Interaction Notes

Note 383

February 1980

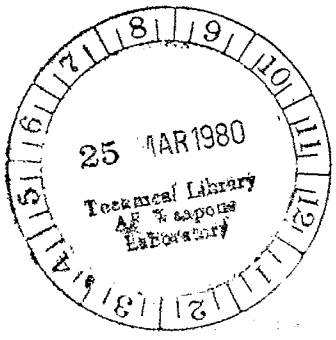
TIME-DOMAIN TECHNIQUES IN THE SINGULARITY EXPANSION METHOD

J. T. Cordaro
University of New Mexico

W. A. Davis
Virginia Polytechnic Institute and State University

ABSTRACT

This paper presents new methods of determining the electromagnetic poles and natural modes of a structure as characterized in the singularity expansion method (SEM). The method is based on the time-domain scattering equation which can be cast in the form of a matrix difference equation. The homogeneous solution of the difference equation is a series of exponentials as found in the SEM representation. The pole and natural mode solutions may be obtained either from the determinant of a matrix sum, which is similar to the current frequency domain search method, or by an eignevalue approach as in systems theory. The latter has shown promise for efficient SEM computations. An example of each approach is presented.



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J. T. Cordaro
University of New Mexico

and

W. A. Davis
Virginia Polytechnic Institute and State University

Introduction

Since the original presentation on the singularity expansion method (SEM) in 1971 (1), several researchers have endeavored to obtain poles and natural modes of several standard structures [2,3].

Primarily two techniques have been used to determine the poles. Frequency domain SEM is the most popular method with the poles determined by searching for the values of complex frequency which makes the determinant of an associated matrix problem zero. This process is extremely slow, often requiring a minute per pole. An alternate method has been to determine the time-domain response either by a time-domain solution method or by the inverse Fourier transform of frequency-domain solutions, and to obtain the poles from the time response by Prony's method.

This paper presents new methods of determining the poles and natural modes using what we shall call time-domain SEM. The method is based on the time-domain scattering equation, but side-steps the calculation of the time response. We will review the time-domain equations and their numerical representation. The numerical equations form a matrix difference equation with real matrix coefficients. The homogeneous solution of the difference equations is a series of exponentials characteristic of the SEM representation. We will review SEM and consider both the time-domain SEM matrix form of solution and an equivalent eigenvalue form of solution. To illustrate the

methods, we will present both the transmission line and wire scatterer problems.

The Singularity Expansion Method

The singularity expansion method (SEM) [4] represents the natural current response of a finite-size perfectly conducting object in free space by a sum of complex exponentials. This sum involves four parameters: natural frequencies, natural modes, coupling coefficients, and the incident field transform evaluated at the natural frequencies. When SEM is used to describe a scattering problem, the incident field is assumed to be known. This leaves the other three parameters to be computed. At least two different approaches have been used to find these remaining parameters.

The first approach depends on solving an integral equation of the form

$$\int_V \overset{\cong}{\Gamma}(\bar{r},\bar{r}';s)\cdot\tilde{\bar{J}}(r',s)\,dv' = \tilde{\bar{I}}'(\bar{r},\bar{s}) \quad , \quad \bar{r},\ \bar{r}' \in V \tag{1}$$

where $\tilde{\bar{I}}'$ is some forcing function, $\tilde{\bar{J}}$ is the current density response function, and $\overset{\cong}{\Gamma}$ is the kernel of the integral equation. A natural mode $\bar{\nu}_\alpha$ satisfies the equation

$$\int_V \overset{\cong}{\Gamma}(\bar{r},\bar{r}';s_\alpha)\cdot\bar{\nu}_\alpha(\bar{r}')\,dv' = 0 \tag{2}$$

where s_α is the corresponding natural frequency. This equation is converted to a matrix equation

$$(\tilde{\Gamma}_{n,m}(s_\alpha))\cdot(\nu_n)_\alpha = (0_n) \tag{3}$$

using the method of moments. Since the kernel is required to be singular at s_α the natural frequencies can be found from

$$\det((\tilde{\Gamma}_{n,m}(s_\alpha))) = 0. \tag{4}$$

When posed this way, the problem of finding the natural frequencies of an object is reduced to that of finding the zeros of a complicated function of s. Tesche [2] used this approach in his SEM analysis of the thin-wire scatterer. He found two layers of poles and speculated that other layers might exist. Several other techniques for finding solutions of Eq.(4) are discussed in [4]. Of interest here is a method due to Singaraju, Giri, and Baum who found a third layer of poles for the thin-wire problem [5]. Once the natural frequencies are known the natural modes can be computed easily from Eq.(3). The coupling coefficients can be computed as inner products of the natural modes and incident field as detailed in [4]. It is fair to say that the burden of computing the SEM parameters with this frequency domain approach is in finding solutions of Eq.(4).

A second approach has been used successfully to find the SEM parameters of a thin wire. In this approach the transient current response is generated using a time domain code. The SEM parameters are computed from these transient currents using Prony's method. Van Blaricum and Mittra [6] found natural frequencies and Pearson and Robertson [7] found both natural frequencies and natural modes that compare well with Tesche's results. The natural frequencies found were all from the first layer. There is no reason why this time domain method cannot be generalized and used to find the natural frequencies for more complicated objects. However, the method is indirect in that the transient response at several points on the body must be known before the SEM parameters can be found. This is the appropriate method for analyzing experimentally

measured transient data but not for analyzing theoretical models.

Time Domain SEM

For theoretical models, it is possible to avoid the time response computation and go directly to the SEM problem from the time domain viewpoint. The time-domain magnetic field integral equation for perfect electric conductors is given by [8]

$$\bar{J}_S(\bar{r},t) = 2\hat{n}\times\bar{H}^{inc} + \frac{\hat{n}}{2\pi}\times\oint_S \left\{\frac{1}{c}\,\frac{\partial}{\partial\tau}\,\bar{J}_S(\bar{r}',\tau) + \frac{\bar{J}_S(\bar{r}',\tau)}{R}\right\}\Bigg|_{\tau=t-R/c} \times \frac{\bar{R}}{R^2}\,ds' \qquad (5)$$

where S is closed surface, $R = |\bar{r}-\bar{r}'|$, and c is the speed of light. If S is thin, or approaches the same, then the electric field integral equation is appropriate.

$$\hat{n}\times\frac{\partial\bar{E}^{inc}}{\partial t} = \hat{n}\times\left[\mu\,\bar{\bar{I}}\,\frac{\partial}{\partial t^2} - \frac{1}{\varepsilon}\,\nabla\nabla\right]\cdot\int_S \frac{\bar{J}_S(\bar{r}',\tau)}{4\pi R}\Bigg|_{\tau=t-R/c} ds'. \qquad (6)$$

Both (5) and (6) may be put into the time-domain form of (1) given by

$$\int_{S,t} \bar{\bar{\Gamma}}(\bar{r},\bar{r}';\ t-t')\cdot\bar{J}_S(\bar{r}',t')\,dt'\,ds' = \bar{I}'(\bar{r},t) \qquad (7)$$

where the dyadic Green's function is a distribution in time containing ordinary functions and the Dirac delta, doublet, and triplet distributions. The process of digitizing (7) in the time-domain usually makes use of pulse expansion functions and finite difference derivatives to represent the effect of the doublet and triplet distributions. The expansion might take the form

$$\bar{J}_S(\bar{r},t) = \sum_m \sum_{n=1}^{N} \bar{J}_{mn}\,P_{\Delta_t}(t-m\Delta_t)\,S_n(\bar{r}) \qquad (8)$$

where P_{Δ_t} is a unit pulse of width Δ_t and S_n is unity over the n^{th} surface patch. The size of S_n and Δ_t are usually related. From this expansion and the finite difference operators, we may write either (5) or (6) in the form

$$J_{m+1} = \sum_{i=0}^{M} A_i \, J_{m-i} + F_m \quad , \quad m = 1,2,3,... \tag{9}$$

where J_m and F_m are vectors representing the current and forcing function over the structure at the m^{th} time step. The matrices A_i relate the spatial components of the current.

The natural frequencies and natural modes can be computed directly from this time domain model. To see how this works consider setting

$$J_{m+1} = Z_\alpha^{m+1} \nu_\alpha \quad , \qquad m = 1,2,3,... \tag{10}$$

and

$$F_m = 0$$

in equation (9). Here $Z_\alpha = \exp(s_\alpha \Delta t)$ and ν_α is a vector of N complex numbers spatially describing the natural mode. The implication is that an incident field has previously interacted with the structure and only the homogeneous time period remains. It follows that

$$Z_\alpha^{m+1} \nu_\alpha = \sum_{i=0}^{M} A_i \, Z_\alpha^{m-i} \nu_\alpha \, . \tag{11}$$

On rearranging, it is seen that Z_α and ν_α must satisfy

$$\left(I - \sum_{i=0}^{N} A_i \, Z_\alpha^{-(i+1)}\right) \nu_\alpha = 0 \tag{12}$$

where I is the NxN identity matrix. This equation is the discrete time analog of equation (3) above. The natural frequencies can be found from

$$\det\left(I - \sum_{i=0}^{N} A_i \, Z_\alpha^{-(i+1)}\right) = 0 \tag{13}$$

which is computationally simpler than equation (4). The elements of the matrix in (13) are easier to compute than those in (4) and since $Z_\alpha = e^{s_\alpha \Delta t}$, the frequency range of the possible s_α is explicitly limited by the time sampling rate. But it appears that solution of (13) still requires a search.

There is an alternative to this search. Define a state vector X_m by

$$X_m^T = [J_m^T, J_{m-1}^T, \ldots, J_{m-N}^T], \tag{14}$$

where T denotes transpose, and state transition matrix Φ by

$$\Phi = \begin{bmatrix} -A_0 & -A_1 & \cdots & -A_N \\ I & 0 & \cdots & 0 \\ 0 & I & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & & 0\ I & 0 \end{bmatrix}. \tag{15}$$

Then it follows that

$$X_{m+1} = \Phi \, X_m \quad , \quad m \geq 1 . \tag{16}$$

Now let Z_α be a non-zero eigenvalue of Φ and E_α a corresponding eigenvector. To simplify notation it is assumed that the eigenvalues of Φ are distinct. It is easy to verify that any eigenvector E_α is in the form

$$E_\alpha^T = \left[\nu_\alpha^T, \; Z_\alpha^{-1} \, \nu_\alpha^T, \; \ldots, \; Z_\alpha^{-N} \, \nu_\alpha^T\right] \tag{17}$$

by simply substituting this relation into (16) and using (12). More

importantly this shows that the eigenvalues Z_α of the Φ matrix yield the natural frequencies since $Z_\alpha = e^{s_\alpha \Delta t}$ and the first N components of the eigenvectors of Φ are the corresponding unnomalized natural mode vectors. Thus the problem of finding natural frequencies is transformed to an eigenvalue problem.

The Transmission Line

The simplest example of this method is the shorted transmission line of length L. For this problem, we have the equations

$$\frac{\partial I}{\partial t} = -\frac{1}{L}\frac{\partial V}{\partial z}$$

and

$$\frac{\partial V}{\partial t} = -\frac{1}{C}\frac{\partial I}{\partial z}$$

where $LC = 1/c^2$. Eliminating I we have

$$\frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = LV = 0 \tag{18}$$

Expanding V as in (8),

$$V(z,t) \approx \sum_m \sum_{n=1}^{N} V_{mn} P_{\Delta_t}(t-m\Delta_t)\ P\ (z-n\Delta). \tag{19}$$

For consistency in the time and space sample, we set $\Delta = c\Delta_t = L/(N+1)$. We have also imposed the short circuit constraint requiring $V_{m,0}$ and $V_{m,N+1}$ to be zero. In this instance we are looking for the homogeneous solution given by the solution of (18) and treat the problem as though $V_{m,0}$ or $V_{m,N+1}$ was non-zero at some past time.

Substituting (19) into (18) and using finite differences for the derivatives,

$$LV\Big|_{\substack{t=m\Delta_t \\ z=n\Delta}} = \frac{1}{2\Delta^2}\,[V_{m,n+1} + V_{m,n-1} - V_{m-1,n}]. \tag{20}$$

If $V_m = [V_{m,1}, \ldots, V_{m,N}]^T$, then

$$V_{m+1} = A_o V_m - V_{m-1}$$

with A_o given by unity in the super- and sub- diagonals. Eq. (12) takes the form

$$\begin{bmatrix} (1+Z^2) & -Z & 0 & \cdots & \cdot & \cdot \\ -Z & (1+Z^2) & -Z & 0 & \cdots & \cdot \\ 0 & -Z & & & & \cdot \\ \cdot & 0 & & & & \cdot \\ \cdot & \cdot & & & & 0 \\ \cdot & \cdot & & & & -Z \\ 0 & \cdot & \cdots & 0 & -Z & (1+Z^2) \end{bmatrix} \nu_\alpha = 0 \tag{21}$$

It is easily shown that the determinant of this matrix is given by

$$\text{Det} = \frac{Z^{2(N+1)}-1}{Z^2-1}$$

or

$$Z_\alpha = \exp\left[j\,\frac{\alpha\pi}{N+1}\right] \quad , \alpha \neq i(N+1).$$

Thus

$$s_\alpha = \frac{\alpha c \pi}{L} \tag{22}$$

which can be obtained directly from (18). Similar results may be obtained for arbitrary terminations with a percentage error up to about 100/N due to the discretization error in the finite difference description of the termination constraints.

The Wire Scatter

The transmission line problem has demonstrated the determinant method of (13). We will now consider the eigenvalue method of (16) for the

wire problem. Using the thin wire approximation for the current I on the wire, (6) may be written as

$$\frac{\partial}{\partial t} E_z^{inc}(z,t) = -\frac{1}{\varepsilon} L \int_0^L \frac{1}{8\pi^2 R} I(z', t - |z - z'|/c)\, dz' \tag{23}$$

where $R = [a^2 + (z-z')^2]^{\frac{1}{2}}$. Expanding I as

$$I(z,t) \simeq \sum_{m=-\infty}^{\infty} \sum_{n=1}^{N} I_{mn} P_\Delta(ct-m\Delta) P_\Delta(z-n\Delta) \tag{24}$$

and using finite differences, (23) becomes

$$\varepsilon \frac{\partial}{\partial t} E_z^{inc}(p\Delta, m\Delta/c) = \frac{1}{2\Delta^2} \sum_{n=0}^{N+1} G_{|p-n|} [I_{(q+1-|p-n|),n} + I_{(q-1-|p-n|),n}$$

$$- I_{(q-|p-n|),n+1} - I_{(q-|p-n|),n-1}] \tag{25}$$

where

$$G_n = \int_{(n-\frac{1}{2})\Delta}^{(n+\frac{1}{2})\Delta} \frac{dz}{8\pi^2 (a^2+z^2)^{\frac{1}{2}}}$$

and I_{-1}, I_0, I_{N+1}, and I_{N+2} are zero. Eq. (25) may be put in the form of (9)

$$I_{m+1} = \sum_{i=0}^{N} A_i I_{m-i} + F_m \quad , \qquad m = 1,2,3, \ldots \tag{26}$$

where I_m represents the vector of $[I_{m,n}]$ and F_m is a vector of the n^{th} time sample of the incident time derivative times the ratio $2\varepsilon\Delta^2/G_0$. Each of

the A_i are sparce and contain at most four non-zero diagonals. After the incident field has past, F_m is zero and we may expand I_{m+1} in a series of the power terms given in (10) to obtain the SEM form of (12). From (12) we may proceed with either the determinant search method or the eigenvalue method to determine the poles of the system.

Considering the simple case of N equal to unity, Eq. (26) becomes

$$I_{m+1} = (2G_1/G_0 - 1)I_{m-1} + F_m \tag{27}$$

where I_m and F_m are scalars. From the procedure of Eq. (11) we obtain the polynomial

$$Z_\alpha^2 + (1 - 2G_1/G_0) = 0 \tag{28}$$

or

$$Z_\alpha = \pm j(1-2G_1/G_0)^{\frac{1}{2}}$$

$$= \pm j\ 1.146 \tag{29}$$

for a radius to length ratio of 0.01. The quantity Z_α was introduced in (10) as $Z_\alpha = \exp(s_\alpha \Delta_t)$ and s_α may be obtained from Z_α by the complex natural logarithm. Scaling s_α by $L/(c\pi)$ to obtain s'_α we have

$$s'_\alpha = s_{1,1} = -\ 0.087 + j\ 1.0 \tag{30}$$

where $s_{1,1}$ designates the first pole in the first layer of poles. The value obtained by both Singaraju, Giri, and Baum [5] and Tesche [2] is $-\ 0.082 + j0.926$ which differs from (30) by less than 7.5 percent in both the

real and imaginary parts. This accuracy was not expected a priori for a current expansion of only one unknown. However, partial explanation of this accuracy is the goodness of fit for simple sinusoidal current expansions on thin wires [2] and the approximation relationship between piecewise sinusoids and finite differences [9].

We have computed the poles and natural modes using the eigenvalue approach for the number of unknowns varying from one to ten. The total computation time for all ten cases was less than nine seconds for as many as 35 meaningful left half-plane poles representing more than 4 layers. Additional eigenvalues generated by the routine represented additional zeroes of the eigenvalue method to within machine accuracy or terms outside of the Nyquist sampling range defined by an equivalent sampling interval of $2\Delta_t$. The factor of 2 arises from the polynomial in Z having only even terms as in (28). We present the data for the N = 18 case in Fig. 1 along with the results of Singaraju, Giri, and Baum [5]. The computed poles show at least six layers with several other poles imbedded in digital noise to the left. Though we are limited on the imaginary axis by the Nyquist sampling rate, we have been able to exceed the three layers computed by Singaraju, Giri, and Baum. It should also be noted that our total computation time was approximatly one minute as compared to approximately one minute per pole in Singaraju, Giri, and Baum's work. To be fair, their computation times may have been drastically reduced with 18 unknowns. They adjusted the number of unknowns to ten per wavelength for each pole. Similar accuracy would also be expected. The eigenvalue method is currently limited by the machine storage since no effort has been made to take advantage of the sparce matrix form. Computational time may also be reduced by the use of sparce matrix methods.

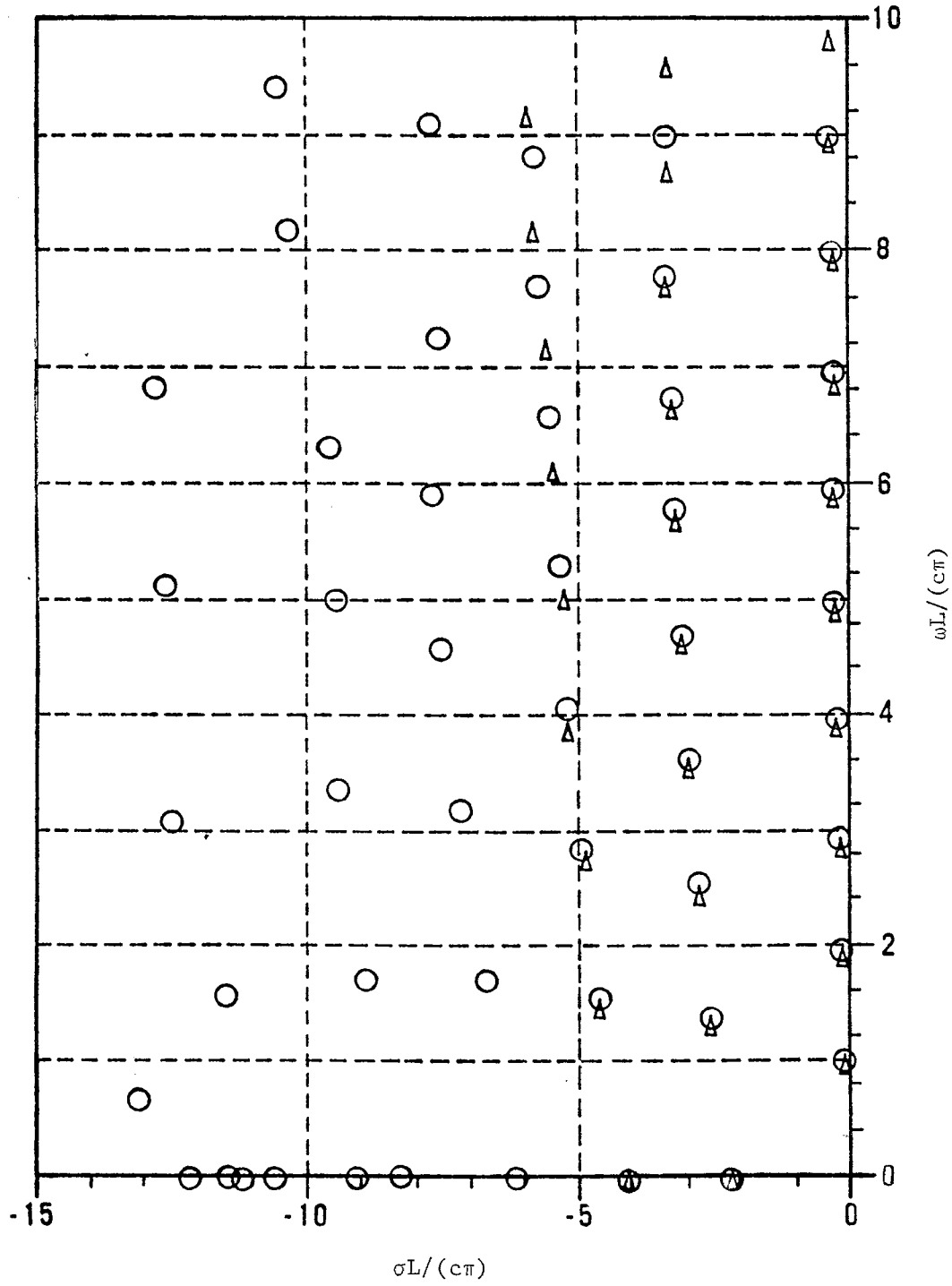

Figure 1. Plot of the normalized poles of a short-circuited dipole for a radius to length ratio of 0.01. The eigenvalue data is represented by "o" for 18 unknowns. The results of Singaraju, Giri, and Baum are given by "Δ"

Conclusions

We have presented an alternative method for the computation of SEM parameters. The method provides a self-consistent relationship between the time and spatial sampling and in particular relates the spatial sampling to the Nyquist sampling rate limitations of the SEM pole computations. This time-domain SEM formulation may be solved in a manner analogous to the frequency - domain search method or as an eigenvalue problem. The latter recognizes the difference equation form of the problem which may be cast in a state vector format to obtain system eigenvalues and thus the poles. The method has been found to be computationally efficient, providing several layers of poles. Reasonable accuracy has also been found for low order approximation opening the way for low cost SEM computations.

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