered by a spherical segment of an infinitesimally thin conducting layer of sheet resistance R_s extending over the surface $r = a$, $\alpha < \theta \leq \pi$. The structure is excited by a magnetic dipole located outside the sphere at $r = r_o$, $\theta = \pi$ $(r_o > a)$. The axis of the magnetic dipole coincides with the symmetry axis of the aperture in the spherical shell. It is assumed that all field quantities vary with time as exp(st) and that the sphere is electrically small, so that a quasistatic analysis is appropriate. Furthermore, it is assumed that over the frequency range of interest, the loading in the aperture is electrically as well as physically thin so that the equivalent sheet impedance model is appropriate. It is easy to show in general that as the frequency is increased, the effect of the enclosure geometry becomes evident before that of the skin depth in the loading material; this point is elaborated in Section VI.

We shall represent the magnetic field $\overline{H}$ in terms of an azimuthal vector potential $\overline{A} = A_\phi \overline{a}_\phi$, where A_ϕ is independent of ϕ and

$$\begin{aligned}
\overline{H} &= \nabla \times A_\phi \overline{a}_\phi \\
H_r &= \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A_\phi) \\
H_\theta &= -\frac{1}{r} \frac{\partial}{\partial r} (r A_\phi) \\
H_\phi &= 0
\end{aligned} \tag{2.1}$$

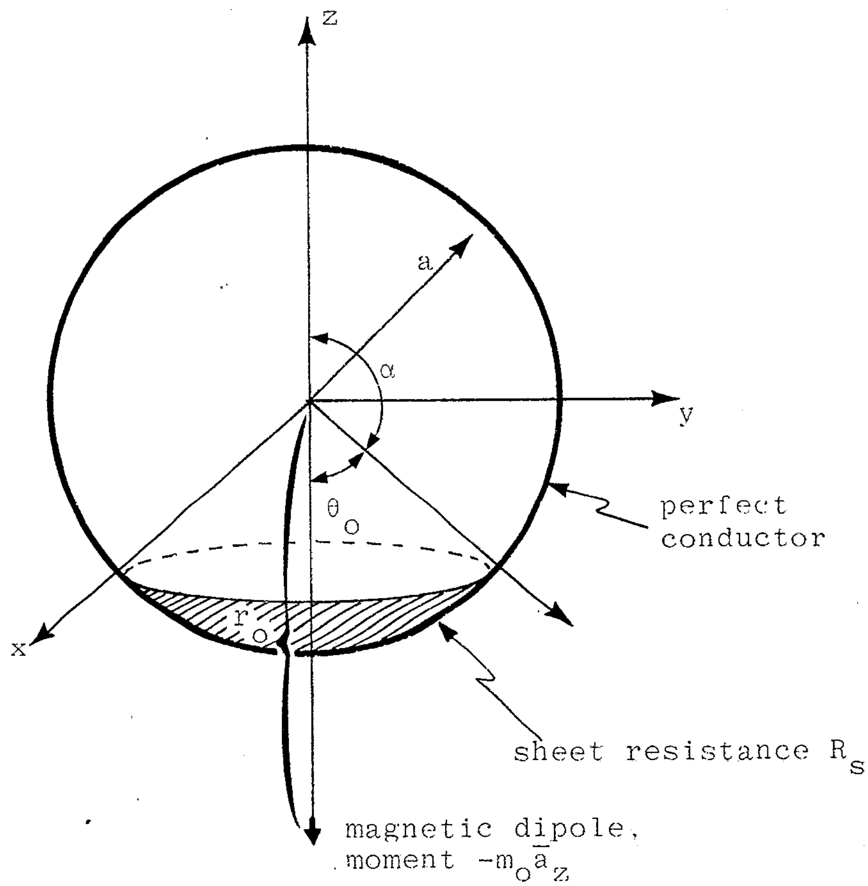

Figure 2.1. Geometry of the problem

Furthermore, for $r \neq a$ and $(r,\theta) \neq (r_o,\pi)$, A_ϕ satisfies

$$\nabla^2 A_\phi - \frac{A_\phi}{r^2 \sin^2\theta} = 0 \tag{2.2}$$

The vector potential A_ϕ is conveniently written as the sum of a "primary" and an "induced" contribution as

$$A_\phi = A_\phi^p + A_\phi^i \tag{2.3}$$

where

$$\begin{aligned}
r \le r_o: \quad A_\phi^p &= \frac{m_o}{4\pi r_o^2} \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{r}{r_o}\right)^n P_n^1(\cos\theta) \\
r \ge r_o: \quad A_\phi^p &= \frac{m_o}{4\pi r_o^2} \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{r}{r_o}\right)^{-n-1} P_n^1(\cos\theta) \\
r \le a: \quad A_\phi^i &= \frac{m_o}{4\pi r_o^2} \sum_{n=1}^{\infty} a_n \left(\frac{r}{a}\right)^n P_n^1(\cos\theta) \\
r \ge a: \quad A_\phi^i &= \frac{m_o}{4\pi r_o^2} \sum_{n=1}^{\infty} a_n \left(\frac{r}{a}\right)^{-n-1} P_n^1(\cos\theta)
\end{aligned} \tag{2.4}$$

$P_n^1(.)$ denotes the associated Legendre function of degree n and order 1 and m_o denotes the magnetic dipole moment of the source. The coefficients a_n are to be determined.

Over the perfectly conducting surface of the shell, the normal magnetic field component H_r must vanish. This can be guaranteed by forcing A_ϕ itself to vanish there, yielding the relation

$$\sum_{n=1}^{\infty} b_n P_n^1(\cos\theta) = 0 \qquad (0 \le \theta < \alpha) \tag{2.5}$$

where

$$b_n \equiv (-1)^{n-1} \left(\frac{a}{r_o}\right)^n + a_n \tag{2.6}$$

On the resistive cap, the normal component of the magnetic field must be continuous and [2,7]

$$\nabla_s \cdot (\overline{H}_{out} - \overline{H}_{in}) = \frac{-s\mu_o}{R_s} H_r \qquad (r = a,\ \alpha < \theta \leq \pi) \tag{2.7}$$

where $\nabla_s \cdot \overline{F}$ is, on a spherical surface of radius a,

$$\nabla_s \cdot \overline{F} = \frac{1}{a \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta\ F_\theta) + \frac{1}{a \sin\theta} \frac{\partial F_\phi}{\partial\phi} \tag{2.8}$$

Now substituting from eqs. (2.1) into (2.7) and using (2.8), we obtain the relation

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left[\sin\theta \left(\frac{\partial A_{\phi,out}}{\partial r} - \frac{\partial A_{\phi,in}}{\partial r} - \frac{s\mu_o}{R_s} A_\phi \right)\right] = 0 \tag{2.9}$$

$$(r = a,\ \alpha < \theta \leq \pi)$$

from which we conclude that

$$\left.\frac{\partial A_\phi}{\partial r}\right|_{r=a+} - \left.\frac{\partial A_\phi}{\partial r}\right|_{r=a-} = \frac{s\mu_o}{R_s} A_\phi \tag{2.10}$$

$$(r = a,\ \alpha < \theta \leq \pi)$$

Substitution of eq. (2.4) into eq. (2.10) yields the relation

$$\begin{aligned} \sum_{n=1}^{\infty} (2n + 1)\, b_n P_n^1(\cos\theta) &+ \frac{s\mu_o a}{R_s} \sum_{n=1}^{\infty} b_n P_n^1(\cos\theta) \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{a}{r_o}\right)^n (2n + 1)\, P_n^1(\cos\theta) \\ &\equiv G(\theta) \qquad (\alpha < \theta \leq \pi) \end{aligned} \tag{2.11}$$

Equations (2.5) and (2.11) constitute a set of dual series equations from which the coefficients b_n can be determined. We shall be most interested in the induced magnetic field near the center of the sphere; this quantity depends only on the coefficient b_1. Specifically, the ratio of the vector potential near the center of the sphere to that which would exist there if the sphere were absent is simply $(r_o/a)b_1$. The solution of the dual series equations is addressed in the next two sections of this note. In Section III, the dual series equations are reduced to a single inhomogeneous Fredholm equation of the second kind; and in Section IV, this Fredholm equation is used to obtain both exact (numerical) and approximate (variational) solutions for the coefficients b_n.

III. REDUCTION OF DUAL SERIES EQUATIONS TO AN INTEGRAL EQUATION

We begin by expressing the coefficients b_n in terms of an unknown function h(u) as [8]

$$b_n = \frac{-\sqrt{\pi}\ \Gamma(n)}{\Gamma(n + 1/2)} \int_0^{\pi} H(u - \alpha)h(u) \cos^3 \frac{u}{2} P_{n-1}^{(1/2,3/2)}(\cos u)\, du \tag{3.1}$$

in which $P_{n-1}^{(\alpha,\beta)}$ denotes a Jacobi polynomial and H is the unit step function. We define two functions $F_1(\theta)$ and $F_2(\theta)$ over the interval $0 \leq \theta \leq \pi$ as

$$\begin{aligned} F_1(\theta) &= \sum_{n=1}^{\infty} b_n P_n^1(\cos\theta) \\ F_2(\theta) &= \sum_{n=1}^{\infty} (2n + 1)\, b_n P_n^1(\cos\theta) \end{aligned} \tag{3.2}$$

so that the dual series equations (2.5) and (2.11) become respectively

$$\begin{aligned} &F_1(\theta) = 0 && (0 \leq \theta < \alpha) \\ &\frac{s\mu_o a}{R_s} F_1(\theta) + F_2(\theta) = G(\theta) && (\alpha < \theta \leq \pi) \end{aligned} \tag{3.3}$$

The function $F_1(\theta)$ is obtained by substitution of eq. (3.1) into eq. (3.2), reversal of the order of integration and summation, and use of the relations [9]

$$P_n^1(\cos\theta) = -\tfrac{1}{2}(n + 1) \sin\theta\ P_{n-1}^{(1,1)}(\cos\theta) \tag{3.4}$$

and

$$\sqrt{\frac{\pi}{2}}\,(\cos\frac{u}{2})^{2\beta+1}\sum_{n=0}^{\infty}\frac{n!\,\Gamma(\alpha+\beta+n+1)(2n+\alpha+\beta+1)}{\Gamma(\alpha+n+1)\Gamma(\beta+n+1/2)}\,\cdot$$

$$P_n^{(\alpha,\beta)}(\cos\theta)\,P_n^{(\alpha-\frac{1}{2},\beta+\frac{1}{2})}(\cos u) = (\sin\frac{\theta}{2})^{-2\alpha}\,\frac{H(\theta-u)}{\sqrt{\cos u-\cos\theta}} \tag{3.5}$$

We obtain

$$F_1(\theta) = \frac{1}{\sqrt{2}}\cot\frac{\theta}{2}\int_{\alpha}^{\pi} h(u)\,\frac{H(\theta-u)du}{\sqrt{\cos u-\cos\theta}} \tag{3.6}$$

which is indeed zero when $\theta < \alpha$, satisfying the first of eqs. (3.3).

In a similar fashion, using the relations

$$\frac{d}{du}\left[(\sin\frac{u}{2})^{2\alpha}\,P_n^{(\alpha,\beta)}(\cos u)\right] = \frac{1}{2}(n+\alpha)\sin u(\sin\frac{u}{2})^{2\alpha-2}\,\cdot$$

$$P_n^{(\alpha-1,\beta+1)}(\cos u) \tag{3.7}$$

and

$$\sqrt{\frac{\pi}{2}}\,(\sin\frac{u}{2})^{2\alpha+1}\sum_{n=0}^{\infty}\frac{n!\,\Gamma(\alpha+\beta+n+1)(2n+\alpha+\beta+1)}{\Gamma(\alpha+n+3/2)\Gamma(\beta+n+1)}\,\cdot$$

$$P_n^{(\alpha,\beta)}(\cos\theta)\,P_n^{(\alpha+\frac{1}{2},\beta-\frac{1}{2})}(\cos u) = (\cos\frac{\theta}{2})^{-2\beta}\,\frac{H(u-\theta)}{\sqrt{\cos\theta-\cos u}} \tag{3.8}$$

we find that

$$F_2(\theta) = \sqrt{2}\tan\frac{\theta}{2}\int_{\alpha}^{\pi} h(u)\cot^2\frac{u}{2}\,\frac{d}{du}\left[\frac{H(u-\theta)}{\sqrt{\cos\theta-\cos u}}\right]du$$

$$= -\frac{1}{\sqrt{2}}\sec^2\frac{\theta}{2}\,\frac{d}{d\theta}\int_{\alpha}^{\pi} h(u)\cot^2\frac{u}{2}\,\frac{H(u-\theta)\sin u\,du}{\sqrt{\cos\theta-\cos u}} \tag{3.9}$$

The second of eqs. (3.3) is thus written

$$- \frac{1}{\pi} \frac{d}{d\theta} \int_{\theta}^{\pi} \frac{h(u) \cot^2 \frac{u}{2} \sin u \, du}{\sqrt{\cos\theta - \cos u}} = \frac{\sqrt{2}}{\pi} \cos^2 \frac{\theta}{2} G(\theta)$$

$$- \cos^3 \frac{\theta}{2} \csc \frac{\theta}{2} \frac{s\mu_o a}{\pi R_s} \int_{\alpha}^{\theta} \frac{h(u) \, du}{\sqrt{\cos u - \cos\theta}} \qquad (3.10)$$

$$(\alpha < \theta \leq \pi)$$

Equation (3.10) is an integral equation for the unknown function h(u), but it is not in a particularly convenient form. To convert it into a more useful form, we make use of the integral equation/solution pair [10]

$$\int_{x}^{b} \frac{f(t) \, dt}{\sqrt{\cos x - \cos t}} = g(x) \qquad (0 \leq a < x < b \leq \pi)$$

$$f(t) = - \frac{1}{\pi} \frac{d}{dt} \int_{t}^{b} \frac{g(u) \sin u \, du}{\sqrt{\cos t - \cos u}} \qquad (a < t < b) \qquad (3.11)$$

to obtain

$$\cot \frac{x}{2} h(x) + \frac{s\mu_o a}{2\pi R_s} \int_{\alpha}^{\pi} \cot \frac{u}{2} h(u) K(x,u) \, du$$

$$= \frac{1}{\pi\sqrt{2}} \tan \frac{x}{2} \int_{x}^{\pi} \frac{\cot \frac{t}{2} G(t) \sin t \, dt}{\sqrt{\cos x - \cos t}} \qquad (3.12)$$

$$(\alpha < x \leq \pi)$$

The kernel K(x,u) is given by

$$K(x,u) = \tan \frac{x}{2} \tan \frac{u}{2} \int_{\max(x,u)}^{\pi} \frac{\cot^2 \frac{t}{2} \sin t \, dt}{\sqrt{\cos x - \cos t} \sqrt{\cos u - \cos t}}$$

$$= \tan \frac{x}{2} \tan \frac{u}{2} \ln \left| \frac{\cos \frac{u}{2} - \cos \frac{x}{2}}{\cos \frac{u}{2} + \cos \frac{x}{2}} \right|$$

$$- \sec \frac{x}{2} \sec \frac{u}{2} \ln \left| \frac{\sin \frac{1}{2}(x - u)}{\sin \frac{1}{2}(x + u)} \right| \quad (\alpha < x, \; u \leq \pi) \qquad (3.13)$$

We shall find it convenient in what follows to extend the domain of definition of $K(x,u)$ to $(0,\pi) \times (0,\pi)$.

Equation (3.12) is the integral equation which must be solved for $h(u)$ in order to evaluate the coefficients b_n. We shall take up the problem of determining the coefficients b_n by solving this integral equation in the next section.

IV. SOLUTION OF THE FREDHOLM EQUATION

The exact solution of the integral equation (3.12) is not known, so that numerical or variational techniques must be employed in order to solve it. Both numerical and variational approaches are described in this section.

4.1 Numerical solution for coefficients b_n

Equation (3.1) for the coefficients b_n can be "inverted," using the orthogonality property of the Jacobi polynomials [9]

$$\int_o^\pi \left(\sin \frac{x}{2}\right)^{2\alpha} \left(\cos \frac{x}{2}\right)^{2\beta} P_{m-1}^{(\alpha,\beta)}(\cos x)\, P_{n-1}^{(\alpha,\beta)}(\cos x) \sin x \, dx$$

$$= \delta_{mn} \frac{2\Gamma(n+\alpha)\,\Gamma(n+\beta)}{\Gamma(n)(\alpha+\beta+2n-1)\,\Gamma(n+\alpha+\beta)} \tag{4.1}$$

in which δ_{mn} is the Kronecker delta-function. We obtain

$$h(u) \cot \frac{u}{2} H(u-\alpha) = -\frac{1}{\sqrt{\pi}} \cos \frac{u}{2} \sin u \sum_{n=1}^{\infty} \frac{\Gamma(n+2)}{\Gamma(n+1/2)} \cdot b_n P_{n-1}^{(1/2,3/2)}(\cos u) \tag{4.2}$$

Now multiply the integral equation (3.12) through by the step function $H(x-\alpha)$ and substitute equation (4.2) as appropriate. There results

$$\cos \frac{x}{2} \sin x \sum_{n=1}^{\infty} \frac{\Gamma(n+2)}{\Gamma(n+1/2)} b_n P_{n-1}^{(1/2,3/2)}(\cos x) + \frac{s\mu_o a}{2\pi R_s} H(x-\alpha) \cdot \sum_{n=1}^{\infty} \frac{\Gamma(n+2)}{\Gamma(n+1/2)} b_n \int_o^\pi P_{n-1}^{(1/2,3/2)}(\cos u) K(x,u) \cos \frac{u}{2} \sin u \, du$$

$$= \frac{-1}{\sqrt{2\pi}} H(x-\alpha) \tan \frac{x}{2} \int_x^\pi \frac{\cot \frac{t}{2} G(t) \sin t \, dt}{\sqrt{\cos x - \cos t}} \qquad (0 \le x \le \pi) \tag{4.3}$$

in which the order of summation and integration in the second term has been reversed.

The integral on the left-hand side in eq. (4.3) can be evaluated by writing K(x,u) in its integral form and making use of the relations [9]

$$\int_0^u \left(\sin\frac{\theta}{2}\right)^{2\alpha} P_{n-1}^{(\alpha,\beta)}(\cos\theta)\,\frac{\sin\theta\, d\theta}{\sqrt{\cos\theta - \cos u}}$$

$$= \frac{2^{-\alpha}\pi^{\frac{1}{2}}\Gamma(n+\alpha)}{\Gamma(n+\alpha+1/2)}\,(1-\cos u)^{\alpha+\frac{1}{2}}\, P_{n-1}^{(\alpha+\frac{1}{2},\beta-\frac{1}{2})}(\cos u) \qquad (4.4)$$

and

$$\int_v^\pi \left(\cos\frac{x}{2}\right)^{2\beta} P_{n-1}^{(\alpha,\beta)}(\cos x)\,\frac{\sin x\, dx}{\sqrt{\cos v - \cos x}}$$

$$= \frac{2^{-\beta}\pi^{\frac{1}{2}}\Gamma(n+\beta)}{\Gamma(n+\beta+1/2)}\,(1+\cos v)^{\beta+\frac{1}{2}}\, P_{n-1}^{(\alpha-\frac{1}{2},\beta+\frac{1}{2})}(\cos v) \qquad (4.5)$$

We find that

$$\int_0^\pi P_{n-1}^{(1/2,3/2)}(\cos u)\, K(x,u)\,\cos\frac{u}{2}\,\sin u\, du$$

$$= \frac{\pi}{n+1/2}\,\cos\frac{x}{2}\,\sin x\, P_{n-1}^{(1/2,3/2)}(\cos x) \qquad (4.6)$$

and eq. (4.3) thus becomes

$$\cos\frac{x}{2}\,\sin x \sum_{n=1}^{\infty} \frac{\Gamma(n+2)}{\Gamma(n+1/2)}\, b_n P_{n-1}^{(1/2,3/2)}(\cos x)$$

$$+ \frac{s\mu_o a}{2R_s}\, H(x-\alpha)\,\cos\frac{x}{2}\,\sin x \sum_{n=1}^{\infty} \frac{\Gamma(n+2)}{\Gamma(n+3/2)}\, b_n P_{n-1}^{(1/2,3/2)}(\cos x)$$

$$= \frac{-1}{\sqrt{2\pi}}\, H(x-\alpha)\,\tan\frac{x}{2} \int_x^\pi \frac{\cot\frac{t}{2}\, G(t)\,\sin t\, dt}{\sqrt{\cos x - \cos t}} \qquad (4.7)$$

$$(0 \le x \le \pi)$$

If we now multiply eq. (4.7) through by the factor

$$\sin \frac{x}{2} \cos^2 \frac{x}{2} P_{n-1}^{(1/2,3/2)}(\cos x)$$

and integrate with respect to x over the interval $0 \leq x \leq \pi$, making use of the orthogonality property in eq. (4.1), there results a system of linear equations for the coefficients b_n, viz.

$$\tilde{b}_m + \frac{s\mu_o a}{3R_s} \sum_{n=1}^{\infty} \tilde{b}_n Q_{mn} = \tilde{b}_m^{(\infty)} \qquad (m \geq 1) \tag{4.8}$$

where

$$\tilde{b}_m \equiv \frac{r_o}{a} \sqrt{\frac{3}{2}} \left[\frac{m(m+1)}{2m+1}\right]^{\frac{1}{2}} b_m \tag{4.9}$$

$$\tilde{b}_m^{(\infty)} = \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \left(-\frac{a}{r_o}\right)^{n-1} \left[n(n+1)(n+\tfrac{1}{2})\right]^{\frac{1}{2}} Q_{mn}$$

$$Q_{mn} = \frac{3}{2} \left[\frac{m+1}{m(m+\frac{1}{2})}\right]^{\frac{1}{2}} \left[\frac{n+1}{n(n+\frac{1}{2})}\right]^{\frac{1}{2}} \frac{\Gamma(m+1)}{\Gamma(m+\frac{1}{2})} \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \cdot$$

$$\int_{\alpha}^{\pi} \sin \frac{x}{2} \cos^3 \frac{x}{2} P_{m-1}^{(1/2,3/2)}(\cos x)\, P_{n-1}^{(1/2,3/2)}(\cos x) \cdot \sin x \, dx \tag{4.10}$$

$$= \frac{-3}{2\pi} \left[\frac{1}{m(m+\frac{1}{2})(m+1)n(n+\frac{1}{2})(n+1)}\right]^{\frac{1}{2}} \cdot \left\{\frac{1}{2} \tan \frac{\alpha}{2} \left[\cos(m-n)\alpha + \cos(m+n+1)\alpha\right] + \left(mn + \frac{m+n}{2}\right) \frac{\sin(m-n)\alpha}{m-n} - \left(mn + \frac{m+n+1}{2}\right) \cdot \frac{\sin(m+n+1)\alpha}{m+n+1}\right\} \qquad (m \neq n)$$

$$Q_{mm} = \frac{3}{2\pi m \; (m + \frac{1}{2})(m + 1)} \left\{ m(m + 1)(\pi - \alpha) \right.$$

$$\left. + \; (m^2 + m + \frac{1}{2}) \frac{\sin(2m+1)\alpha}{2m + 1} - \frac{1}{2} \tan \frac{\alpha}{2} \left[1 + \cos(2m+1)\alpha\right] \right\}$$

It will be noted that

$$\tilde{b}_1 = \frac{r_o}{a} b_1 \tag{4.11}$$

(cf. discussion following eq. (2.11)). Furthermore, the coefficients $b_m^{(\infty)}$ are those appropriate for an open or unloaded aperture ($R_s = \infty$).

We have reduced the problem of evaluating the coefficients b_n to that of solving the system of equations (4.8). The coefficients so obtained can be considered exact, since they can be calculated to essentially arbitrary accuracy on a digital computer. Numerical results will be considered in Section V.

4.2 Variational expressions for coefficients b_n

It is well known that if K(x,u) is symmetric and if

$$\begin{aligned} f(x) + \lambda \int_\alpha^\pi f(u)\, K(x,u)\, du &= g(x) \qquad (\alpha \le x \le \pi) \\ f_a(x) + \lambda \int_\alpha^\pi f_a(u)\, K(x,u)\, du &= h_a(x) \qquad (\alpha \le x \le \pi) \end{aligned} \tag{4.12}$$

Then a variational expression for the quantity

$$I = \int_\alpha^\pi f(x)\, h_a(x)\, dx \tag{4.13}$$

is [11]

$$I = \frac{\int_\alpha^\pi f(x)h_a(x)\,dx \int_\alpha^\pi f_a(x)g(x)\,dx}{\int_\alpha^\pi f(x)f_a(x)dx + \lambda \int_\alpha^\pi\int_\alpha^\pi f(x)f_a(u)K(x,u)\,dxdu} \tag{4.14}$$

Identifying K(x,u) with the kernel given in eq. (3.13), setting $\lambda = s\mu_o a/2\pi R_s$ and

$$f(x) = h(x) \cot \frac{x}{2} H(x - \alpha)$$

$$g(x) = \frac{1}{\pi\sqrt{2}} H(x - \alpha) \tan \frac{x}{2} \int_x^\pi \frac{\cot \frac{t}{2} G(t) \sin t\, dt}{\sqrt{\cos x - \cos t}} \tag{4.15}$$

$$h_a(x) = \frac{-\sqrt{\pi}\,\Gamma(n)}{2\Gamma(n + \frac{1}{2})} \cos \frac{x}{2} \sin x\, P_{n-1}^{(1/2,\ 3/2)} (\cos x)$$

and using as a trial solution

$$f(x) \simeq g(x)$$

$$f_a(x) \simeq h_a(x) \tag{4.16}$$

which is exact in the limit $R_s \to \infty$, we obtain after some manipulation the following variational formula for the coefficients $\tilde{b}_n$:

$$\tilde{b}_n \simeq \frac{\tilde{b}_n^{(\infty)}}{1 + s\tau_o F_n} \tag{4.17}$$

in which

$$\tau_o = \frac{\mu_o a}{3R_s} \tag{4.18}$$

$$F_n = \frac{1}{\tilde{b}_n^{(\infty)}} \sum_{m=1}^{\infty} Q_{mn} \tilde{b}_m^{(\infty)} \tag{4.19}$$

In the special case $n = 1$, $a/r_o = 0$, we have

$$F_1 = \frac{1}{Q_{11}} \sum_{m=1}^{\infty} Q_{m1}^2 \tag{4.20}$$

Curves of the "time-constant factor" F_1 as a function of $\theta_o = \pi - \alpha$ for three values of a/r_o are shown in Figure 4.1. Numerical values of F_1 are also given in Table I. As one might expect, F_1 increases essentially linearly with θ_o for "moderate" values of θ_o [2]. When $a/r_o = 0$, $\theta_o \leq 90^\circ$,

$$F_1 \simeq \frac{5\theta_o}{4\pi} \tag{4.21}$$

If we interpret the quantity $\mu_o a/3$ as the equivalent inductance of the spherical cavity, then the equivalent resistance of the cap is

$$R_{eq} = \frac{R_s}{F_1} \simeq \frac{4\pi R_s}{5\theta_o} \qquad (\theta_o \lesssim 90^\circ)$$

so that

$$\tau_1 = \tau_o F_1 = \frac{L_{eq}}{R_{eq}}$$

where $L_{eq} = \mu_o a/3 = \mu_o V/S$, in which V and S are respectively the volume and the surface area of the spherical shell.

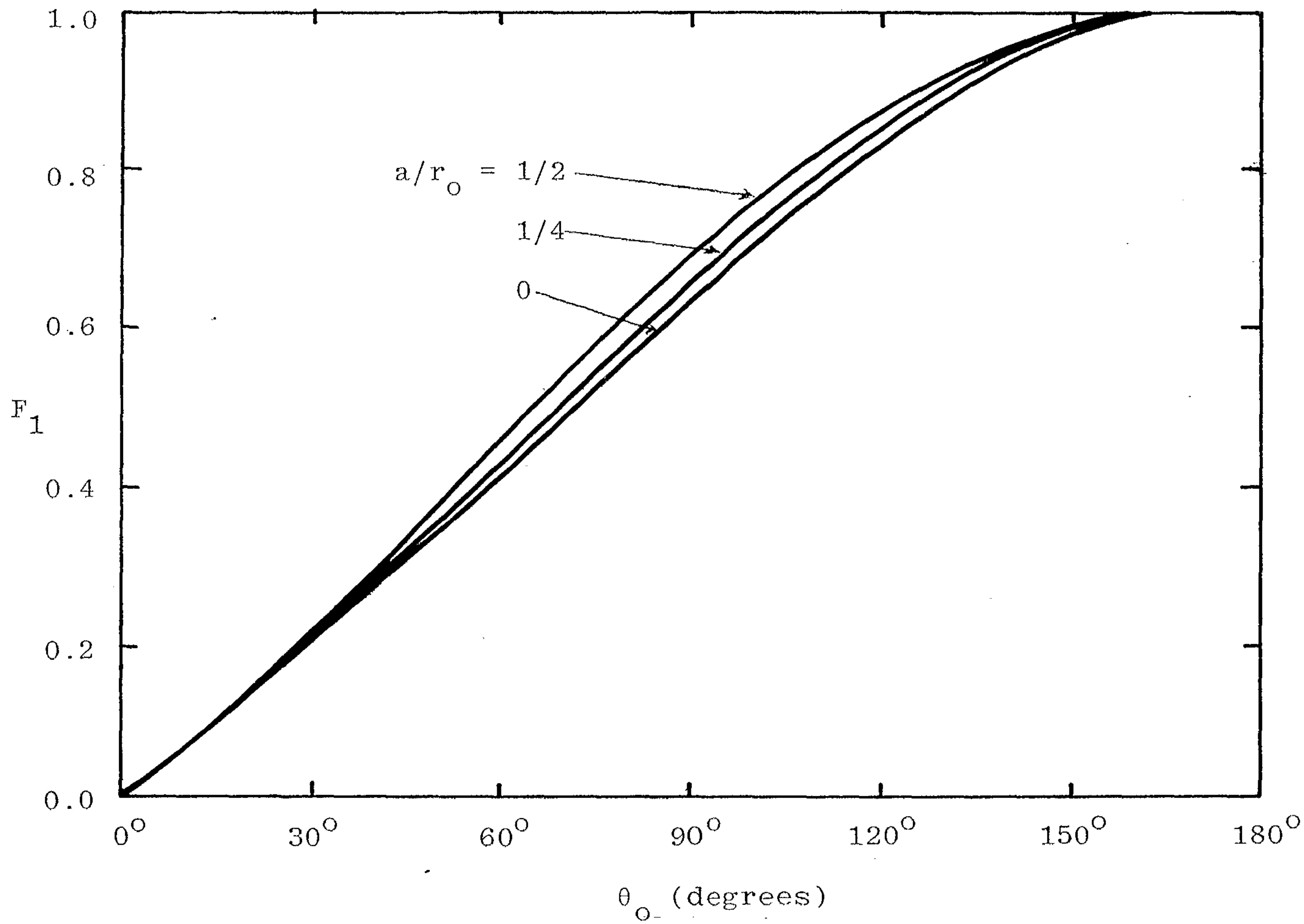

Figure 4.1. Time-constant factor F_1 vs. θ_o; a/r_o = 0, 1/4, 1/2

Table 1. F_1 vs. θ_o ($0^o \leq \theta_o \leq 90^o$) for a/r_o = 0.00, 0.25, 0.50

θ_o	$a/r_o = 0$	$a/r_o = 0.25$	$a/r_o = 0.50$
0^o	0.0000	0.0000	0.0000
5^o	0.0345	0.0345	0.0346
10^o	0.0692	0.0693	0.0696
15^o	0.1036	0.1039	0.1055
20^o	0.1390	0.1397	0.1418
25^o	0.1736	0.1749	0.1786
30^o	0.2086	0.2108	0.2168
35^o	0.2433	0.2466	0.2552
40^o	0.2784	0.2831	0.2944
45^o	0.3135	0.3198	0.3341
50^o	0.3485	0.3567	0.3739
55^o	0.3834	0.3937	0.4139
60^o	0.4186	0.4311	0.4539
65^o	0.4531	0.4681	0.4936
70^o	0.4868	0.5044	0.5321
75^o	0.5326	0.5430	0.5720
80^o	0.5583	0.5798	0.6101
85^o	0.5923	0.6160	0.6472
90^o	0.6274	0.6525	0.6838

V. NUMERICAL RESULTS

In this section we present curves of $b_1^{(\infty)}$ as a function of $\theta_o = \pi - \alpha$ and of a/r_o, and compare the exact and variational solutions for the quantity $\tilde{b}_1/\tilde{b}_1^{(\infty)}$.

In Figure 5.1 are shown curves of $\tilde{b}_1^{(\infty)}$ as a function of θ_o for $a/r_o = 0$, 1/4, and 1/2. Figure 5.2 shows curves of $\tilde{b}_1^{(\infty)}$ as a function of a/r_o for $\theta_o = 30^o$, 60^o, and 90^o. It is evident from these plots that, as one would certainly expect, $\tilde{b}_1^{(\infty)}$ is an increasing function of both θ_o and a/r_o.

Extensive computations of $\tilde{b}_1/\tilde{b}_1^{(\infty)}$ have been carried out for various values of θ_o and a/r_o, as a function of normalized frequency $\omega\tau_o$. For $\theta_o \leq 90^o$ and $a/r_o \leq 1/2$, the dependence of $\tilde{b}_1/\tilde{b}_1^{(\infty)}$ on a/r_o is very weak; this is to be expected from the results shown in Figure 4.1. Figure 5.3 shows curves of $|\tilde{b}_1/\tilde{b}_1^{(\infty)}|$ vs. $\omega\tau_o$ for $a/r_o = 0$ and various values of θ_o. Both the exact and the variationally derived results are shown and it will be noted that the agreement between these is quite good, indicating the utility of the variational results.

An approximate expression for $\tilde{b}_1$, useful for engineering purposes, is

$$\tilde{b}_1 \simeq \tilde{b}_1^{(\infty)}(\theta_o, \frac{a}{r_o})(1 + s\tau_1)^{-1} \tag{5.1}$$

in which

$$\tau_1 \simeq \frac{5\mu_o a\theta_o}{12\pi R_s} = \frac{L_{eq}}{R_{eq}} \tag{5.2}$$

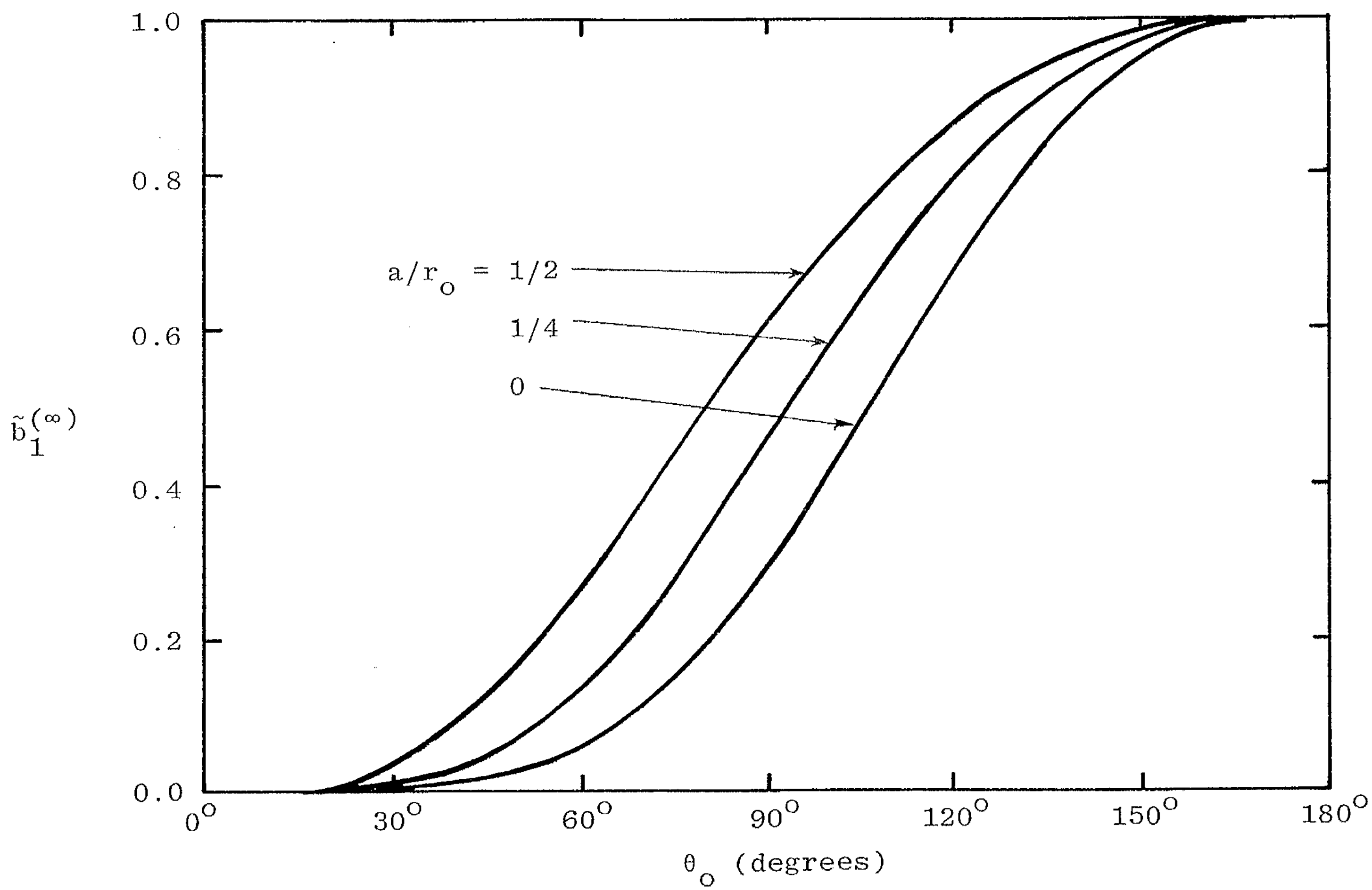

Figure 5.1. $\tilde{b}_1^{(\infty)}$ vs. θ_o for $a/r_o = 0, 1/4, 1/2$

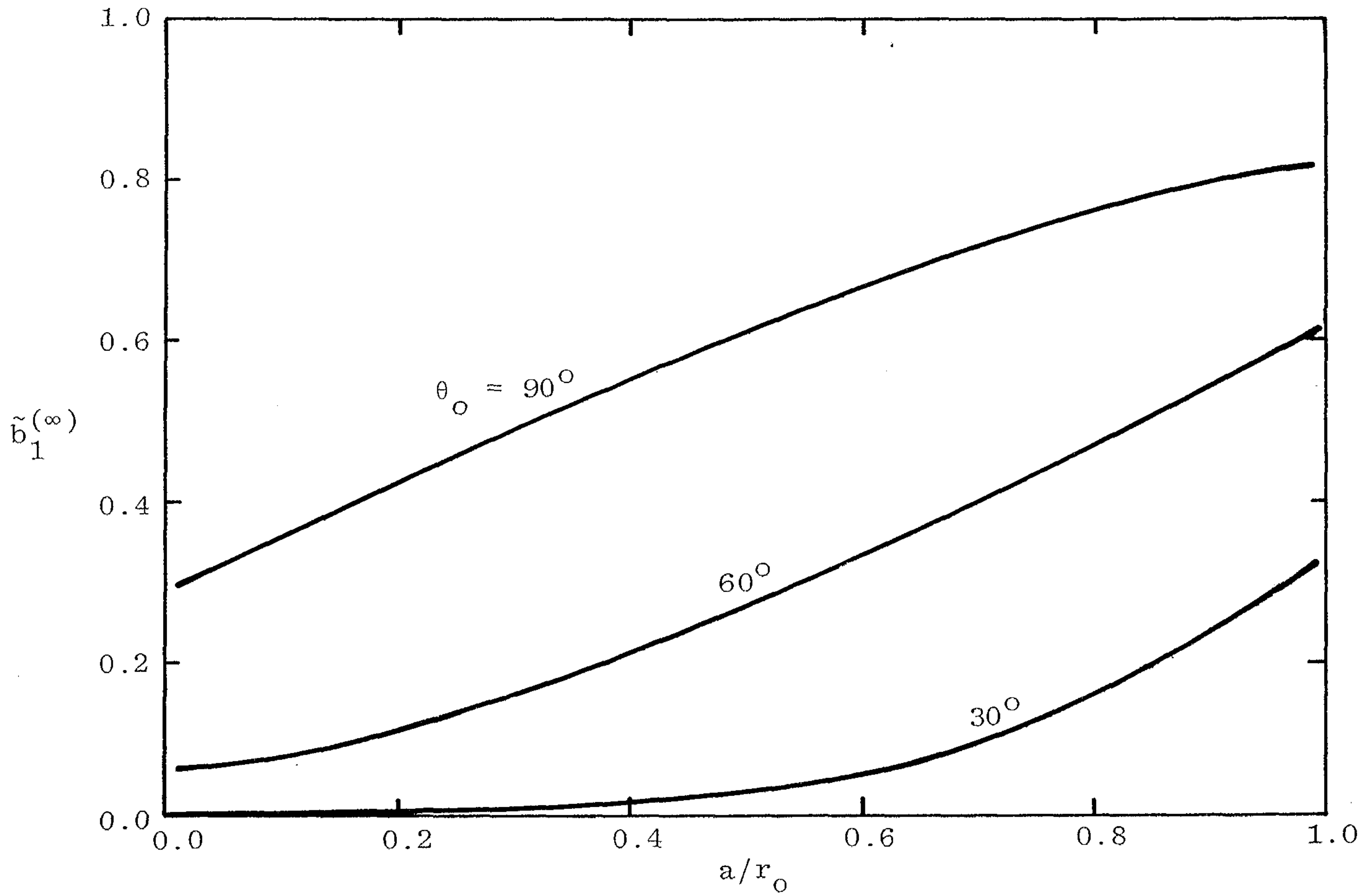

Figure 5.2. $\tilde{b}_1^{(\infty)}$ vs. a/r_o for $\theta_o = 30^\circ, 60^\circ, 90^\circ$

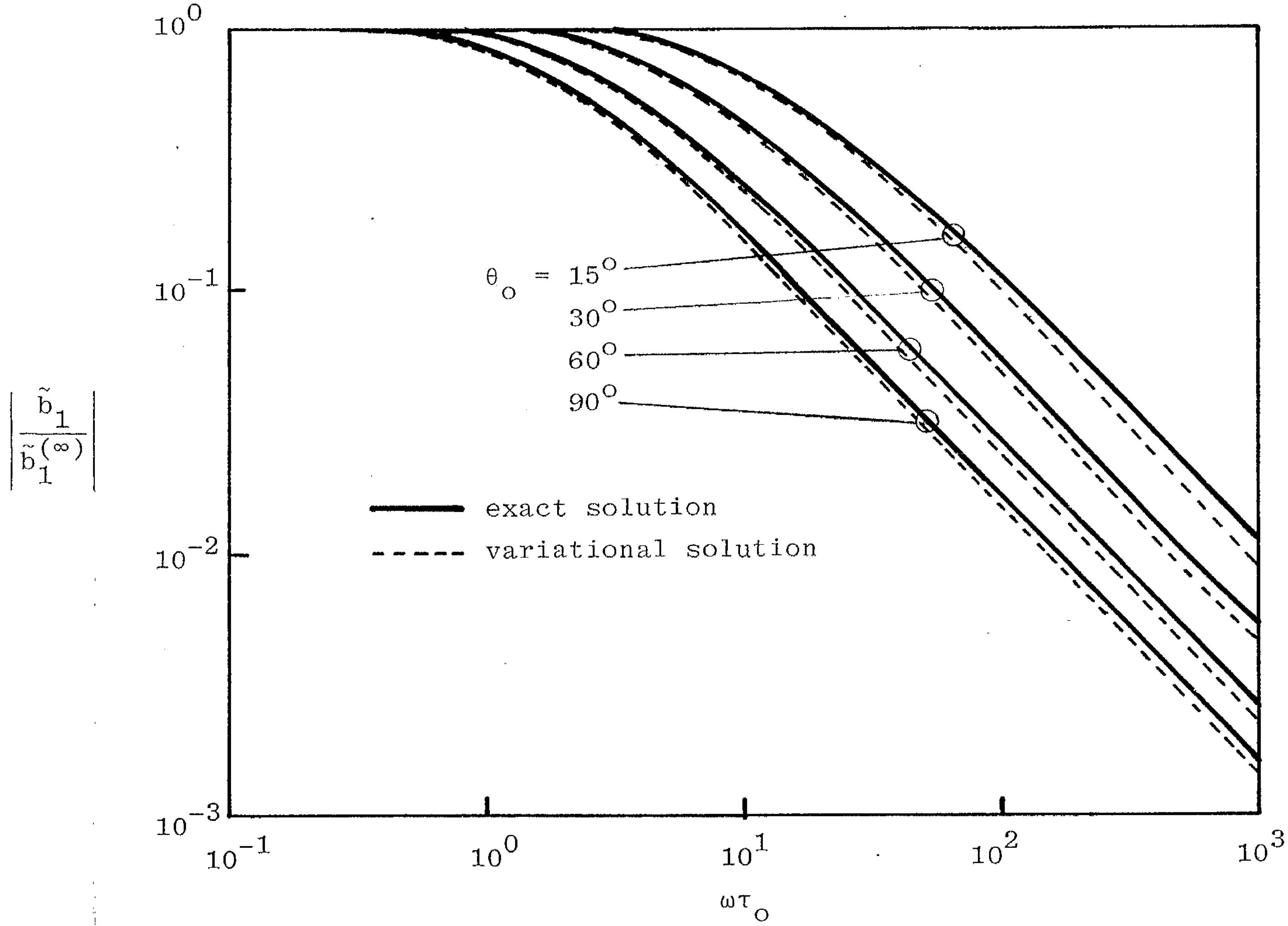

Figure 5.3. Magnitude of $\tilde{b}_1/\tilde{b}_1^{(\infty)}$ vs. $\omega\tau_o$ for $a/r_o = 0$, $\theta_o = 15^o, 30^o, 60^o, 90^o$

and $\theta_o \lesssim 90^o$, $a/r_o \lesssim 1/2$. This approximate result indicates that the time constant τ_1 is essentially independent of a/r_o when $a/r_o \lesssim 1/2$; this is to be expected because unless the source dipole is very close to the surface of the sphere, the current distribution on the resistive cap is essentially that for the case $a/r_o = 0$.

VI. CONCLUDING REMARKS

In this note we have formulated and solved, both numerically and by variational means, the problem of the interaction between an axial magnetic dipole and a spherical shell with a resistively loaded aperture. It has been shown that the effect of the shell on the magnetic field at its center can be approximately factored into two parts, one of which depends only on the sheet resistance of the loading and the aperture angle θ_o, and the other only upon the angle θ_o and the source distance factor a/r_o.

It has been assumed in the analysis that the resistive sheet is electrically thin. This assumption is valid only for frequencies below that at which the diffusion time constant for the loading,

$$\tau_d = \mu_o \sigma d^2 \tag{6.1}$$

is such that $\omega\tau_d < 1$. In equation (6.1), σ and d denote respectively the conductivity and the thickness of the loading material. When $\omega\tau_d = 1$, the normalized frequency $\omega\tau_1$ is given approximately by

$$\omega\tau_1\Big|_{\omega\tau_d=1} \approx \frac{5\theta_o a}{12\pi d} \tag{6.2}$$

from which we conclude that, since $a/d \gg 1$ in any realistic situation, $\omega\tau_1 \gg 1$ when $\omega\tau_d = 1$ except for very small aperture angles θ_o. Thus the macroscopic enclosure effect is manifest at frequencies far below that at which skin-depth effects become important.

The determination of the properties of the aperture loading material via frequency-domain measurements involves primarily the experimental determination of the critical frequency $\omega_c = 1/\tau_1$. Having found this frequency by measurement, one could then use Figure 4.1 and Table 1 to determine R_s knowing θ_o, a/r_o, and the sphere radius a. This issue will be discussed further in Part II.

REFERENCES

1. K. F. Casey, "Static Electric and Magnetic Field Penetration of a Spherical Shield Through a Circular Aperture," Interaction Notes, Note 381, February 1980.

2. K. F. Casey, "Quasistatic Electromagnetic Penetration of a Mesh-Loaded Circular Aperture," Interaction Notes, Note 387, March 1980.

3. K. F. Casey, "EMP Penetration Through Advanced Composite Skin Panels," Interaction Notes, Note 315, December 1976.

4. K. F. Casey, "Electromagnetic Shielding by Advanced Composite Materials," Interaction Notes, Note 341, January 1978.

5. C. D. Skouby, "Electromagnetic Effects of Advanced Composites," Report No. MDC-FR-75-1, McDonnell-Douglas Corporation, January 1975.

6. "Attenuation Measurements for Enclosures, Electromagnetic Shielding, for Electronic Test Purposes, Methods of," MIL-STD-285, June 1956.

7. R. W. Latham and K.S.H. Lee, "Magnetic-Field Leakage into a Semi-Infinite Pipe," Can. J. Phys., Vol. 46, pp. 1455-1462, 1968, and also Interaction Notes, Note 10, August 1967.

8. I. N. Sneddon, Mixed Boundary Value Problems in Potential Theory, North-Holland, Amsterdam (1966), pp. 173 ff.

9. ibid., pp. 54 ff.

10. ibid., pp. 40-42.

11. R. F. Harrington, Field Computation by Moment Methods, MacMillan, New York (1968), pp. 18, 19.