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On the Physical Realizability of Broadband Lumped-Parameter Equivalent Circuits for Energy-Collecting Structures

L. Wilson Pearson
Donald R. Wilton

University of Kentucky
Lexington, KY

The practical implementation of equivalent circuits which represent the energy delivered to an arbitrary load impedance by an antenna or a port on a scatterer is discussed. Particular attention is paid to the positive-real function issues associated with the equivalent admittance of the structure when evolved from the point of view of the Singularity Expansion Method. A proof is given which establishes that the eigenadmittances (reciprocal eigenvalues) of a structure are positive-real functions. The connection between this fact and the positive-realness of the admittances associated with individual current eigenmode contributions to total current is discussed. An _ad hoc_ approach to approximating these admittances with simpler pole-pair positive-real admittances is given for high-Q structures. A brief survey of the applicable circuit synthesis algorithms and their features is provided. Some fundamental considerations in the implementation of the sources representing the energy coupled from the incident wave are presented.

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ON THE PHYSICAL REALIZABILITY OF BROADBAND LUMPED-PARAMETER EQUIVALENT CIRCUITS FOR ENERGY-COLLECTING STRUCTURES

I. INTRODUCTION

Schelkunoff presented, to our knowledge, the first development relative to physically realizable equivalent circuits for radiating structures [1]. While his theory offers a great deal of insight into the resonance and anti-resonance phenomena which antennas manifest, the theory provides no means for the actual physical determination of equivalent circuits for radiating or receiving structures. More recently, Weber and Toulios provided a means for some equivalent circuit modeling of antennas on a broadband transient basis by using a Laplace domain curve-fitting technique [2].

In 1971, Baum introduced the Singularity Expansion Method (SEM) [3]. SEM provides a natural resonance representation for the current on an antenna or scattering structure. Such a complex resonance representation admits to the possibility of applying general formal circuit synthesis procedures which can be implemented in specific cases, provided the SEM description for the case in question is available. However, a number of questions have arisen relative to the completeness of the representation. A great deal of work in the SEM area has proceeded since Baum's introduction of it. The three summary works by Baum [4,5,6] provide excellent bibliographies and give a nearly up-to-date summary of what has proceeded. In a later report [7], Baum relates the singularity expansion to eigenfunction analysis of integral operators. This relationship bears heavily on the work reported here. Also in a later note, he develops a formal theory for equivalent circuit synthesis for radiating and energy collecting structures [8].

Hess has used SEM analysis to develop a transfer function model for determining EMP coupling by means of equivalent transfer function circuits [9]. The present method goes beyond the work of Hess in that it accounts for the possibility of loading at the terminals of the receiving structure. More recently, Schaubert has used measured transient response data which was subsequently analyzed, using a Prony-type algorithm, for the determination of the complex natural resonances in a network synthesis [10]. This latter work, though ad hoc in character, is likely to be quite important in elaborations on the synthesis methods presented here. The use of measured data allows the possibility of structures which are too complex to be modeled on a theoretical or numerical basis. Schaubert, however, limits his concern to the determination of the admittance or impedance part of the equivalent circuit; he does not consider the source problem.

In this work, we have explored some of the ramifications of Baum's development in reference [8]. In particular, we have emphasized the physical realizability of the circuits developed with passive elements. As a consequence, the dominant issue which arises is that of the positive-realness (PR-ness) of the impedance or admittance representations which result. In a companion work [11], the results of an extensive numerical study emphasizing PR properties of admittances for straight wires and wire loop structures are presented. In subsequent paragraphs here, we shall draw on the results of reference [11] from time to time as evidence supporting various assertions.

In the following, we develop the formal theory of equivalent network synthesis based on the singularity expansion. This development essentially follows Baum [8]. It is reiterated here for the sake of completeness and for the sake of introducing what we feel to be a somewhat simplified notation. We have chosen to limit our discussion to that of synthesis of the admittance of the antenna or energy collector, and to electric field integral equation formulations. These results may be generalized to embrace impedance-type synthesis and also alternative integral equations for the structures. Indeed, Baum's formal development in reference [8] includes this level of generality. We have chosen, for the sake of clarity, to restrict the scope of the present development.

II. FORMAL DEVELOPMENT OF THE EQUIVALENT CIRCUIT: TERMINAL EIGENELEMENTS

In this section we develop a formal equivalent circuit for an energy-collecting structure. By an energy-collecting structure, we mean either a single-port antenna or a scatterer with an inadvertent penetration which can be identified in terms of a port.

The objective of the development is indicated in Figure 1a. There, a generic electromagnetic energy collecting structure, loaded by a frequency dependent admittance $\tilde{Y}_L(s)$, is indicated. (We shall use the complex frequency variable s--the Laplace transform variable--to indicate frequency thoughout this development. A tilde ~ over a function denotes a Laplace transformed quantity.) We wish to replace the influence of the driven energy collecting structure on the admittance $\tilde{Y}_L(s)$ with a Norton equivalent circuit, as shown in the right portion of Figure 1a. The elements in the equivalent circuit are derivable by the steps indicated in Figures 1b and 1c. First, the equivalent admittance of the structure $\tilde{Y}_L(s)$ is obtainable by treating the structure as a transmitting antenna. To do this, we introduce a frequency dependent voltage $\tilde{V}(s)$ across the port of the structure, determine the port current $\tilde{I}(s)$, and find the ratio $\tilde{I}(s)/\tilde{V}(s)$. This ratio

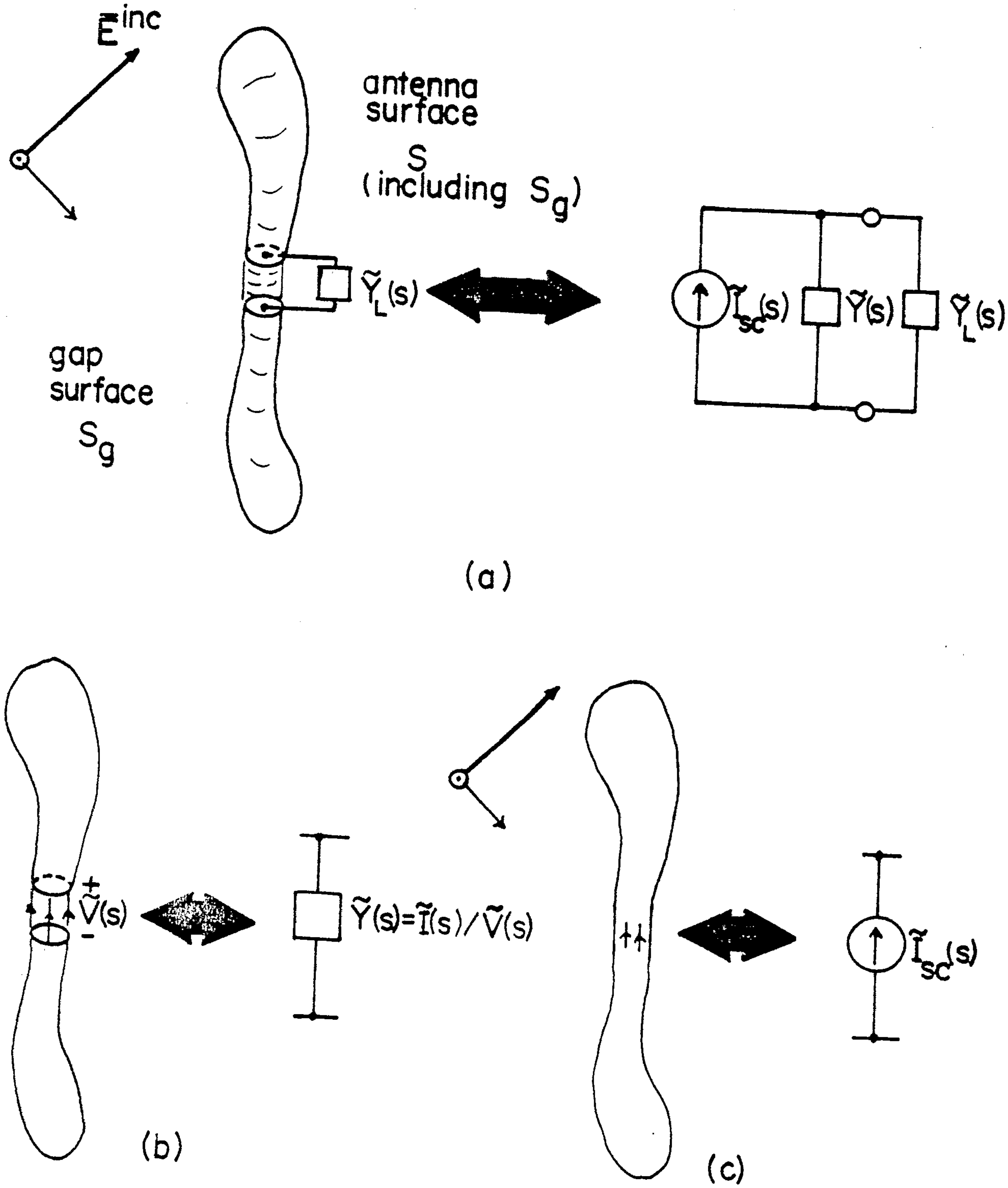

Figure 1. Development of the Norton equivalent circuit for an energy-collecting structure: (a) the equivalence sought; (b) the admittance problem; and (c) the current source problem.

is the admittance in question. The source term in the equivalent circuit is determined by shorting the port and treating the resulting scattering problem. From it, we obtain the net current at the port, $\tilde{I}_{sc}(s)$, which becomes the generator in the equivalent circuit.

The formal SEM equivalent circuit results from representing the solutions to the two boundary value problems indicated in Figures 1b and 1c in terms of the singularity expansion. In the present development, we use the eigenfunction expansion for the solutions to the respective problems as an intermediate step toward the development of the singularity expansion. The intermediate forms of the eigenfunction expansions and properties associated therewith subsequently prove useful in the development of practical equivalent circuits.

Either of the boundary value problems in question may be formally cast as an integral equation for the surface current on the surface of the energy collecting structure. The electric field integral equation for the structure is representable as

$$\langle \tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) \; ; \; \tilde{\bar{J}}(\bar{r}',s)\rangle = \tilde{\bar{E}}^{inc}(\bar{r},s) \; , \qquad \bar{r} \in S \; , \tag{1}$$

where the dyadic kernel is given by

$$\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) = [s\mu - \frac{1}{s\varepsilon} \text{ grad div}] \, \bar{\bar{I}} \, e^{-s|\bar{r}-\bar{r}'|/c}/[4\pi|\bar{r}-\bar{r}'|/c] \; , \tag{2}$$

with

μ = permeability of the medium,

ε = permittivity of the medium, and

$c = (\mu\varepsilon)^{-\frac{1}{2}}$ = velocity of light in the medium.

The term $\tilde{\bar{E}}^{inc}(\bar{r},s)$ represents the tangential component of the incident field on s exciting the object. The bracket notation is used to indicate a symmetric product--a surface integral over the extent of the scatterer surface, viz.,

$$<\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) \; ; \; \tilde{\bar{J}}(\bar{r}',s)> = \iint_S \tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) \cdot \tilde{\bar{J}}(\bar{r}',s) \, dS' \; . \tag{3}$$

The term $\tilde{\bar{E}}^{inc}(\bar{r},s)$ is replaced with a field due to the impressed voltage $\tilde{V}(s)$ in the case of the transmitting problem of Figure 1b.

A formal solution to the integral equation (1) above is given in terms of the eigenfunction expansion for the surface current $\tilde{\bar{J}}(\bar{r},s)$, which is

$$\tilde{\bar{J}}(\bar{r},s) = \sum_n \frac{1}{\lambda_n(s)} \frac{<\tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{E}}^{inc}(\bar{r},s)>}{<\tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{J}}_n(\bar{r},s)>} \tilde{\bar{J}}_n(\bar{r},s) \tag{4}$$

for symmetric kernels $\tilde{\bar{\bar{\Gamma}}}$ as in the case of the electric field integral equation. The $\tilde{\bar{J}}_n(\bar{r},s)$ and the $\lambda_n(s)$ are the eigenfunction/eigenvalue pairs for the integral operator. Namely,

$$<\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) \; ; \; \tilde{\bar{J}}_n(\bar{r}',s)> = \lambda_n(s) \, \tilde{\bar{J}}_n(\bar{r},s) \; . \tag{5}$$

The admittance of the structure is determined by solving the antenna problem indicated in Figure 1b. The incident field is taken to be the quasi-static EMF across the gap width Δ due to the impressed voltage $\tilde{V}(s)$,

$$\tilde{\bar{E}}^{inc}(\bar{r},s) = \tilde{V}(s)/\Delta \; \hat{a}_g \; , \qquad \bar{r} \in S_g \; ,$$

with $\hat{a}_g$ a unit vector oriented from one gap surface to the other on S_g, defining the reference direction for the EMF, as depicted in Figure 2. In terms of the eigenfunction expansion, this solution may be written as

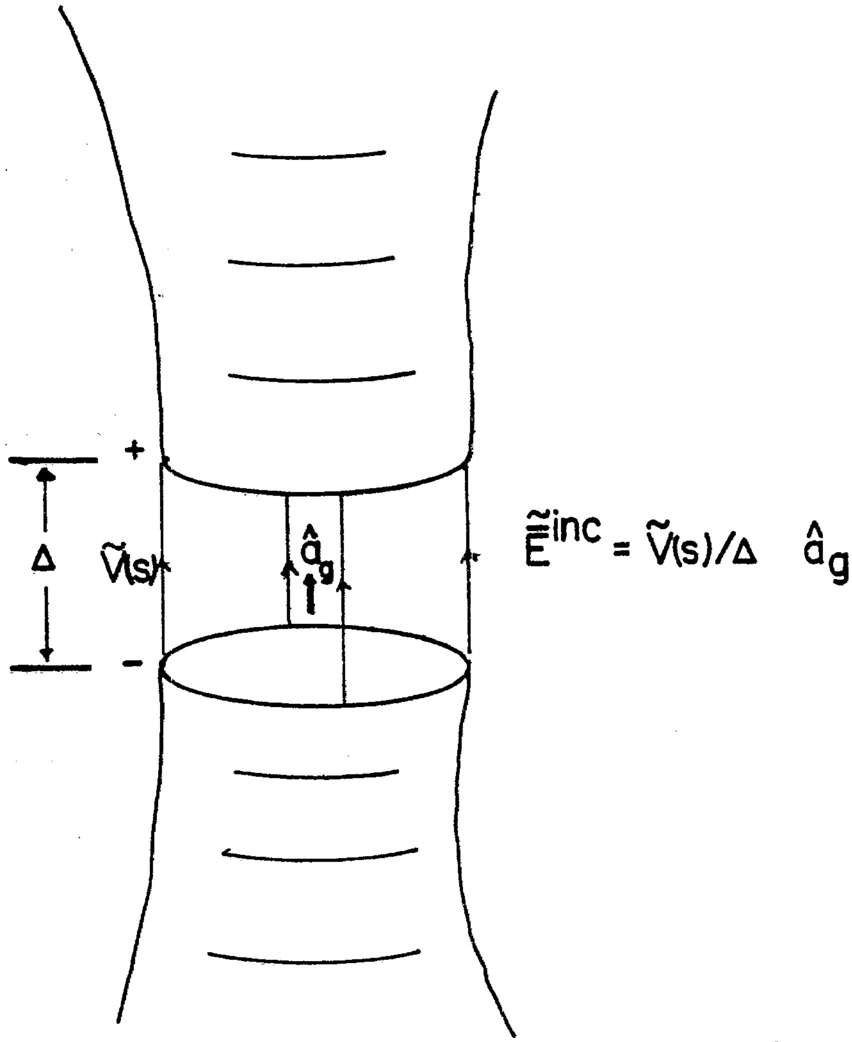

Figure 2. Detail of the feed gap for the admittance problem.

$$\tilde{\bar{J}}^{ant}(\bar{r},s) = \frac{\tilde{V}(s)}{\Delta} \sum_n \frac{1}{\lambda_n(s)} \frac{\langle \tilde{\bar{J}}_n(\bar{r}',s) \; ; \; \hat{a}_g \rangle_{S_g}}{\langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{J}}_n(\bar{r},s) \rangle} \tilde{\bar{J}}_n(\bar{r},s) \; . \tag{6}$$

The S_g on the bracket of the coupling integration in the numerator indicates that the extent of the integration is limited to the surface area comprising the gap region of the structure. The net current flowing through the gap region is representable as an azimuthal integration around the gap of the expression (6). This integration may be stated formally as $1/\Delta$ times the integration over S_g. Thus,

$$\begin{aligned} \tilde{I}(s) &= \frac{1}{\Delta} \langle \hat{a}_g \; ; \; \tilde{\bar{J}}^{ant}(\bar{r},s) \rangle_{S_g} \\ &= \frac{\tilde{V}(s)}{\Delta^2} \sum_n \frac{1}{\lambda_n(s)} \frac{\langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \hat{a}_g \rangle^2_{S_g}}{\langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{J}}_n(\bar{r},s) \rangle} \; . \end{aligned}$$

The eigenfunction expansion of the admittance $\tilde{Y}(s)$ follows directly:

$$\begin{aligned} \tilde{Y}(s) &= \tilde{I}(s)/\tilde{V}(s) \\ &= \sum_n \frac{1}{\lambda_n(s)} \frac{\langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \hat{a}_g \rangle^2_{S_g}}{\Delta^2 \langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{J}}_n(\bar{r},s) \rangle} \qquad (7) \\ &= \sum_n \tilde{Y}_n(s) \; . \end{aligned}$$

We introduce the notation $\tilde{Y}_n(s)$ to represent the elements of the summation in equation (7). We term these $\tilde{Y}_n$ "terminal eigenadmittances". We use this terminology to explicitly distinguish these terms from the eigenadmittance quantities which Baum defines in reference [7].

The short circuit current for the boundary value problem of Figure 1c is formally representable as an eigenfunction expansion

$$\tilde{\tilde{J}}^{sc}(\bar{r},s) = \Sigma \frac{1}{\lambda_n(s)} \frac{<\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \tilde{\tilde{E}}^{inc}(\bar{r},s)>}{<\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \tilde{\tilde{J}}_n(\bar{r},s)>} \tilde{\tilde{J}}_n(\bar{r},s) \; . \tag{8}$$

From this form, a short circuit current can be formed if one uses an integration identical to the one above.

$$\begin{aligned}
\tilde{I}^{sc}(s) &= \frac{1}{\Delta} <\tilde{\tilde{J}}^{sc}(\bar{r},s) \; ; \; \hat{a}_g>_{S_g} \\
&= \sum_n \frac{1}{\lambda_n(s)} \frac{<\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \tilde{\tilde{E}}^{inc}(\bar{r},s)> <\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \hat{a}_g>}{\Delta <\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \tilde{\tilde{J}}_n(\bar{r},s)>} \\
&= \sum_n \tilde{I}_n^{sc}(s)
\end{aligned} \tag{9}$$

Here we introduce the designation "terminal eigensource currents" to denote the $\tilde{I}_n^{sc}(s)$.

A useful form results from representing the terminal eigensource currents in terms of the terminal eigenadmittances and related voltage sources. The currents $\tilde{I}_n^{sc}$ may be rewritten as

$$\tilde{I}_n^{sc}(s) = \tilde{Y}_n(s) \; \tilde{V}_n(s) \; , \tag{10}$$

with

$$\tilde{V}_n(s) = \Delta \; \frac{<\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \tilde{\tilde{E}}^{inc}(\bar{r},s)>}{<\tilde{\tilde{J}}_n(\bar{r},s) \; ; \; \hat{a}_g>} \; . \tag{11}$$

The expressions above lead to the equivalent circuit representation shown in Figure 3. This circuit is a recasting of the equivalent circuit sought in Figure 1a. The recasting results from pairing the current sources and the admittance contributions due to individual elements in the eigenset. The form of the circuit suggests transforming from a Norton to a Thevenin form for each terminal eigensource/eigenadmittance pair, as shown in Figure 3b. The elements of this circuit are given, respectively, by equations (7) and (11). A caution must be given in regard to applying these forms when $\hat{a}_g$ lies at the null of a mode. Note that if the symmetric product $\langle \tilde{\tilde{J}}_n ; \hat{a}_g \rangle_{S_g} = 0$, $\tilde{Y}_n$ vanishes quadratically in this product while $\tilde{V}_n$ has a reciprocal singularity. Thus, for such a terminal location, although $\tilde{V}_n \to \infty$, the leg of the circuit contributes nothing at frequencies where $\langle \tilde{\tilde{J}}_n ; \hat{a}_g \rangle_{S_g} = 0$. If this occurs at a pole, the branch of the circuit associated with that pole vanishes.

III. FORMAL DEVELOPMENT OF THE EQUIVALENT CIRCUIT: SINGULARITY EXPANSION OF ELEMENTS

The singularity expansion for surface current quantities on a scattering object can be obtained from the eigenfunction expansion by performing the residue series expansion of expressions such as equation (4) in terms of the poles or complex natural resonances of the object. These poles are the complex frequencies s_{ni} at which the eigenvalues of the integral operator vanish, viz.,

$$\{s_{ni} : \lambda_n(s_{ni}) = 0\} .$$

Recently, we have uncovered evidence that branch points may be present in the complex s-plane, and that at least two eigenvalues are degenerate at these frequencies. Marin and Latham [12] argue the absence of branch points in the complex plane representation of the eigenvalues. However, their argument is based on an initial assumption of distinct eigenvalues, whereas, in fact, the

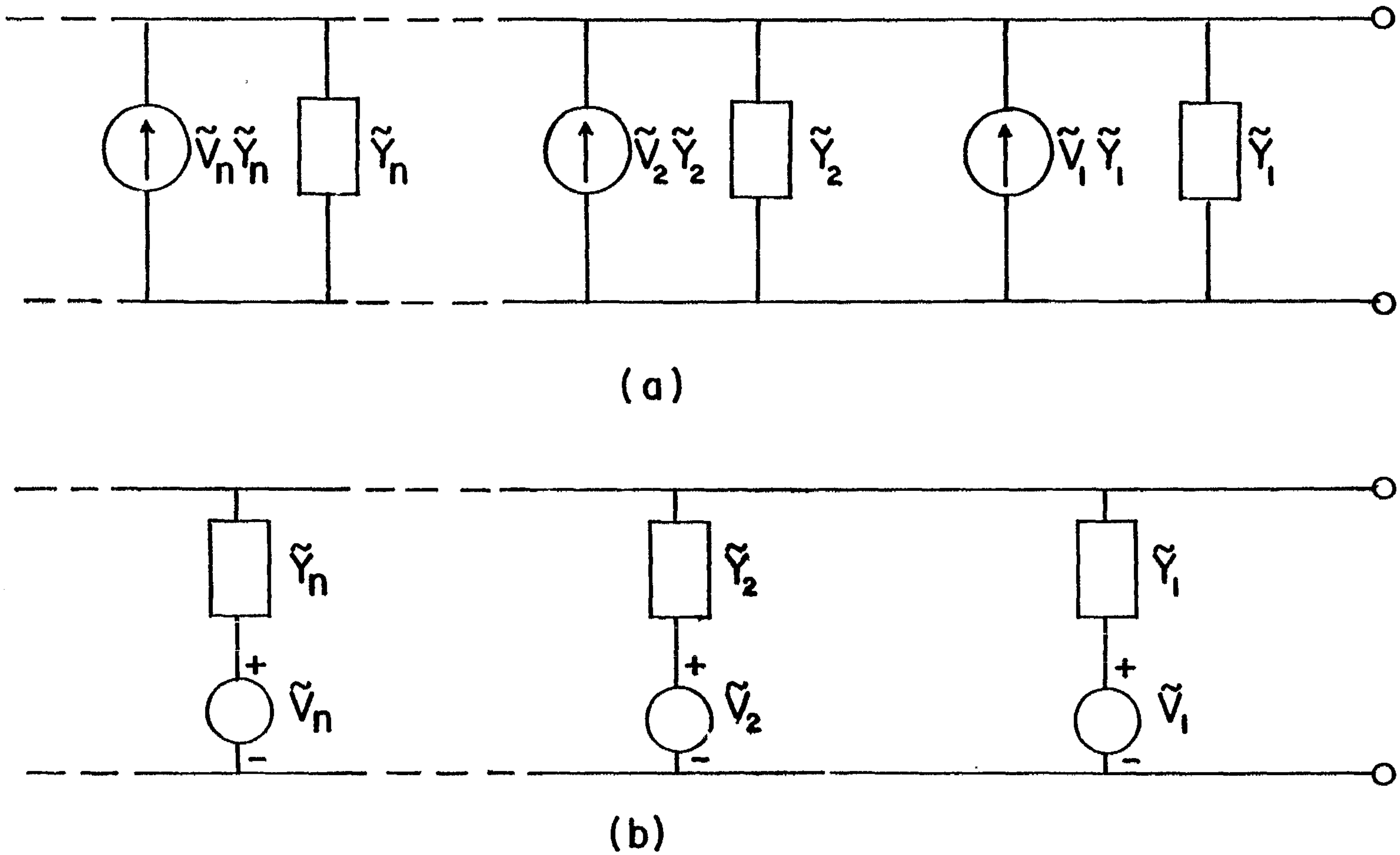

Figure 3. The Norton equivalence circuit organized into eigenelement pairs: (a) canonical Norton form; and (b) Thevenin eigenset branches.

branch points actually originate from the eigenvalue degeneracies [13]. However, they correctly argue that for an object immersed in a lossless medium the overall representation of the current is free of branch integral contributions. There appears to be no inconsistency here, for in the summation over all the eigenvalues as indicated in equation (4), the branch integral contributions contributed by all of the eigenvalues associated with a given branch sum to zero. The evidence of such a phenomenon occurring in singularity expansion representations for scattering objects is, at this writing, sketchy, although we plan to elaborate on this subject in the future. Because the presence of branch integrals in individual eigenvalue representations is potentially highly significant in terms of the physical realizability of the admittances shown in Figure 3, we shall explicitly include branch integral contributions in our admittance representations for the moment.

The reciprocal eigenvalue factor which appears in equation (4) may be expanded in a residue series expansion [6] to yield

$$\frac{1}{\lambda_n(s)} = \sum_i \left[\frac{r_{ni}}{s - s_{ni}} + \frac{r_{ni}}{s_{ni}}\right] + \tilde{b}_n(s) + \tilde{e}_n(s) \ . \tag{12}$$

The residues in this expansion are given by

$$\begin{aligned} r_{ni} &= [d\lambda_n(s_{ni})/ds]^{-1} \\ &= \left\{\frac{\langle \tilde{\bar{J}}_n(\bar{r},s) \ ; \ d\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s_{ni})/ds \ ; \ \tilde{\bar{J}}_n(\bar{r}',s)\rangle}{\langle \tilde{\bar{J}}_n(\bar{r},s) \ ; \ \tilde{\bar{J}}_n(\bar{r},s)\rangle}\right\}^{-1} \ . \end{aligned} \tag{13}$$

This form results from applying residue calculus to the Rayleigh quotient representation of the eigenvalues. The $\tilde{b}_n(s)$ are contributions to the eigenvalue expansion arising from branch integrals and the $\tilde{e}_n(s)$ are entire functions which, according to the Mittag-Leffler theorem, may also be required. Neither of these types of functions are represented explicitly here, but are included in the expansion above for the sake of completeness. It appears, to date, that for practical circuit synthesis it is necessary that both of these terms be zero. No completely conclusive arguments have been put forth that they are zero in the general situation.

The use of the expansion (12) and the residue definition (13) in the terminal eigenadmittances of (7) yields the following:

$$\tilde{Y}_n(s) = \sum_i \beta_{ni} \frac{\langle \tilde{\bar{J}}_n(\bar{r},s_{ni}) \; ; \; \hat{a}_g \rangle^2_{S_g}}{\Delta^2} \left[\frac{1}{s - s_{ni}} + \frac{1}{s_{ni}}\right] + \tilde{Y}^{bi}_n(s) + \tilde{Y}^{ent}_n(s) \; , \tag{14}$$

where

$$\beta_{ni} = \langle \tilde{\bar{J}}_n(\bar{r},s_{ni}) \; ; \; d\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s_{ni})/ds \; ; \; \tilde{\bar{J}}_n(\bar{r}',s_{ni})\rangle^{-1} \; .$$

The $\tilde{Y}^{bi}_n$ and the $\tilde{Y}^{ent}_n$ are, respectively, admittance contributions from the branch integral function $\tilde{b}_n$ in (12) and from the entire function $\tilde{e}_n$. We emphasize that for numerically or experimentally derived SEM descriptions neither of these terms are explicitly identifiable. By defining

$$A_{ni} = \beta_{ni} \frac{\langle \tilde{\bar{J}}_n(\bar{r},s_{ni}) \; ; \; \hat{a}_g \rangle^2_{S_g}}{\Delta^2} \; ,$$

we may compactly write the expansion (14) as

$$\tilde{Y}_n(s) = \sum_i A_{ni} \left[\frac{1}{s - s_{ni}} + \frac{1}{s_{ni}}\right]$$

$$= \sum_i \tilde{Y}_{ni}(s) \ . \qquad (15)$$

The branch integral and entire function terms are dropped here and henceforth in expressions for $\tilde{Y}_n$. The potential consequences of the presence of a non-zero $\tilde{Y}_n^{bi}$ are discussed in Section IV. The $\tilde{Y}_{ni}(s)$ are termed modified pole admittances*. We note that $\tilde{\bar{J}}_{ni}(\bar{r},s_{ni})$ solves

$$\langle \tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s_{ni}) \ ; \ \tilde{\bar{J}}_n(\bar{r}',s_{ni})\rangle = 0$$

and hence is the natural mode of the object associated with the pole s_{ni}.

It is evident from (10) and (11) that the terminal eigensource currents may be expanded in the same fashion to yield

$$\tilde{I}_n^{sc}(s) = \sum_i \tilde{V}_n(s_{ni}) \ A_{ni} \left[\frac{1}{s - s_{ni}} + \frac{1}{s_{ni}}\right]$$

$$+ \tilde{I}_n^{bi}(s) + \tilde{I}_n^{ent}(s)$$

$$= \sum_i \tilde{V}_{ni} \ \tilde{Y}_{ni} + I_n^{bi}(s) + I_n^{ent}(s) \ . \qquad (16)$$

We term the $\tilde{V}_{ni}$ pole source voltages. Through precisely the same reasoning as that used on the eigensource current/eigenadmittance pairs, we interpret (15) and (16) as representing parallel combinations of legs comprising series connections of $\tilde{Y}_{ni}(s)$ and $\tilde{V}_{ni}(s)$.

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*We use the term "modified" to be consistent with Baum in [8]. There he defines an "unmodified" pole admittance $A_{ni}/(s - s_{ni})$, as well.

IV. POSITIVE-REAL CONSIDERATIONS

The fundamental condition which dictates the physical realizability of any impedance or admittance is that it be a positive-real (PR) function of the complex frequency. This condition results from the fact that a passive structure (or network) can never dissipate a negative amount of energy. Since the admittance function (7) represents that manifested by a passive energy-collecting object, it must be a PR function. If this admittance is to be approximately realized by a lumped-parameter circuit, it is necessary that the summation of (7) be truncated to some finite n. Further, each $\tilde{Y}_n(s)$ embraces a summation over a potentially infinite set of poles, as expressed in (14). This summation, too, must be truncated for circuit realization. The circuit realization will encompass a combination of circuits representing terms in either the summation of (14) or of (7). Therefore, two issues emerge relative to the way in which this approximate realization is approached: 1) Is it possible to truncate the summation(s) of (14) and (7) in such a way as to maintain PR properties, and 2) are the elemental terms in the summand of (14) or the $\tilde{Y}_n(s)$ of (7) individually PR? This section deals with these issues, as we presently understand them.

Based on the discussion of the PR-ness considerations of the terminal eigenadmittances found in the Appendix, it is useful to postulate that the terminal eigenadmittances are PR functions. At the present writing, this postulate is subject to verification--indeed, it must be verified on a case-at-a-time basis. Nevertheless, it is useful to think in terms of PR terminal eigenadmittances, and modify our thinking should exceptions occur.

To have terminal eigenadmittances that are positive-real functions of frequency is a particularly robust result. First of all, it provides a means for truncating the summation given in (7). Since any terminal eigenadmittance term is by itself PR, then any finite sum of terminal eigenadmittances is also PR. Thus, the summation in (7) may be truncated to produce a PR approximation to $\tilde{Y}(s)$. The truncation may be based either on temporal or on spatial frequency of the excitation. The lumped-parameter circuit synthesis problem thus reduces to one of synthesizing terminal eigenadmittances, such as that expressed in (14).

The synthesis of the terminal eigenadmittances would be carried out in the most favorable fashion if the terms in the summation of (15) could be dealt with a conjugate pole-pair at a time. Conjugate pairs of modified pole admittances are required to provide the conjugate symmetry associated with PR functions. Synthesizing these terms on a pole-pair basis limits the constituent admittances which are dealt with to second order in s. Unfortunately, a number of counterexamples to the PR-ness of these conjugate pole-pair terms are described in reference [11].

For some structures, the poles associated with a given eigenvalue comprise a finite collection. An analytically tractable example of a structure manifesting a finite collection of exterior poles associated with each eigenvalue is the sphere [3]. One of the authors of the present paper (DRW) has recently engaged in a limited study of the eigenvalue/pole associations present for the straight wire scatterer. The evidence uncovered in this numerical study of the eigenvalues for the wire indicates a collection of poles distributed along an arc in the s-plane, as manifested by the sphere. On the other hand, a wire-loop structure evidences an infinite number of poles associated with each eigenvalue [14]. The sphere and wire structures may be differentiated from the loop on a topological basis: Both the sphere and the wire are simply-connected structures, while the loop is a multiply-connected structure.

For structures where the pole set is finite for a given eigenvalue, the summation indicated in (14) is a finite summation. This resolves the truncation issue in the summation expressions for terminal eigenadmittances. In the case of the loop, there is a dominant collection of poles which is of finite extent (the so-called type 1 and type 2 poles [14]), while the infinite collection is systematic in its form (the so-called type 3 poles). The numerical study reported in reference [11] indicates that the sum in (14) converges to a positive-real value for a reasonable number of type 3 poles retained in the summation of (14).

Perhaps more important than the truncation issue in determining the positive-realness of (14) is the role of the two terms $\tilde{Y}_n^{bi}$ and $\tilde{Y}_n^{ent}$. There is no evidence to date that an entire function contribution apart from a constant ever arises in the SEM representation. On the other hand, it seems likely that for many geometries branch integral terms do appear in the singularity expansion for reciprocal eigenvalues. When the eigenadmittances are all summed together, the branch integral contributions must vanish since the terminal admittance is known to be meromorphic. However, the branch integrals must be retained in the terminal eigenadmittance representation in order to insure their PR-ness. The numerical study reported in reference [11] indicates that the terminal eigenadmittance derived from a straight wire driven at its one-quarter point manifests a non-PR eigenadmittance for the third eigenset. Investigation of this lack of PR-ness indicates that the departure from PR is sufficiently drastic that it is not explainable in terms of numerical error in the derived SEM parameters. It is conceivable that the failing for this particular case occurs because of the absence of the branch integral in the finite pole sum taken from the numerical SEM data or due to failure of $\tilde{Y}_n(s)$ to be PR. The issues of the occurrence of branch integrals and their consequences on PR properties of terminal eigenadmittances bear further investigation. This investigation is substantially complicated by the fact that most problems which are treated by means of the Singularity Expansion Method are handled on a numerical basis. Indeed, our early investigations of the occurrence of branch integrals seem to indicate that their presence results from asymmetry in the structure. Thus, structures which are highly symmetrical, such as the sphere and the loop, would not be expected to yield branch integrals in their analytically obtained eigenvalues.

The numerical study reported in [11] indicates that for both the straight wire scatterer and for the loop, the single dominant resonant pole-pair (i.e., the pole-pair located nearest the $j\omega$ axis) is almost PR when taken alone. Furthermore, this single pole is the dominant contributor to the summation portion of the terminal eigenadmittance represented in (14). If we choose the indices +1 and -1 to denote the conjugate constituents of the dominant pole-pair in the residue expansion for the reciprocal eigenvalues and for the terminal eigenadmittances, we may write

$$\tilde{Y}_n(s) \cong \tilde{Y}_n^{res}(s)$$

$$= \beta_{n1} \frac{\langle \tilde{\tilde{J}}_n(\bar{r}, s_{n1}) \; ; \; \hat{a}_g \rangle_{S_g}^2}{\Delta^2} \left[\frac{1}{s - s_{n1}} + \frac{1}{s_{n1}} \right]$$

$$+ \beta_{n1}^* \frac{\langle \tilde{\tilde{J}}_n(\bar{r}, s_{n1}^*) \; ; \; \hat{a}_g \rangle_{S_g}^2}{\Delta^2} \left[\frac{1}{s - s_{n1}^*} + \frac{1}{s_{n1}^*} \right] . \qquad (17)$$

We emphasize that the presumption of this approximation is an empirical one, based on observation of the straight wire and loop results alone. The occurrence of a significant branch integral contribution to $\tilde{Y}_n$ is certainly one means whereby this result might be invalid. For the numerical examples tested in [9], these "almost PR" admittances can be adjusted to be approximated by a PR admittance. The adjustment required is negligible in magnitude in these two examples. Thus, we are able to write the approximation

$$\tilde{Y}_n(s) \cong G_n + \tilde{Y}_n^{res}(s) ,$$

where G_n is a small, positive conductance added to adjust $\tilde{Y}_n^{res}$ to be PR. If this approximation is possible with some degree of generality, it constitutes a particularly robust result. Namely, the terminal eigenadmittances for a structure may be approximately represented in terms of second-order admittances comprising an adjusted pole-pair admittance. Such a second-order admittance is relatively simple to realize.

V. APPLICABLE CIRCUIT SYNTHESIS TECHNIQUES

The problem of translating a positive-real rational function, such as one derives for a pole-pair or for an eigenset of poles, falls in the realm of classical network synthesis. The applicable synthesis methods have existed for a long time [15,16,17]. To the end of providing a realization of equivalent admittances in terms of RLC networks, we present a brief overview of the methods available. Although there exist methods which include ideal transformers and/or gyrators in the realization, we avoid these realizations since many applications preclude the use of these devices (e.g., due to power-handling considerations).

The classical approach to the synthesis of a lossy one-port is the Brune method [15,16]. The Brune method realizes the sought-after admittance or impedance in terms of a ladder network. This method was significant at the time that it was originally formulated, because it was the first successful effort to realize a general lossy PR one-port. The use of the ladder topology introduces a constraint which results in an undesirable element in the realization. This undesirable element appears either as a perfectly-coupled transformer or a negative inductance. Neither of these elements is realizable without the use of active devices, of course. Therefore, the Brune procedure offers little benefit for present purposes.

The Bott-Duffin procedure was the first procedure developed which allowed the synthesis of a general PR one-port admittance or impedance with a network comprising only resistors, capacitors, and inductors. The Bott-Duffin procedure is useful for formal circuits in the present work because of its generality and its relative amenability to automated implementation. It is somewhat doubtful, however, that it is useful for the synthesis of implementable circuits. This doubt stems from the fact that the Bott-Duffin network takes the form of a balanced bridge for all frequencies [16]. Indeed, there is a measure of arbitrariness in the network in that any impedance which one chooses may be inserted in the so-called detector leg of the bridge. That the network constitutes a balanced bridge produces a high measure of sensitivity of the network response to errors in component values in the network. This fact is dramatically demonstrated in the present context by an example given in reference [11]. Balabanian gives some transformations of the Bott-Duffin topology which result from different choices of the arbitrary impedance in the bridge leg of the network [16]. It is conceivable that some or all of these transformations can provide a utilitarian synthesis by suppressing the sensitivity to component errors.

The Miyata synthesis procedure [15,16] offers a more practical synthesis method than either the Brune or the Bott-Duffin procedures. The circuits which result from this procedure are always RLC and would seem to present less sensitivity to component errors. The implementation of this method is a bit more complicated than the implementation of either of the previous two, because no single format of the synthesis is applicable to a general PR function. The method must be adapted on what is quite nearly an ad hoc basis to make adjustments in the particular form of the rational function with which one deals [15].

The fourth alternative available as a synthesis procedure is the Darlington method [17]. This method realizes the admittance in terms of a lossless two-port which is terminated in a resistance. Although the Darlington procedure produces networks which are relatively insensitive to component error, it has the drawback that it may or may not result in a purely RLC network. The method for some admittances or impedances can introduce either gyrators or ideal transformers.

VI. SOURCE SYNTHESIS CONSIDERATIONS

While positive real considerations constitute the principal issue in the realizability of the admittance constituents of the equivalent circuit, we must consider the scheme of implementation carefully in postulating the means whereby the associated generator elements might be realized. We consider two generator forms here: namely, the eigenvoltage sources given by (11) and the pole voltage sources given by (16).

Most cases of practical interest for broadband circuit synthesis involve an energy collector which is excited by what we shall term a "factorable" incident field. That is, a field which can be written in the form

$$\tilde{\bar{E}}^{inc}(\bar{r},s) = \bar{F}(\bar{r})\ \tilde{p}(\bar{r},s)\ \tilde{f}(s)\ . \qquad (18)$$

Here, $\bar{F}(\bar{r})$ indicates a spatial distribution factor, $\tilde{p}(\bar{r},s)$ is a propagation factor which links space and time dependencies, and $\tilde{f}(s)$ is the Laplace transform of the time history of the wave. For example, a plane wave with a time history which is a Dirac function, $\delta(t)$, may be written

$$\tilde{\bar{E}}^{inc}(\bar{r},s) = \bar{E}_0 \cdot (\bar{\bar{I}} - \hat{n}\hat{n})\ e^{-s\hat{p}\cdot\bar{r}/c}\ \ 1 .$$

The factor 1 is the transform of $\delta(t)$, $\bar{E}_0$ is a constant polarization vector, and $\hat{p}$ is a unit vector in the direction of propagation. If the plane wave carries a time history $f(t) = \mathcal{L}^{-1}\{\tilde{f}(s)\}$, then the representation is

$$\tilde{\bar{E}}^{inc}(\bar{r},s) = \bar{E}_0 \cdot (\bar{\bar{I}} - \hat{n}\hat{n})\ e^{-s\hat{p}\cdot\bar{r}/c}\ \ \tilde{f}(s) .$$

A highly desirable scheme of implementation for the eigensource form would introduce $f(t)$ into only a single port. This concept is understood more clearly with a reference to Figure 4. A voltage $v(t) = \sqrt{N}\ f(t)$ is introduced at the port labeled "INPUT". The introduction of $f(t)$ at a single port allows the use of the network with any input waveform which adheres to the bandwidth limitations inherent in the circuit design. The input is divided into N identical signals $f(t)$, where N is the number of constituent eigensources. These sources would be processed by transfer functions $\tilde{G}_n(s)$ which modify the spectrum of $\tilde{f}(s)$ to produce $\tilde{V}_n(s)$ as given by (11). This transfer function is identifiable from (11) and (18). Viz.,

$$\tilde{V}_n(s) = \Delta \frac{\langle \tilde{\bar{J}}_n(\bar{r},s)\ ;\ \bar{F}(\bar{r})\ \tilde{p}(\bar{r},s)\rangle}{\langle \tilde{\bar{J}}_n(\bar{r},s)\ ;\ \hat{a}_g\rangle}\ \tilde{f}(s)$$

$$= \tilde{G}_n(s)\ \tilde{f}(s) . \tag{19}$$

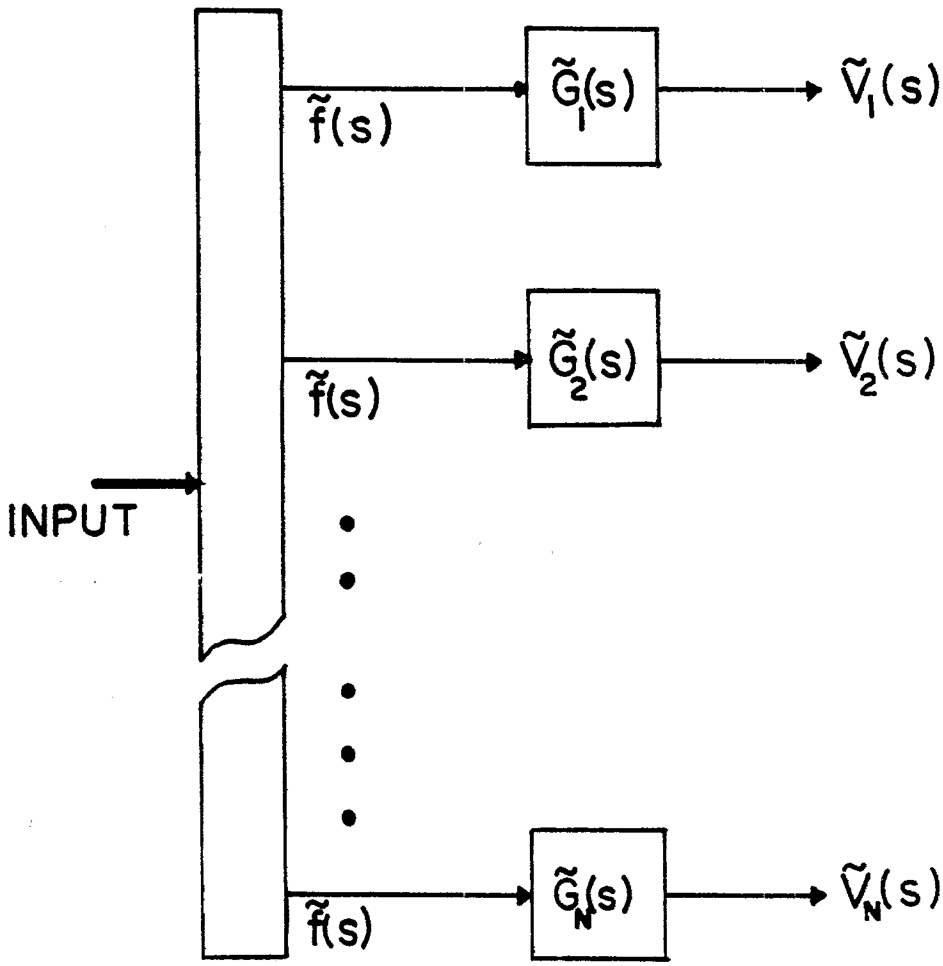

Figure 4. A practical format for realizing the source voltages. The single input allows variation in excitation waveshape.

In this implementation, the issue becomes, then, the realizability of the $\tilde{G}_n(s)$. Note that the $\tilde{G}_n$ depend on the source configuration that produces the incident field--i.e., on parameters $\bar{F}$ and $\tilde{p}$. If, for example, in the case of plane wave excitation, the angle of incidence is changed, then the transfer functions $\tilde{G}_n$ also change.

If the circuit may be limited to a realization of the energy collecting properties of the object for a single given excitation configuration, then the collection of transfer functions $\tilde{G}_n(s)$ is fixed. It is conceivable that through the well understood approximation and synthesis methods for two-port networks, the $\tilde{G}_n(s)$ might be realized, at least in an approximate fashion.

Another practical possibility is the realization of transfer functions which are constant amplitude and linear phase shift with respect to frequency. That is, they are realizable in terms of an attenuation and a time delay. Clearly, such a realization is possible only if the $\tilde{G}_n(s)$ in (19) are independent of frequency, at least in an approximate sense. We shall explore this possibility by means of the example of the straight wire scatterer.

The eigenmodes of the straight wire scatterer are observed to be essentially independent of frequency and can be approximated by trigonometric functions. For the coordinate system indicated in Figure 5, we write the net longitudinal current eigenmodes on the wire as

$$\tilde{I}_n(z,s) \cong \begin{cases} \cos\frac{n\pi}{2h}z \,, & n \text{ odd} \\ \sin\frac{n\pi}{2h}z \,, & n \text{ even} \,. \end{cases} \tag{20}$$

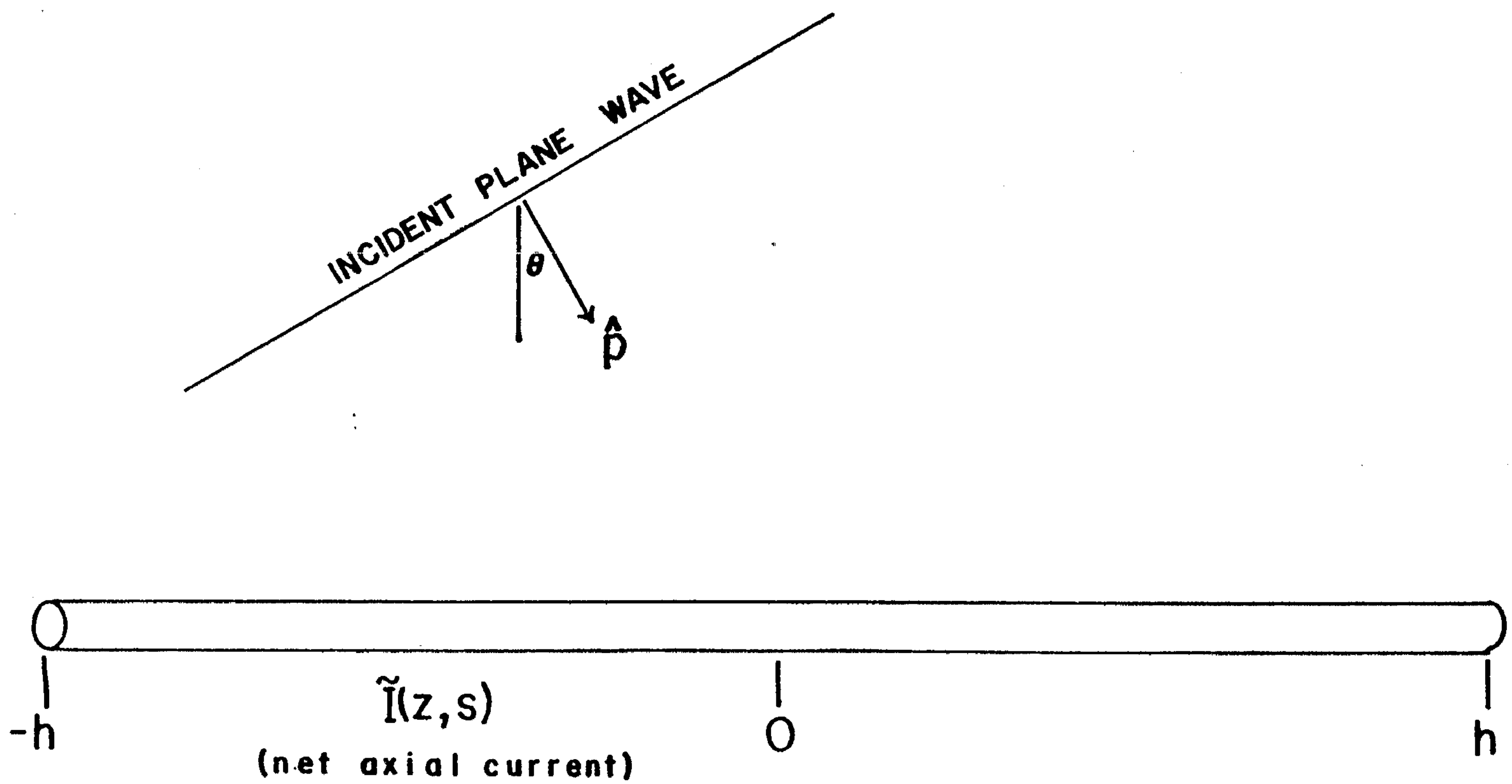

Figure 5. Incident wave geometry for the straight wire sources.

The z component of the incident plane wave is expressible on the wire as

$$\tilde{E}_z^{inc}(z,s) = \tilde{f}(s)\, E_0 \cos\theta\, e^{-sz\sin\theta/c} \quad . \tag{21}$$

Thus,

$$\tilde{G}_n(s) = \frac{E_0\cos\theta \int_{-h}^{h} \tilde{I}_n(z,s)\, e^{-sz\sin\theta/c}\, dz}{\tilde{I}_n(z_g,s)} \quad . \tag{22}$$

Evaluation of the integral for the modes approximated by (20) yields

$$\tilde{G}_n(s) \cong \frac{4h\, E_0\cos\theta}{n\pi \quad s^2\left\{\left[\frac{2h\sin\theta}{n\pi c}\right]^2 + 1\right\}} \begin{cases} (-1)^{(n-1)/2}\, \dfrac{\cosh(sh\sin\theta)/c}{\cos(n\pi z_g/h)} \; , & n \text{ odd} \\ (-1)^{n/2}\, \dfrac{\sinh(sh\sin\theta)/c}{\sin(n\pi z_g/h)} \; , & n \text{ even} \; . \end{cases} \tag{23}$$

Recall that we indicate in Section II that a voltage source and hence the associated $\tilde{G}_n$ is not needed when the port lies at a null of the mode. Hence, we never evaluate (23) in a situation that produces a zero in the denominator.

It is clear in (23) that the source transfer functions are significantly dependent on frequency and that the nature of this frequency dependence changes with the incident angle θ. Only in the broadside incidence case, $\theta = 0$, are the $\tilde{G}_n(s)$ frequency independent. This indicates in a concrete fashion what one is tempted to discern from (19): The frequency dependence in the source transfer functions stems from phase delay as the incident field propagates across the object. This dependence exhibited appears physically realizable, however: the polynomial denominator in terms of a resonant circuit and the hyperbolic functions in terms of time delays. The time delays are recognizable by factoring a common time advance and shifting the time origin to suppress this advance.

Another tempting alternative is to exploit the frequency independent character of the pole voltage sources of (16). However, the admittances associated therewith are realizable on a conjugate pair basis. Thus, the two pole terms must be combined. We adopt the notation that i and -i constitute the indices for a conjugate pair of poles for a given eigenfunction. Thus, the conjugate pair admittance is

$$\tilde{Y}^{cp}_{ni}(s) = \tilde{Y}_{ni}(s) + \tilde{Y}_{n,-i}(s)$$

and the pole current source is

$$\tilde{I}^{cp}_{ni}(s) = \tilde{Y}_{ni}(s)\ \tilde{V}_{ni} + \tilde{Y}_{n,-i}(s)\ \tilde{V}_{n,-i}\ . \qquad (24)$$

Baum [8] points out that only for the case that $\tilde{V}_{ni}$ is real such that $\tilde{V}_{ni} = \tilde{V}_{n,-i}$ does a frequency independent source result. He also points out that factoring (24) to obtain a voltage source in series with $\tilde{Y}^{cp}_{ni}(s)$ leads to frequency dependent voltage sources. The nature of the frequency dependence again changes with the incident field configuration since the $\tilde{V}_{ni}$ do.

The observations and comments in this section delve into only the more obvious approaches to generator synthesis. The approximation and synthesis of the transfer function form in (19) will possibly prove fruitful for a fixed spatial form of excitation. The simple attenuator/delay representation is likely to be useful for electrically-thin objects* when the incident wave traverses the electrically-thin direction.

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*Electrically-thin in terms of the shortest significant wavelength in the excitation spectrum.

VII. CONCLUSIONS

Several issues are discussed in this work relative to the practical realizability of equivalent circuits of energy collecting structures where the circuits are derived from the singularity expansion of the scattering response on the structure. The principal issue presented is whether or not component impedance or admittance functions which result are positive-real (and hence realizable) functions. A general result is that the terminal eigenadmittance quantities defined herein are positive-real functions of frequency.

We introduce the concept of approximating terminal eigenadmittances by the pole admittance of the most highly resonant pole in each eigenset. This approach is supported as to its practicability in reference [11] through the examples of the straight wire scatterer and the wire loop. It is emphasized that this approximation is *ad hoc* in character and must be tested for any specific structure to which it is applied. The cases cited produced admittances which were "almost positive-real"--which could be shifted negligibly to be made positive-real. The quality of the approximation and the near PR-ness of the resulting admittance is conceivably a consequence of the high Q character of the structures considered. For a low Q structure such as a spherical scatterer, it is not clear whether or not the admittance contribution of the dominant resonance pole in the eigenset will suffice or prove realizable.

Some of the practical considerations in implementing the source portions of the equivalent circuits are discussed. In particular, the desirability of formulating aspect and/or frequency independent transfer functions from the incident waveform's time history to the individual sources is suggested. Indications are that each of these features results only when geometric degeneracies are present and are not true in general. If the frequency and aspect dependent character of these transfer functions is borne out, the upshot is that the transfer functions must be realized through the approximation and realization procedures for two-port circuit synthesis. This realization will be specific to a given angle of incidence. Although a more general result is desirable, in many applications, such as the modeling of energy coupling into a scatterer which functions as a shield as well, the single-aspect-angle circuit can provide a useful worst case model.

It is suggested herein that for some structures--in particular those devoid of appreciable symmetry--branch integrals may constitute a portion of the terminal eigenadmittance representation along with the usual SEM pole terms. Though these branch integrals do not appear in the ultimate current representation, they can well compromise the realizability of the eigen-admittance segments into which the circuit realization is parceled. The evidence available regarding these branch integrals is sketchy, at best. Further work is warranted in this area, too.

In summary, the practical realizability of circuits which model the energy collecting properties of equivalent circuits over a broad range of frequencies appears possible. In candor, the procedures and understanding of realizability issues for a general structure must be declared primitive, at present.

VIII. ACKNOWLEDGMENTS

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Appendix

Consider a conducting scatterer S with unit surface normal $\hat{n}$ illuminated by an incident tangential electric field $\bar{E}^{inc}(\bar{r},t)$. A surface current $\bar{J}(\bar{r},t)$ is induced on S, producing a scattered electric field $\bar{E}^{sc}(\bar{r},t)$ which satisfies the boundary condition

$$\bar{E}^{inc}(\bar{r},t) + \bar{E}^{sc}_{tan}(\bar{r},t) = 0 , \qquad r \varepsilon S . \tag{A.1}$$

By the equivalence theorem, the scatterer may be replaced by a surface current $\bar{J}(\bar{r},t)$ on S radiating in free space. This current, when radiating in the absence of the sources of the incident field, produces the scattered field $\bar{E}^{sc}$ exterior to S.

In the absence of other sources, the total energy U radiated by the current distribution $\bar{J}$ is positive-semidefinite;

$$U(t) = \int_{-\infty}^{t} W(t)\ dt \geq 0 \tag{A.2}$$

where U is the energy and W(t) is the total power radiated by $\bar{J}$ at time t and is computed as follows:

$$\begin{aligned} W(t) &= -<\bar{E}^{sc}(\bar{r},t)\ ;\ \bar{J}(r,t)> \\ &= <\bar{E}^{inc}(\bar{r},t)\ ;\ \bar{J}(\bar{r},t)> . \end{aligned} \tag{A.3}$$

Although the current $\bar{J}(\bar{r},t)$ and the incident field $\bar{E}^{inc}(\bar{r},t)$ are related by an integral equation (1), we do not make use of this fact until later.

Next, suppose the incident electric field and induced current have the form

$$\bar{J}(\bar{r},t) = \tilde{\bar{J}}(\bar{r},s)\, e^{st} + \tilde{\bar{J}}^*(\bar{r},s)\, e^{s^* t}$$

$$= 2\, \mathrm{Re}\{\tilde{\bar{J}}(\bar{r},s)\, e^{st}\} , \qquad \text{(A.4a)}$$

and

$$\bar{E}^{inc}(\bar{r},t) = \tilde{\bar{E}}^{inc}(\bar{r},s)\, e^{st} + \tilde{\bar{E}}^{inc*}(\bar{r},s)\, e^{s^* t}$$

$$= 2\, \mathrm{Re}\{\tilde{\bar{E}}^{inc}(\bar{r},s)\, e^{st}\} , \qquad \text{(A.4b)}$$

where $s = \sigma + j\omega$ is a <u>fixed</u> complex number with $\sigma > 0$, $\tilde{\bar{E}}^{inc}(\bar{r},s)$ is an arbitrary complex vector function of position, and $\tilde{\bar{J}}(\bar{r},s)$ is the spatial distribution of the resulting complex current response. The excitation is assumed to begin at $t = -\infty$ at which time there is no initial energy in the system. Since $e^{\sigma t} = 0$ for $t = -\infty$, both $\bar{E}^{inc}$ and $\bar{J}$ are zero at $t = -\infty$ and there is no transient term. That is, the forced response (A.4a) is the total response.

The power radiated W(t) is now

$$W(t) = \langle\tilde{\bar{J}}(\bar{r},s)\ ;\ \tilde{\bar{E}}^{inc}(\bar{r},s)\rangle\, e^{2st}$$

$$+ \langle\tilde{\bar{J}}^*(\bar{r},s)\ ;\ \tilde{\bar{E}}^{inc*}(\bar{r},s)\rangle\, e^{2s^* t}$$

$$+ [\langle\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}^*(\bar{r},s)\rangle + \langle\tilde{\bar{E}}^{inc*}(\bar{r},s)\ ;\ \tilde{\bar{J}}(\bar{r},s)\rangle]\, e^{2\sigma t}$$

$$= 2\, \mathrm{Re}\{\langle\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}(\bar{r},s)\rangle\, e^{2st}$$

$$+ \langle\tilde{\bar{E}}(\bar{r},s)\ ;\ \tilde{\bar{J}}^*(\bar{r},s)\rangle\, e^{2\sigma t}\} .$$

Hence, the total energy radiated, from (A.2), is

$$U(t) = \mathrm{Re}\{<\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}(\bar{r},s)>\ e^{2st}/s$$

$$+ <\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}^{*}(\bar{r},s)>\ e^{2\sigma t}/\sigma\}\ . \qquad (A.5)$$

If we write the first of the above products in polar form,

$$<\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}(\bar{r},s)>/s = |<\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}(\bar{r},s)>/s|\ e^{j\phi}\ ,$$

then U(t) can be written as

$$U(t) = e^{2\sigma t}\ [|<\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}(\bar{r},s)>/s|\ \cos(2\omega t + \phi)$$

$$+ \mathrm{Re}\{<\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}^{*}(\bar{r},s)>/\sigma\}] \geq 0\ . \qquad (A.6)$$

We consider the cases $\omega = 0$ and $\omega \neq 0$ separately.

CASE I, $\omega = 0$.

$$U(t) = 2\ <\mathrm{Re}\{\tilde{\bar{E}}^{inc}(\bar{r},s)\}\ ;\ \mathrm{Re}\{\tilde{\bar{J}}(\bar{r},s)\}>\ e^{2\sigma t}/\sigma\ ,$$

whence

$$<\mathrm{Re}\{\tilde{\bar{E}}^{inc}(\bar{r},s)\}\ ;\ \mathrm{Re}\{\tilde{\bar{J}}(\bar{r},s)\}> \geq 0\ . \qquad (A.7)$$

Since $\tilde{\bar{E}}^{inc}(\bar{r},s)$ is arbitrary, we could replace it with $j\tilde{\bar{E}}^{inc}(\bar{r},s)$ and the corresponding current response would be $j\tilde{\bar{J}}(\bar{r},s)$. Hence, from (A.7) it follows that

$$<\mathrm{Re}\{j\tilde{\bar{E}}^{inc}(\bar{r},s)\}\ ;\ \mathrm{Re}\{j\tilde{\bar{J}}(\bar{r},s)\}> = <\mathrm{Im}\{\tilde{\bar{E}}^{inc}(\bar{r},s)\}\ ;\ \mathrm{Im}\{\tilde{\bar{J}}(\bar{r},s)\}> \geq 0\ . \qquad (A.8)$$

Equations (A.7) and (A.8) imply that

$$\mathrm{Re}\{<\tilde{\bar{E}}^{inc}(\bar{r},s)\ ;\ \tilde{\bar{J}}^{*}(\bar{r},s)>\} \geq 0\ . \qquad (A.9)$$

CASE II, $\omega \neq 0$.

Since $\min\{\cos(\omega t + \phi)\} = -1$, we conclude from (A.6) that

$$\mathrm{Re}\{<\tilde{\bar{E}}^{inc}(\bar{r},s) \; ; \; \tilde{\bar{J}}^{*}(\bar{r},s)>\}/\sigma \geq \left|<\tilde{\bar{E}}^{inc}(\bar{r},s) \; ; \; \tilde{\bar{J}}(\bar{r},s)>/s\right| \geq 0 \; . \qquad \text{(A.10)}$$

Thus, we again obtain

$$\mathrm{Re}\{<\tilde{\bar{E}}^{inc}(\bar{r},s) \; ; \; \tilde{\bar{J}}^{*}(\bar{r},s)>\} \geq 0 \; , \qquad \sigma > 0 \qquad \text{(A.11)}$$

as a necessary condition on the current on a passive scatterer.

Using (A.11), we may derive a condition similar to the positive-real condition on driving-point immittances in circuit theory. The condition applies to the dyadic kernel of (1). Recognizing that the quantities $\tilde{\bar{E}}^{inc}(\bar{r},s)$ and $\tilde{\bar{J}}(\bar{r},s)$ in the foregoing are the transformed field quantities in (1), we write

$$\mathrm{Re}\{<\tilde{\bar{J}}^{*}(\bar{r},s) \; ; \; \tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s) \; ; \; \tilde{\bar{J}}(\bar{r},s)>$$

$$= <\tilde{\bar{J}}^{*}(\bar{r},s) \; ; \; \mathrm{Re}\{\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s)\} \; ; \; \tilde{\bar{J}}(\bar{r},s)> \geq 0 \; . \qquad \text{(A.12)}$$

This result is obtained with the benefit of the dyadic reciprocity condition

$$\tilde{\bar{\bar{\Gamma}}}^{\dagger}(\bar{r},\bar{r}',s) = \tilde{\bar{\bar{\Gamma}}}^{*}(\bar{r},\bar{r}',s) = \tilde{\bar{\bar{\Gamma}}}^{*}(\bar{r}',\bar{r},s) \; .$$

Thus, $\mathrm{Re}\{\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s)\}$ is positive-semidefinite for $\sigma > 0$. The condition (A.12) is the operator counterpart to one of the so-called positive-real conditions for immittance matrices of multiport networks. We may establish counterparts to the remaining two conditions defining PR-ness for multiports by examining the dyadic kernel $\tilde{\bar{\bar{\Gamma}}}$. The conditions may be stated as follows:

(i) $\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s)$ is analytic for $\mathrm{Re}\{s\} > 0$;

(ii) $\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s)$ is real for real positive s;

with the positive-semidefinite condition of (A.12) restated in the notation

(iii) $\mathrm{Re}\{\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s)\} \geq 0$, $\mathrm{Re}\{s\} > 0$.

The immittance matrix counterparts to these conditions are necessary and sufficient that rational polynomial immittance matrices completely represent a network comprising passive elements. While we presently cannot establish sufficiency, the three conditions above are necessary for $\tilde{\bar{\bar{\Gamma}}}(\bar{r},\bar{r}',s)$ to represent a passive scattering process.

Not only the dyadic kernel, but also any input immittance quantity is positive-real when the object is excited as an antenna. To see this, we begin with (A.11) with the excitation specialized to a field impressed across a gap surface S_g, namely,

$$\tilde{\bar{E}}^{inc}(\bar{r},s) = \begin{cases} \hat{a}_g \tilde{V}(s)/\Lambda , & \bar{r} \in S_g \\ 0 , & \bar{r} \notin S_g . \end{cases} \qquad (A.13)$$

Then (A.11) becomes

$$\begin{aligned} \mathrm{Re}\{\tilde{V}(s) \langle \hat{a}_g ; \tilde{\bar{J}}^*(\bar{r},s)\rangle_{S_g} /\Lambda\} &= \mathrm{Re}\{\tilde{V}(s)\ \tilde{I}^*(s)\} \\ &= \mathrm{Re}\{\tilde{V}(s)\ \tilde{Y}^*(s)\ \tilde{V}^*(s)\} \\ &= |\tilde{V}(s)|^2\ \mathrm{Re}\{\tilde{Y}(s)\} \geq 0 , \qquad \sigma > 0 . \qquad (A.14) \end{aligned}$$

This implies that

$$\mathrm{Re}\{\tilde{Y}(s)\} \geq 0 , \qquad \sigma > 0 . \qquad (A.15)$$

The remaining two positive-real conditions follow from observation. Thus the input admittance from the short circuit boundary value problem is positive-real. Similarly, one may show that the input impedance obtained from the open circuit boundary value problem [8] is also positive-real.

Next, suppose that the dyadic kernel $\tilde{\tilde{\Gamma}}$ has an eigenvalue λ_n and corresponding eigenvector $\tilde{\bar{J}}_n$ satisfying (5). If we choose as an excitation

$$\tilde{\bar{E}}^{inc}(\bar{r},s) = \lambda_n(s)\ \tilde{\bar{J}}_n(\bar{r},s) \ ,$$

the response current is $\tilde{\bar{J}}_n(\bar{r},s)$ and (A.11) becomes

$$\mathrm{Re}\{\lambda_n(s)\ \langle\tilde{\bar{J}}_n(\bar{r},s)\ ;\ \tilde{\bar{J}}_n^*(\bar{r},s)\rangle\}$$

$$= \mathrm{Re}\{\lambda_n(s)\}\ \left\|\tilde{\bar{J}}_n(\bar{r},s)\right\|^2 \geq 0 \ , \qquad \sigma > 0 \ .$$

This implies that

$$\mathrm{Re}\{\lambda_n(s)\} \geq 0 \ , \qquad \sigma > 0 \ . \tag{A.16}$$

By observation, the remaining PR conditions may be established for the eigenvalues with the exception of the case of a $\lambda_n(s)$ with a branch point in the right half plane. We may admit even such an eigenvalue to an "extended" PR class where the "extended" implies "analytic except for branch points" in the right half plane.

Thus, eigenvalues (eigenimpedances) are positive-real. Since reciprocals of positive-real quantities are also positive-real, then the reciprocal eigenvalues (eigenadmittances) must also be positive-real;

$$\mathrm{Re}\ \frac{1}{\lambda_n(s)} \geq 0 \ , \qquad \sigma > 0 \ . \tag{A.17}$$

For scalar positive-real quantities, such as input immittances or eigenimmittances, it is well-known that the positive-real conditions are also equivalent to the following conditions [18]:

(a) $\tilde{f}(s)$ has no poles or zeros for $\sigma > 0$;

(b) poles of $\tilde{f}(s)$ on the imaginary axis must be simple and their residues must be real and positive; and

(c) $\mathrm{Re}\{\tilde{f}(j\omega)\} \geq 0$, $\omega \varepsilon (0,\infty)$,

where $\tilde{f}(s)$ is an arbitrary PR function. The last of these conditions is particularly important in testing PR-ness because it requires that $\tilde{f}(s)$ be examined only on the $j\omega$ axis.

The PR-ness of the terminal eigenadmittances may be explored based on the PR-ness of the eigenadmittances $1/\lambda_n(s)$ established above and on (7), which is restated here for convenience as

$$\tilde{Y}_n(s) = \frac{1}{\lambda_n(s)} \left\{ \frac{\langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \hat{a}_g \rangle^2_{S_g}}{\Delta^2 \; \langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{J}}_n(\bar{r},s) \rangle} \right\} . \tag{A.18}$$

It is convenient to define $\gamma_n(s) = 1/\lambda_n(s)$ and

$$\tilde{F}_n(s) = \frac{\langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \hat{a}_g \rangle^2_{S_g}}{\Delta^2 \; \langle \tilde{\bar{J}}_n(\bar{r},s) \; ; \; \tilde{\bar{J}}_n(\bar{r},s) \rangle} , \tag{A.19}$$

such that

$$\tilde{Y}_n(s) = (\gamma^r_n F^r_n - \gamma^i_n F^i_n) + j(\gamma^r_n F^i_n + \gamma^i_n F^r_n) ,$$

where the superscripts denote respective real and imaginary parts. It follows that PR-ness of $\tilde{Y}_n(s)$ hinges on the adherence to

$$\gamma^r_n F^r_n \geq \gamma^i_n F^i_n , \qquad \mathrm{Re}\{s\} \geq 0 . \tag{A.20}$$

Because of the complexity of (A.19), it is difficult to draw general conclusions regarding the satisfaction of (A.20). Clearly, one might test (A.20) on a numerical basis, but to do so would be computationally costly since an eigenvalue problem would need to be solved. For the case that the eigenmodes $\tilde{\tilde{J}}_n$ are pure real for $s = j\omega$, it follows directly from (A.19) and (A.20) that the terminal eigenadmittances $\tilde{Y}_n(s)$ are PR. This is the case for both the loop and sphere geometries, but the real modes for these structures devolve from symmetry degeneracies. On the other hand, the first few natural modes on the straight wire exhibit small real parts for s <u>near</u> the $j\omega$ axis, and (A.20) is likely to be satisfied.

With the respective high-Q and low-Q extremes of the straight wire and the sphere likely yielding PR terminal eigenadmittances, one might be tempted to draw broad conclusions. However, some common topological feature, such as convexity, might bear on the results for these two special cases, thereby qualifying any general conclusions which one might draw.

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