Interaction Notes

Note 377

A Procedure for Constructing Single Port Equivalent Circuits from the SEM Solution

September 1978

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Abstract

In this report we examine the realizability considerations of the short circuit admittance formulation for a linear thin-wire antenna. General realizability considerations are discussed for the pole and modified-pole admittance formulations. Using numerically calculated natural frequencies and natural modes and analytically calculated coupling coefficients, realizability conditions are discussed for the pole and modified-pole admittances. Circuit realization for the pole-admittance form is shown; the realizability of the modified pole-admittance form is also discussed.

I. Introduction

This report deals with the construction of equivalent circuits from their singularity expansion method (SEM) parameters. Simply stated, if the natural frequencies of the scatterer/radiator are known, a procedure for constructing equivalent circuits involving physically realizable circuit elements such as resistors, inductors and capacitors, along with voltage/current sources, will be discussed.

The question which naturally arises is, why would one want to construct an equivalent circuit for an electromagnetic scattering/radiating problem. Under certain circumstances, such a representation could be helpful in providing

1. Physical insight
2. Computational convenience
3. Capability of using well-established circuit transformation techniques
4. Combination of electromagnetic analysis with physical circuit elements, transmission lines, etc., which are constructed as part of an antenna or scatterer
5. Use of existing computerized circuit analysis programs
6. Physical construction of equivalent circuits for use in pulsers for special types of EMP simulators

The electromagnetic theoretist's urge to construct equivalent circuits from Maxwell's equations is nothing new. James Clerk Maxwell[1] himself alludes to lumped parameters such as inductors and capacitors in his now famous work. Transmission lines have been thought of in terms of distributed circuit parameters consisting of inductors, capacitors and resistors for small sections of transmission lines.[2] The first serious attempt to construct equivalent circuits from Maxwell's equations was made by Gabriel Kron.[3] His procedure was to expand Maxwell's equations in suitable orthogonal curvilinear coordinate systems and identify the capacitance and inductance for each differential element. There are several drawbacks in this procedure:

1. Even free space is thought of as consisting of resistors, inductors, capacitors, and possibly ideal transformers. This leads to artificial resonances in the radiation region,

2. Resistors as circuit elements occur primarily because of the conduction current. Radiation from a scatterer/antenna may not be treated very effectively,

3. In treating large electromagnetic problems, one needs large core storage in the digital computer, and finally,

4. When the free space is artifically terminated (this has to be done because of the finite size of the core storage associated with computing machines), one has to deal with artificial reflection from the outer boundary. Although this last problem might be overcome by putting impedance loading at the outer boundary, its effectiveness over a wide band of frequencies is not known very well.

Using the natural frequencies (resonances), Schelkunoff[4] has attempted to construct lumped parameter circuits for the input impedance of antennas at each of the natural frequencies. This was based on function-theoretic techniques and the assumption that impedances are analytic functions in the complex frequency plane. He also conjectured on the existence of certain representations for driving point and transfer impedances. In a recent paper, Bucci and Franceschetti[5] have constructed driving point admittance equivalent circuits for a spheroidal antenna in a dispersive medium and have also calculated the transient response using the natural modes in spherical coordinates. In this technique, however, gyrators had to be used in the equivalent circuits.

In a recent report, Baum[6] has taken the singularity expansion representation of the response of antennas and scatterers in free space and exhibited equivalent circuit representations. This representation included voltage and current sources where appropriate. This report is an outgrowth of Baum's original work with the express purpose of testing the theory on a simple body and to discuss some aspects of realizability not covered earlier.

II. Basic Theory

In this chapter, we will examine some of the basic principles associated with the SEM and derive relevant formulae for the driving point admittance.

2.1. Preliminaries of SEM

SEM is a generalization[7] of circuit concepts, where natural frequencies, natural modes, and coupling coefficients are used to represent the pole terms in the response to antenna/scattering problems. Using the electric field integral equation, letting $\tilde{E}(\vec{r},s)$ represent the incident or source electric field, $\tilde{J}(\vec{r},s)$ the response current, and $\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)$ the impedance kernel, we can write

$$\left\langle \tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s) \; ; \; \tilde{J}(\vec{r}\,',s) \right\rangle = \tilde{E}(\vec{r},s) \tag{2.1}$$

where the tilde (~) denotes the bilateral Laplace transformed quantity. In equation (2.1), the impedance kernel $\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s)$ is defined via the free-space dyadic Green's[8] function as

$$\begin{aligned}\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s) &= s\mu_o \, \tilde{\overleftrightarrow{G}}_o(\vec{r},\vec{r}\,';s) \\ &= s\mu_o(\overleftrightarrow{1} - \frac{1}{\gamma^2} \nabla\nabla) \, \tilde{G}_o(\vec{r},\vec{r}\,';s)\end{aligned} \tag{2.2}$$

with the scalar free space Green's function $\tilde{G}_o(\vec{r},\vec{r}\,';s)$ defined as

$$\tilde{G}_o(\vec{r},\vec{r}\,';s) = \frac{e^{-\gamma|\vec{r}-\vec{r}\,'|}}{|\vec{r} - \vec{r}\,'|} \tag{2.3}$$

and $\overleftrightarrow{1}$ is the identity dyadic, μ_o the free space permeability and $\gamma = s/c$ the complex propagation constant, s the complex radian-frequency, and c the speed of light in free space.

If the antenna/scatterer is perfectly conducting,

$$\tilde{\overleftrightarrow{\Gamma}}(\vec{r},\vec{r}\,';s) \equiv \tilde{\overleftrightarrow{Z}}_{pc}(\vec{r},\vec{r}\,';s) \tag{2.4}$$

where the subscript pc denotes perfectly conducting. If the scatterer/antenna is impedance loaded,

$$\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) = \tilde{\vec{\vec{Z}}}_{pc}(\vec{r},\vec{r}\,';s) + \tilde{\vec{\vec{Z}}}_{\ell}(\vec{r},\vec{r}\,';s)\ \delta(\vec{r} - \vec{r}\,') \qquad (2.5)$$

where the subscript ℓ represents the loading term. In (2.1) $<;>$ indicates a dot product along with (line, surface or volume as the case may be) integration over common coordinates.

Natural frequencies s_α of the scatterer are defined as the locations in the complex s plane where the response $\tilde{\vec{J}}(\vec{r},s)$ has a pole while the natural mode is the solution of

$$\left\langle \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\ ;\ \vec{\nu}_\alpha(\vec{r}\,') \right\rangle = \vec{0} \qquad (2.6)$$

or

$$\left\langle \vec{\mu}_\alpha\ ;\ \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s) \right\rangle = \vec{0} \qquad (2.7)$$

at the natural frequency s_α and $\vec{\nu}_\alpha(\vec{r})$ $(\vec{\mu}_\alpha(\vec{r}))$ is the right (left) natural mode. In terms of the natural frequencies and natural modes, one can write the response as

$$\tilde{\vec{J}}(\vec{r},s) = \sum_\alpha \frac{\tilde{\eta}_\alpha(s)\ \vec{\nu}_\alpha(\vec{r})}{(s - s_\alpha)} + \tilde{\vec{W}}_p(\vec{r},s) \qquad (2.8)$$

where $\tilde{\eta}_\alpha(s)$ is the coupling coefficient and $\tilde{\vec{W}}_p(\vec{r},s)$ contains singularities other than poles, such as essential singularities and branch points. The nature of $\tilde{\vec{W}}_p$ for an arbitrary scatterer has not been established in any general way; for finite size objects in free space with sufficiently simple electromagnetic description it has been shown that $\tilde{\vec{W}}_p$ is an entire function, i.e., has no singularities in the finite s plane.[9]

2.2 Definition of the Driving Point Admittance for Antennas

In order that we may construct equivalent circuits for an antenna/scatterer we need to define at least one port on the body. If, for the time being, we consider an antenna/scatterer with a single port, at the terminals of this port we can construct either a Thevenin or a Norton equivalent circuit as shown in figures 2.1 and 2.2, respectively. Here $\tilde{Z}_a(\tilde{Y}_a)$ is the driving point impedance (admittance) of the antenna/scatterer while $\tilde{Z}_T(\tilde{Y}_T)$ is the impedance (admittance) associated with the terminals. Terminal impedance (admittance) includes the loading one might have in the antenna gap, V_s and I_s are the voltage and current sources respectively. Note that $\tilde{Z}_T(\tilde{Y}_T)$ will not enter into our analysis and may even be nonlinear.

The driving point impedance $\tilde{Z}_a(s)$ is defined in the circuit analysis as

$$\tilde{Z}_a(s) = \frac{\tilde{V}_{oc}(s)}{\tilde{I}_{sc}(s)} \tag{2.9}$$

where $\tilde{V}_{oc}(s)$ is the open circuit voltage while $\tilde{I}_{sc}(s)$ is the short circuit current with the convention that one is looking into the port from the exterior. The driving point admittance $\tilde{Y}_a(s)$ is defined as

$$\tilde{Y}_a(s) = \tilde{Z}_a^{-1}(s) \tag{2.10}$$

In generalizing this concept to antennas/scatterers, one defines a set of ports (terminals) on the antenna/scatterer (figure 2.3). The gap region at the ports is assumed to be small at all wavelengths of interest which implies that the electric field is uniform in the gap. It should be pointed out that for an antenna, a definable gap region already exists. However, for a scatterer, the creation of a gap region is artificial.

If we specify the gap electric field $\vec{E}_g(\vec{r},t)$ as conservative in the gap, it can be expressed as the negative gradient of the potential Φ_g. The gap voltage $V_g(t)$ is defined as

$$V(t) = \Phi_g(\vec{r},t)\Big|_{\vec{r}\in s_+} - \Phi_g(\vec{r},t)\Big|_{\vec{r}\in s_-} \tag{2.11}$$

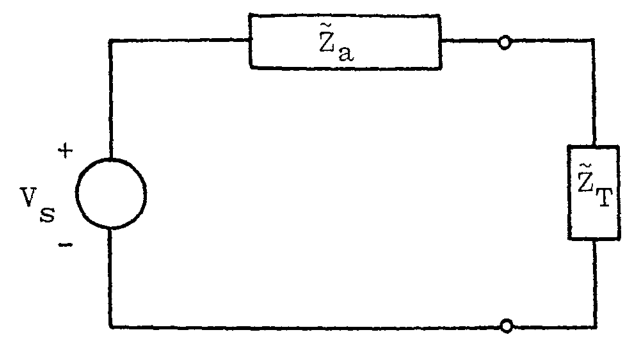


Figure 2.1. Thevenin Equivalent Circuit

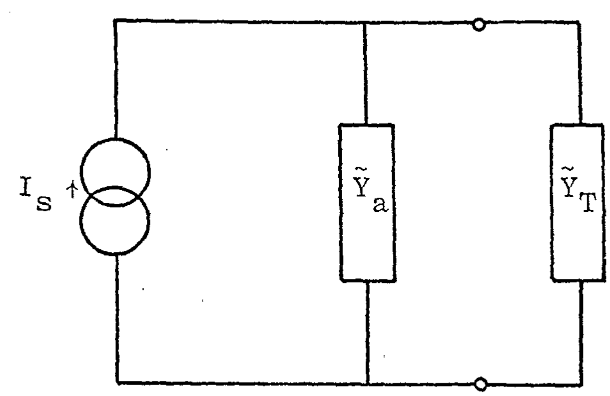
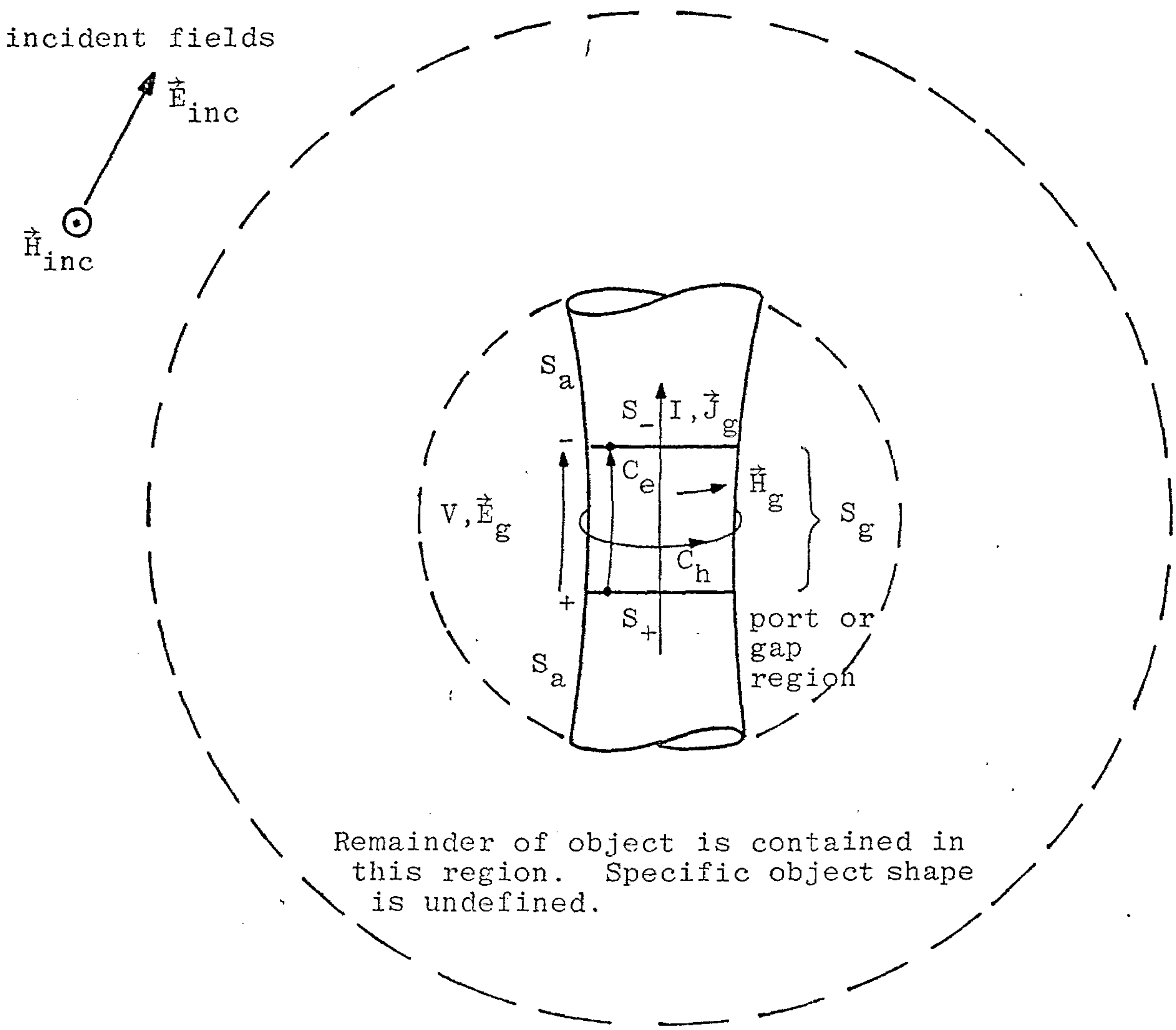


Figure 2.2. Norton Equivalent Circuit



The gap tangential electric field, voltage, and current are shown with a convention to directly give the antenna impedance as $\tilde{V}_{oc}/\tilde{I}_{sc}$.

Figure 2.3. Antenna or Scatterer with Single Port

or

$$V(t) = -\int_{C_e} \vec{E}_g(\vec{r},t) \cdot \vec{1}_e(\vec{r})\, d\ell \tag{2.12}$$

where s_+ and s_- are equipotential surfaces on opposite sides of the gap, C_e is the contour from s_+ to s_-, while $\vec{1}_e(\vec{r})$ is the unit vector tangential to the contour C_e.

If the gap width is Δ, and the source (gap electric field) parallel to the z axis, we can write without any loss of generality

$$\vec{E}_g(\vec{r},t) = -\frac{1}{\Delta} V(t)\, \vec{1}_Z \tag{2.13}$$

$$\vec{e}_g(\vec{r},t) = -\frac{1}{\Delta} \vec{1}_Z \tag{2.14}$$

If we assume that the displacement current through the gap is small, and that most of the current is due to charge motion, then the current I(t) through the gap can be written as

$$I(t) = \left\langle \vec{J}(\vec{r},t) \,;\, \vec{e}_g \right\rangle_g \tag{2.15}$$

where the subscript g indicates that the integration is over the gap region and $\vec{J}(\vec{r},t)$ is the current density. Hence the admittance $\tilde{Y}_a(s)$ is given by

$$\tilde{Y}_a(s) = -\frac{\tilde{I}(s)}{\tilde{V}(s)} = -\frac{\tilde{I}(s)\,\tilde{V}(-s)}{\tilde{V}(s)\,\tilde{V}(-s)} \tag{2.16}$$

or

$$\tilde{Y}_a(s) = -\frac{\left\langle \tilde{\vec{J}}(\vec{r},s) \,;\, \vec{e}_g(\vec{r}) \right\rangle_g}{\tilde{V}(s)} \tag{2.17}$$

In a similar fashion the antenna impedance can also be defined. For this definition the reader is referred to the earlier work.[6]

2.3 Short-Circuit Boundary Value Problem

The short-circuit boundary value problem is defined as the antenna or scatterer with the impedance loading in the gap region short circuited. Although both SEM and eigenmode expansion method (EEM) can be used for the short-circuit boundary value problem, we will discuss here SEM form only; and further details on EEM form can be obtained in the earlier work.[6] We use the subscript sc to denote short-circuit quantities as

$\alpha_{sc} = (\beta_{sc}, \beta'_{sc})$:	Index set for short-circuit natural frequencies (β'_{sc} represents the layer while β_{sc} the pole in that layer)
$\vec{\nu}_{\alpha_{sc}}$:	Right short-circuit natural mode
$\vec{\mu}_{\alpha_{sc}}$:	Left short-circuit natural mode
$\tilde{\eta}'_{\alpha_{sc}}$:	Admittance coupling coefficients
$\tilde{\eta}_{\alpha_{sc}}$:	Short-circuit current coupling coefficients
$s_{\alpha_{sc}}$:	Short-circuit natural frequencies

Assuming that the short-circuit boundary value problem has only first order poles, the short-circuit current can be represented as[10]

$$\tilde{\vec{J}}_{sc}(\vec{r},s) = \sum_{\alpha_{sc}} \frac{\tilde{\eta}_{\alpha_{sc}}(s)\,\vec{\nu}_{\alpha_{sc}}(\vec{r})}{(s - s_{\alpha_{sc}})} + \tilde{\vec{W}}_{sc}(\vec{r},s) \tag{2.18}$$

where

$$\tilde{\eta}_{\alpha_{sc}}(s) = \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r})\ ;\ \tilde{\vec{E}}_{inc}(\vec{r},s) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r})\ ;\ \frac{\partial}{\partial s}\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}}\ ;\ \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.19}$$

with the subscript g indicating integration over the gap while a+g indicates integration over the antenna plus the gap. Note that even though the numerator of (2.19) indicates integration over the gap region alone, since the incident field is zero everywhere on the perfectly conducting scatterer except in the gap, this specification is superfluous.

Returning to (2.18) and (2.19), we can rewrite them in terms of admittance coupling coefficients as

$$\tilde{\vec{J}}_{sc}(\vec{r}) = \frac{\tilde{V}_g(s)}{Z_o}\left\{\sum_{\alpha_{sc}} \frac{\tilde{\eta}'_{\alpha_{sc}}\ \vec{\nu}_{\alpha_{sc}}(\vec{r})}{(s - s_{\alpha_{sc}})} + \tilde{\vec{W}}'_{sc}(\vec{r},s)\right\} \tag{2.20}$$

with

$$\eta'_{\alpha_{sc}}(s) \equiv -Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r})\ ;\ \vec{e}_g(\vec{r}) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r})\ ;\ \frac{\partial}{\partial s}\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}';s)\Big|_{s=s_{\alpha_{sc}}}\ ;\ \vec{\nu}_{\alpha_{sc}}(\vec{r}') \right\rangle_{a+g}} \tag{2.21}$$

Note that if in (2.19) and (2.21) the dyadic operator $\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}';s)$ is symmetric, the left natural mode $\vec{\mu}_{\alpha_{sc}}(\vec{r})$ is equal to the right natural mode $\vec{\nu}_{\alpha_{sc}}(\vec{r})$.

Using the antenna admittance formula given by (2.17) we can write the admittance $\tilde{Y}_a(s)$ as

$$\tilde{Y}_a(s) = \frac{1}{Z_o}\left\{\sum_{\alpha_{sc}} \frac{\tilde{a}_{\alpha_{sc}}}{(s - s_{\alpha_{sc}})} + \tilde{Y}_{sce}(s)\right\} \tag{2.22}$$

with

$$\tilde{a}_{\alpha_{sc}} \equiv Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r})\ ;\ \vec{e}_g(\vec{r}) \right\rangle_g \left\langle \vec{\nu}_{\alpha_{sc}}(\vec{r})\ ;\ \vec{e}_g \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r})\ ;\ \frac{\partial}{\partial s}\tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}';s)\Big|_{s=s_{\alpha_{sc}}}\ ;\ \vec{\nu}_{\alpha_{sc}}(\vec{r}) \right\rangle_{a+g}} \tag{2.23}$$

$$\tilde{Y}_{sce}(s) \equiv \left\langle \tilde{\vec{\vec{W}}}'_{sc}(\vec{r},s) \; ; \; \vec{e}_g \right\rangle = \text{admittance entire function} \tag{2.24}$$

If we assume, without any loss of generality, that the gap electric field is parallel to the z axis and that the gap is of width Δ, from (2.14) and (2.23) we have

$$\tilde{a}_{\alpha_{sc}} = \frac{Z_o}{\Delta^2} \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \vec{1}_z \right\rangle_g \left\langle \vec{\nu}_{\alpha_{sc}}(\vec{r}) \; ; \; \vec{1}_z \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.25}$$

and

$$\tilde{Y}_{sce}(s) = -\frac{1}{\Delta} \left\langle \tilde{\vec{\vec{W}}}'_{sc}(\vec{r},\vec{r}\,';s) \; ; \; \vec{1}_z \right\rangle_g \tag{2.26}$$

One should note, however, that under the definitions (2.22) and (2.25), a delta gap is not permissible. This is not any restriction at all since a closer examination of (2.25) reveals that $\tilde{a}_{\alpha_{sc}}$ is independent of the gap.

We can also define pole admittance $\tilde{Y}_{a_{\alpha_{sc}}}(s)$ as

$$\tilde{Y}_{a_{\alpha_{sc}}}(s) \equiv \frac{1}{Z_o} \frac{\tilde{a}_{\alpha_{sc}}}{(s - s_{\alpha_{sc}})} \tag{2.27}$$

where $\tilde{a}_{\alpha_{sc}}$, the pole admittance coupling coefficient, is given by (2.25) while the entire-function contribution is not included. If we are interested in the admittance in the neighborhood of some natural frequency, it would be sufficient to compute the pole admittance at the nearest natural frequency (this could lead to significant errors if the admittance coupling coefficients corresponding to other natural frequencies are not negligible).

Various conclusions can be drawn regarding the coefficients $\tilde{a}_{\alpha_{sc}}$ and properties of the admittance function. We will postpone this discussion until the next chapter.

2.4 Short-Circuit Boundary Value Problem with Sources

The admittance form discussed in section 2.4 is independent of the forcing function, i.e., the incident field. If we are to use the equivalent circuits to their fullest, they should include source functions. The source electric field in the form of general incident field can be defined as

$$
\begin{aligned}
\tilde{\vec{E}}_s(\vec{r},s) &= \tilde{\vec{E}}_{inc}(\vec{r},s) = E_o \sum_p \tilde{f}_p(s)\, \tilde{\vec{\delta}}_p(\vec{r},s) \\
\tilde{\vec{E}}_s(\vec{r},t) &= \tilde{\vec{E}}_{inc}(\vec{r},t) = E_o \sum_p \tilde{f}_p(t) * \tilde{\vec{\delta}}_p(\vec{r},t)
\end{aligned}
\tag{2.28}
$$

where E_o is a constant, $\tilde{f}_p(s)$ ($\tilde{f}_p(t)$) is the generalized incident wave form and $\tilde{\vec{\delta}}_p(\vec{r},s)$ ($\tilde{\vec{\delta}}_p(\vec{r},t)$) the generalized spatial form, while $*$ represents the convolution with respect to the time coordinates.

In terms of the generalized incident (source) fields, the short circuit current $\tilde{\vec{J}}_{sc}(\vec{r},s)$ and the coupling coefficent $\tilde{\eta}_{\alpha_{sc}}(s)$ can be rewritten as

$$
\tilde{\vec{J}}_{sc}(\vec{r},s) = \sum_{\alpha_{sc}} \frac{\tilde{\eta}_{\alpha_{sc}}(s)\, \vec{\nu}_{\alpha_{sc}}(\vec{r})}{(s - s_{\alpha_{sc}})} + \tilde{\vec{W}}_{sc}(\vec{r},s) \tag{2.29}
$$

with

$$
\tilde{\eta}_{\alpha_{sc}}(s) = \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{E}}_{inc}(\vec{r},s) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.30}
$$

$$
= \frac{E_o \left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \sum_p \tilde{f}_p(s)\, \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.31}
$$

$$= \frac{E_o \sum\limits_p \tilde{f}_p(s) \left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.32}$$

Note that in (2.32) various simplifications result if the natural modes $\tilde{\vec{\nu}}_{\alpha_{sc}}(\vec{r})$ for the scatterer (antenna) are the same as the expansion functions $\tilde{\vec{\delta}}_p(\vec{r},s)$, under the assumption that the natural modes are orthogonal.

Using (2.15) and (2.29), the short circuit current $\tilde{I}(s)$ can be written as

$$\tilde{I}_{sc}(\vec{r}) = \sum_{\alpha_{sc}} \frac{\tilde{\eta}_{\alpha_{sc}}(s) \left\langle \tilde{\vec{\nu}}_{\alpha_{sc}}(\vec{r}) \; ; \; \vec{e}_g \right\rangle}{(s - s_{\alpha_{sc}})} + \left\langle \tilde{\vec{W}}'_{sc}(\vec{r},s) \; ; \; \vec{e}_g \right\rangle \tag{2.33}$$

with

$$\tilde{\eta}_{\alpha_{sc}}(s) = \frac{E_o \sum\limits_p \tilde{f}_p(s) \left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p(\vec{r},s) \right\rangle_{a+g}}{\left\langle \tilde{\vec{\mu}}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.34}$$

or equivalently

$$\tilde{\eta}_{\alpha_{sc}}(s) = \frac{E_o}{Z_o} \sum_p Z_o \frac{\tilde{f}_p(s)}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \vec{e}_g(\vec{r}) \right\rangle}$$

$$\frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \; ; \; \tilde{\vec{\delta}}_p \right\rangle_{a+g} \left\langle \vec{\nu}_{\alpha_{sc}} \; ; \; \vec{e}_g \right\rangle_g}{\left\langle \vec{\nu}_{\alpha_{sc}}(\vec{r}) \; ; \; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r},\vec{r}\,';s)\Big|_{s=s_{\alpha_{sc}}} \; ; \; \vec{\nu}_{\alpha_{sc}}(\vec{r}\,') \right\rangle_{a+g}} \tag{2.35}$$

Using (2.35), (2.23) one can rewrite (2.33) as

$$\tilde{I}_{sc}(\vec{r}) = \sum_p \tilde{f}_p(s) \left\{ \frac{1}{Z_o} \sum_{\alpha_{sc}} \tilde{V}_{\alpha_{sc},p} \frac{\tilde{a}_{\alpha_{sc}}}{(s - s_{\alpha_{sc}})} + \tilde{V}_{sce,p}(s)\, \tilde{Y}_{sce}(s) \right\} \tag{2.36}$$

with

$$\tilde{V}_{\alpha_{sc},p} = E_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \;;\; \tilde{\vec{\vec{\delta}}}_p(\vec{r}, s_{\alpha_{sc}}) \right\rangle_{a+g}}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \;;\; \vec{e}_g(\vec{r}) \right\rangle_g} \tag{2.37}$$

called the voltage source coefficent, and

$$\tilde{a}_{\alpha_{sc}} = Z_o \frac{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \;;\; \vec{e}_g \right\rangle_g \left\langle \vec{\nu}_{\alpha_{sc}}(\vec{r}) \;;\; \vec{e}_g \right\rangle_g}{\left\langle \vec{\mu}_{\alpha_{sc}}(\vec{r}) \;;\; \frac{\partial}{\partial s} \tilde{\vec{\vec{\Gamma}}}(\vec{r}, \vec{r}\,'; s) \Big|_{s=s_{\alpha_{sc}}} \;;\; \vec{\nu}_{\alpha_{sc}}(\vec{r}) \right\rangle_{a+g}} \tag{2.38}$$

known as the normalized admittance residue. As defined in (2.27), the pole admittance $\tilde{Y}_{a_{\alpha_{sc}}}(s)$ is given by

$$\tilde{Y}_{a_{\alpha_{sc}}}(s) = \frac{1}{Z_o} \frac{\tilde{a}_{\alpha_{sc}}}{(s - s_{\alpha_{sc}})} \tag{2.39}$$

In terms of the pole admittance $\tilde{Y}_{a_{\alpha sc}}(s)$, the short circuit current $\tilde{I}_{sc}(\vec{r})$ can be written as

$$\tilde{I}_{sc}(\vec{r}) = \sum_p \tilde{f}_p(s) \left\{ \sum_{\alpha_{sc}} \tilde{V}_{\alpha_{sc},p}\, \tilde{Y}_{a_{\alpha_{sc}}}(s) + \tilde{V}_{sce,p}(s)\, \tilde{Y}_{sce}(s) \right\} \tag{2.40}$$

or

$$\tilde{I}_{sc}(\vec{r}) = \sum_{\alpha_{sc}} \left\{ \sum_p \tilde{f}_p(s)\, \tilde{V}_{\alpha_{sc},p} \right\} \tilde{Y}_{a_{\alpha_{sc}}}(s) + \left\{ \sum_p \tilde{f}_p(s)\, \tilde{V}_{sce,p}(s) \right\} \tilde{Y}_{sce}(s) \tag{2.41}$$

where $\sum_p \tilde{f}_p(s)\tilde{V}_{\alpha_{sc},p}$ is interpreted as a voltage for pole α or pole source. Knowing the sources and admittances, one can construct the equivalent circuits and their proper sources.

3. Some Concepts of Synthesis

In the last chapter we derived the formulae for the admittance and a representation of the short circuit current when sources were present in terms of the mode admittances. Since sources are generally "simple," and problem dependent, we will not discuss them here. Admittances are another story. However, they have several interesting properties which can be exploited.

3.1 Properties of the Driving Point Admittance

The general form of the driving point admittance is given by

$$\tilde{Y}_{d.p.}(s) = \sum_{\alpha} \frac{\tilde{a}_{\alpha}}{(s - s_{\alpha})} + \text{(other terms)} \tag{3.1}$$

where s_α is the αth pole, $\tilde{a}_\alpha$ is the residue at the αth pole, and the subscript d.p. denotes driving point admittance. In (3.1) other terms include singularities other than poles. For convenience, we let the complex frequency s be of the form

$$s = \Omega + i\omega \tag{3.2}$$

where Ω is the real part of s (Re(s)) while ω is the imaginary part (Im(s)) of s. We also assume that the objects into whose ports we are looking are completely passive. If[11]

1. The driving point admittance $\tilde{Y}_{d.p.}(s)$ is analytic for $\Omega > -a$ where $a \geq 0$,

2. If s is real ($\omega = 0$) this implies $\tilde{Y}_{d.p.}(s)$ is real and hence by Schwartz reflection principle

$$\tilde{Y}_{d.p.}(\bar{s}) = \bar{\tilde{Y}}_{d.p.}(s) \tag{3.3}$$

where (-) implies complex conjugation,

3. Zeros and poles on the imaginary s axis are simple,

4. At a zero on the imaginary axis, $d\tilde{Y}_{d.p.}/ds$ is a real positive constant while at a pole on the imaginary axis the residue is a real positive constant,

then $\tilde{Y}_{d.p.}(s)$ is a positive real (PR) function. Since, if $\tilde{Y}_{d.p.}(s)$ is a PR function, $\tilde{Z}_{d.p.}(s) \equiv 1/\tilde{Y}_{d.p.}(s)$ is also PR, we conclude that if the driving point admittance is PR, so is the driving point impedance and vice versa.

3.2 Alternative Representations of the Admittance

If, for the time being, we assume that the admittance can be represented purely by the poles, we can write

$$\tilde{Y}_{d.p.}(s) = \sum_{\alpha} \frac{\tilde{a}_{\alpha}}{s - s_{\alpha}} \tag{3.4}$$

Since this is PR, $\tilde{Y}_{d.p.}(s)$ satisfies all of the conditions discussed in section 3.1. If the number of poles a given representation has is infinite (generally the case because all of the natural frequencies are excited), there is no a priori knowledge that (3.4) converges. In addition, knowing the low frequency $(s \to 0)$ and/or the high frequency $(s \to \infty)$ behavior of $\tilde{Y}_{d.p.}(s)$, one can write more efficient formulations.

Mittag-Leffler's theorem[12,13] in the functions of a complex variable states that a function which is analytic in the finite part of the complex plane except at simple poles can be replaced by any other function having the same poles and residues at the poles differing from the original function by an entire function. Assuming that one of the $s_{\alpha} = 0$, we can write (3.4) as

$$\tilde{Y}_{d.p.}(s) = \tilde{a}_{\infty} s + \frac{\tilde{a}_o}{s} + \sum_{\alpha} \frac{\tilde{a}_{\alpha}}{s - s_{\alpha}} \tag{3.5}$$

where

$$\tilde{a}_{\infty} = \lim_{s\to\infty} \frac{\tilde{Y}_{d.p.}(s)}{s} \quad |\arg(s)| < \pi/2 \tag{3.6}$$

Here either $\tilde{a}_o$ or $\tilde{a}_\infty$ could be zero. If we assume that none of the s_α are zero (if this happens in addition to $\tilde{a}_o$ not being zero, $\tilde{Y}_{d.p.}$ is not PR) we can simplify (3.5) under various conditions. If $\tilde{Y}_{d.p.}(0) = 0$, $\tilde{a}_o = 0$, and

$$\tilde{Y}_{d.p.}(s) = \tilde{a}_\infty s + \sum_\alpha \frac{\tilde{a}_\alpha}{s - s_\alpha} \tag{3.7}$$

$\tilde{a}_\infty$ represents a capacitor of $\tilde{a}_\infty$ farads in parallel with the pole admittances. In general, for a truncated set of poles one can set $\tilde{a}_\infty = 0$ since in the right half plane $\lim_{s\to\infty} \tilde{Y}_{d.p.}(s) \to$ constant. If $\tilde{a}_\infty$ is set to zero

$$\tilde{Y}_{d.p.}(s) = \frac{\tilde{a}_o}{s} + \sum_\alpha \frac{\tilde{a}_\alpha}{s - s_\alpha} \tag{3.8}$$

with $\tilde{a}_o$ representing an inductor of $1/a_o$ henries inductance in parallel with the other pole admittances. If $\mathrm{Re}(s_\alpha) \leq 0$, i.e., all of the poles are in the left half s plane or on the imaginary ($j\omega$) axis and if the poles occur in conjugate pairs, (3.7) and (3.8) are generally realizable. There are certain special conditions under which these are not realizable[14,15] and we will discuss these conditions later.

Returning to (3.4), this can also be written as

$$\tilde{Y}'_{d.p.}(s) = \sum_\alpha \left\{\frac{\tilde{a}_\alpha}{s - s_\alpha} + \frac{\tilde{a}_\alpha}{s_\alpha}\right\} = \sum_\alpha \frac{\tilde{a}_\alpha s}{s - s_\alpha} \tag{3.9}$$

known as the modified pole admittance with the entire-function contribution modified by $\sum_\alpha - \tilde{a}_\alpha/s_\alpha$. If we restrict our attention to PR functions, $\tilde{a}_\alpha/s_\alpha$ along with its complex conjugate degenerates to a conductance in parallel with the individual pole admittance; however, in the modified entire function this may not necessarily be the case. If the driving point admittance approaches a constant value at high frequencies, the entire-function contribution would be zero and this happens to be the case for a thin cylindical antenna.

Another type of expansion which is useful is analogous to the Foster's canonical form for lumped constant elements in the form of an infinite product.[16,17] If $\tilde{Y}_{d.p.}(0) = 0$, we can write

$$\tilde{Y}_{d.p.}(s) = sC \prod_{\alpha} \frac{1 - s/s'_{\alpha_{oc}}}{1 - s/s_{\alpha_{sc}}} \tag{3.10}$$

while if $\tilde{Y}_{d.p.}(\infty) = 0$

$$\tilde{Y}_{d.p.}(s) = \frac{1}{sL} \prod_{\alpha} \frac{1 - s/s'_{\alpha_{oc}}}{1 - s/s_{\alpha_{sc}}} \tag{3.11}$$

where subscript oc represents open-circuit while sc the short-circuit quantities.

Using Cauchy's integral formula, Bode[18] derived various integral representations. If the resistive part $R(\omega)$ of the impedance (admittance) is known on the imaginary ($j\omega$) axis, the reactive part can be written as

$$X(\omega) = \frac{2\omega}{\pi} \int_{o}^{\infty} \frac{R(\chi) - R(\omega)}{\chi^2 - \omega^2} \, d\omega \tag{3.12}$$

where poles on the $j\omega$ axis and at ∞ are indented around. This implies that no series inductors or capacitors are allowed. It should be noted that if $R(\omega)$ and $X(\omega)$ are known on the imaginary ($j\omega$) axis, using analytic continuation techniques one can analytically extend these values along an arc into the left (right) half-plane[12,13] as long as the arc does not pass through any of the singularities. Hence, if one is trying to construct the equivalent circuits from measured data, we only need to make one set of good measurements and use the above principles to construct an SEM representation.

3.3 Network Representation of the Pole Admittance

It is an accepted fact that it is easier to represent the driving point admittance (impedance) in an analytical representation than in a network representation which is physically realizable. In this section we will consider the representations (3.4) and discuss the condition under which this is physically realizable.

Rewriting (3.4) here for convenience,

$$\tilde{Y}_{d.p.}(s) = \sum_{\alpha} \frac{\tilde{a}_\alpha}{s - s_\alpha} \tag{3.13}$$

For the present we will assume that $\tilde{Y}_{d.p.}(s)$ is PR. If we consider the conjugate pole pair, for each pole pair the admittance $\tilde{Y}_{d.p._{\alpha,cp}}$ can be written as

$$\tilde{Y}_{d.p._{\alpha,cp}}(s) = \frac{\tilde{a}_\alpha}{s - s_\alpha} + \frac{\bar{\tilde{a}}_\alpha}{s - \bar{s}_\alpha} \tag{3.14}$$

or

$$\tilde{Y}_{d.p._{\alpha,cp}}(s) = \frac{2\ \mathrm{Re}(\tilde{a}_\alpha)s - (\bar{s}_\alpha \tilde{a}_\alpha + s_\alpha \bar{\tilde{a}}_\alpha)}{s^2 - 2\ \mathrm{Re}(s_\alpha)s + |s_\alpha|^2} \tag{3.15}$$

where $|s_\alpha|^2$ represents the square of the magnitude of s_α. If we represent the real part of $\tilde{a}_\alpha$ as $\tilde{a}_{\alpha_R}$, the imaginary part by $\tilde{a}_{\alpha I}$, the real part of s_α by Ω_α and the imaginary part by ω_α,

$$\tilde{Y}_{d.p._{\alpha,cp}}(s) = \frac{2\ \tilde{a}_{\alpha_R} s + 2(\tilde{a}_{\alpha_R}|\Omega_\alpha| - \tilde{a}_{\alpha_I}\omega_\alpha)}{s^2 + 2|\Omega_\alpha|s + |s_\alpha|^2} \tag{3.16}$$

If the residues $\tilde{a}_\alpha$ are purely real, the numerator of (3.16) can be simplified further.

Consider a general representation of the form

$$\tilde{Y}_{d.p.}(s) = \frac{c_1 s + c_2}{s^2 + c_3 s + c_4} \tag{3.17}$$

where c_i is real and ≥ 0. It is easy to show that,[14] for (3.17) to be PR it is sufficient if $c_1 c_3 \geq c_2 \geq 0$. Applying this condition to (3.16), for $\tilde{Y}_{d.p._{\alpha,cp}}(s)$ to be PR it is sufficient if

$$\frac{\tilde{a}_{\alpha_R}}{|\tilde{a}_{\alpha_I}|} \geq \frac{\omega_\alpha}{|\Omega_\alpha|} \tag{3.18a}$$

where $\omega_\alpha > 0$. This condition simply requires that for poles of higher order on the first layer the admittance coupling coefficient be mostly real. Geometrical interpretation[14] of this condition requires that the complex admittance residue lie within the shaded area shown in figure (3.1). If $c_2 \not\geq 0$, then the requirement for PR is

$$\omega^2 (c_2 - c_1 c_3) \leq c_2 c_4 \tag{3.18b}$$

or

$$\omega^2 \left(\tilde{a}_{\alpha_R} |\Omega_\alpha| + \tilde{a}_{\alpha_I} \omega_\alpha\right) \geq \left(\tilde{a}_{\alpha_I} \omega_\alpha - \tilde{a}_{\alpha_R} |\Omega_\alpha|\right)\left(\Omega_\alpha^2 + \omega_\alpha^2\right) \tag{3.18c}$$

This is a much more general condition than (3.18a) and hence more stringent. Note that this condition will have to be satisfied for each pole pair and it is possible that this condition may be satisfied only on part of the $j\omega$ axis for a given pole pair, which implies that the equivalent circuit is appropriate only for a portion of the $j\omega$ axis.

Constructing a lumped parameter circuit for (3.17), it will be of the form shown in figure (3.2). Comparing the coefficients, we obtain

$$L = \frac{1}{c_1} \text{ henries}$$

$$C = \frac{c_1^3}{c_1^2 c_4 - c_2(c_1 c_3 - c_2)} \text{ farads}$$

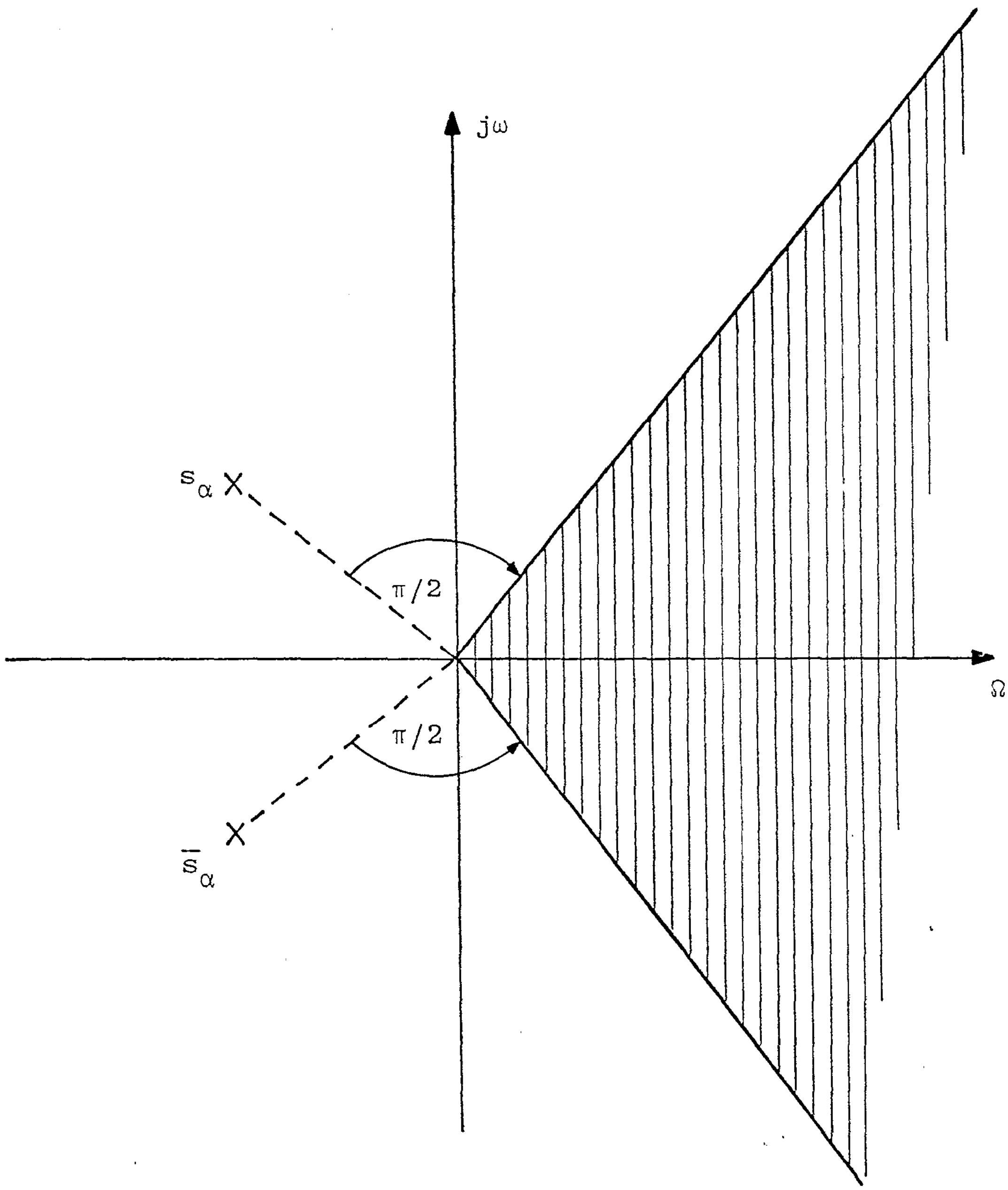
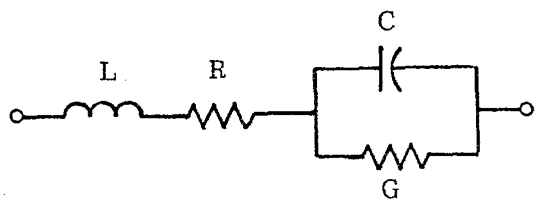


Figure 3.1. Geometrical Interpretation of the Realizability Condition (Admittance Residue $\tilde{a}_\alpha$ and its Conjugate $\bar{\tilde{a}}_\alpha$ Should Lie in the Shaded Area)



$$L = \frac{1}{2\tilde{a}_{\alpha_R}} \text{ henries}$$

$$C = \frac{2\tilde{a}^3_{\alpha_R}}{\omega^2_\alpha\left(\tilde{a}^2_{\alpha_R} + \tilde{a}^2_{\alpha_I}\right)} \text{ farads}$$

$$R = \frac{\tilde{a}_{\alpha_R}|\Omega_\alpha| + \tilde{a}_{\alpha_I}\omega_\alpha}{2\tilde{a}^2_{\alpha_R}} \text{ ohms}$$

$$G = \frac{2\tilde{a}^2_{\alpha_R}\left(\tilde{a}_{\alpha_R}|\Omega_R| - \tilde{a}_{\alpha_I}\omega_\alpha\right)}{\omega^2_\alpha\left(\tilde{a}^2_{\alpha_R} + \tilde{a}^2_{\alpha_I}\right)} \text{ mhos}$$

Figure 3.2. Synthesis of the Pole Admittance for Poles Not on the Boundary

$$R = \frac{c_1 c_3 - c_2}{c_1^2} \text{ ohms}$$

$$G = \frac{c_1^2 c_2}{c_1^2 c_4 - c_1 c_2 c_3 + c_2^2} \text{ mhos} \qquad (3.19)$$

with

$$\frac{R}{L} = \frac{c_1 c_3 - c_2}{c_1}$$

$$\frac{G}{C} = \frac{c_2}{c_1} \qquad (3.20)$$

Note that if $c_1 c_3 \geq c_2$, all of the quantities in (3.19) are ≥ 0 and hence physically realizable. In terms of the quantities in (3.16) we have

$$L = \frac{1}{2\tilde{a}_{\alpha_R}} \text{ henries} \qquad (3.20a)$$

$$C = \frac{2\tilde{a}_{\alpha_R}^3}{\omega_\alpha^2\left(\tilde{a}_{\alpha_R}^2 + \tilde{a}_{\alpha_I}^2\right)} \text{ farads} \qquad (3.20b)$$

$$R = \frac{\tilde{a}_{\alpha_R}|\Omega_\alpha| + \tilde{a}_{\alpha_I}\omega_\alpha}{2\tilde{a}_{\alpha_R}^2} \text{ ohms} \qquad (3.20c)$$

$$G = \frac{2\tilde{a}_{\alpha_R}^2\left(\tilde{a}_{\alpha_R}^2|\Omega_\alpha| - \tilde{a}_{\alpha_I}\omega_\alpha\right)}{\omega_\alpha^2\left(\tilde{a}_{\alpha_R}^2 + \tilde{a}_{\alpha_I}^2\right)} \text{ mhos} \qquad (3.20d)$$

with

$$\frac{R}{L} = \frac{\tilde{a}_{\alpha_R}^2|\Omega_\alpha| + \tilde{a}_{\alpha_I}\omega_\alpha}{\tilde{a}_{\alpha_R}} \qquad (3.20e)$$

$$\frac{G}{C} = \frac{\tilde{a}_{\alpha_R}|\Omega_\alpha| - \tilde{a}_{\alpha_I}\omega_\alpha}{\tilde{a}_{\alpha_R}} \tag{3.20f}$$

Note that if the admittance residue is real and the poles are on the $j\omega$ axis, $G = 0$ and we obtain a simple RLC series circuit. Alternatively, if

$$\tilde{a}_{\alpha_R}\omega_\alpha = \tilde{a}_{\alpha_I}\omega_\alpha \tag{3.21}$$

the conductance G is zero and we obtain a simple series RLC circuit. Note that condition (3.21) implies that the residue must fall on the boundary of the shaded area in figure 3.1.

If the simple pole is on the negative real axis, the realization procedure is slightly different. Assuming a simple pole,

$$\tilde{Y}_{d.p._{\alpha,Re}}(s) = \frac{\tilde{a}_\alpha}{s + |\Omega_R|} = \frac{1}{s/\tilde{a}_\alpha + |\Omega_R|/\tilde{a}_\alpha} \tag{3.22}$$

If $\tilde{a}_\alpha$ is complex, (3.24) is not PR and hence not realizable; however, by conjugate symmetry, $\tilde{a}_\alpha$ must be real. If $\tilde{a}_\alpha > 0$ ($\tilde{a}_\alpha = 0$ being a trivial case) (3.24) can be realized as shown in figure 3.3 with

$$L = 1/\tilde{a}_\alpha \text{ henries} \tag{3.23}$$

$$R = |\Omega_R|/\tilde{a}_\alpha \text{ ohms} \tag{3.24}$$

with

$$\frac{R}{L} = |\Omega_R| \tag{3.25}$$

Note that in figures 3.1 and 3.2 the sources, when present, appear as a voltage source in series with the proper admittance.

$$L = 1/\tilde{a}_{\alpha} \quad \text{henries}$$

$$R = |\Omega_R|/\tilde{a}_{\alpha} \quad \text{ohms}$$

Figure 3.3. Synthesis of the Pole Admittance for Poles on the Negative Real Axis

3.4 Network Representation of the Modified Pole Admittances

Rewriting (3.9) here for convenience,

$$\tilde{Y}'_{d.p.}(s) = \sum_{\alpha} \left(\frac{\tilde{a}_{\alpha}}{s_{\alpha}}\right) \frac{s}{s - s_{\alpha}} \equiv \sum_{\alpha} \frac{\tilde{a}'_{\alpha} s}{s - s_{\alpha}} \tag{3.26}$$

Considering a conjugate pole pair, for each pole pair we have

$$\tilde{Y}'_{d.p._{\alpha,cp}}(s) = \frac{\tilde{a}'_{\alpha} s}{s - s_{\alpha}} + \frac{\bar{\tilde{a}}'_{\alpha} s}{s - \bar{s}_{\alpha}} \tag{3.27}$$

or

$$\tilde{Y}'_{d.p._{\alpha,cp}}(s) = \frac{2\tilde{a}'_{\alpha_R} s^2 + 2\left(\tilde{a}'_{\alpha_R}|\Omega_{\alpha}| - \tilde{a}'_{\alpha_I}\omega_{\alpha}\right)s}{s^2 + 2s|\Omega_{\alpha}| + |s_{\alpha}|^2} \tag{3.28}$$

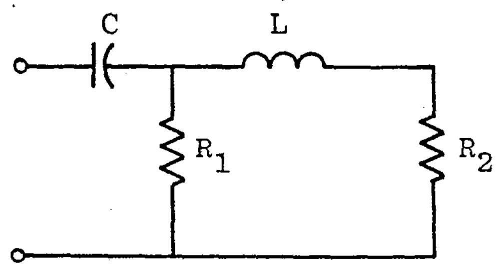
where the fact that $\text{Re}(s_{\alpha}) \leq 0$ has been used along with the assumption that $\tilde{a}_R$ and $\tilde{a}_I \geq 0$, this however does not lead to any loss of generality.

Consider a general modified admittance representation of the form

$$\tilde{Y}'_{d.p.} = \frac{c_1 s^2 + c_2 s}{s^2 + c_3 s + c_4} \tag{3.29}$$

The representation shown in (3.29) is PR if and only if all $c_i \geq 0$ and $c_2 c_3 \geq c_1 c_4$. Note that figure 3.4 is appropriate for $c_2 > c_3$ and c_3, $c_4 \geq 0$. If $c_2 < c_3$ with c_3, $c_4 \geq 0$ the inductor in the figure should be replaced by a capacitor. This later representation is perfectly acceptable if the poles under consideration are on the negative real axis. In terms of the c_i's, the element values in figure 3.4 are given by

$$C = \frac{c_2}{c_4} \tag{3.30a}$$



$$C = \frac{2\left(\tilde{a}'_{\alpha_R}|\Omega_\alpha| - \tilde{a}'_{\alpha_I}\omega_\alpha\right)}{\left(|\Omega_\alpha|^2 + \omega_\alpha^2\right)}$$

$$L = \frac{\left(\tilde{a}'_{\alpha_R}|\Omega_\alpha| - \tilde{a}'_{\alpha_I}\omega_\alpha\right)}{2(\omega_\alpha)^2\left(\tilde{a}'^{2}_{\alpha_R} + \tilde{a}'^{2}_{\alpha_I}\right)}$$

$$R_1 = \frac{1}{2\tilde{a}'_{\alpha_R}}$$

$$R_2 = \frac{\tilde{a}'_{\alpha_R}\left(|\Omega_\alpha|^2 - \omega_\alpha^2\right) - 2\tilde{a}'_{\alpha_I}|\Omega_\alpha|\omega_\alpha}{2\omega_\alpha^2\left(\tilde{a}'^{2}_{\alpha_R} + \tilde{a}'^{2}_{\alpha_I}\right)}$$

Figure 3.4. Synthesis of the Modified Pole Admittance for Poles Not on the Boundary

$$L = \frac{1}{c_2 - c_1 c_3\left(1 - \frac{c_1 c_4}{c_2 c_3}\right)} \tag{3.30b}$$

$$R_1 = \frac{1}{c_1} \tag{3.30c}$$

$$R_2 = \frac{(c_2 c_3 - c_1 c_4)}{c_2^2 - c_1(c_2 c_3 - c_1 c_4)} \tag{3.30d}$$

For the present case, making the appropriate substitutions into (3.30) from (3.29) we obtain

$$C = \frac{2\left(\tilde{a}'_{\alpha_R}|\Omega_\alpha| - \tilde{a}'_{\alpha_I}\omega_\alpha\right)}{\left(|\Omega_\alpha|^2 + \omega_\alpha^2\right)} \tag{3.31a}$$

$$L = \frac{\left(\tilde{a}'_{\alpha_R}|\Omega_\alpha| - \tilde{a}'_{\alpha_I}\omega_\alpha\right)}{2\left(\omega_\alpha\right)^2\left(\tilde{a}'^{\,2}_R + \tilde{a}'^{\,2}_I\right)} \tag{3.31b}$$

$$R_1 = \frac{1}{2\tilde{a}'_{\alpha_R}} \tag{3.31c}$$

$$R_2 = \frac{\tilde{a}'_{\alpha_R}\left(|\Omega_\alpha|^2 - \omega_\alpha^2\right) - 2\tilde{a}'_I|\Omega_\alpha|\omega_\alpha}{2\omega_\alpha^2\left(\tilde{a}'^{\,2}_R + \tilde{a}'^{\,2}_I\right)} \tag{3.31d}$$

Representations shown in (3.31) are physically realizable only if

$$2|\Omega_\alpha|\left(\tilde{a}'_{\alpha_R}|\Omega_\alpha| - \tilde{a}'_{\alpha_I}\omega_\alpha\right) \geq \tilde{a}'_{\alpha_R}\left(\Omega_\alpha^2 + \omega_\alpha^2\right) \tag{3.32}$$

along with

$$2|\Omega_\alpha| \quad , \quad |s_\alpha|^2 \geq 0 \tag{3.33}$$

which is a tautology.

If the pole is on the negative real axis, assuming that it is a simple pole, we can write

$$\tilde{Y}'_{d.p._{\alpha,Re}}(s) = \left(\frac{\tilde{a}_\alpha}{s_\alpha}\right)\frac{s}{s - s_\alpha}$$

$$= \left(\frac{\tilde{a}_\alpha}{|\Omega_\alpha|}\right)\frac{s}{s + |\Omega_\alpha|} \qquad (3.34)$$

By conjugate symmetry, $\tilde{a}_\alpha$ is required to be real. If $\tilde{a}_\alpha$ is negative, circuit representation of (3.34) is not physically realizable. A circuit realization for $\tilde{a}_\alpha > 0$ of (3.34) is shown in figure 3.5 with

$$C = \frac{\tilde{a}_\alpha}{|\Omega_\alpha|} \text{ farads} \qquad (3.35)$$

$$R = \frac{1}{\tilde{a}_\alpha} \text{ ohms} \qquad (3.36)$$

This representation is simply a specialization of the case shown in figure 3.2.

$$C = \frac{\tilde{a}_\alpha}{|\Omega_\alpha|} \text{ farads}$$

$$R = \frac{1}{\tilde{a}_\alpha} \text{ ohms}$$

Figure 3.5. Synthesis of the Modified Pole Admittances for Poles on the Real Axis

4. Admittance Networks for a Cylindrical Antenna

We will now consider a well known problem of a thin cylindrical antenna for the modified and the unmodified forms of the equivalent circuits. For the present, we discuss the analytical calculations using the natural frequencies calculated earlier.[19] Shown in table 4.1 are the numerical values for the natural frequencies while figure 4.1 shows these geometrically. Natural frequencies s_α as shown in figure 4.1 or table 4.1 are normalized as s'_α with $s'_\alpha = s_\alpha \ell/\pi c$, where ℓ is the total length of the antenna. We normalize the complex frequency s' in the same fashion. We assume that the natural mode $I_\alpha(z)$ is real and is of the form

$$I_\alpha(z) = \sin\left(\frac{\alpha\pi}{\ell} z\right) \qquad \alpha = 1,2,... \tag{4.1}$$

where α is the order of the mode in the first layer.

The integral equation which we will be dealing with is the Pocklington form of the integral equation given by

$$\left(\frac{d^2}{dz^2} - \frac{s^2}{c^2}\right)\int_o^\ell \frac{\tilde{I}(z',s)}{4\pi R}\, e^{-sR/c}\, dz' = -s\varepsilon_o\, \tilde{E}_z^{inc}(z,s) \tag{4.2}$$

where

$$R^2 = (z - z')^2 + a^2 \tag{4.3}$$

After considerable algebraic manipulations, it can be shown that the denominator of the coupling coefficients given by $\left\langle I_\alpha\, ,\, \frac{d\mathcal{Z}}{ds}\Big|_{s=s_\alpha}\, ,\, I_\alpha\right\rangle$ can be approximated as

$$\left\langle I_\alpha\, ,\, \frac{d\tilde{\mathcal{Z}}}{ds}\Big|_{s=s_\alpha}\, ,\, I_\alpha\right\rangle \approx \frac{\Omega s_\alpha}{2\pi c^2}\left[\left\langle I_\alpha(z)\, ,\, I_\alpha(z)\right\rangle + O\left(\frac{1}{\Omega}\right)\right] \tag{4.4}$$

where Ω is the fatness factor defined by $\Omega = 2\ln(\ell/a)$. If we assume that I_α is of the form (4.1), we can rewrite (4.4) as

α	Layer 1	Layer 2	Layer 3
1	-.0828+j.9251	$-2.1687+j.349\times10^{-11}$	$-4.0993+j.394\times10^{-7}$
2	-.1212+j1.9117	-2.500+j1.3329	-4.5142+j1.4979
3	-.1491+j2.8835	-2.7342+j2.4680	-4.8285+j2.7472
4	-.1713+j3.8741	-2.9146+j3.5334	-5.0693+j3.8894
5	-.1909+j4.8536	-3.0454+j4.5757	-5.2851+j5.0070
6	-.2080+j5.8453	-3.1640+j5.6097	-5.4647+j6.0811
7	-.2240+j6.8286	-3.2659+j6.6221	-5.6277+j7.1478
8	-.2383+j7.8212	-3.3562+j7.6405	-5.772+j8.1901
9	-.2522+j8.8068	-3.4376+j8.6466	-5.9045+j9.2351
10	-.2648+j9.8001	-3.5108+j9.6555	

Table 4.1. Pole locations $s'_\alpha = \left[\frac{s_\alpha L}{\pi c}\right]$ in the complex frequency plane for the thin wire of $\Omega = 10.6$ determined by the contour integration method.[19]

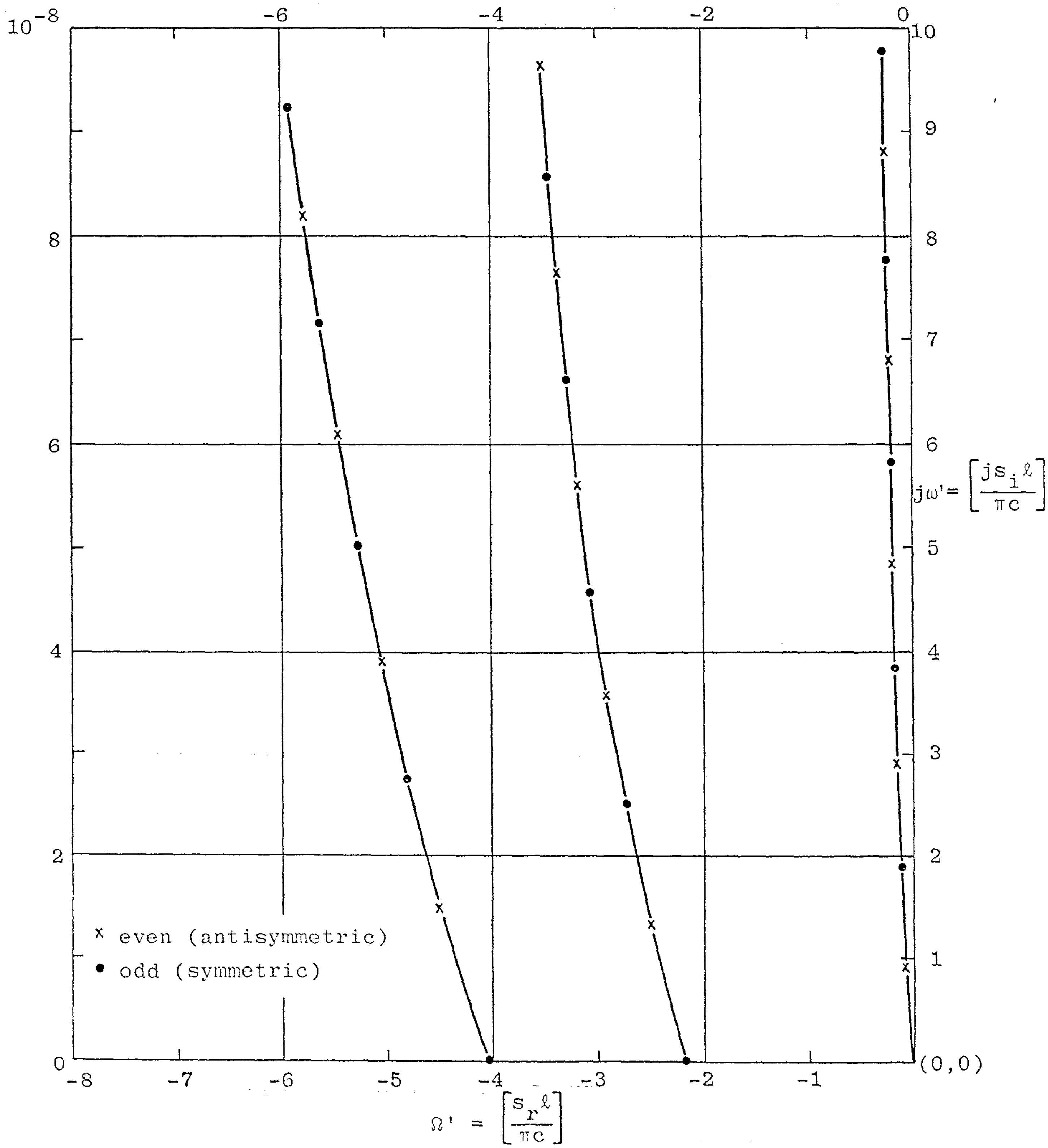


Figure 4.1. Plot of Pole Locations $[s_\alpha \ell/\pi c]$ in the Normalized Complex Frequency Plane for the Thin Wire of Ω = 10.6, Found by Contour Integration Method[19]

$$\left\langle I_\alpha \, , \, \frac{d\tilde{\mathscr{Z}}}{ds}\bigg|_{s=s_\alpha} \, , \, I_\alpha \right\rangle \approx \frac{\Omega s_\alpha \ell}{\pi c^2} \tag{4.5}$$

where terms of order Ω^{-1} are neglected. It may be argued that terms of the order Ω^{-1} play an important role in the positive real properties of the admittance functions. In our opinion, this does represent the best approximation.

4.1 Pole Admittance Formulation for a Cylindrical Antenna

The short circuit current $\tilde{I}(z)$ can be written as

$$\tilde{I}(z,s) = \frac{2\ell}{\Omega Z_o} \sum_\alpha \frac{\left\langle I_\alpha \, , \, \tilde{E}_z^{inc} \right\rangle}{\left[\left\langle I_\alpha \, , \, I_\alpha \right\rangle + 0\left(\frac{1}{\Omega}\right)\right]} \frac{1}{(s - s_\alpha')} \tag{4.6}$$

where, as defined before, s' is the normalized complex radian frequency. Note that as $s' \rightarrow 0$, $\tilde{I}(z,s) \rightarrow$ a constant value. This is not physical for a cylindrical antenna.

Using (2.22) and (4.6), we can write the input admittance at the center of the antenna as

$$\tilde{Y}(s') = \frac{1}{\Delta^2} \frac{2\ell}{\Omega Z_o} \sum_\alpha \frac{\left\langle I_\alpha \, , \, 1 \right\rangle_g^2}{\left[\left\langle I_\alpha \, , \, I_\alpha \right\rangle + 0\left(\frac{1}{\Omega}\right)\right]} \frac{1}{(s' - s_\alpha')} \tag{4.7}$$

Neglecting terms of order Ω^{-1}, we have

$$\tilde{Y}(s) = \frac{1}{\Delta^2} \frac{4}{\Omega Z_o} \sum_\alpha \left\langle I_\alpha \, , \, 1 \right\rangle_g^2 \frac{1}{(s' - s_\alpha')} \tag{4.8}$$

We note that

$$\left\langle I_\alpha \, , \, 1 \right\rangle_g^2 = \begin{cases} \left(\frac{2\ell}{\alpha\pi}\right)^2 \sin^2\left(\frac{\alpha\pi}{2\ell} \Delta\right) & \alpha = \text{odd} \\ \\ 0 & \alpha = \text{even} \end{cases} \tag{4.9}$$

If the gap is small, i.e., $\Delta << 2\ell/\alpha\pi$ (note that the gap has to be small compared to the frequency of interest), we can approximate (4.9) as

$$\left\langle I_\alpha , 1 \right\rangle_g^2 = \begin{cases} \Delta^2 & \alpha = \text{odd} \\ 0 & \alpha = \text{even} \end{cases} \quad \Delta << \frac{2\ell}{\alpha\pi} \tag{4.10}$$

Hence, for a small gap approximation, (4.8) can be simplified to read

$$\tilde{Y}(s') = \frac{4}{\Omega Z_o} \sum_\alpha \frac{1}{(s' - s'_\alpha)} \qquad \alpha = \text{odd} \tag{4.11}$$

where if a finite sum is considered, s' cannot be larger than the largest s'_α. We note that the pole admittance coupling coefficients are real and positive. Hence they satisfy the realizability considerations discussed in section 3. This implies that for each conjugate pole pair, a network representation is realizable. Shown in figures 4.2 through 4.6 are the real and imaginary parts of the pole admittance for each conjugate pole pair. Figure 4.7 shows the cumulative sum of the real and imaginary parts of the admittance for nine conjugate pole pairs. Note that at zero frequency, the admittance reaches a constant value which is not physical. In addition, results for $\omega' > 9$ should not be trusted because only nine pole pairs were considered in our calculations. Comparing the admittance values with the King-Middleton[20] theory, one finds that the peaks and valleys agree within 10%. Shown in figure 4.8 are the network realizations for each of those pole pairs. Note that all the element values are positive and real, $\tilde{Y}(s) \to \frac{-4}{\Omega Z_O} \sum_\alpha \frac{1}{s'_\alpha}$ as $s \to 0$, which is a constant. This behavior at $s = 0$ does not agree with a cylindrical antenna admittance.

Returning to (4.6), we can rewrite this as

$$\tilde{I}_{sc}(z) = \frac{4}{\Omega Z_o} \sum_\alpha \frac{\left\langle I_\alpha , \tilde{E}_z^{inc} \right\rangle}{(s' - s'_\alpha)} \tag{4.12}$$

Note that the numerator term $\left\langle I_\alpha , E_z^{inc} \right\rangle$ is simply the source term. Comparing the remaining terms with the admittance equation (4.11), they are found to be identical. Note that this holds only

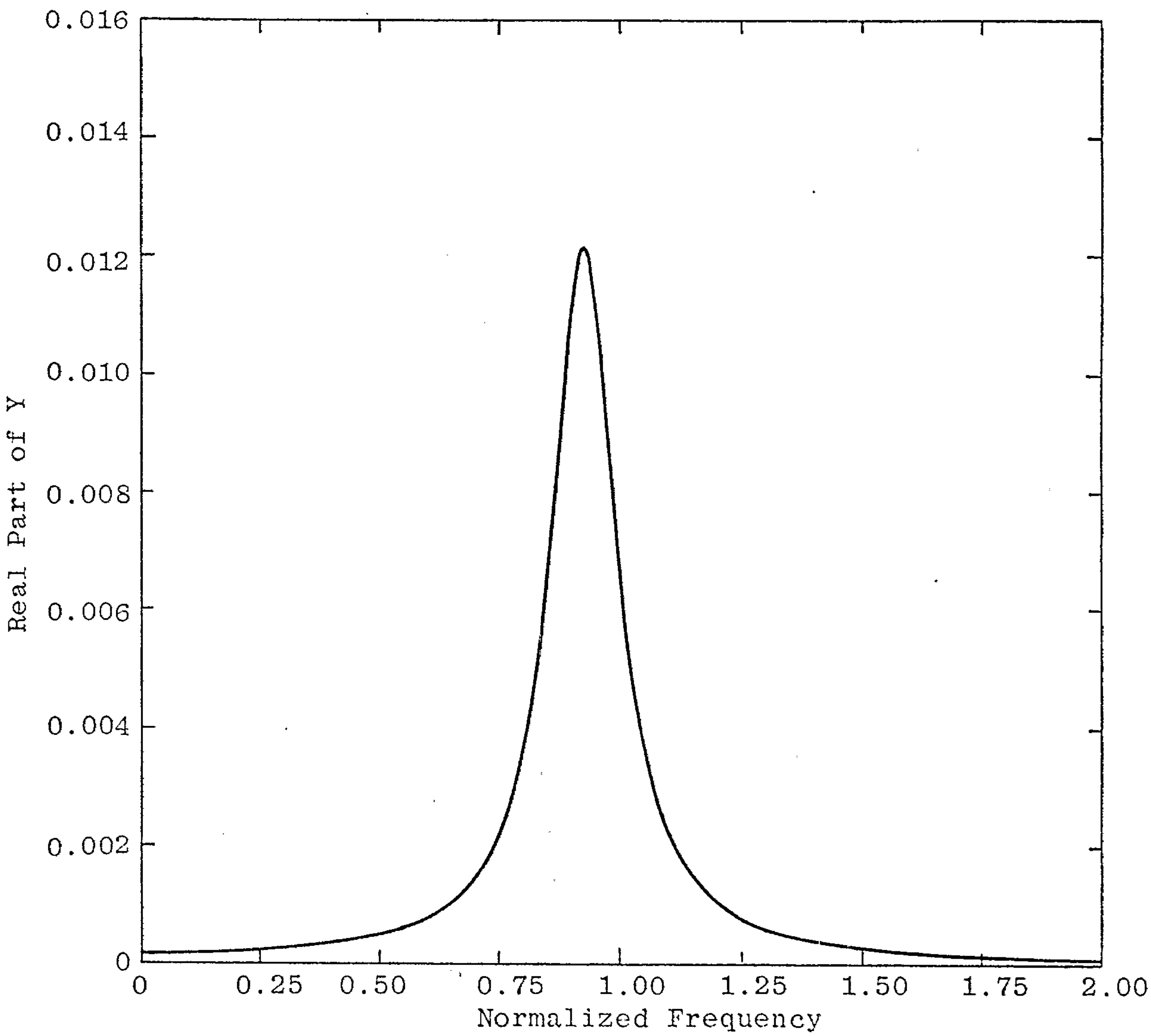


Figure 4.2a. Real Part of the Pole Admittance for the First Conjugate Pole Pair (Ω = 10.6)

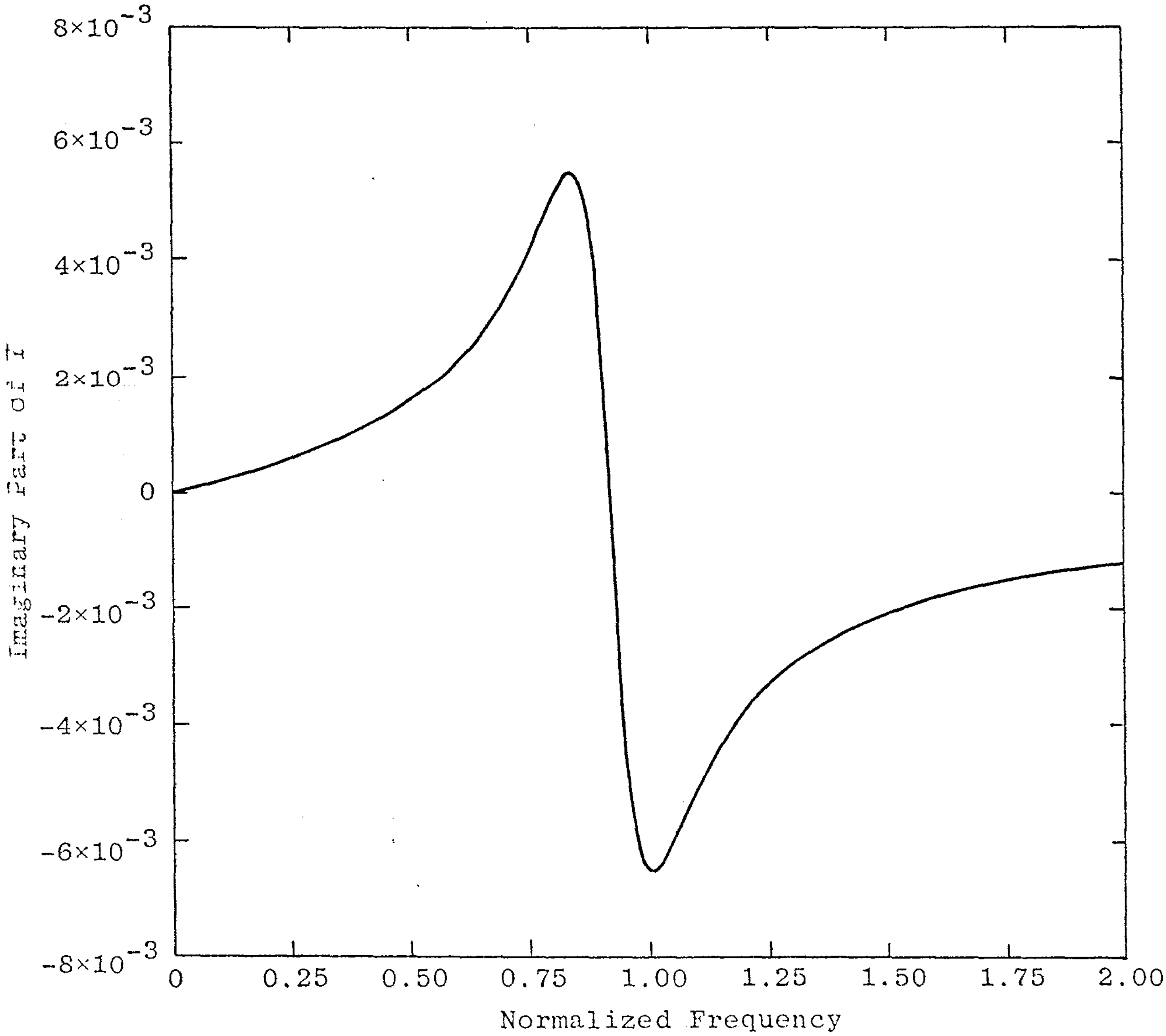


Figure 4.2b. Imaginary Part of the Pole Admittance for the First Conjugate Pole Pair (Ω = 10.6)

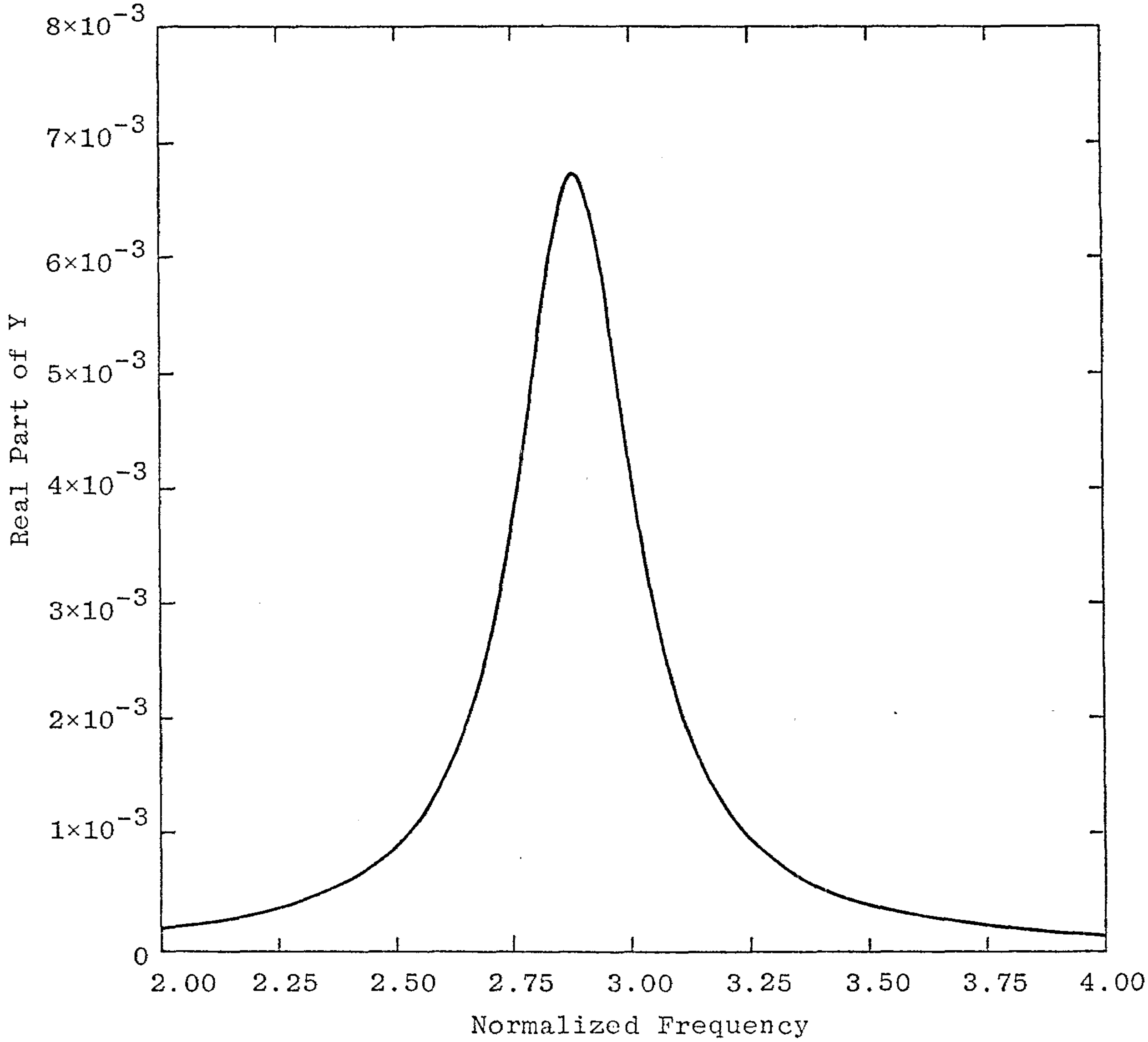


Figure 4.3a. Real Part of the Pole Admittance for the Third Conjugate Pole Pair (Ω = 10.6)

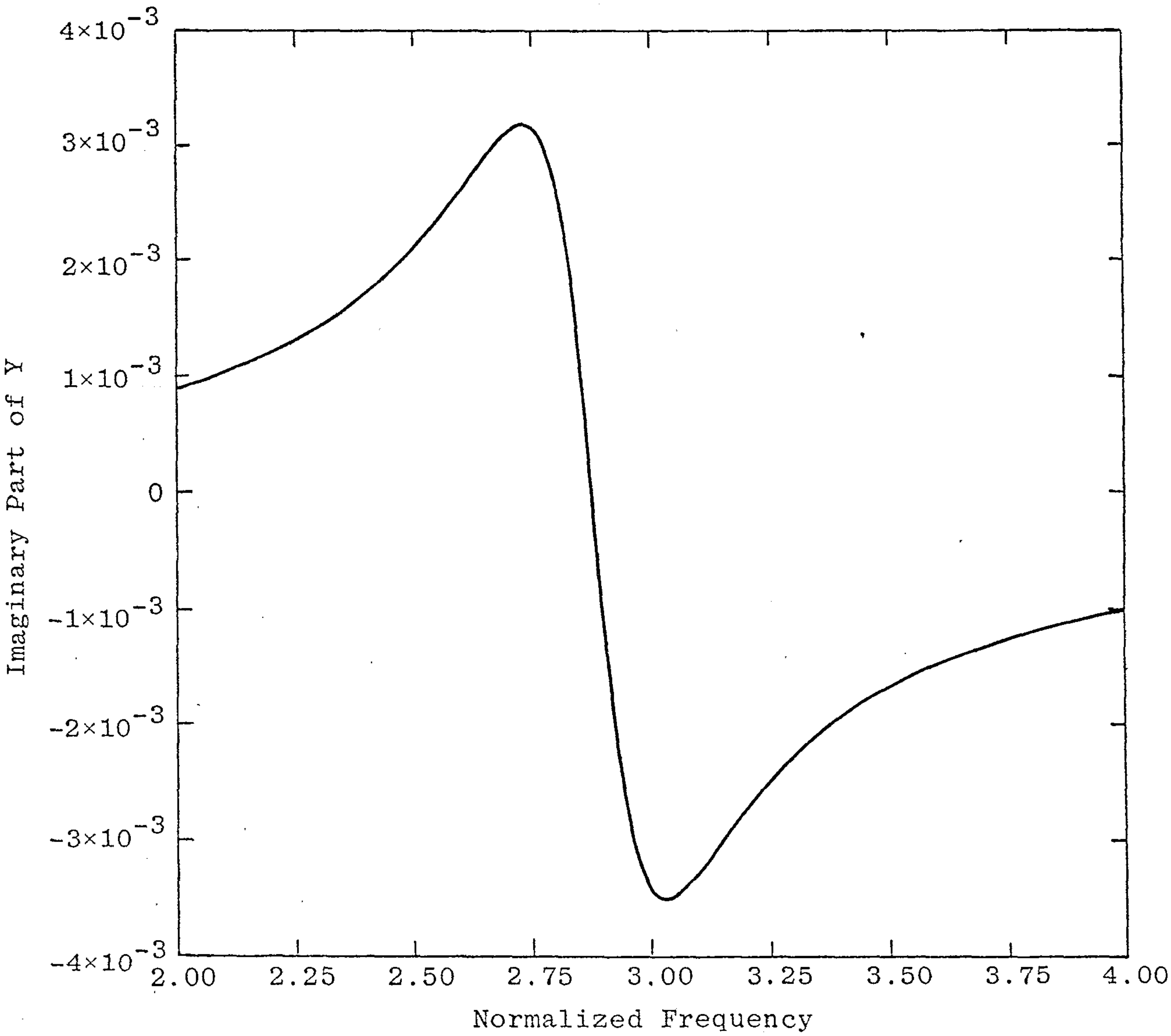


Figure 4.3b. Imaginary Part of the Pole Admittance for the Third Conjugate Pole Pair (Ω = 10.6)

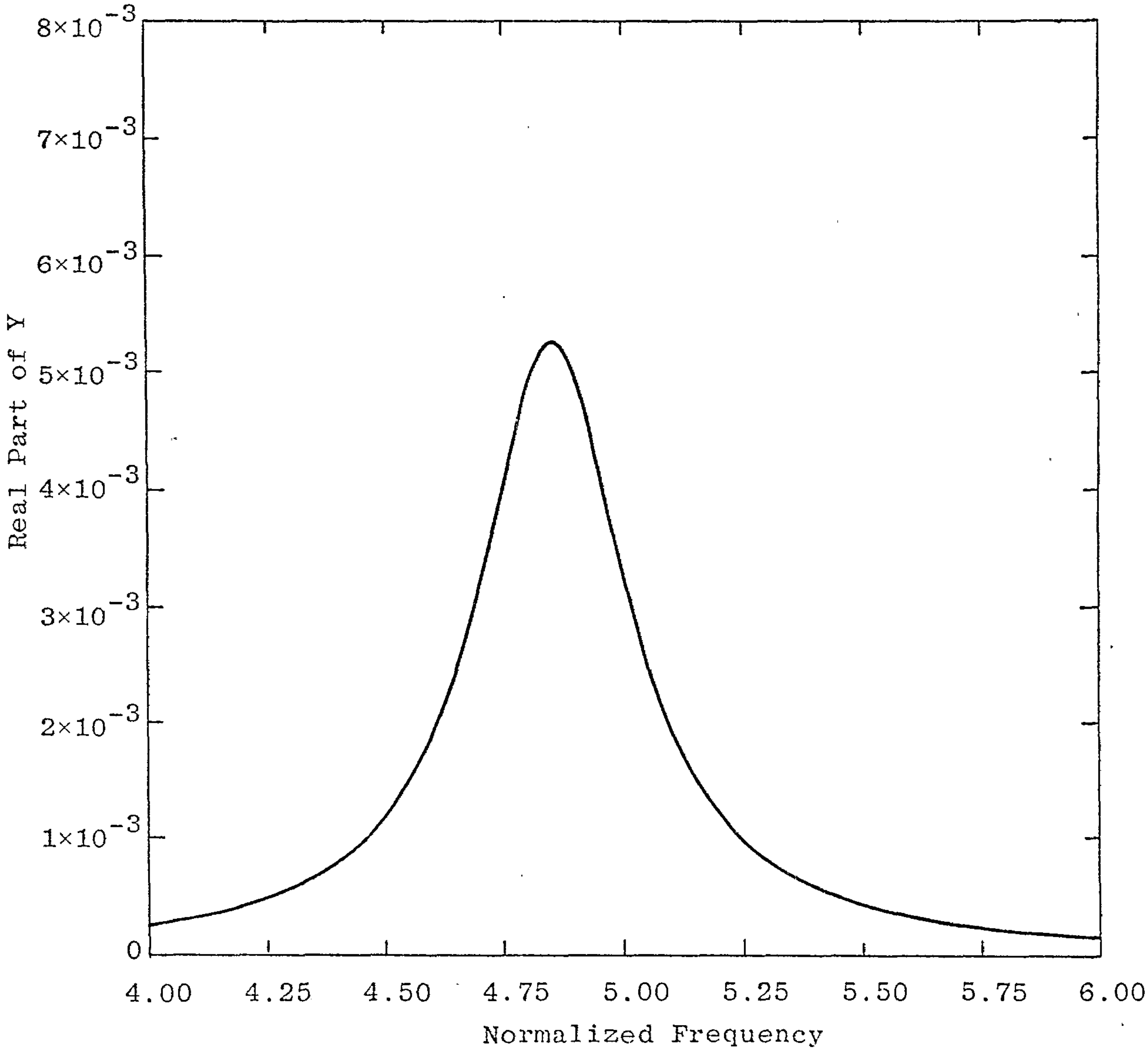


Figure 4.4a. Real Part of the Pole Admittance for the Fifth Conjugate Pole Pair (Ω = 10.6)

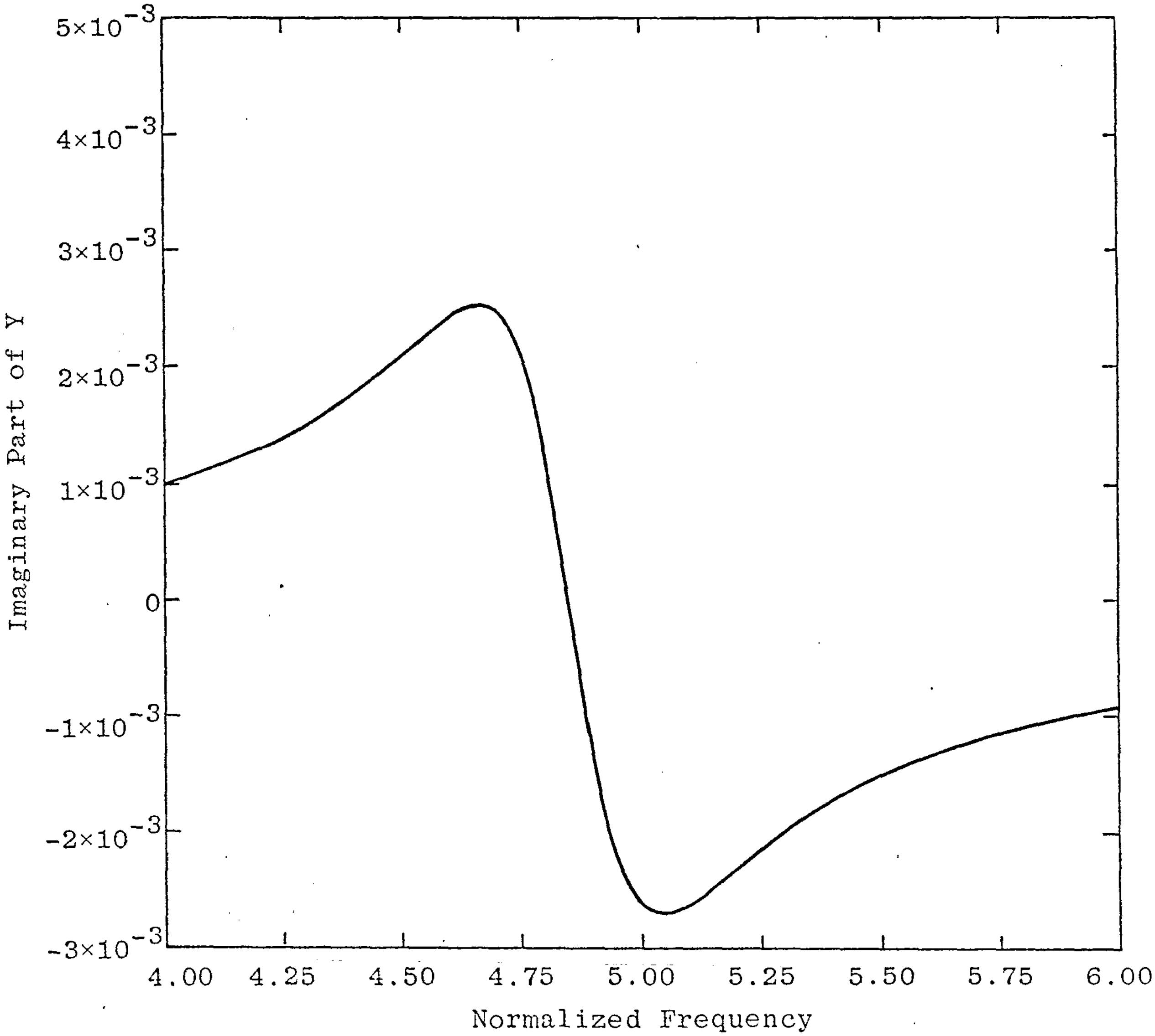


Figure 4.4b. Imaginary Part of the Pole Admittance for the Fifth Conjugate Pole Pair (Ω = 10.6)

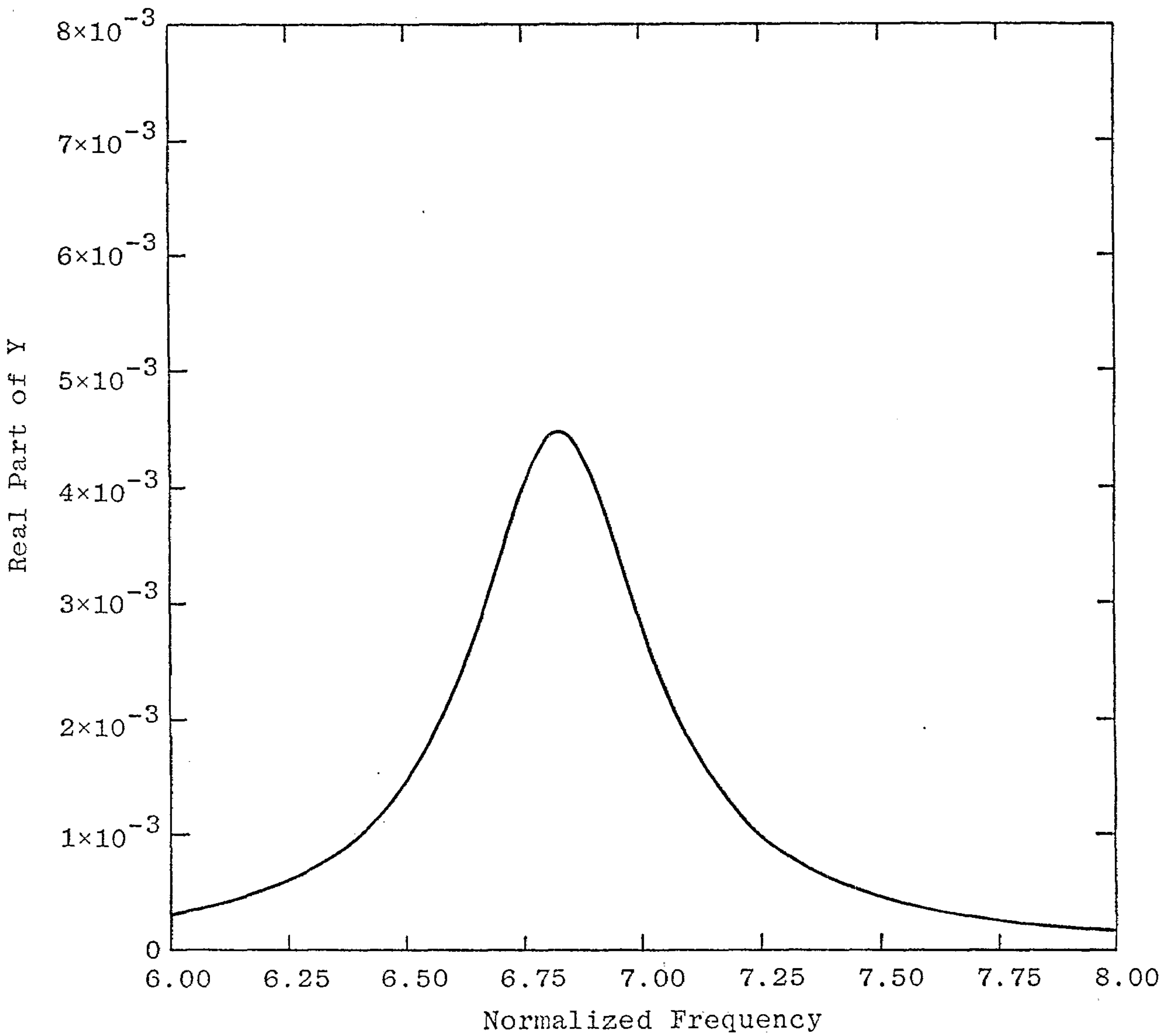


Figure 4.5a. Real Part of the Pole Admittance for the Seventh Conjugate Pole Pair (Ω = 10.6)

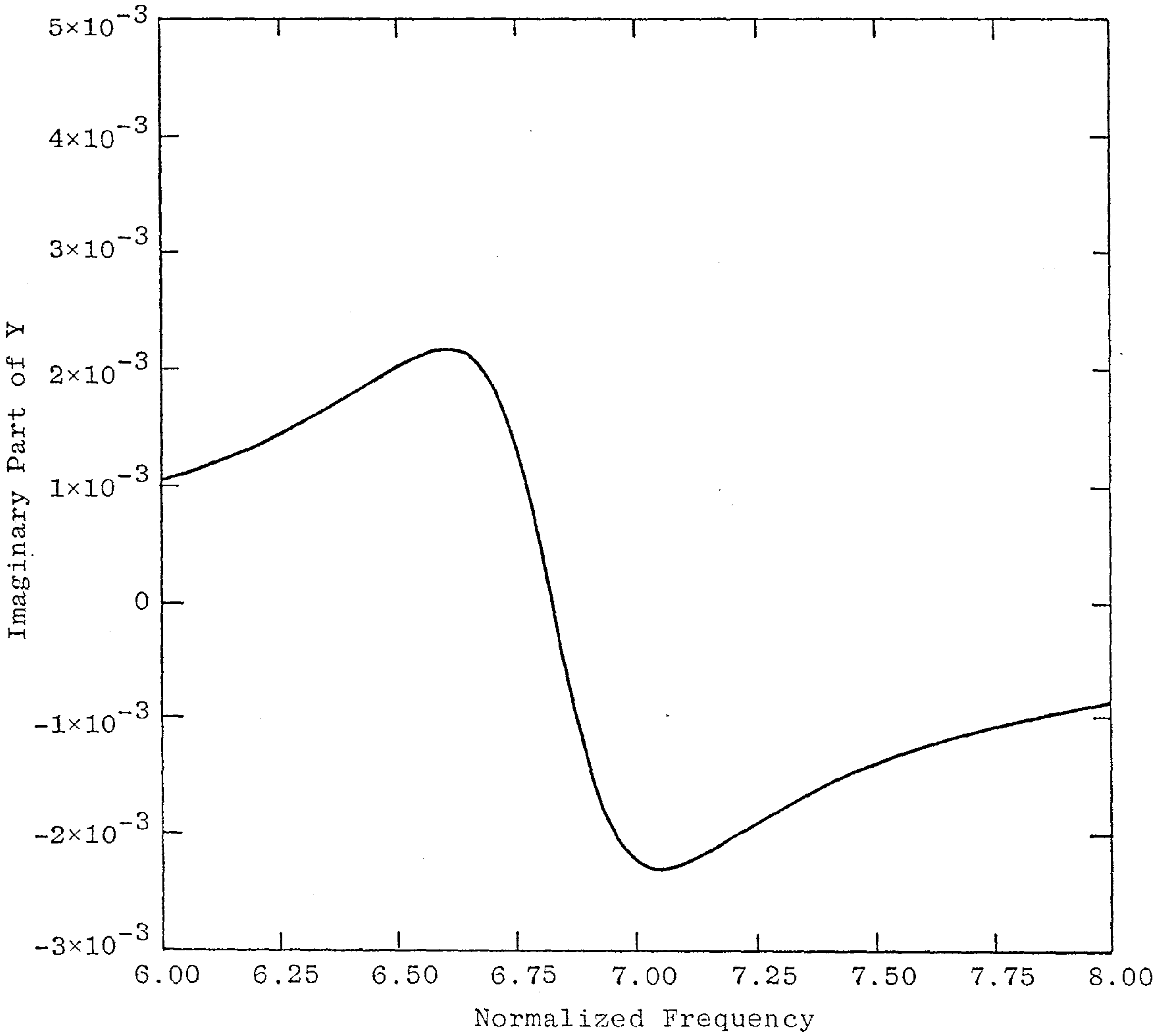


Figure 4.5b. Imaginary Part of the Pole Admittance for the Seventh Conjugate Pole Pair (Ω = 10.6)

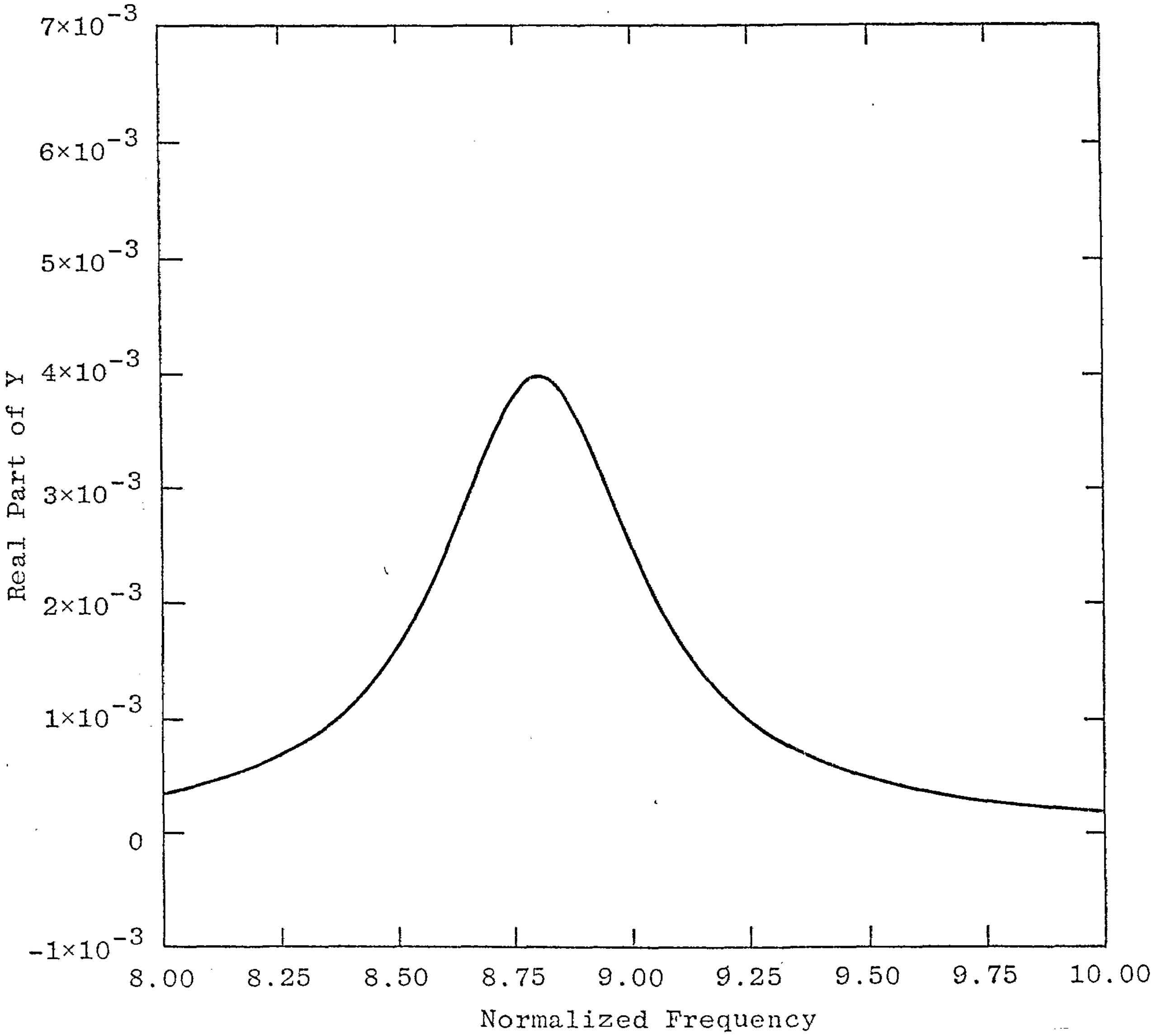


Figure 4.6a. Real Part of the Pole Admittance for the Ninth Conjugate Pole Pair (Ω = 10.6)

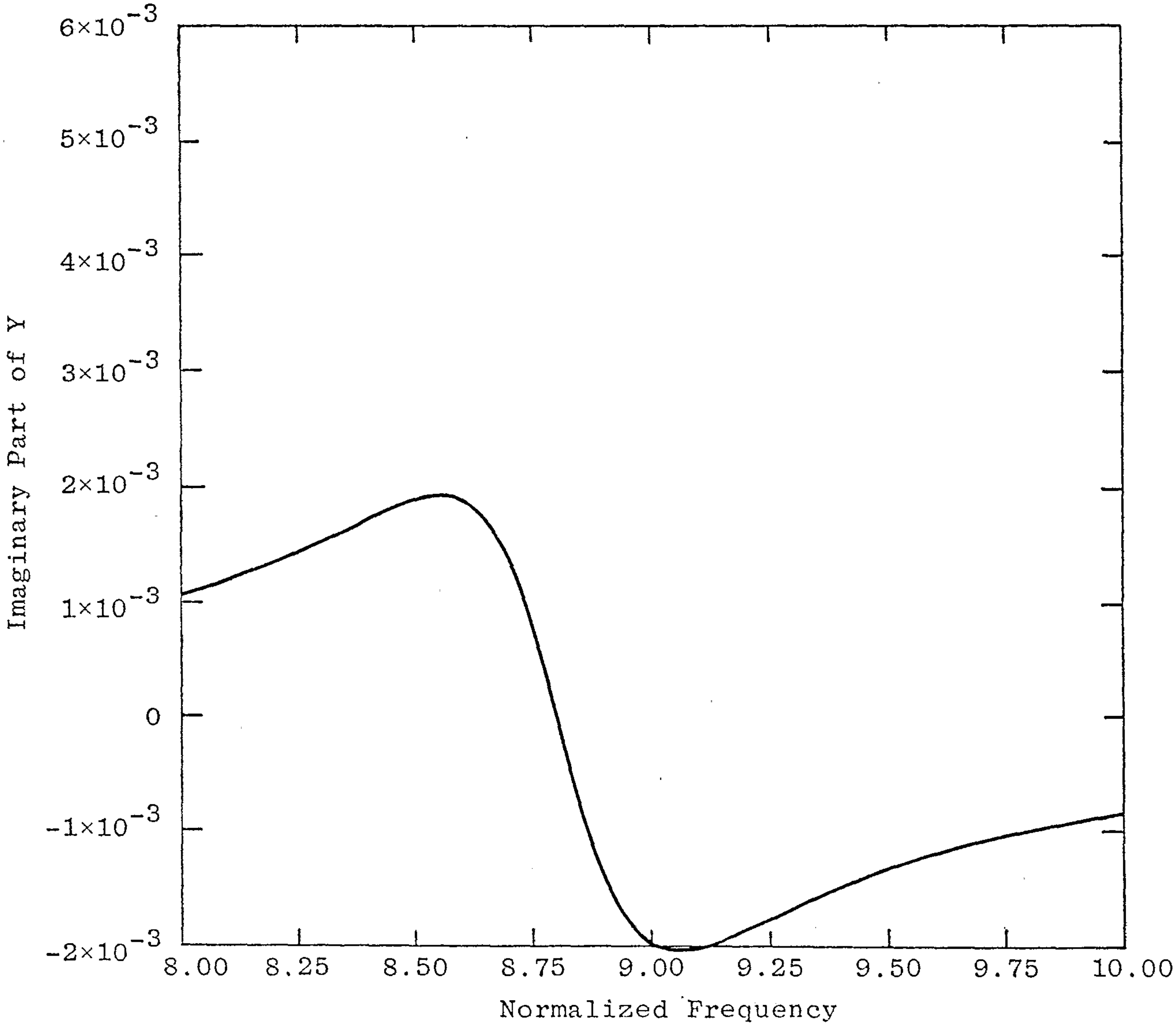


Figure 4.6b. Imaginary Part of the Pole Admittance for the Ninth Conjugate Pole Pair (Ω = 10.6)

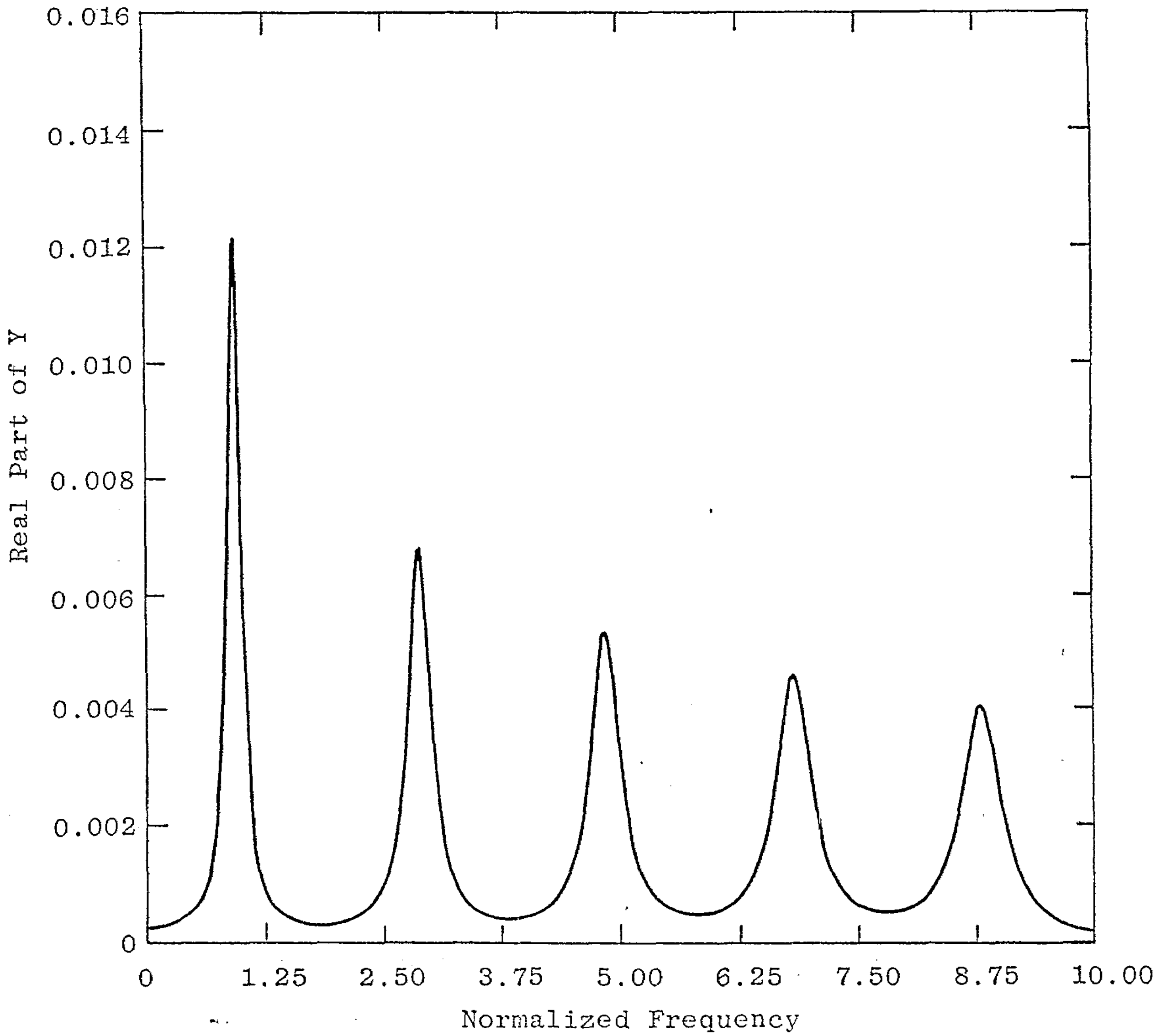
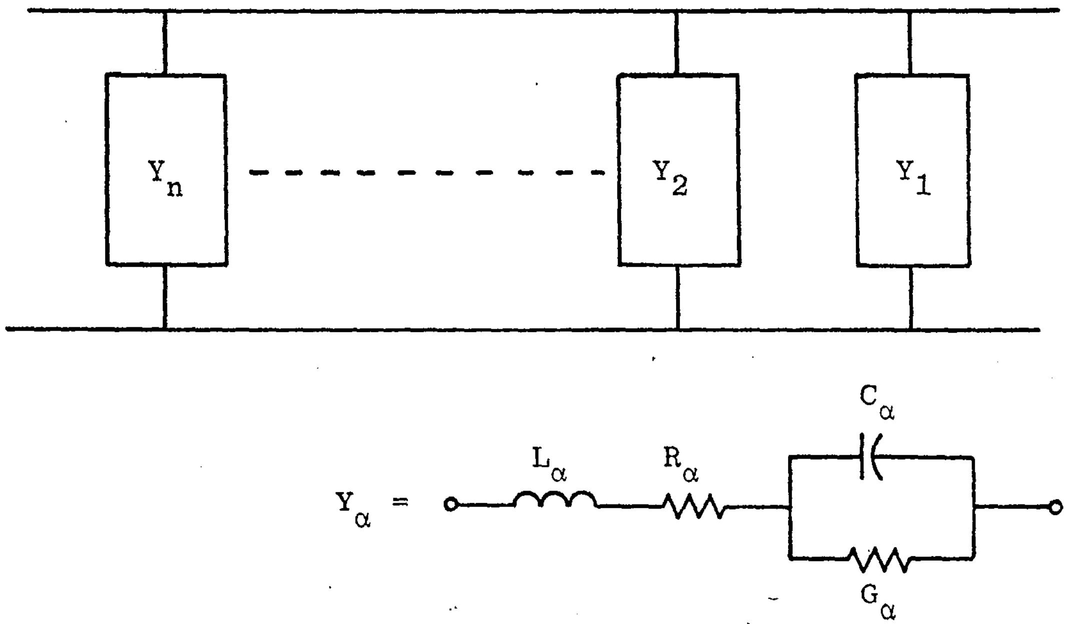


Figure 4.7a. Real Part of the Pole Admittance for the First Nine Conjugate Pole Pairs (Ω = 10.6)



$$L_\alpha = \frac{\Omega Z_o}{8}$$

$$C_\alpha = \frac{8}{\Omega Z_o}\,\frac{1}{\omega_\alpha^2}$$

$$R_\alpha = \Omega\,\frac{|\Omega_\alpha|\,Z_o}{8}$$

$$G_\alpha = \frac{8}{\Omega Z_o}\,\frac{|\Omega_\alpha|}{\omega_\alpha^2}$$

Figure 4.8. Network Realization for the Pole Admittance Formulation (Note that the element values are in terms of the normalized s plane (s') quantities)

at the center of the dipole antenna. One interpretation is that in the small gap approximation, if one eliminates the source term, the remaining term is the pole admittance. This has been found to be true in the case of a cylindrical antenna and a Helix.[21] Since the natural mode is real, the coefficients for the voltage coefficients for the conjugate pair circuit are constant and real.

4.2 Modified Pole Admittance Formulation for a Cylindrical Antenna

We write the short-circuit current for this case as

$$\tilde{I}_{sc}(z) = \frac{2\ell}{\Omega Z_o} \sum_\alpha \frac{\left\langle I_\alpha , E_z^{inc} \right\rangle}{\left[\left\langle I_\alpha , I_\alpha \right\rangle + 0\left(\frac{1}{\Omega}\right)\right]} \frac{s'}{s'_\alpha (s' - s'_\alpha)} \tag{4.13}$$

As $s' \to 0$, $\tilde{I}(z) \to 0$ uniformly, which is the condition an electric dipole is required to satisfy. The input admittance at the center of the antenna is

$$\tilde{Y}(s') = \frac{1}{\Delta^2} \frac{2\ell}{\Omega Z_o} \sum_\alpha \frac{\left\langle I_\alpha , 1 \right\rangle_g^2}{\left[\left\langle I_\alpha , I_\alpha \right\rangle + 0\left(\frac{1}{\Omega}\right)\right]} \frac{s'}{s'_\alpha (s' - s'_\alpha)} \qquad \alpha = 1,3,\ldots \tag{4.14}$$

Neglecting terms of order Ω^{-1}; we can rewrite (4.14) as

$$\tilde{Y}(s') = \frac{1}{\Delta^2} \frac{4}{\Omega Z_o} \sum_\alpha \left\langle I_\alpha , 1 \right\rangle_g^2 \frac{s'}{s'_\alpha (s' - s'_\alpha)} \qquad \alpha = 1,3,\ldots \tag{4.15}$$

Using the small gap approximation given by (4.10), we can rewrite (4.15) as

$$\tilde{Y}(s') = \frac{4}{\Omega Z_o} \sum_\alpha \frac{s'}{s'_\alpha (s' - s'_\alpha)} \qquad \alpha = 1,3,\ldots \tag{4.16}$$

Note that as $s' \to 0$, $\tilde{Y}(s') \to 0$, which is the condition the input admittance of a cylindrical antenna is required to satisfy. As $s' \to \infty$, $\tilde{Y}(s') \to$ a constant value, which also satisfies the

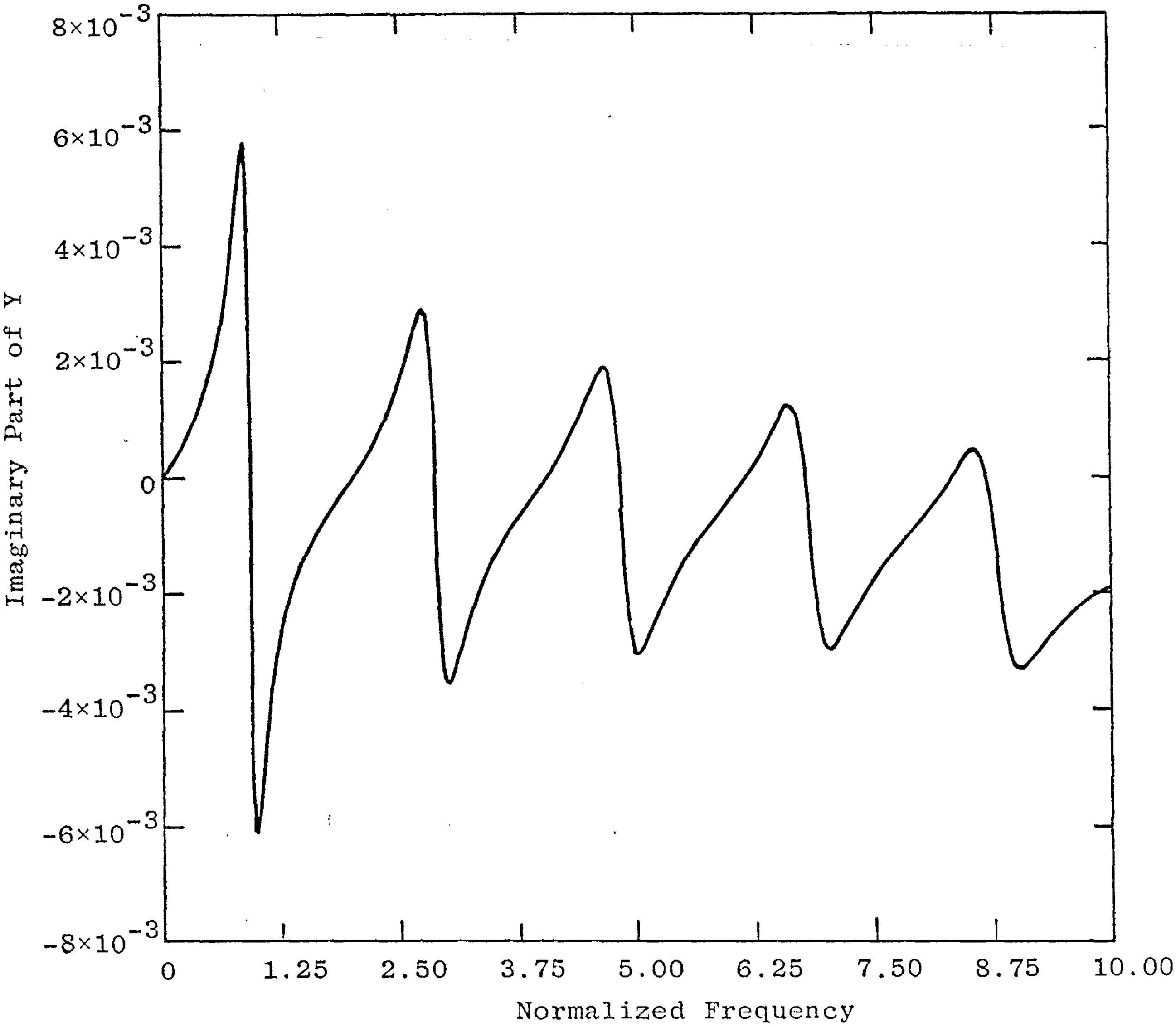


Figure 4.7b. Imaginary Part of the Pole Admittance for the First Nine Conjugate Pole Pairs (Ω = 10.6)

requirements for a cylindrical antenna. Comparing the pole admittance formulation (4.11) with the modified pole admittance formulation, one notes that the two formulations are identical at the pole locations.

Rewriting (4.13) for the small gap approximation, we obtain

$$\tilde{I}_{sc}(z) = \frac{4}{\Omega Z_o} \sum_\alpha \left\langle I_\alpha \, , \, \tilde{E}_z^{inc} \right\rangle \frac{s'}{s'_\alpha (s' - s'_\alpha)} \tag{4.17}$$

Comparing the terms other than the forcing function with the admittance given by (4.16), we find them to be identical. As mentioned earlier, this appears to be true for all cases considered by these authors. As discussed earlier, the voltage coefficients for the conjugate pair circuit are constant and real.

We note that (4.16) is obtained from (4.11) by way of

$$\tilde{Y}'(s') = \frac{4}{\Omega_f Z_o} \sum_\alpha \left[\frac{1}{s - s'_\alpha} + \frac{1}{s'_\alpha} \right] \tag{4.18}$$

If we compare the pole admittances with the modified pole admittances, the modified pole admittances are obtained from the pole admittances by the addition of $(1/s'_\alpha)$. Note that the real part of s'_α is negative, and the real part of the modified pole admittance is obtained from the real part of the pole admittance by the addition of $-2|\Omega'_\alpha|/(\Omega_\alpha^2 + \omega_\alpha^2)$. Although this yields the correct behavior for low frequencies, this will make the real part of the modified pole admittance negative for high frequencies, thereby making the modified pole admittances non-P.R. and hence non-realizable. An interesting result is that if the pole admittances (modified pole admittances) are P.R., corresponding modified pole admittance will not be P.R. assuming that the pole admittance coupling coefficients have positive real part. Note that resistive padding techniques[21] are available to correct the non-P.R. property of the modified pole admittances. Note that one can also consider the natural frequencies according to the eigenvalues of an appropriate integral equation and construct equivalent circuits. Another alternative is to consider the open-circuit boundary value problem and construct the equivalent circuits. Although not discussed here, a final alternative is to construct equivalent circuits from measured data, where the required SEM quantities are found from the measured time or frequency domain data. This is a powerful tool for structures that are too complicated to model in an analytical or computer model.

5. Conclusions

In this work, we discussed the application of the SEM representation to the short-circuit boundary-value formulation of a dipole antenna. General realization constraints were discussed along with the example of a cylindrical antenna. It has been shown that under the analytical formulation of the coupling coefficients, the pole admittances (but not the modified pole admittances) are positive real and hence realizable. Network realization for the pole admittances is exhibited.

From this work, it is clear that at least for a cylindrical antenna under the analytical formulation, network realizations are possible. Approximations were made in analytically evaluating the coupling coefficients, and we estimated the effect of these approximations to be of second order. Until an "accurate" numerical evaluation is made, one cannot conclusively say much about the realization procedure for the "exact" formulation.

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