Interaction Notes

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QUASISTATIC ELECTROMAGNETIC PENETRATION OF A MESH-LOADED CIRCULAR APERTURE

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Abstract

In this note are described the results of an investigation concerning the low-frequency electromagnetic penetration of an impedance-loaded circular aperture in a perfectly conducting ground plane. The loading impedance is that representing a bonded-junction wire mesh; a special case of this loading impedance is that which represents a purely resistive sheet impedance. The existence of a non-zero contact resistance between the aperture loading and the rim of the aperture is also included in the formulation. Both the relevant magnetostatic and electrostatic boundary value problems are formulated as dual integral equations. It is shown that the solution of each of these sets of dual integral equations can be expressed in terms of the solution of a single Fredholm integral equation of the second kind, to which both numerical and variational solutions are obtained. The effectiveness of the loading in reducing the field penetration through the aperture can be described in an approximate but accurate way with simple formulas and equivalent circuits.

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I. INTRODUCTION

The problem of determining the electromagnetic penetration through a loaded aperture in a conducting plane surface has not received a great deal of attention, although the problem is one of much practical interest for shielding applications. Latham and Lee developed quasi-static magnetic boundary conditions for a resistive sheet [1,2], and Baum and Singaraju [3] have considered Babinet's principle as it applies to problems of this type. However, the aperture-penetration problem seems not to have been considered.

When the aperture loading is a bonded-junction wire mesh screen, the sheet impedance is actually a dyadic operator. Furthermore, in contrast to purely resistive loading, the mesh allows electric-field penetration in the dc limit. These features add complication and interest to the aperture-penetration problem. In this note we describe the mesh boundary conditions to be applied in the low-frequency limit and consider the canonical problems of quasi-static electric and magnetic penetration of a mesh-loaded circular aperture in an infinite perfectly conducting plane. This particular geometry is analytically tractable and the results are indicative of those to be expected when more general aperture shapes are considered.

The principal results to be obtained are those which describe the effect of the aperture loading upon its equivalent "imaged" electric and magnetic polarizabilities and upon the penetrant flux.

For ease of practical application, these results are expressed in terms of simple equivalent circuits and variationally derived approximate formulas.

In the next section of this note we discuss the quasi-static boundary conditions appropriate for a bonded-junction wire mesh. These boundary conditions are then used in Sections III and IV to formulate the relevant magnetic field (Section III) and electric field (Section IV) interaction problems. Both of these problems are shown to reduce to that of solving a certain integral equation, which is discussed in Section V. A numerical solution technique is outlined and variational expressions are obtained for the dipole moments and penetrant fluxes. Equivalent circuits based on these variational expressions are given. Numerical results are discussed in Section VI and the exact and variational solutions are compared; the agreement is found to be very good. The results of the study are summarized in Section VII, which concludes the note.

II. QUASI-STATIC BOUNDARY CONDITIONS FOR A BONDED-JUNCTION MESH SCREEN

A wire mesh screen with bonded junctions can be described electromagnetically by an equivalent sheet impedance operator $\overline{\overline{Z}}_s$ when the mesh dimensions are small in comparison to the wavelength. The operator $\overline{\overline{Z}}_s$ relates the space-averaged tangential electric field to the space-averaged surface current density on the screen, viz.

$$(\overline{\overline{I}} - \overline{n}\overline{n}) \cdot \overline{E}_s = \overline{\overline{Z}}_s \cdot \overline{J}_s \tag{2.1}$$

in which $\overline{\overline{I}}$ denotes the unit dyad and $\overline{n}$ is the unit vector normal to the screen surface. $\overline{E}_s$ and $\overline{J}_s$ are respectively the electric field and surface current density in the screen surface, averaged over a single mesh. The tangential electric field is assumed to be continuous through the screen surface.

When the meshes are square, the equivalent sheet impedance is [4] (the time dependence exp(st) is assumed)

$$\overline{\overline{Z}}_s = (Z_w' a_s + sL_s)(\overline{\overline{I}} - \overline{n}\overline{n}) - \frac{a_s^2}{sC_s} \nabla_s \nabla_s \tag{2.2}$$

in which a_s denotes the mesh size, Z_w' is the internal impedance per unit length of the mesh wires, and ∇_s denotes the "surface" del operator. The parameters L_s and C_s are given by

$$L_s = \frac{\mu_o a_s}{2\pi} \ln\left(1 - e^{-2\pi r_w/a_s}\right)^{-1}$$

$$\frac{1}{C_s} = \frac{1}{2\pi\varepsilon_o(\varepsilon_{r1} + \varepsilon_{r2})a_s} \ln\left(1 - e^{-2\pi r_w/a_s}\right)^{-1} \qquad (2.3)$$

where r_w is the radius of the mesh wires, μ_o and ε_o are respectively the permeability and permittivity of free space, and ε_{r1} and ε_{r2} denote the relative permittivities of the media on either side of the screen.* The internal impedance per unit length Z'_w is

$$Z'_w = R'_w \frac{\sqrt{s\tau_w}\, I_o(\sqrt{s\tau_w})}{2I_1(\sqrt{s\tau_w})} \qquad (2.4)$$

in which R'_w is the dc resistance per unit length of the mesh wires, τ_w is the diffusion time constant of the wire material, and I_ν denotes the modified Bessel function of the first kind. Denoting by μ_w and σ_w the permeability and conductivity of the wire material, we have

$$R'_w = \frac{1}{\pi r_w^2 \sigma_w}$$

$$\tau_w = \mu_w \sigma_w r_w^2 \qquad (2.5)$$

Substitution of eq. (2.2) into eq. (2.1) and use of the continuity equation

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*It is assumed that the thicknesses of the dielectrics on either side of the mesh are at least $a_s/2$ [4].

$$\nabla_s \cdot \overline{J}_s + s\rho_s = 0 \tag{2.6}$$

yields

$$(\overline{\overline{I}} - \overline{n}\overline{n}) \cdot \overline{E}_s = Z_s \overline{J}_s + \frac{a_s^2}{C_s} \nabla_s \rho_s \tag{2.7}$$

where we have introduced the abbreviated notation

$$Z_s \equiv Z_w' a_s + sL_s \tag{2.8}$$

Eq. (2.7) can be rearranged as

$$\overline{J}_s = \frac{1}{Z_s}(\overline{\overline{I}} - \overline{n}\overline{n}) \cdot \overline{E}_s - \frac{a_s^2}{Z_s C_s} \nabla_s \rho_s \tag{2.9}$$

which may be interpreted as indicating that the surface current density contains a "drift" component $Z_s^{-1}(\overline{\overline{I}} - \overline{n}\overline{n}) \cdot \overline{E}_s$ and a "diffusion" component $a_s^2 Z_s^{-1} C_s^{-1} \nabla_s \rho_s$ resulting from the gradient of the surface charge density.

<u>Quasi-static magnetic boundary conditions</u>

A boundary condition involving only magnetic fields may be obtained from eq. (2.7) by taking the surface curl of both sides and using the result that

$$\overline{n} \cdot \nabla_s \times E_s = -s\mu_o \overline{n} \cdot \overline{H} \tag{2.10}$$

which is a consequence of Faraday's law. We find

$$\overline{n} \cdot \nabla_s \times \overline{J}_s = \frac{-s\mu_o}{Z_s} \overline{n} \cdot \overline{H} \tag{2.11}$$

or, equivalently,

$$\overline{n} \cdot \nabla_s \times \left[\overline{n} \times (\overline{H}_2 - \overline{H}_1)\right] = \frac{-s\mu_o}{Z_s} \overline{n} \cdot \overline{H} \tag{2.12}$$

from which it is easy to show that

$$\nabla_s \cdot (\overline{H}_2 - \overline{H}_1) = \frac{-s\mu_o}{Z_s} \overline{n} \cdot \overline{H} \tag{2.13}$$

$\overline{H}_2$ and $\overline{H}_1$ denote the magnetic fields on either side of the screen surface and $\overline{n}$ is taken to be positive in the direction from the "1" side to the "2" side. Furthermore, $\overline{n} \cdot \overline{H}$ is continuous through the screen surface.

In the quasi-static limit, the magnetic field in a current-free region can be calculated from the magnetic scalar potential V_m via

$$\overline{H} = -\nabla V_m \tag{2.14}$$

so that eq. (2.13) is written in terms of V_m as

$$\nabla_s^2 (V_{m2} - V_{m1}) = \frac{-s\mu_o}{Z_s} \frac{\partial V_m}{\partial n} \tag{2.15}$$

This result, together with the requirement that

$$\frac{\partial V_{m2}}{\partial n} = \frac{\partial V_{m1}}{\partial n} = \frac{\partial V_m}{\partial n} \tag{2.16}$$

at the mesh surface, constitutes the quasi-static magnetic boundary condition for the bonded mesh. A related result applicable to the case in which the sheet impedance is purely resistive has been derived by Latham and Lee [1].

It is important to note that the result in eq. (2.15) is a necessary but not sufficient boundary condition. As a consequence, an additional condition on $V_{m2} - V_{m1}$ is required in order to specify this quantity completely. This additional condition is developed in Section III, in which we formulate the quasi-magnetostatic boundary value problem for a mesh-loaded circular aperture.

Electrostatic boundary conditions

In the electrostatic limit, there is no current flow and the electric field can be derived from the scalar potential V via

$$\bar{E} = -\nabla V \tag{2.17}$$

Setting $\bar{J}_s = 0$ and substituting eq. (2.17) into eq. (2.7) yields

$$\nabla_s (V + \frac{a_s^2}{C_s} \rho_s) = 0 \tag{2.18}$$

which may be integrated at once to yield

$$V - V_o = - \frac{a_s^2}{C_s} \rho_s \tag{2.19}$$

where V_o is a constant. The surface charge density ρ_s may be expressed in terms of the normal derivatives of the scalar potential on either side of the mesh as

$$\rho_s = -\varepsilon_2 \frac{\partial V_2}{\partial n} + \varepsilon_1 \frac{\partial V_1}{\partial n} \tag{2.20}$$

yielding

$$V = V_o + \frac{a_s^2}{C_s} (\varepsilon_2 \frac{\partial V_2}{\partial n} - \varepsilon_1 \frac{\partial V_1}{\partial n}) \tag{2.21}$$

which is the required electrostatic boundary condition. The other condition follows from the fact that the tangential electric field, and thus the scalar potential itself, must be continuous through the mesh surface:

$$V_2 = V_1 = V \tag{2.22}$$

In the applications to mesh-loaded aircraft windows which we shall consider in Section IV, the mesh itself is embedded in the window material (typically Lucite or Plexiglas). However, it is convenient to consider only the fields outside the window material. Assuming that the medium on either side of the window is free space, we may readily show that the appropriate boundary condition is

$$V = V_o + \frac{a_s^2 \varepsilon_o}{C_s} \left(\frac{\partial V_2}{\partial n} - \frac{\partial V_1}{\partial n} \right) \tag{2.23}$$

where

$$\frac{a_s^2 \varepsilon_o}{C_s} = \frac{a_s}{4\pi \varepsilon_{rw}} \ln\left(1 - e^{-2\pi r_w / a_s}\right)^{-1} \tag{2.24}$$

in which ε_{rw} denotes the relative permittivity of the window material.

The constant V_o can be shown to be equal to the potential of the mesh wires themselves. Since the junctions are bonded, the mesh structure is an equipotential in the static limit. It is important to note, however, that this does not imply that the space-averaged potential V is constant in the mesh surface. It will, in general, be a function of position.

We turn in the next two sections to the formulation of the static (or quasi-static) boundary value problems which characterize the field penetration of a mesh-loaded circular aperture in an infinite ground plane. The quasi-static magnetic problem is formulated in Section III and the electrostatic problem in Section IV.

III. FORMULATION OF THE QUASI-STATIC MAGNETIC BOUNDARY VALUE PROBLEM

In this section we shall formulate the relevant quasi-static magnetic boundary value problem for the determination of the magnetic polarizability and flux penetration of a mesh-loaded circular aperture in a perfectly conducting plane surface. It will be shown that the problem reduces to that of solving a Fredholm integral equation of the second kind and that the physical quantities of interest can be expressed in terms of moments of the solution of this equation.

The geometry of the problem is shown in figure 3.1. The perfectly conducting plane sheet is located at $z = 0$ in circular-cylindrical coordinates (ρ,ϕ,z), and the z-axis passes through the center of the aperture, whose radius is a. A uniform quasi-static magnetic field $H_o\bar{a}_x$ is applied in the region $z > 0$. The aperture is loaded with a bonded-junction wire mesh. It is assumed that the meshes are square and that the area of a single mesh, a_s^2, is small in comparison to the aperture area πa^2, in order to justify our modeling of the screen by an equivalent sheet impedance Z_s. It is further assumed that there exists a net dc contact resistance R_c between the outer edge of the mesh and the rim of the aperture. Such a contact resistance could arise because of aging of the junction between the mesh loading and the surrounding conductor, or because of improper installation of the mesh loading.

The fact that the quasi-static magnetic boundary condition given in eq. (2.15) is necessary but not sufficient indicates that

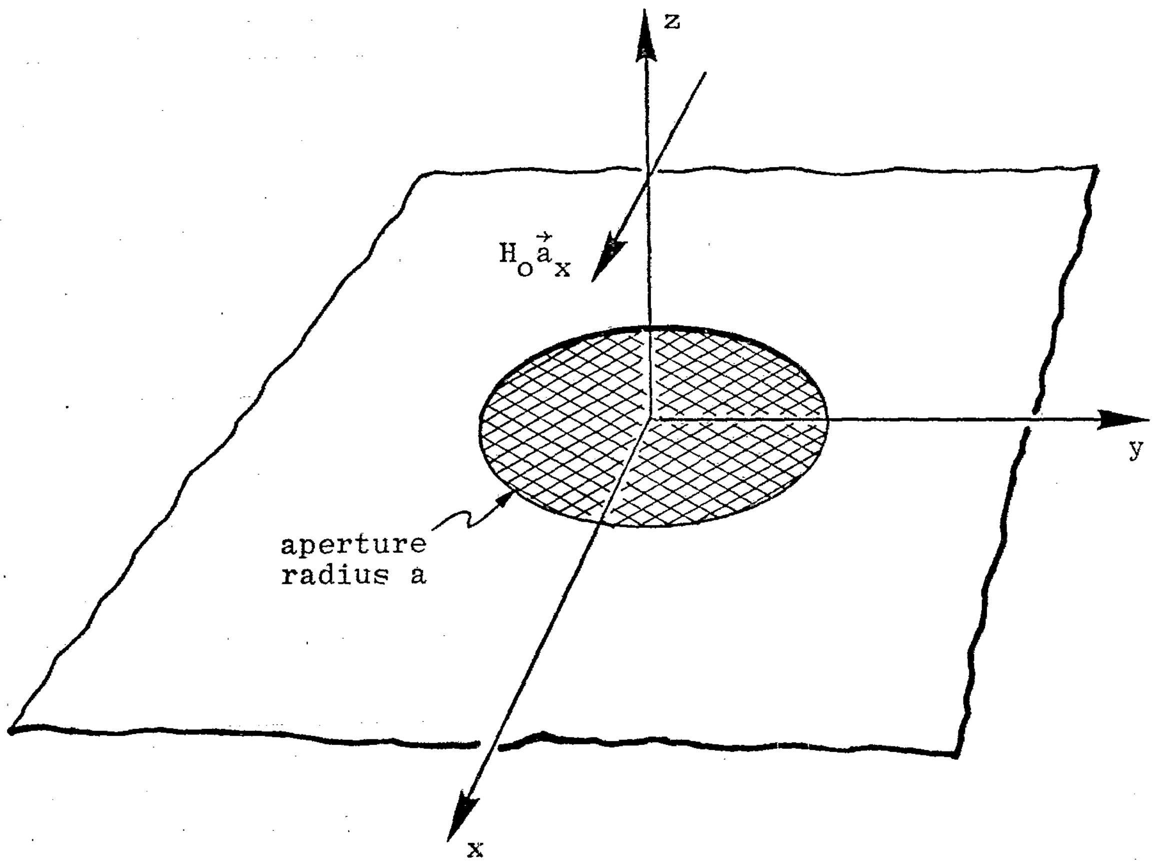

Figure 3.1. Geometry of the magnetostatic problem

an additional condition must be applied in order to obtain a unique solution to the problem we wish to solve. Such a condition can be found by evaluating the line integral of the electric field $\overline{E}$ around the closed path C shown in figure 3.2. The region $\rho < a-w$, $z = 0$ contains the mesh loading, which is described by the sheet impedance Z_s; the region $a-w < \rho < a$, $z = 0$ contains a material of sheet resistance R_s; and the region $\rho > a$, $z = 0$ is the perfectly conducting plane. The width w of the junction region between the mesh loading and the rim of the aperture will be allowed to approach zero, so that the magnetic flux linking the closed path C will vanish in this limit. We thus obtain

$$\oint_C \overline{E} \cdot \overline{d\ell} = -\Delta\phi \; R_s w \frac{\partial J_{s\rho}}{\partial \phi} - a\Delta\phi \; Z_s J_{s\phi} = 0 \tag{3.1}$$

where the surface current density component $J_{s\phi}$ is evaluated at $\rho = a-w$, and $w \to 0+$.* The contact resistance R_c is defined as

$$R_c = \lim_{\substack{R_s \to \infty \\ w \to 0}} \frac{R_s w}{2\pi a} \tag{3.2}$$

The boundary condition just inside the rim of the aperture is therefore

$$2\pi R_c \frac{\partial J_{s\rho}}{\partial \phi} + Z_s J_{s\phi} = 0 \qquad \text{at } \rho = a-, \; z = 0 \tag{3.3}$$

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*It is easy to show from the continuity equation (2.6) that the component $J_{s\rho}$ must be continuous at $\rho = a$.

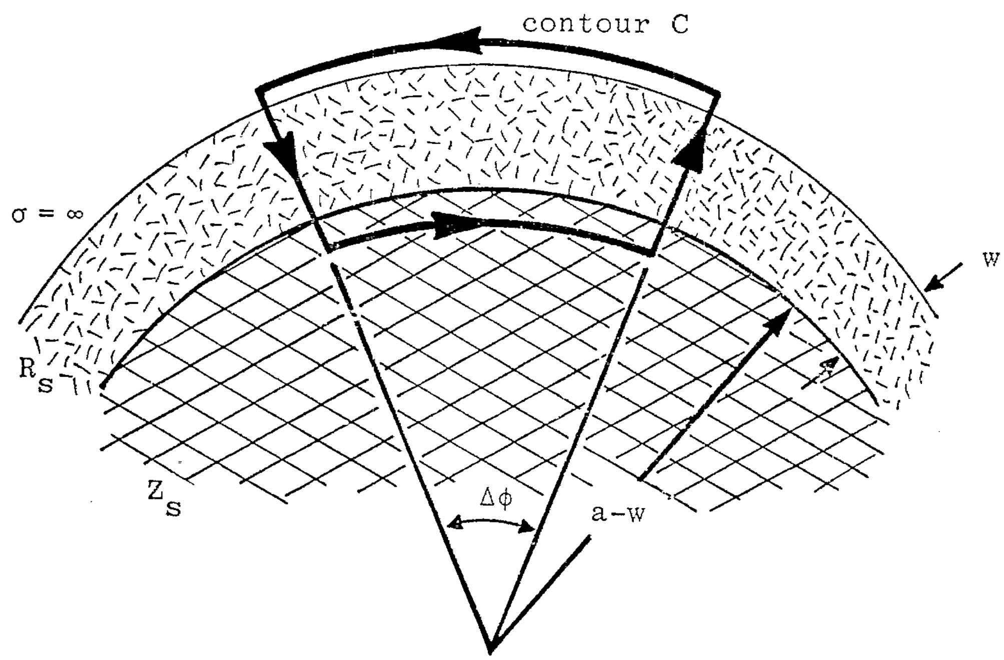

Figure 3.2. Contour for determination of auxiliary boundary condition

If the magnetic field is expressed in terms of the magnetic scalar potential V_m as $\overline{H} = -\nabla V_m$, then the surface current density components are given in terms of the discontinuity in V_m, ΔV_m, as

$$J_{s\rho} = \frac{1}{\rho} \frac{\partial}{\partial \phi} \Delta V_m$$
$$J_{s\phi} = - \frac{\partial}{\partial \rho} \Delta V_m \tag{3.4}$$

Thus the boundary condition of eq. (3.3) is given in terms of ΔV_m by

$$2\pi R_c \frac{\partial^2}{\partial \phi^2} \Delta V_m = Z_s a \frac{\partial}{\partial \rho} \Delta V_m \qquad \text{at } \rho = a- \tag{3.5}$$

We shall express the magnetic scalar potential V_m in terms of Fourier-Bessel integrals as

$$z > 0: \quad V_m = V_{m2} = -H_o \rho \cos\phi + \cos\phi \int_0^\infty F_m(\lambda) J_1(\lambda\rho)\, e^{-\lambda z} \frac{d\lambda}{\lambda}$$
$$z < 0: \quad V_m = V_{m1} = -\cos\phi \int_0^\infty F_m(\lambda) J_1(\lambda\rho)\, e^{\lambda z} \frac{d\lambda}{\lambda} \tag{3.6}$$

where J_ν denotes the Bessel function of order ν and $F_m(\lambda)$ is to be determined. It will be noted that the requirement that the normal component of the magnetic field $H_z = -\partial V_m/\partial z$ be continuous at $z = 0$ is satisfied by this representation. Specifically,

$$H_z(\rho,\phi,0) = \cos\phi \int_0^\infty F_m(\lambda) J_1(\lambda\rho)\, d\lambda \quad (0 \leq \rho < \infty) \tag{3.7}$$

and since the surface $z = 0$ is a perfect conductor for $\rho > a$, we require that

$$\int_o^\infty F_m(\lambda)J_1(\lambda\rho)\,d\lambda = 0 \qquad (\rho > a) \tag{3.8}$$

In the region $0 \leq \rho < a$, $z = 0$, $\Delta V_m = V_{m2} - V_{m1}$ must be a solution of the equation (cf. eq. (2.15))

$$\nabla_s^2(\Delta V_m) = \frac{s\mu_o}{Z_s} H_z \qquad (0 \leq \rho < a) \tag{3.9}$$

The general solution of eq. (3.9) is easily shown to be

$$\Delta V_m = (A\rho + \frac{B}{\rho}) \cos\phi - \cos\phi \frac{s\mu_o}{Z_s} \int_o^\infty F_m(\lambda)J_1(\lambda\rho) \frac{d\lambda}{\lambda^2} \quad (0 \leq \rho < a) \tag{3.10}$$

in which A and B are constants to be determined. Now since ΔV_m cannot be singular at $\rho = 0$, we set $B = 0$; and the constant A is determined from eq. (3.5):

$$A = (2\pi R_c + Z_s)^{-1} s\mu_o \int_o^\infty F_m(\lambda)\left[\frac{2\pi R_c}{Z_s}\frac{J_1(\lambda a)}{\lambda a} + J_1'(\lambda a)\right]\frac{d\lambda}{\lambda} \tag{3.11}$$

From eq. (3.6), we have

$$\Delta V_m = -H_o\rho \cos\phi + 2\cos\phi \int_o^\infty F_m(\lambda)J_1(\lambda\rho) \frac{d\lambda}{\lambda} \quad (0 \leq \rho < \infty) \tag{3.12}$$

so that, using eqs. (3.10) and (3.12), we obtain

$$\int_o^\infty F_m(\lambda)J_1(\lambda\rho)\left[1 + \frac{s\mu_o}{2Z_s\lambda}\right]\frac{d\lambda}{\lambda} = \frac{1}{2}(H_o + A)\rho \qquad (0 \leq \rho < a) \tag{3.13}$$

in which the constant A is given by eq. (3.11).

Equations (3.8) and (3.13) constitute a pair of dual integral equations from which the unknown function $F_m(\lambda)$ can be determined.

These equations can be put into a more convenient form by defining normalized variables ξ and u and a new transform function $A_m(u)$ by

$$\xi = \rho/a$$
$$u = \lambda a \qquad (3.14)$$
$$A_m(u) = \frac{1}{a}(H_o + A)^{-1} F_m(\lambda)$$

We thus obtain as dual integral equations for $A_m(u)$

$$\int_o^\infty A_m(u)J_1(\xi u)(1 + \frac{\beta_m}{u}) \frac{du}{u} = \frac{\xi}{2} \qquad (0 \leq \xi < 1)$$
$$\qquad (3.15)$$
$$\int_o^\infty A_m(u)J_1(\xi u)du = 0 \qquad (\xi > 1)$$

in which

$$\beta_m = \frac{s\mu_o a}{2Z_s} \qquad (3.16)$$

denotes the magnetic loading parameter of the aperture. The limiting case $\beta_m \to 0$ represents an open or unloaded aperture, while the limit $\beta_m \to \infty$ corresponds to an aperture which is completely blocked. The quantity H_o+A is expressed in terms of $A_m(u)$ by

$$\frac{H_o}{H_o + A} = 1 - \frac{s\mu_o a}{2\pi R_c + Z_s} \int_o^\infty A_m(u)\left[\frac{2\pi R_c}{Z_s} \frac{J_1(u)}{u} + J_1'(u)\right]\frac{du}{u} \qquad (3.17)$$

This completes the formulation of the quasi-static magnetic boundary-value problem. When $A_m(u)$ is determined from eqs. (3.15), the constant A can be found using eq. (3.17) and $F_m(\lambda)$ evaluated using eq. (3.14). We now turn to the reduction of the dual integral equations for $A_m(u)$ to a single Fredholm equation of the second kind, and show how the physical quantities of interest can be expressed in terms of the solution to this Fredholm equation.

The reduction procedure is given by Sneddon [5] and briefly outlined below. Consider the dual integral equations

$$\begin{aligned} &\int_0^\infty u^{-2\alpha}\left[1 + k(u)\right]A(u)J_\nu(\xi u)\,du = F(\xi) \quad (0 \leq \xi < 1) \\ &\int_0^\infty A(u)J_\nu(\xi u)\,du = 0 \quad (\xi > 1) \end{aligned} \tag{3.18}$$

in which α and ν are parameters and $k(u)$ and $F(\xi)$ are known functions. The unknown function $A(u)$ is expressed in terms of another function $h_1(t)$ as

$$A(u) = 2^{-\alpha}\, u^{1+\alpha}\int_0^1 t^{1+\alpha}\, J_{\nu-\alpha}(tu)h_1(t)\,dt \tag{3.19}$$

where $h_1(t)$ satisfies the inhomogeneous Fredholm equation of the second kind

$$h_1(x) + \int_0^1 h_1(u)K(x,u)\,du = H(x) \quad (0 \leq x \leq 1) \tag{3.20}$$

The kernel $K(x,u)$ is given by

$$K(x,u) = x^{-\alpha}\, u^{1+\alpha}\int_0^\infty tk(t)J_{\nu-\alpha}(xt)J_{\nu-\alpha}(ut)\,dt \tag{3.21}$$

and the free term $H(x)$ is

$$H(x) = 2^{2\alpha}x^{-2\alpha}I_{\frac{1}{2}\nu,-\alpha}F(x) \tag{3.22}$$

in which $I_{\eta,\gamma}$ is the Erdélyi-Kober operator defined by

$$I_{\eta,\gamma} f(x) = \frac{2x^{-2(\gamma+\eta)}}{\Gamma(\gamma)} \int_0^x (x^2 - u^2)^{\gamma-1} u^{2\eta+1} f(u)\, du \qquad (3.23)$$

when $\gamma > 0$. When $\gamma < 0$,

$$I_{\eta,\gamma} f(x) = x^{-2(\gamma+\eta)-1} D_x^n\, x^{2(n+\eta+\gamma)+1}\, I_{\eta,\gamma+n} f(x) \qquad (3.24)$$

where n is a positive integer such that $n+\gamma > 0$ and

$$D_x g(x) = \frac{1}{2} \frac{d}{dx} \left[\frac{1}{x}\, g(x)\right] \qquad (3.25)$$

Now identifying $A(u) = A_m(u)$, $\nu = 1$, $\alpha = 1/2$, $k(u) = \beta_m/u$, and $F(\xi) = \xi/2$, and carrying out the various manipulations, we find that

$$A_m(u) = \frac{2u}{\pi} \int_0^1 g(t) \sin ut\, dt \qquad (3.26)$$

where g(t) satisfies

$$g(t) + \frac{\beta_m}{\pi} \int_0^1 g(\tau)\, \ell n \left|\frac{t+\tau}{t-\tau}\right| d\tau = t \qquad (0 \le t \le 1) \qquad (3.27)$$

The magnetostatic problem thus reduces to that of solving eq. (3.27). We shall consider the solution of this equation in Section V. We conclude this section by giving expressions in terms of g(t) for the various physical quantities of interest.

The magnetic field in the aperture, $H_z(\rho,\phi,0)$ is

$$H_z(\rho,\phi,0) = \frac{-2H_o}{\pi} \left(\frac{H_o + A}{H_o}\right) \cos\phi \frac{d}{d\xi} \int_\xi^1 \frac{g(t)dt}{\sqrt{t^2-\xi^2}} \qquad (0 \le \xi < 1) \quad (3.28)$$

The equivalent "imaged" magnetic polarizability of the aperture, α_m, is

$$\alpha_m = \frac{1}{H_o} \int_{-\pi}^{\pi} \cos\phi \; d\phi \int_o^a H_z(\rho,\phi,0) \; \rho^2 d\rho$$

$$= \alpha_{mo} \left(\frac{H_o + A}{H_o} \right) 3 \int_o^1 tg(t) \; dt \qquad (3.29)$$

where $\alpha_{mo} = 4a^3/3$ is the magnetic polarizability of an unloaded circular aperture of radius a. The magnetic flux Φ_m linking the aperture is

$$\Phi_m = \mu_o \int_{-\pi/2}^{\pi/2} d\phi \int_o^a H_z(\rho,\phi,0) \; \rho d\rho$$

$$= \Phi_{mo} \left(\frac{H_o + A}{H_o} \right) 2 \int_o^1 g(t) \; dt \qquad (3.30)$$

in which $\Phi_{mo} = \mu_o a^2 H_o$ is the magnetic flux linking the aperture when $\beta_m = 0$. The factor $(H_o+A)/H_o$ is found from eq. (3.17) to be

$$\frac{H_o + A}{H_o} = \frac{1 + \dfrac{2\pi R_c}{Z_s}}{1 + \dfrac{2\pi R_c}{Z_s} + \left(1 - \dfrac{2\pi R_c}{Z_s}\right) \dfrac{4\beta_m}{\pi} \displaystyle\int_o^1 tg(t) \; dt} \qquad (3.31)$$

Thus the normalized magnetic polarizability $\hat{\alpha}_m = \alpha_m/\alpha_{mo}$ and the normalized penetrant flux $\hat{\Phi}_m = \Phi_m/\Phi_{mo}$ are given by

$$\hat{\alpha}_m = \frac{I_1(\beta_m)}{1 + \left(\dfrac{1 - r_c}{1 + r_c}\right)\left(\dfrac{4\beta_m}{3\pi}\right) I_1(\beta_m)} \qquad (3.32)$$

$$\hat{\Phi}_m = \frac{I_o(\beta_m)}{1 + \left(\frac{1 - r_c}{1 + r_c}\right)\left(\frac{4\beta_m}{3\pi}\right) I_1(\beta_m)} \tag{3.33}$$

in which $r_c = 2\pi R_c/Z_s$ and

$$I_\ell(\beta_m) = (\ell + 2) \int_o^1 t^\ell g(t)\, dt \tag{3.34}$$

are simply moments of the solution of the Fredholm equation (3.27).

We turn now to the formulation of the electrostatic boundary-value problem.

IV. FORMULATION OF THE ELECTROSTATIC BOUNDARY VALUE PROBLEM

In this section we shall formulate the relevant electrostatic boundary value problem for the determination of the electric polarizability and flux penetration of a mesh-loaded circular aperture in a perfectly conducting plane surface. It will be shown that the problem reduces to that of solving the same Fredholm equation which arose in connection with the quasi-static magnetic boundary value problem considered in the previous section. The electric polarizability and the penetrant electric flux will be shown to be expressible in terms of a single moment of the solution to this Fredholm equation. We shall consider the two cases in which the mesh wires are either connected to, or isolated from, the aperture rim.

The geometry of the problem, which is shown in figure 4.1, is identical to that considered in the previous section except that the applied field is now a uniform electrostatic field $E_o\bar{a}_z$. The electric scalar potential V is expressed in terms of Fourier-Bessel integrals as

$$\begin{aligned} z \geq 0: \quad V = V_2 &= -E_o z + \int_0^\infty F(\lambda) J_o(\lambda\rho)\, e^{-\lambda z}\, d\lambda \\ z \leq 0: \quad V = V_1 &= \int_0^\infty F(\lambda) J_o(\lambda\rho)\, e^{\lambda z}\, d\lambda \end{aligned} \tag{4.1}$$

in which $F(\lambda)$ is to be determined. It will be noted that the requirement that the potential be continuous at z = 0 is satisfied by this representation. Specifically,

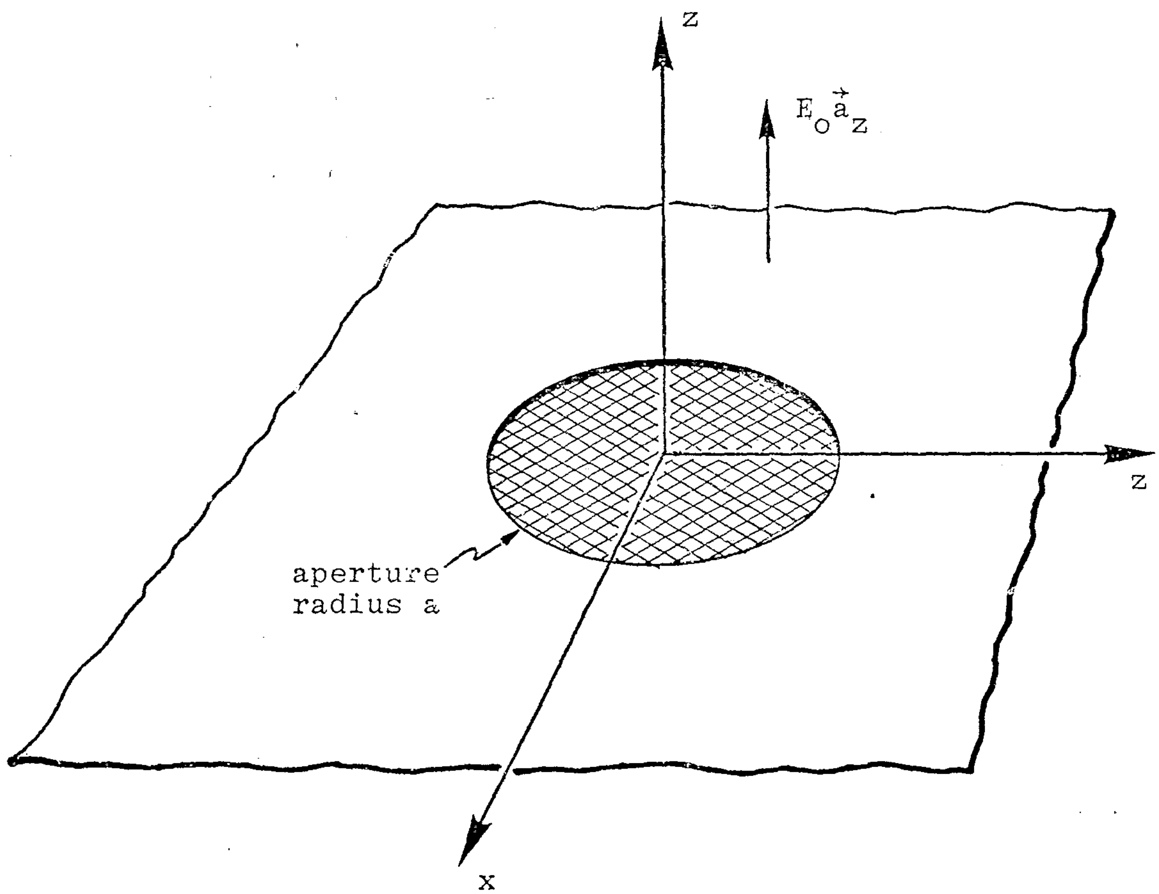

Figure 4.1. Geometry of the electrostatic problem

$$V(\rho,0) = \int_0^\infty F(\lambda)J_0(\lambda\rho)\, d\lambda \qquad (0 \le \rho < \infty) \tag{4.2}$$

It is assumed that the conducting plane is at potential V = 0, so that

$$\int_0^\infty F(\lambda)J_0(\lambda\rho)\, d\lambda = 0 \qquad (\rho \ge a) \tag{4.3}$$

The surface charge density ρ_s in the plane z = 0 is, from eqs. (4.1),

$$\rho_s = \varepsilon_0 E_0 + 2\varepsilon_0 \int_0^\infty F(\lambda)J_0(\lambda\rho)\, \lambda d\lambda \tag{4.4}$$

and from the electrostatic mesh boundary condition of eq. (2.19), we obtain for the region $0 \le \rho < a$, $z = 0$

$$\rho_s = -\frac{C_s}{a_s^2}\int_0^\infty F(\lambda)J_0(\lambda\rho)\, d\lambda + \rho_{so} \tag{4.5}$$

in which ρ_{so} is a constant. This constant is related to the potential V_o of the mesh wires by

$$\rho_{so} = \frac{C_s}{a_s^2} V_o \tag{4.6}$$

The total charge Q on the mesh can be computed from eq. (4.4) or (4.5); it is, via eq. (4.5),

$$Q = \pi a^2 \rho_{so} - \frac{2\pi C_s}{a_s^2}\int_0^\infty F(\lambda)d\lambda \int_0^a \rho J_0(\lambda\rho)\, d\rho$$

$$= \pi a^2 \rho_{so} - \frac{2\pi C_s a}{a_s^2}\int_0^\infty F(\lambda)J_1(\lambda a)\, \frac{d\lambda}{\lambda} \tag{4.7}$$

Now from eqs. (4.4) and (4.5) we obtain the relation valid for $0 \leq \rho < a$

$$\int_0^\infty F(\lambda) J_0(\lambda\rho)\left(1 + \frac{C_s}{2\varepsilon_0 a_s^2 \lambda}\right) \lambda d\lambda = -\frac{E_0}{2} + \frac{\rho_{so}}{2\varepsilon_0} \quad (0 \leq \rho < a) \tag{4.8}$$

Equations (4.3) and (4.8) constitute a pair of dual integral equations from which the function $F(\lambda)$ can be determined. Defining normalized variables ξ and u and a new transform function $A(u)$ by

$$\xi = \rho/a$$

$$u = \lambda a$$

$$A(u) = -\frac{2}{a^2}\left(E_0 - \frac{\rho_{so}}{\varepsilon_0}\right)^{-1} F(\lambda) \tag{4.9}$$

we obtain as dual integral equations for $A(u)$

$$\begin{aligned} &\int_0^\infty A(u) J_0(\xi u)\left(1 + \frac{\beta_e}{u}\right) u du = 1 \quad (0 \leq \xi < 1) \\ &\int_0^\infty A(u) J_0(\xi u)\, du = 0 \quad (\xi > 1) \end{aligned} \tag{4.10}$$

in which

$$\beta_e = \frac{C_s a}{2\varepsilon_0 a_s^2} \tag{4.11}$$

denotes the electric loading parameter of the aperture. The limit $\beta_e \to 0$ corresponds to an open or unloaded aperture; the limit $\beta_e \to \infty$ represents an aperture which is completely blocked.

As in the previous section, we reduce the problem of solving the dual integral equations (4.10) to that of solving an inhomogeneous Fredholm equation of the second kind. Specifically, we can show that

$$A(u) = \frac{2}{\pi} \int_0^1 h(t) \sin ut \, dt \qquad (4.12)$$

in which h(t) satisfies the integral equation

$$h(t) + \frac{\beta_e}{\pi} \int_0^1 h(\tau) \, \ell n \left| \frac{t+\tau}{t-\tau} \right| d\tau = t \qquad (0 \leq t \leq 1) \qquad (4.13)$$

It will be noted that h(t) satisfies the same integral equation as does g(t) for the quasi-static magnetic problem, except for the change in the definition of the parameter β_e. Thus the electrostatic and quasi-static magnetic problems are effectively solved simultaneously when the solution to the integral eq. (4.13) is found.

There remains the problem of determining the unknown constant $\rho_{so} = C_s V_o / a_s^2$. As was mentioned at the beginning of this section, we distinguish two cases: the mesh screen either is or is not connected to the rim of the aperture. In the first case, when the mesh wires make electrical contact with the surrounding conductor, their potential is the same as that of the conductor, which is zero. Thus $V_o = 0$ and

$$\rho_{so} = 0, \text{ when the mesh wires are connected to the conductor} \qquad (4.14)$$

In the second case, when the mesh wires are isolated from the surrounding conductor, the total charge on the mesh must be zero. Thus, from eqs. (4.7), (4.9), and (4.12), we obtain

$$\rho_{so} = \varepsilon_o E_o \frac{4\beta_e}{3\pi} I_1(\beta_e) \left[\frac{4\beta_e}{3\pi} I_1(\beta_e) - 1\right]^{-1}, \tag{4.15}$$

when the mesh wires are isolated from the conductor

The potential in the aperture $V(\rho,0)$ is expressed in terms of $h(t)$ as

$$V(\rho,0) = -\frac{E_o a}{\pi}\left(1 - \frac{\rho_{so}}{\varepsilon_o E_o}\right)\int_\xi^1 \frac{h(t)dt}{\sqrt{t^2-\xi^2}} \tag{4.16}$$

and the equivalent "imaged" electric polarizability of the aperture α_e is given by

$$\alpha_\epsilon = -\frac{2\pi}{E_o}\int_o^a \rho V(\rho,0)\, d\rho$$

$$= \alpha_{eo}\left(1 - \frac{\rho_{so}}{\varepsilon_o E_o}\right) I_1(\beta_e) \tag{4.17}$$

in which $\alpha_{eo} = 2a^3/3$ is the electric polarizability of an open circular aperture of radius a. The total electric flux Φ_e penetrating the aperture is

$$\Phi_e = -2\pi\varepsilon_o \int_o^a \rho \left.\frac{\partial V_1}{\partial z}\right|_{z=0} d\rho$$

$$= \Phi_{eo}\left(1 - \frac{\rho_{so}}{\varepsilon_o E_o}\right)\frac{4}{\pi}\int_o^1 \frac{th(t)dt}{\sqrt{1 - t^2}}$$

$$= \Phi_{eo}\left(1 - \frac{\rho_{so}}{\varepsilon_o E_o}\right)\left[1 - \frac{4\beta_e}{3\pi} I_1(\beta_e)\right] \tag{4.18}$$

where $\Phi_{eo} = \pi a^2 \varepsilon_o E_o/2$ denotes the electric flux which would penetrate the unloaded aperture. The relation

$$\frac{4}{\pi} \int_o^1 \frac{th(t)dt}{\sqrt{1 - t^2}} = 1 - \frac{4\beta_e}{\pi} \int_o^1 th(t)\, dt \qquad (4.19)$$

which has been used in the last line of eq. (4.18) can be derived either directly from the integral equation (4.13) or by equating the expressions for the charge Q calculated from eqs. (4.4) and (4.5).

We conclude this section by presenting the formulas for the normalized electric polarizability $\hat{\alpha}_e = \alpha_e/\alpha_{eo}$ and the normalized penetrant electric flux $\hat{\Phi}_e = \Phi_e/\Phi_{eo}$ in the two cases described above.

Case 1: the mesh is connected to the aperture rim

$$\hat{\alpha}_e = \hat{\alpha}_{ec} = I_1(\beta_e) \qquad (4.20a)$$

$$\hat{\Phi}_e = \hat{\Phi}_{ec} = 1 - \frac{4\beta_e}{3\pi} I_1(\beta_e) = 1 - \frac{4\beta_e}{3\pi} \hat{\alpha}_{ec} \qquad (4.20b)$$

Case 2: the mesh is isolated from the aperture rim

$$\hat{\alpha}_e = \hat{\alpha}_{ei} = \frac{I_1(\beta_e)}{1 - \frac{4\beta_e}{3\pi} I_1(\beta_e)} = \frac{\hat{\alpha}_{ec}}{1 - \frac{4\beta_e}{3\pi} \hat{\alpha}_{ec}} = \frac{\hat{\alpha}_{ec}}{\hat{\Phi}_{ec}} \qquad (4.21a)$$

$$\hat{\Phi}_e = \hat{\Phi}_{ei} = 1 \qquad (4.21b)$$

It is thus apparent that all the electrostatic quantities of interest can be expressed in terms of the single moment I_1 of the solution of the Fredholm eq. (4.13). We address the problem of solving this equation in the next section.

V. SOLUTION OF THE FREDHOLM INTEGRAL EQUATION

It has been shown in the previous two sections that both the magnetostatic and the electrostatic problems reduce to that of solving an inhomogeneous Fredholm equation of the second kind, viz.

$$F(t) + \frac{\beta}{\pi} \int_0^1 \ln\left|\frac{t+\tau}{t-\tau}\right| F(\tau)\, d\tau = t \qquad (0 \leq t \leq 1) \tag{5.1}$$

in which $F(t) = g(t)$ and $\beta = \beta_m$ for the magnetostatic problem and $F(t) = h(t)$ and $\beta = \beta_e$ for the electrostatic problem. We shall take up the solution of this integral equation in this section.

We begin by extending the domain of definition of $F(t)$ to $(-1,1)$ and that of the kernel $\ln|(t+\tau)/(t-\tau)|$ to $(-1,1) \times (-1,1)$; the extension of $F(t)$ is such that $F(-t) = -F(t)$. Then we find that $F(t)$ satisfies

$$F(t) - \frac{\beta}{\pi} \int_{-1}^1 F(\tau) \ln|t-\tau|\, d\tau = t \qquad (-1 \leq t \leq 1) \tag{5.2}$$

It has been shown by Davis [6] that an efficient approach to the problem of determining the eigenvalues of a homogeneous equation of the form of eq. (5.2) is to expand the unknown function $F(t)$ in a series of Legendre polynomials of odd degree. We shall apply this approach to find the solution of the inhomogeneous eq. (5.2). Let

$$F(t) = \frac{1}{\sqrt{3}} \sum_{n=1}^{\infty} \sqrt{4n-1}\, P_{2n-1}(t)\, F_n \tag{5.3}$$

in which P_k denotes the Legendre polynomial of degree k and the coefficients F_n are to be determined. Substituting eq. (5.3) into (5.2), multiplying through the result by $P_{2n-1}(t)$ and integrating with respect to t (interchanging orders of integration and summation as appropriate), we obtain the system of linear equations

$$F_m + \frac{\beta}{\pi} \sum_{n=1}^{\infty} K_{mn} F_n = \delta_{m,1} \qquad (m \geq 1) \tag{5.4}$$

in which $\delta_{m,1} = 1$ if $m = 1$ and $\delta_{m,1} = 0$ otherwise; and

$$K_{mn} = -\frac{1}{2}(4m - 1)^{\frac{1}{2}}(4n - 1)^{\frac{1}{2}} \int_{-1}^{1}\int_{-1}^{1} P_{2m-1}(t)P_{2n-1}(\tau)\ \ln|t-\tau|\ dt d\tau$$

$$= \frac{(4m - 1)^{\frac{1}{2}}(4n - 1)^{\frac{1}{2}}}{(n + m)(n + m - 1)[1 - 4(m - n)^2]} \tag{5.5}$$

Equation (5.4) can be solved for the expansion coefficients F_n on a computer. The normalized quantities describing the effect of the loading on the electric and magnetic polarizabilities and the penetrant fluxes are expressed in terms of these coefficients as

$$\hat{\alpha}_m = F_1^{(m)}\left[1 + \frac{4\beta_m}{3\pi}\left(\frac{Z_s - 2\pi R_c}{Z_s + 2\pi R_c}\right) F_1^{(m)}\right]^{-1} \tag{5.6a}$$

$$\hat{\Phi}_m = \left[1 + \frac{4\beta_m}{3\pi}\left(\frac{Z_s - 2\pi R_c}{Z_s + 2\pi R_c}\right) F_1^{(m)}\right]^{-1} .$$

$$\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{4n-1}{3}\right)^{\frac{1}{2}} F_n^{(m)} \frac{(2n - 3)!!}{2^{n-1}n!} \tag{5.6b}$$

$$\hat{\alpha}_{ec} = F_1^{(e)} \qquad (5.7a)$$

$$\hat{\Phi}_{ec} = 1 - \frac{4\beta_e}{3\pi} F_1^{(e)} \qquad (5.7b)$$

$$\hat{\alpha}_{ei} = F_1^{(e)} \left[1 - \frac{4\beta_e}{3\pi} F_1^{(e)}\right]^{-1} \qquad (5.7c)$$

$$\hat{\Phi}_{ei} = 1 \qquad (5.7d)$$

in which the (e) superscripts imply $\beta = \beta_e$ and the (m) superscripts imply $\beta = \beta_m$. Also,

$$(2n - 3)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 3)$$

$$(1)!! = (-1)!! = 1 \qquad (5.8)$$

The results given in eqs. (5.6) and (5.7) above constitute the exact solutions for the physical quantities of interest, in that they can be calculated to any desired accuracy from eq. (5.4).

It is also useful to obtain approximate expressions for these quantities. Such approximate expressions may be obtained by applying variational methods. It is well known that if F(t) is a solution of eq. (5.2) and that if $F_a(t)$ is a solution of

$$F_a(t) - \frac{\beta}{\pi} \int_{-1}^{1} F_a(\tau)\, \ell n|t - \tau| d\tau = h(t) \qquad (5.9)$$

then a variational formula for the quantity

$$\int_{-1}^{1} F(t)\, h(t)\, dt$$

is

$$\int_{-1}^{1} F(t)h(t)dt = \frac{\int_{-1}^{1} F(x)h(x)dx \int_{-1}^{1} F_a(x)xdx}{\int_{-1}^{1} F(x)F_a(x)dx - \frac{\beta}{\pi}\int_{-1}^{1}\int_{-1}^{1} F(x)F_a(y)\ln|x-y|dxdy} \tag{5.10}$$

Using $F(x) \simeq x$ and $F_a(x) \simeq h(x)$, which are the exact solutions of eqs. (5.2) and (5.9) in the limit $\beta \to 0$, eq. (5.10) becomes

$$\int_{-1}^{1} F(t)h(t)dt \simeq \frac{\left[\int_{-1}^{1} xh(x)dx\right]^2}{\int_{-1}^{1} xh(x)dx - \frac{\beta}{\pi}\int_{-1}^{1}\int_{-1}^{1} xh(y)\ln|x-y|dxdy} \tag{5.11}$$

We therefore obtain the following approximations to the integrals required for the evaluation of the physical quantities of interest:

$$3\int_{0}^{1} tF(t)dt = \frac{3}{2}\int_{-1}^{1} tF(t)dt \simeq \left(1 + \frac{3\beta}{2\pi}\right)^{-1} \tag{5.12a}$$

$$2\int_{0}^{1} F(t)dt = \int_{-1}^{1} F(t)\mathrm{sgn}(t)dt \simeq \left[1 + \frac{4\beta}{3\pi}\left(\ln 2 + \frac{1}{2}\right)\right]^{-1} \tag{5.12b}$$

$$\frac{4}{\pi}\int_{0}^{1} \frac{tF(t)dt}{\sqrt{1-t^2}} = \frac{2}{\pi}\int_{-1}^{1} \frac{tF(t)dt}{\sqrt{1-t^2}} \simeq \left(1 + \frac{4\beta}{3\pi}\right)^{-1} \tag{5.12c}$$

Approximate expressions for $\hat{\alpha}_m$, $\hat{\Phi}_m$, $\hat{\alpha}_e$, and $\hat{\Phi}_e$ obtained from the defining eqs. (3.32), (3.33), (4.15), (4.17), and (4.18) and the variational expressions in eqs. (5.12) are

$$\hat{\alpha}_m \simeq \left[1 + \frac{3\beta_m}{2\pi} + \frac{1 - r_c}{1 + r_c}\,\frac{4\beta_m}{3\pi}\right]^{-1} \tag{5.13a}$$

$$\hat{\Phi}_m \simeq \left[\frac{1 + \frac{3\beta_m}{2\pi}}{1 + \frac{4\beta_m}{3\pi}(\ln 2 + \frac{1}{2})}\right]\hat{\alpha}_m \qquad (5.13b)$$

$$\hat{\alpha}_{ec} \simeq \left(1 + \frac{3\beta_e}{2\pi}\right)^{-1} \qquad (5.13c)$$

$$\hat{\Phi}_{ec} \simeq \left(1 + \frac{4\beta_e}{3\pi}\right)^{-1} \qquad (5.13d)$$

$$\hat{\alpha}_{ei} \simeq \left(1 + \frac{\beta_e}{6\pi}\right)^{-1} \qquad (5.13e)$$

The result for $\hat{\Phi}_{ec}$ is obtained from the second of the lines in eq. (4.18). Also, since

$$\frac{3}{2\pi} = 0.47746\ldots \qquad (5.14a)$$

$$\frac{4}{3\pi}(\ln 2 + \frac{1}{2}) = 0.50638\ldots \qquad (5.14b)$$

we have

$$\hat{\Phi}_m \simeq \hat{\alpha}_m \qquad (5.15)$$

These approximate variational expressions can be used to represent the aperture and its loading by equivalent circuits. The quantity $-s\Phi_m$, for example, is the maximum electromotive force (emf) which could be induced in any loop behind the aperture. Defining an open-circuit voltage V_{oc}^m as

$$V_{oc}^m = s\Phi_m \qquad (5.16)$$

and noting that when the aperture is unloaded,

$$\Phi_m = \mu_o a^2 H_o$$

$$= \left(\frac{\mu_o a}{2}\right)\left(2H_o a\right)$$

$$= L_a I_{sc}^m \qquad (5.17)$$

in which $L_a = \mu_o a/2$ is an equivalent aperture inductance and $I_{sc}^m = 2H_o a$ is the short-circuit current flowing across the aperture when it is shorted, we have

$$V_{oc}^m = I_{sc}^m \; sL_a \hat{\Phi}_m$$

$$\simeq I_{sc}^m \; sL_a \frac{Z_1 Z_2}{Z_1 Z_2 + sL_a(Z_1 + Z_2)} \qquad (5.18)$$

in which

$$Z_1 = \frac{3\pi}{8}\,(Z_s + 2\pi R_c) \qquad (5.19a)$$

$$Z_2 = 6\pi Z_s \qquad (5.19b)$$

The relationship in eq. (5.18) can be represented by the equivalent circuit shown in figure 5.1.

Similarly, we can construct an equivalent circuit for the electric-field penetration. We define a short-circuit current I_{sc}^e by

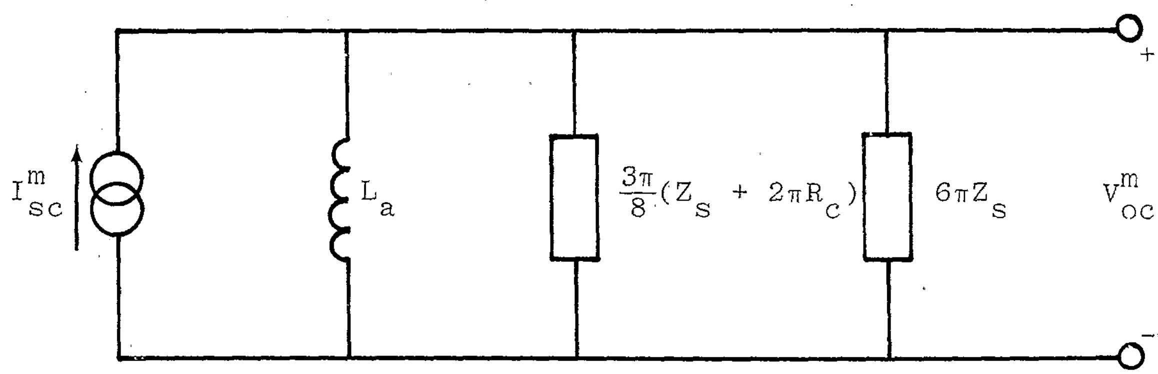

Figure 5.1. Equivalent circuit for quasi-static magnetic field penetration. The short-circuit current $I_{sc}^{m} = 2H_o a$ and the aperture inductance $L_a = \mu_o a/2$.

$$I^e_{sc} = s\Phi_e$$

$$= s\pi a^2 \varepsilon_o \frac{E_o}{2} \hat{\Phi}_e$$

$$\simeq \frac{s\pi a^2 \varepsilon_o E_o/2}{1 + \frac{8}{3}\frac{a\varepsilon_{rw}}{a_s \ell n\ Q}} \tag{5.20}$$

where $Q = (1 - e^{-2\pi r_w/a_s})^{-1}$. Now $\Phi_{eo} = \pi a^2 \varepsilon_o E_o/2$ can be written as the product of a capacitance C_1 and a voltage V_1. Thus

$$I^e_{sc} = sC_1V_1 \left(1 + \frac{C_1}{C_2}\right)^{-1} \tag{5.21}$$

where

$$\frac{C_1}{C_2} = \frac{8}{3}\frac{a\varepsilon_{rw}}{a_s \ell n\ Q} \tag{5.22}$$

It is not possible to specify C_1 and V_1 separately; but it is possible to represent eq. (5.21) by an equivalent circuit of the form shown in figure 5.2.

An alternate descriptor of the electric field shielding effectiveness of the aperture loading would be a "charge transfer frequency" Ω_T such as is used in cable coupling calculations [7]. We would have, simply,

$$\Omega_T = s\left(1 + \frac{8}{3}\frac{a\varepsilon_{rw}}{a_s \ell n\ Q}\right)^{-1} \tag{5.23}$$

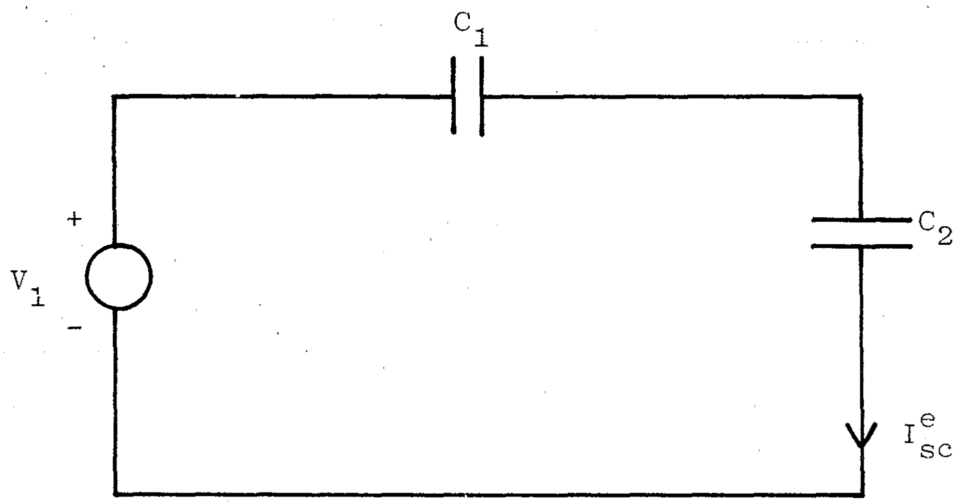

Figure 5.2. Equivalent circuit for quasi-static electric field penetration. The product $C_1V_1 = \pi a^2 \varepsilon_o E_o/2$. The ratio C_1/C_2 is given in eq. (5.22).

where $I_{sc}^{e} = \Omega_T \Phi_{eo}$. The quantity $\Omega_T/s = \left(1 + \frac{8}{3} \frac{a\varepsilon_{rw}}{a_s \ell n\, Q}\right)^{-1}$ is the charge coupling coefficient.

In the next section we present the results of representative numerical calculations in order to compare the exact and the variationally derived solutions for the physical quantities of interest.

VI. NUMERICAL RESULTS

Extensive numerical calculations of the quantities $\hat{\alpha}_m$, $\hat{\Phi}_m$, $\hat{\alpha}_e$, and $\hat{\Phi}_e$ have been carried out using the approach outlined in the previous section in eqs. (5.3) to (5.7). Some of these exact results are discussed in this section and compared to those obtained by variational means given in eqs. (5.13).

For simplicity we shall confine our attention to the special case $R_c = 0$ and concentrate on the normalized magnetic aperture polarizability $\hat{\alpha}_m$. In this case the variational expression for $\hat{\alpha}_m$ is written

$$\hat{\alpha}_m = \frac{1 + s\tau_s}{1 + s(\tau_s + \tau_a)} \tag{6.1}$$

in which

$$\tau_s = \frac{L_s}{R_s}$$
$$\tau_a = \frac{17}{6\pi}\frac{L_a}{R_s} \tag{6.2}$$

Defining $\tau_o = \tau_s + \tau_a$, we have

$$\hat{\alpha}_m = \frac{1 + s\tau_s}{1 + s\tau_o} \tag{6.3}$$

An asymptotic Bode plot of the magnitude of $\hat{\alpha}_m(j\omega)$ as a function of normalized frequency $\omega\tau_o$ is shown in figure 6.1.

In figure 6.2 are shown exact (solid) and variational (dashed) curves of $|\hat{\alpha}_m(j\omega)|$ vs. $\omega\tau_o$ for various values of the parameter τ_s/τ_o. The parameter $\beta_m = sL_a/Z_s$ is written in terms of τ_o and τ_s, when $s = j\omega$, as

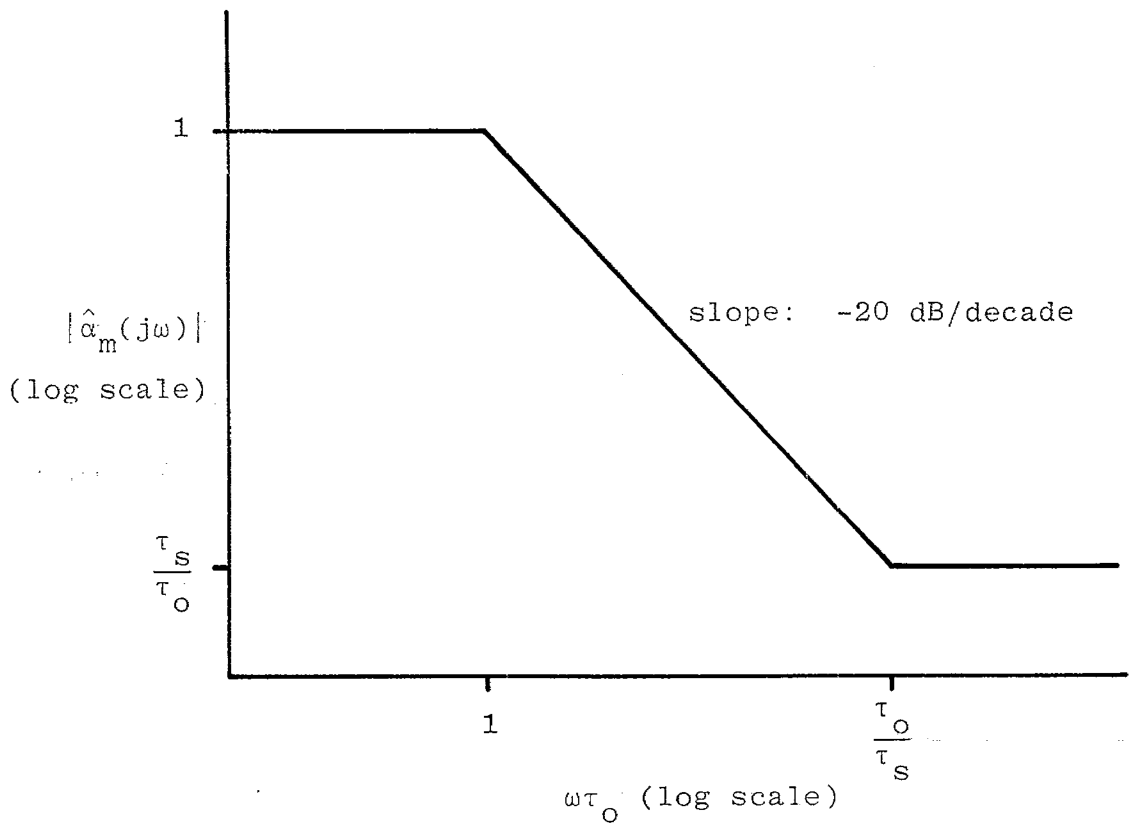

Figure 6.1. Asymptotic Bode plot of $|\hat{\alpha}_m(j\omega)|$ as a function of normalized frequency $\omega\tau_o$.

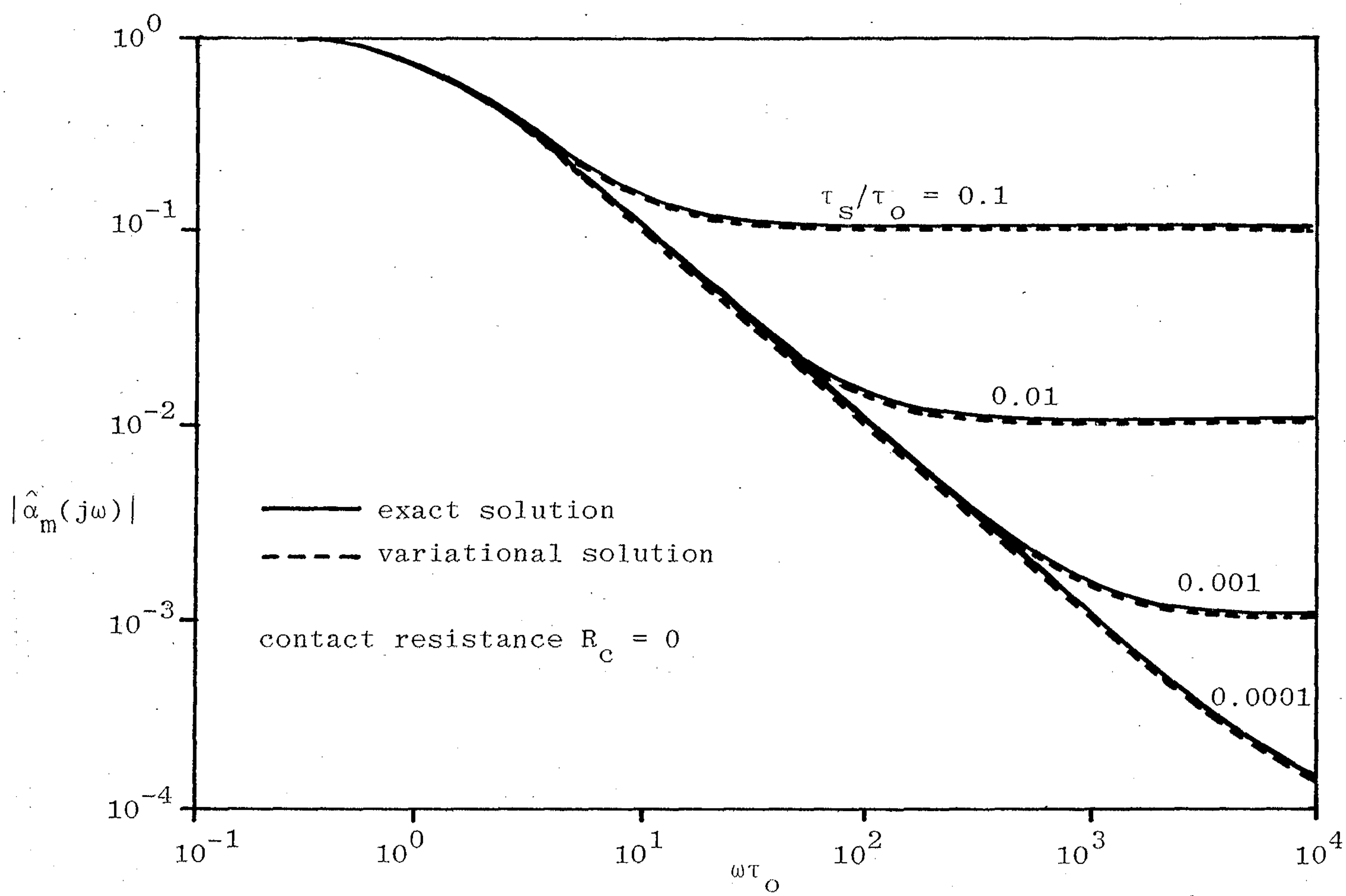

Figure 6.2. $|\hat{\alpha}_m(j\omega)|$ as a function of normalized frequency $\omega\tau_o$. Solid curves show the exact solution and dashed curves the variational solution.

$$\beta_m = \left(\frac{6\pi}{17}\right)\frac{j\omega\tau_o(1 - \tau_s/\tau_o)}{1 + j\omega\tau_o(\tau_s/\tau_o)} \tag{6.4}$$

The agreement between the exact and variationally derived expressions for $|\hat{\alpha}_m|$ is very good. Similar agreement has been found for $\hat{\Phi}_m$, $\hat{\alpha}_e$, and $\hat{\Phi}_e$, from which we conclude that the variationally derived expressions are adequate for the practical utilization of the results developed in this note.

The excellent agreement between the exact results and those obtained by variational means can be at least partly explained by consideration of the poles and zeros in the complex β-plane of, for example, the coefficient F_1 (cf. eqs. (5.3) and (5.4)). The poles are, of course, the eigenvalues of the integral equation (5.2). The first twenty of these poles and zeros have been calculated and are given in tables 6.1 and 6.2, together with approximate formulas for their locations. Denoting the poles by β_k, we have

$$\beta_k \simeq -\pi(k - \frac{3}{8}) - \frac{1}{8\pi k} \quad (k = 1,2,\ldots) \tag{6.5}$$

and the zeros β_{k1} are approximately located at

$$\beta_{k1} \simeq -\pi\left[(k + 1) - \frac{3}{8}\right] + \frac{15}{8\pi(k + 1)} \quad (k = 1,2,\ldots) \tag{6.6}$$

Now the separation d_k between the k^{th} zero and the $(k + 1)^{st}$ pole, normalized by π, the separation between the k^{th} and $(k + 1)^{st}$ poles for large k, is

$$\frac{d_k}{\pi} \simeq \frac{2}{\pi^2(k + 1)} \quad (k = 1,2,\ldots) \tag{6.7}$$

Table 6.1

EIGENVALUES $-\beta_k/\pi$ AND THEIR APPROXIMATIONS

k	$-\beta_k/\pi$	$k - \frac{3}{8}$	$k - \frac{3}{8} + \frac{1}{8\pi^2 k}$
1	0.63857	0.62500	0.63767
2	1.63143	1.62500	1.63133
3	2.62923	2.62500	2.62922
4	3.62815	3.62500	3.62817
5	4.62751	4.62500	4.62753
6	5.62709	5.62500	5.62711
7	6.62679	6.62500	6.62881
8	7.62657	7.62500	7.62658
9	8.62639	8.62500	8.62641
10	9.62625	9.62500	9.62627
11	10.62614	10.62500	10.62615
12	11.62605	11.62500	11.62606
13	12.62596	12.62500	12.62597
14	13.62590	13.62500	13.62590
15	14.62584	14.62500	14.62584
16	15.62578	15.62500	15.62579
17	16.62574	16.62500	16.62575
18	17.62570	17.62500	17.62570
19	18.62566	18.62500	18.62567
20	19.62563	19.62500	19.62563

Table 6.2

ZEROS $-\beta_{k1}/\pi$ AND THEIR APPROXIMATIONS

k	$-\beta_{k1}/\pi$	$k + \frac{5}{8}$	$k + \frac{5}{8} - \frac{15}{8\pi^2(k+1)}$
1	1.51781	1.62500	1.53001
2	2.55841	2.62500	2.56167
3	3.57628	3.62500	3.57751
4	4.58648	4.62500	4.58700
5	5.59311	5.62500	5.59334
6	6.59777	6.62500	6.59786
7	7.60123	7.62500	7.60125
8	8.60391	8.62500	8.60389
9	9.60604	9.62500	9.60600
10	10.60778	10.62500	10.60773
11	11.60922	11.62500	11.60917
12	12.61044	12.62500	12.61039
13	13.61149	13.62500	13.61143
14	14.61239	14.62500	14.61233
15	15.61318	15.62500	15.61313
16	16.61388	16.62500	16.61382
17	17.61450	17.62500	17.61445
18	18.61505	18.62500	18.61500
19	19.61555	19.62500	19.61550
20	20.61600	20.62500	20.61595

which approaches zero as k increases. Thus all but one of the poles (the k = 1 pole) are approximately cancelled by adjacent zeros, leaving the pole at $\beta_1 = -0.63857\pi = -2.00613...$ as, in a sense, the "dominant" pole. The "equivalent" location of this pole as it appears in the variational expressions is (cf. eq. (5.12a))

$$\beta_1 \simeq -0.66667\pi = -2.09439... \tag{6.8}$$

The dominance of this single eigenvalue leads to the accuracy of the variational expression for F_1 and the consequent accuracy of the variational expressions for $\hat{\alpha}_m$ and for the other normalized polarizabilities and the penetrant fluxes.

VII. SUMMARY AND CONCLUDING REMARKS

In this note we have formulated and solved the quasi-magnetostatic and electrostatic problems of penetration of a mesh-loaded circular aperture in a ground plane of infinite transverse extent. It has been shown that variationally derived expressions for the aperture polarizabilities and penetrant fluxes are excellent approximations to the exact results for these quantities and that the loaded aperture can be represented by simple equivalent circuits.

The application of these results to the analysis of penetration through more realistic apertures (i.e., apertures which are not circular and/or which are not situated in an infinite ground plane) would be facilitated by

1. an assessment of the effect of curvature of the surface containing the aperture
2. knowledge of, or bounds on, the equivalent inductance L_a of non-circular apertures in an infinite ground plane

The first item is not as important as the second, for the effect of surface curvature is small and, in any event, the penetration is maximized for a flat surface [8]. Thus the analysis given in this note should be applicable in an "upper bound" sense. The evaluation of the equivalent inductances of apertures of, say, elliptical or rectangular shape would be useful in developing approximate expressions for, or bounds on, the equivalent inductance as functions of the aperture area and/or perimeter.

Also useful would be a study of a loaded non-circular aperture. Of particular interest in this problem would be the method employed for incorporating the aperture-rim boundary condition in an integral-equation formulation. This problem is presently under investigation.

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