Interaction Notes

Note 364

May 1978

Geometrical Theory of Diffraction in Electromagnetics
Volume I: Geometrical Optics

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Preface

In this series of reports, the geometrical theory of diffraction in electromagnetics is reviewed. It is intended to cover the following subjects:

Volume I:	Geometrical Optics (Mathematics Note 53)
Volume II:	Diffraction by Edge
Volume III:	Diffraction by Smooth Surface
Appendix A:	Differential Geometry (Interaction Note 364)

I would like to take this opportunity to express my most sincere appreciation to Professor J. Boersma, who has contributed significantly to my learning of ray techniques.

Acknowledgement

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COMMON NOTATIONS AND CONVENTIONS

1. The SI units and time convention $\exp -i\omega t$ are used.

2. Unless stated otherwise, the medium is the free space with the wave number $k = \omega(\varepsilon\mu)^{1/2}$.

3. $\vec{E}^i$, $\vec{E}^r$, $\vec{E}^d$, $\vec{E}^t$ and $\vec{E}$ are, respectively, incident, reflected, diffracted, total, and scattered fields. Unless stated otherwise, $\vec{E}^t = \vec{E}^i + \vec{E}$ everywhere.

4. In a two-dimensional problem (no z-variation), $u = E_z$ and $R = -1$ for E-wave, and $u = H_z$ and $R = +1$ for H-wave.

5. $\{i \to r\}$ means to repeat all the terms after the equal sign and change the superscript "i" to "r" in those terms. For example,

$$\vec{E}^t(\vec{r}) = \theta(-\varepsilon^i)\ \vec{E}^i + \{i \to r\}$$

means

$$\vec{E}^t(\vec{r}) = \theta(-\varepsilon^i)\ \vec{E}^i + \theta(-\varepsilon^r)\ \vec{E}^r$$

6. $A^{i,r} = \mp B^{i,r}$ means $A^i = -B^i$ and $A^r = + B^r$.

7. $g(kr) = \dfrac{1}{2\sqrt{2\pi kr}}\, e^{i(kr+\pi/4)}$ which is a "unit" cylindrical wave.

8. The Fresnel function for a real x is defined by

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^\infty e^{it^2}\, dt$$

Its large argument asymptotic expansion is

$$F(x) = \theta(-x) + \hat{F}(x) + O(x^{-3})$$

where θ is a unit step function and

$$\hat{F}(x) = \frac{1}{2x\sqrt{\pi}} \exp i(x^2 + \frac{\pi}{4})$$

CHAPTER 1. PRELIMINARIES

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1.1 Maxwell's Equations

This part of the book discusses the application of ray techniques to electromagnetic edge diffraction at high frequencies. As a preparatory step we will in this chapter review some basic concepts of edge diffractions, and explain some notations, conventions and special functions that will be used frequently later.

The behavior of the electromagnetic field in a continuous medium is governed by Maxwell's equations. Except for a few isolated problems we consider exclusively the electromagnetic field which

(i) propagates in the free space with permittivity ε and permeability μ; and

(ii) has $\exp(-i\omega t)$ time dependence.

Under the above two conditions, Maxwell's equations in SI system of units assume the form

$$\nabla \times \vec{E} = i\omega\mu\vec{H} \tag{1.1}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon\vec{E} + \vec{J} \tag{1.2}$$

$$\nabla \cdot \vec{E} = 0 \tag{1.3}$$

$$\nabla \cdot \vec{H} = 0 \quad . \tag{1.4}$$

Here $\vec{E}$ is the <u>electric field</u> (in volts per meter), $\vec{H}$ is the <u>magnetic field</u> (in amperes per meter), and $\vec{J}$ is <u>current density</u> (in amperes per square meter). All three quantities are functions of spatial variable $\vec{r}$ only, while the common time factor $\exp(-i\omega t)$, as usual, has been suppressed. In a source-free region where $\vec{J} = 0$, Maxwell's equations in (1.1) through (1.4) can be replaced by

$$\begin{cases} (\nabla^2 + k^2)\, \vec{E} = 0 & (1.5) \\ \nabla \cdot \vec{E} = 0 & (1.6) \end{cases}$$

$$\vec{H} = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{1}{ik}\, \nabla \times \vec{E} \tag{1.7}$$

where $k = \omega\sqrt{\mu\varepsilon} = 2\pi/\lambda$ is the wave number and λ is the wavelength. In a given problem we usually concentrate on solving $\vec{E}$ from (1.5) and (1.6), and next calculate $\vec{H}$ from (1.7).

At the surface Σ of a perfect conductor, the boundary conditions are

$$\hat{N} \times \vec{E} = 0 \tag{1.8}$$

$$\hat{N} \times \vec{H} = \vec{J}_s \tag{1.9}$$

where $\hat{N}$ is the outward unit normal of Σ, and $\vec{J}_s$ is the surface current density (in amperes per meter).

In problems involving either unbounded space or containing geometrical singularities, it is possible to derive several mathematically acceptable solutions of Maxwell's equations, only one of which is consistent with anticipated physical phenomena. Therefore in these situations it is necessary to introduce additional physical constraints to ensure the uniqueness of the solution.

In an unbounded space with all sources contained in a finite region, the additional physical constraint that governs the field behavior at infinity is known as radiation condition. It may be stated in any one of two ways described below. (i) Introduction of loss: The lossless free space is regarded as a limiting case of a lossy medium, i.e., $k \to k + i\delta$, $\delta > 0$. In a lossy medium, the field vanishes at infinity. (ii) Sommerfeld radiation condition: At a large distance r for the source region, the field has a phase progressing outward and has an amplitude that decreases at least as rapidly as r^{-1}.

The other situation where the solution of Maxwell's equations may not be unique arises when the configuration of the problem contains sharp edges or tips. The additional physical constraint needed here is the edge condition: the energy stored in the electromagnetic field in any finite region must be finite. In particular, as volume V containing a point on edges or tips contracts to zero, we require

$$\int_V (\varepsilon|\vec{E}|^2 + \mu|\vec{H}|^2)\, dv \to 0 \quad . \tag{1.10}$$

In a given problem, the requirement in (1.10) can often enable us to derive an explicit upper bound for the field behavior near an edge or tip.* For later applications let us consider the configuration of a two-dimensional (no z variation), perfectly conducting wedge in Figure 1-1. The exterior angle of the wedge is $m\pi$. It can be shown that the enforcement of (1.10) leads to the following conclusion about the singular field behavior near the edge as $\rho = \sqrt{x^2 + y^2} \to 0$:

$$E_x, E_y, H_x, H_y = O[\rho^{(1-m)/m}] \tag{1.11a}$$

$$E_z, H_z = O(\rho^{1/m}) \quad . \tag{1.11b}$$

Thus, near the edge, the field components parallel to the edge are always bounded, while transverse components may become singular when $1 < m \leq 2$. (The symbol O is defined in Problem 1-1.)

---

* Explicit edge condition for a wedge was first derived in J. Meixner, "The behavior of electromagnetic fields at edges," Inst. Math. Sci. Res. Rept. EM-72, New York University, New York, 1954. Discussions on edge condition can be found, for example, in D. S. Jones, The Theory of Electromagnetism, Macmillan, New York, 1964, pp. 566-569; R. Mittra and S. W. Lee, Analytical Techniques in the Theory of Guided Waves, Macmillan, New York, 1971, pp. 4-11.

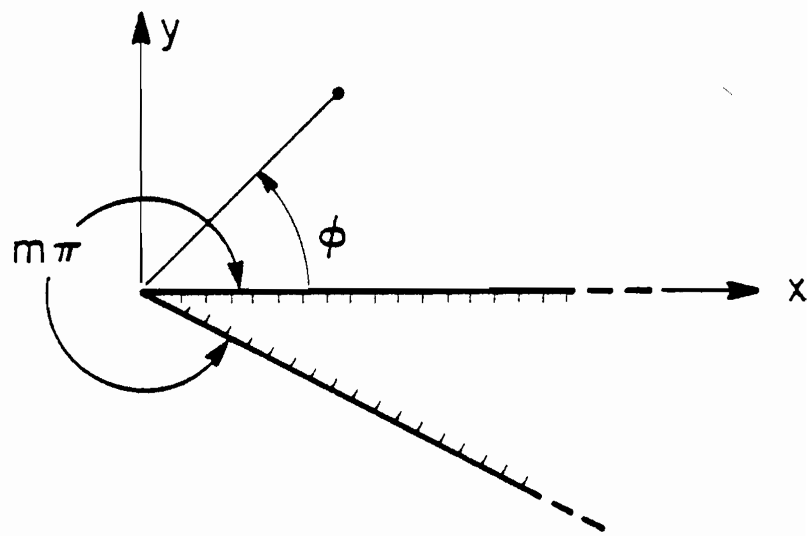

Figure 1-1. A two-dimensional perfectly conducting wedge.

1.2 Asymptotic Expansion

The central theme of this book is to study an asymptotic expansion of the edge diffraction problems at high frequencies. Thus, it is important that the meaning and implication of an "asymptotic expansion" is understood.*

We are interested in the behavior of a function f(k) for large values of k. In the simplest case, f(k) may be conveniently approximated by a formal power series of k, namely,

$$\sum_{m=0}^{\infty} a_m k^{-m} = a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \dots + \frac{a_m}{k^m} + \dots \quad . \tag{2.1}$$

Let us define a partial sum of the series

$$f_M(k) = \sum_{m=0}^{M} a_m/k^m \quad . \tag{2.2}$$

Then (2.1) is said to be the asymptotic (power series) expansion of f(k) if for a fixed M

$$\lim_{k\to\infty} k^M[f(k) - f_M(k)] = 0 \quad . \tag{2.3}$$

We write it with the symbol "$\sim$":

$$f(k) \sim \sum_{m=0}^{\infty} a_m/k^m \quad . \tag{2.4}$$

---

* A. Erdelyi, Asymptotic Expansions, Dover Publications, New York, 1956.
H. Jeffrys, Asymptotic Approximations, Oxford University Press, 1962.
E. Copson, Asymptotic Expansions, Cambridge University Press, 1965.
L. Sirovich, Techniques of Asymptotic Analysis, Springer-Verlag, New York, 1971.

Using the symbol O (big oh), (2.4) may be rewritten as

$$f(k) = \sum_{m=0}^{M} \frac{a_m}{k^m} + O[k^{-(M+1)}] \quad . \tag{2.5}$$

Using the symbol o (small oh), (2.4) may be also written as

$$f(k) = \sum_{m=0}^{M} \frac{a_m}{k^m} + o(k^{-M}) \quad . \tag{2.6}$$

Symbols O and o are defined in Problem 1-1. From the viewpoint of application, we list below several common properties of asymptotic expansions.

First, for a <u>fixed M</u>, the difference between f(k) and $f_M(k)$ can be made arbitrarily small provided that a large enough k is used. This is simply a restatement of (2.3). Readers undoubtedly are familiar with the fact that, in many asymptotic expansions, the first few terms of the series often give surprisingly accurate results even in non-asymptotic regimes.

Second, for a <u>fixed k</u>, the difference between f(k) and $f_M(k)$, however, <u>cannot</u> be made arbitrarily small by increasing M. Unfortunately, in almost all the practical problems in electromagnetic theory, we face problems with a fixed k. In such cases, the higher-order terms of an asymptotic series are useful only in the following sense:

(i) For a fixed k, the error bound of using $f_M(k)$ for f(k) is often given by the magnitude of (M + 2)th term, the first term neglected.

(ii) For a fixed k, the magnitudes of terms in (2.4) usually first decrease until, say, the (Q + 2)th term. Starting from the (Q + 3)th term, the magnitude of each term becomes larger and larger, and the series diverges.

For an asymptotic expansion with the above two characteristics, it is obvious that, for a fixed k, there exists a "best" asymptotic approximation of f(k), which is $f_Q(k)$. To illustrate this point consider the following example:

$$f(k) = \sqrt{\pi}\, k e^{k^2} \operatorname{erfc}(k) \tag{2.7}$$

where erfc(·) is an error function*. Its asymptotic expansion for large positive k is given by

$$f(k) \sim \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(2k^2)^n}, \quad k \to +\infty \tag{2.3}$$

$$= 1 - \frac{1}{2k^2} + \frac{3}{4k^4} - \frac{15}{8k^6} + \frac{105}{16k^8} - \frac{945}{32k^{10}} + \ldots$$

Let us calculate f(k = 2), which has an exact value (after rounding-off at the fourth decimal place).

$$f(k = 2) = 0.9054$$

The partial sum f_M, the error of f_M, and the (M + 1) term are tabulated for M = 0 to 5.

M	$f_M(2)$	Error	(M + 1) term
0	1.0000	+ 0.0946	
1	0.8750	- 0.0304	- 0.1250
2	0.9219	+ 0.0165	+ 0.0469
3	0.8926	- 0.0128	- 0.0293
4	0.9183	+ 0.0129	+ 0.0256
5	0.8894	- 0.0160	- 0.0288

*The error function is defined later in (3.14). Its asymptotic expansion (2.8) may be obtained by using (3.15) and (3.5).

Note that (i) as a function of M, the partial sums oscillate around the exact value; (ii) the fifth term has the minimum magnitude (0.0256); and (iii) the best approximation of f(2) is $f_3(2)$ and the error is 0.0128, which is bounded by the magnitude of the first term neglected (0.0256).

Third, for a given function f(k), the asymptotic power series expansion in (2.4) is unique. This fact may be used to construct the asymptotic series by the successive application of the following limits:

$$a_0 = \lim_{k\to\infty} k^0[f(k)]$$

$$a_1 = \lim_{k\to\infty} k^1[f(k) - a_0]$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_n = \lim_{k\to\infty} k^n[f(k) - \sum_{m=0}^{n-1} a_m k^{-m}] \tag{2.9}$$

A useful consequence of this property is that, if f(k) = 0, each coefficient a_m in (2.4) must be zero individually. This fact will be frequently used in later discussions.

In addition to the power-series in (2.1), the function f(k) may have a more general asymptotic expansion

$$f(k) \sim \sum_{m=0}^{\infty} a_m \phi_m(k) \quad , \quad k \to \infty \tag{2.10}$$

with respect to a set of functions $\{\phi_m\}$ subject to

$$[\phi_{m+1}(k)/\phi_m(k)] \to 0 \quad , \quad k \to \infty \tag{2.11}$$

Then, for a fixed M, we have

$$\lim_{k\to\infty} [f(k) - f_M(k)]/\phi_M(k) = 0 \tag{2.12}$$

which is a general version of (2.3).

1.3 Fresnel Function

A special function that will be used frequently in this book is the Fresnel function F(x) defined by

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^\infty e^{it^2}\, dt, \text{ for real } x \quad . \tag{3.1}$$

We will list below several useful properties of the Fresnel function.

(i) Symmetry relation

$$F(x) + F(-x) = 1 \tag{3.2}$$

(ii) Differentiation

$$\frac{d}{dx} F(x) = \frac{1}{\sqrt{\pi}} \exp i(x^2 + \frac{3\pi}{4}) \tag{3.3}$$

(iii) Series expansion

$$F(x) = \frac{1}{2} - \frac{e^{-i\pi/4}}{\sqrt{\pi}}\, x \sum_{n=0}^{\infty} \frac{(ix^2)^n}{n!\,(2n+1)} \tag{3.4}$$

(iv) Asymptotic expansion

$$F(x) \sim \theta(-x) + \hat{F}(x) + e^{i(x^2+\pi/4)} \frac{1}{2\pi x} \sum_{n=1}^{\infty} \Gamma(n + \tfrac{1}{2})\, (ix^2)^{-n}, \quad |x| \to \infty \tag{3.5}$$

where

$$\theta(y) = \begin{cases} 1, & \text{if } y > 0 \\ 0, & \text{if } y < 0 \end{cases} \tag{3.6}$$

$$\hat{F}(x) = \frac{1}{2(\pi)^{1/2} x} \exp i(x^2 + \frac{\pi}{4}) \tag{3.7}$$

$$\Gamma(n + \tfrac{1}{2}) = \sqrt{\pi}\, (\tfrac{1}{2})\, (\tfrac{3}{2}) \ldots (\tfrac{2n-1}{2}) \tag{3.8}$$

(v) Special values

$$F(-\infty) = 1 \tag{3.9}$$

$$F(0) = \frac{1}{2} \tag{3.10}$$

$$F(+\infty) = 0 \tag{3.11}$$

(vi) Integral representation

$$F(x) = \frac{e^{ix^2}}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - xe^{i\pi/4}} \, dt, \quad x > 0 \tag{3.12}$$

$$= \frac{1}{\pi} e^{i(x^2-\pi/4)} \; x \int_0^{\infty} \frac{e^{-t^2}}{t^2 - i\,x^2} \, dt, \quad x > 0 \quad . \tag{3.13}$$

The proofs of (3.4) and (3.5) are given in Problems 1-2 and 1-3. The trajectory of F(x) in its complex plane using x as the parameter is presented in Figure 1-2. This curve is known as the Cornu Spiral (after A. Cornu).

The Fresnel function may be related to other special functions. Some of the relations are listed below*.

(i) Error function

$$\operatorname{erfc} z = 1 - \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt \tag{3.14}$$

$$F(x) = \frac{1}{2} \operatorname{erfc}(ze^{-i\pi/4}) \tag{3.15}$$

(ii) Real Fresnel integrals

$$C(z) = \int_0^z \cos(\tfrac{\pi}{2} t^2) \, dt \tag{3.16}$$

$$S(z) = \int_0^z \sin(\tfrac{\pi}{2} t^2) \, dt \tag{3.17}$$

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*All the notations are the same as those used in M. Abramowitz and I. A. Stegun, Handbook of Math. Functions. Chapter 7, Dover Pub., Inc., New York, 1965.

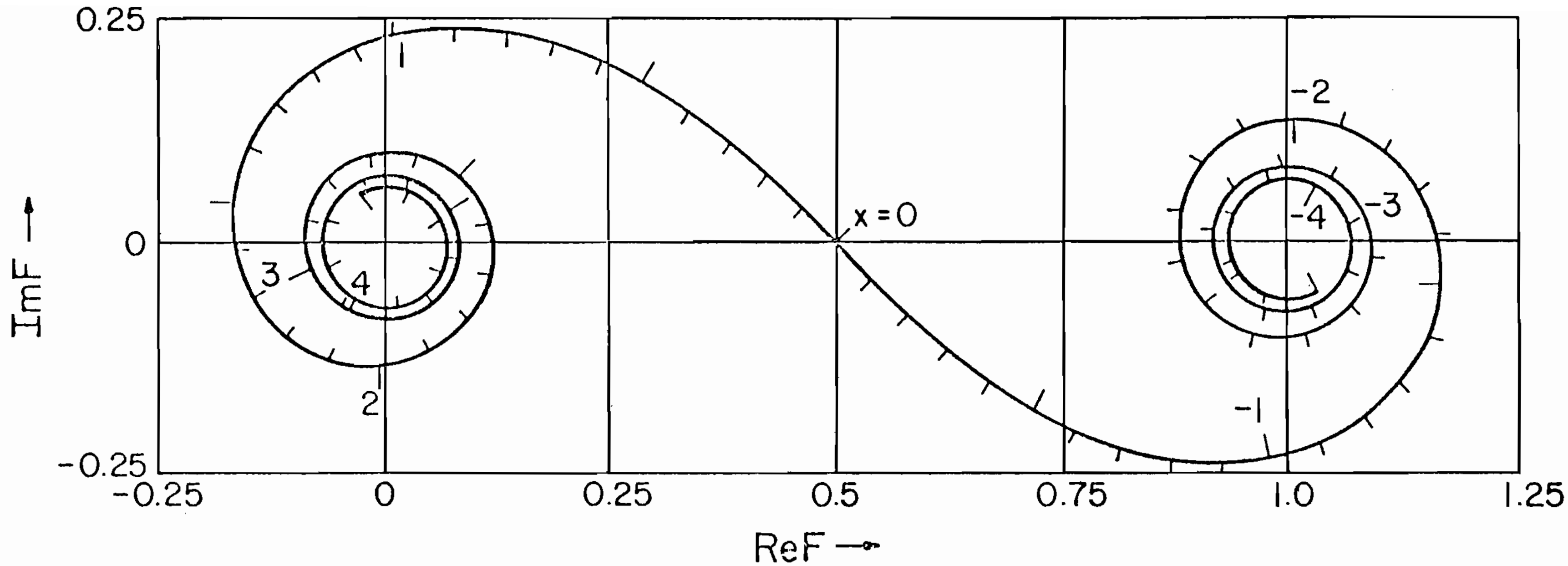

Figure 1-2. Cornu spiral for the Fresnel integral F(x).

$$F(x) = \frac{(1 - i)}{2}\left\{\left[\frac{1}{2} - C(\sqrt{\tfrac{2}{\pi}}\, x)\right] + i\left[\frac{1}{2} - S(\sqrt{\tfrac{2}{\pi}}\, x)\right]\right\} \tag{3.18}$$

(iii) Probability integral

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}\, dt\right) \tag{3.19}$$

$$F(x) = \frac{1}{2}\, e^{ix^2}\, w(xe^{i\pi/4}) \tag{3.20}$$

(iv) Plasma dispersion function*

$$Z(z) = i\sqrt{\pi}\, w(z) \tag{3.21}$$

$$F(x) = \frac{1}{i2\sqrt{\pi}}\, e^{ix^2}\, Z(xe^{i\pi/4}) \tag{3.22}$$

The Fresnel function has been defined in the literature of electromagnetic theory in several different forms. We will list some of them and relate them to the present definition.

(i)

$$F_1(x) = \frac{e^{-ix^2}}{\sqrt{\pi}\, e^{i\pi/4}} \int_{-\infty}^{x} e^{it^2}\, dt \tag{3.23}$$

$$F_1(x) = e^{-ix^2}\, F(-x) \tag{3.24}$$

(ii)

$$F_2(x) = \frac{2}{\sqrt{\pi}}\, e^{-i2x^2} \int_{(1-i)x}^{\infty} e^{-t^2}\, dt \tag{3.25}$$

$$F_2(x) = 2e^{-i2x^2}\, F(\sqrt{2}\, x) \tag{3.26}$$

(iii)

$$G(x) = e^{-ix^2} \int_x^{\infty} e^{it^2}\, dt \tag{3.27}$$

$$G(x) = \sqrt{\pi}\, e^{i\pi/4}\, e^{-ix^2}\, F(x) \tag{3.28}$$

---

*Numericalvalue of $Z(x + iy)$ and its derivative $Z'(x + iy)$ are tabulated for $x = 0(0.1)\ 10$, $y = -10(0.1)\ 10$, in B. D. Fried and S. D. Conte, The Plasma Dispersion Function. Academic Press, New York, 1961.

(iv)

$$Q(y) = \int_y^\infty e^{-t^2}\, dt \tag{3.29}$$

$$Q(xe^{-i\pi/4}) = \sqrt{\pi}\, F(x) \tag{3.30}$$

In the above, the Fresnel function F(x) has been defined for a real argument x. When the argument is a complex number z = x + iy, we define F(z) as the analytical continuation of F(x) in (3.1), namely,

$$F(z) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_z^\infty e^{it^2}\, dt, \text{ for complex } z \tag{3.31}$$

where the path of integration in the complex t-plane is subject to the restriction Arg t $\to \alpha$ with $0 < \alpha < \pi/2$ as $|t| \to \infty$ along the path. All the equations given in this section remain valid when x is replaced by z, except for (3.5) and (3.12). The latter two equations should be modified to read

$$F(z) \sim \gamma(z) + \hat{F}(z)$$

$$+ e^{i(z^2+\pi/4)} \frac{1}{2\pi z} \sum_{n=1}^{\infty} \Gamma(n + \tfrac{1}{2}) \, (iz^2)^{-n}, \quad |z| \to \infty \tag{3.32}$$

where

$$\gamma(z) = \begin{cases} 0 \;, & \text{if } (-\pi/4) < \text{Arg } z < (3\pi/4) \\ 1/2, & \text{if Arg } z = (-\pi/4) \text{ or } (3\pi/4) \\ 1 \;, & \text{if } (3\pi/4) < \text{Arg } z < (7\pi/4) \end{cases} \tag{3.33}$$

and $\hat{F}(z)$ is given by (3.7) after replacing x by z, and

$$F(z) = \frac{e^{iz^2}}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t - ze^{i\pi/4}}\, dt, \quad \text{Im}(ze^{i\pi/4}) > 0 \tag{3.34a}$$

$$= \frac{e^{iz^2}}{2\pi i} \int_C \frac{e^{-it^2}}{t - z}\, dt, \quad \text{Im}(ze^{i\pi/4}) > 0 \tag{3.34b}$$

where contour C in the complex t-plane is the straight line from $t = -\infty\, e^{-i\pi/4}$ to $t = \infty\, e^{-i\pi/4}$.

1.4 Sommerfeld Half-Plane Problem I. Two-Dimensional Case

To facilitate our later discussion of ray techniques, it is desirable for us to have some understanding of the basic edge diffraction phenomenon. Hence, we will in this and the next sections examine the exact solution of the famous Sommerfeld half-plane problem.

The problem of a half plane is sketched in Figure 1-3. A perfectly conducting plane is located at $(x > 0,\ y = 0)$, and illuminated by an incident plane wave. For the present two-dimensional problem, it is convenient to resolve the fields into two modes: E-wave and H-wave, and solve the problem associated with each mode separately. From the Maxwell's equations, it is readily found that the non-zero field components of the E-wave are governed by

$$\text{E-wave:} \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) E_z(x,y) = 0 \quad , \tag{4.1a}$$

$$H_x = \frac{1}{i\omega\mu}\frac{\partial}{\partial y}E_z \ , \quad H_y = \frac{i}{\omega\mu}\frac{\partial}{\partial x}E_z \ , \tag{4.1b}$$

$$H_\rho = \frac{1}{i\omega\mu}\frac{1}{\rho}\frac{\partial}{\partial\phi}E_z \ , \quad H_\phi = \frac{i}{\omega\mu}\frac{\partial}{\partial\rho}E_z \tag{4.1c}$$

where (ρ,ϕ) are the cylindrical coordinates. The corresponding equations for the H-wave are

$$\text{H-wave:} \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) H_z(x,y) = 0 \quad , \tag{4.2a}$$

$$E_x = \frac{i}{\omega\varepsilon}\frac{\partial}{\partial y}H_z \ , \quad E_y = \frac{1}{i\omega\varepsilon}\frac{\partial}{\partial x}H_z \ , \tag{4.2b}$$

$$E_\rho = \frac{i}{\omega\varepsilon}\frac{1}{\rho}\frac{\partial}{\partial\phi}H_z \ , \quad E_\phi = \frac{1}{i\omega\varepsilon}\frac{\partial}{\partial\rho}H_z \quad . \tag{4.2c}$$

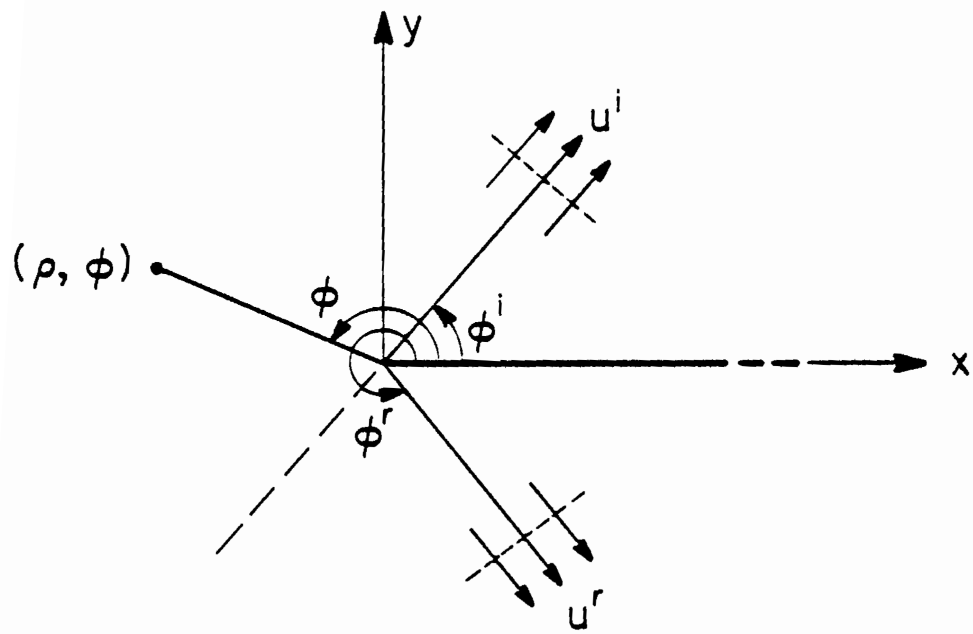

Figure 1-3. A perfectly conducting half plane illuminated by an incident plane wave u^i.

In the following we study the half-plane problem associated with the E-wave and that with the H-wave simultaneously, To do so conveniently, let us introduce a scalar function u such that

$$u(x,y) = \begin{cases} E_z \ , & \text{for E-wave} \\ H_z \ , & \text{for H-wave} \end{cases} \tag{4.3}$$

The incident plane wave is specified by, for all (ρ,ϕ),

$$u^i(\rho,\phi) = e^{ik\rho\cos(\phi^i-\phi)} \quad . \tag{4.4}$$

Note carefully that the plane wave travels into (not comes from) the direction ϕ^i. Thus, ϕ^i is the angle measured counterclockwise from the x-axis to the "head" (not the "tail") of the incident direction. Without loss of generality, we restrict the ranges of ϕ^i, such that

$$0 \leq \phi^i < \pi \quad . \tag{4.5}$$

(This restriction will be relaxed at the end of this section.) The problem at hand is to determine the scattered field u, or the total field u^t defined by

$$u^t(\rho,\phi) = u^i + u \tag{4.6}$$

at an observation point (ρ,ϕ), where $0 < \rho < \infty$ and $0 < \phi < 2\pi$.

The necessary equations and conditions for solving this problem are the wave equation for u^t [cf. (4.1a) and (4.2a)]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) u^t = 0 \ ; \tag{4.7}$$

the boundary condition for u^t

$$\text{E-wave:} \quad u^t = 0, \qquad x > 0 \text{ and } y = 0, \tag{4.8}$$

$$\text{H-wave:} \quad \frac{\partial u^t}{\partial y} = 0, \qquad x > 0 \text{ and } y = 0; \tag{4.9}$$

the edge condition for u^t

$$\frac{\partial u^t}{\partial x}, \ \frac{\partial u^t}{\partial y} = O(\rho^{-1/2}), \quad \rho \to 0 \tag{4.10}$$

and finally, the radiation condition for the scattered field u as $\rho \to \infty$.

Based on the above equations and conditions, the half-plane problem was first solved exactly by A. Sommerfeld in 1896, using a double-valued function (A. Sommerfeld, "Mathematische theorie der diffraktion," Math. Ann., vol. 47, pp. 317-374, 1896). Later, the same solution was obtained by several different methods*. The steps of solution are not the main concern here. We simply give below the exact solution for the total field u^t at an observation point (ρ,ϕ):

$$u^t(\rho,\phi) = F(\xi^i)\, u^i(\rho,\phi) + F(\xi^r)\, u^r(\rho,\phi) \quad . \tag{4.11}$$

The notations used in (4.11) are explained below. $u^r(\rho,\phi)$ is the reflected field from the half plane due to the incidence of u^i in (4.4), and is given by

$$u^r(\rho,\phi) = Re^{ik\rho\cos(\phi^r-\phi)} \quad . \tag{4.12}$$

Here R is the reflection coefficient of u^i from the half plane

$$R = \begin{cases} -1, & \text{for E-wave} \\ +1, & \text{for H-wave} \end{cases} \quad . \tag{4.13}$$

The reflected angle ϕ^r specifies the direction where the reflected field travels into, and is the mirror image of the incident angle ϕ^i with respect to the half plane, i.e.,

* See, e.g., M. Born and E. Wolf, Principles of Optics. 2nd ed., Pergamon Press, New York, 1964, pp. 560-578; B. Noble, Methods Based on the Wiener-Hopf Technique. Pergamon Press, New York, 1958, Chapter 2; R. Mittra and S.W. Lee, Analytical Techniques in the Theory of Guided Waves. MacMillan Co., New York, 1971, pp. 137-147. A method involving elementary steps is given in J. Boersma, "A simple solution of Sommerfeld's half-plane diffraction problem," J. Appl. Science and Engineering A, 2, pp. 187-193, 1977.

$$\phi^r = 2\pi - \phi^i \quad . \tag{4.14}$$

Because of the restriction in (4.5), we note that $\pi < \phi^r \leq 2\pi$. The Fresnel function in (4.11) is defined by

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^{\infty} e^{it^2}\, dt \tag{4.15}$$

which has been studied in detail in Section 1.3. The two functions ξ^i, and ξ^r, called <u>detour parameters</u> of the incident, and reflected fields, respectively, are given by*

$$\xi^{i,r}(\rho,\phi) = \sqrt{2k\rho}\, \sin \frac{1}{2}(\phi^{i,r} - \phi) \quad . \tag{4.16}$$

Several remarks about the exact solution for the total field in (4.11) are in order:

(i) The total field consists of two symmetrical parts: one due to the incident field u^i and the other the reflected field u^r. To emphasize this symmetry, we rewrite (4.11) as

$$u^t(\rho,\phi) = u^{ti} + u^{tr} \tag{4.17a}$$

where

$$u^{ti}(\rho,\phi) = F(\xi^i)\, u^i(\rho,\phi) \tag{4.17b}$$

and u^{tr} is the same as u^{ti} after the superscript "i" is replaced by "r." Alternatively, we write (4.11) as

$$u^t(\rho,\phi) = F(\xi^i)\, u^i(\rho,\phi) + \{i \rightarrow r\} \tag{4.18}$$

where the symbol $\{i \rightarrow r\}$ means to repeat all the terms after the equal sign and to change the superscript "i" into "r" in those terms.

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*Physical implication of detour parameters will be given in Chapter 5. Note also that $A^{i,r} = \mp B^{i,r}$ means $A^i = -B^i$ and $A^r = +B^r$.

(ii) Because of the sine half-angle function, $\xi^{i,r}$ are double-valued functions (with one value being the negative of the other). Thus, the restriction in (4.5) must be observed in order to assure the correct signs for $\xi^{i,r}$.

In the remainder of this section, we will study the exact solution given in (4.18) on the following aspects: (i) the field components transverse to the edge, (ii) the field behavior near the edge, (iii) the field behavior far away from the edge, and (iv) the relaxation of the restriction on ϕ^i in (4.5).

Complete Field Solution. With u^t given in (4.18), all the field components can be calculated from (4.1) and (4.2). We will present the final solution in a simple and suggestive form. This would require some proper arrangements. Let us concentrate on the case of the E-wave. The complete incident field is given by

$$E_z^i(\rho,\phi) = u^i(\rho,\phi) = e^{ik\rho\cos(\phi^i-\phi)} \tag{4.19a}$$

$$H_\rho^i(\rho,\phi) = \sqrt{\frac{\varepsilon}{\mu}}\sin(\phi^i - \phi)\, u^i(\rho,\phi) \tag{4.19b}$$

$$H_\phi^i(\rho,\phi) = -\sqrt{\frac{\varepsilon}{\mu}}\cos(\phi^i - \phi)\, u^i(\rho,\phi) \quad . \tag{4.19c}$$

In particular, with respect to the incident direction $\phi = \phi^i$, the incident field at the edge is

$$E_z^i(0,\phi^i) = 1 \tag{4.20a}$$

$$H_\rho^i(0,\phi^i) = 0 \tag{4.20b}$$

$$H_\phi^i(0,\phi^i) = -\sqrt{\frac{\varepsilon}{\mu}} \quad . \tag{4.20c}$$

Now, note the manipulations [Problems 1-4]

$$\frac{1}{ik\rho}\frac{\partial}{\partial\phi}\left[F(\xi^i)\ u^i(\rho,\phi)\right] = \sin(\phi^i - \phi)\ \left[F(\xi^i) - \hat{F}(\xi^i)\right]\ u^i(\rho,\phi) \quad , \tag{4.21a}$$

$$\frac{1}{ik}\frac{\partial}{\partial\rho}\left[F(\xi^i)\ u^i(\rho,\phi)\right] = \cos(\phi^i - \phi)\ \left[F(\xi^i) - \hat{F}(\xi^i)\right]\ u^i(\rho,\phi)$$

$$+ g(k\rho)\ \chi^i\ u^i(0,\phi^i) \quad . \tag{4.21b}$$

Here $\hat{F}(x)$ is the term of order x^{-1} in the asymptotic expansion of $F(x)$, and is given in (3.7) or

$$\hat{F}(x) = \frac{1}{2(\pi)^{1/2}\ x} \exp i(x^2 + \frac{\pi}{4}) \quad . \tag{4.22}$$

The factor $g(k\rho)$ is a cylindrical wave factor*

$$g(k\rho) = \frac{e^{i(k\rho+\pi/4)}}{2\sqrt{2\pi k\rho}} \quad , \tag{4.23}$$

and χ^i (and χ^r) are defined by

$$\chi^{i,r} = \frac{\sqrt{2k\rho}}{\xi^{i,r}} = \csc \frac{1}{2}(\phi^{i,r} - \phi) \quad . \tag{4.24}$$

---

*The outgoing-wave solution of the two-dimensional wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) G = -\delta(x)\ \delta(y)$$

is given by

$$G(x,y) = \frac{i}{4} H_0^{(1)}(k\rho) = \frac{e^{i(kr+\pi/4)}}{2\sqrt{2\pi k\rho}} \left[1 + O\left(\frac{1}{k\rho}\right)\right]$$

$$= g(k\rho)\left[1 + O(\frac{1}{k\rho})\right]$$

For this reason, $g(\cdot)$ is sometimes known as a "unit" cylindrical wave.

Substituting (4.18) into (4.1) and making use of (4.21), the complete exact solution of the total field for the E-wave is found, namely,

$$E_z^t(\rho,\phi) = F(\xi^i)\, E_z^i(\rho,\phi) + \{i \to r\} \tag{4.25a}$$

$$H_\rho^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\, H_\rho^i(\rho,\phi) + \{i \to r\} \tag{4.25b}$$

$$H_\phi^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\, H_\phi^i(\rho,\phi) + g(k\rho)\, \chi^i\, H_\phi^i(0,\phi^i)$$

$$+ \{i \to r\} \quad . \tag{4.25c}$$

Because of (4.20b) and

$$\hat{F}(\xi^i)\, E_z^i(\rho,\phi) = g(k\rho)\, \chi^i\, E_z^i(\rho = 0,\phi^i)$$

(4.25) may be rewritten as*

$$U^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\, U^i(\rho,\phi) + g(k\rho)\, \chi^i\, U^i(0,\phi^i)$$

$$+ \{i \to r\} \tag{4.26}$$

where U stands for E_z, H_ρ, or H_ϕ. In exactly the same manner, it may be shown that the exact solution of the total field for the H-wave is also given by (4.26) with U representing H_z, E_ρ, or E_ϕ. As a third way to express the exact total field solution for the present half-plane problem, we may rewrite (4.26) in a vector form, namely,

$$\vec{E}^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\, \vec{E}^i(\rho,\phi) + g(k\rho)\, \chi^i[\hat{z}\, E_z^i(0,\phi^i) + \hat{\phi}\, E_\phi^i(0,\phi^i)]$$

$$+ \{i \to r\} \tag{4.27a}$$

$$\vec{H}^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\, \vec{H}^i(\rho,\phi) + g(k\rho)\, \chi^i[\hat{z}\, H_z^i(0,\phi^i) + \hat{\phi}\, H_\phi^i(0,\phi^i)]$$

$$+ \{i \to r\} \tag{4.27b}$$

---

*Note that as $\phi \to \phi^i$, H_ρ^i in (4.19b) becomes zero as $(\phi^i - \phi)$ while χ^i in (4.24) becomes infinite as $(\phi^i - \phi)^{-1}$. Here, however, we use the definition in (4.20b), and the second term in (4.26) is identically zero when $U^i = H_\rho^i$. More discussion on this is given in Chapter 5.

which applies for both the E-wave and the H-wave. Since

$$E_z^i(0,\phi^i) = -E_z^r(0,\phi^r),\ E_\phi^i(0,\phi^i) = +E_\phi^r(0,\phi^r) \quad ,$$

$$H_z^i(0,\phi^i) = +H_z^r(0,\phi^r),\ H_\phi^i(0,\phi^i) = -H_\phi^r(0,\phi^r) \quad ,$$

(4.27) may be rewritten as

$$\vec{E}^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\ \vec{E}^i(\vec{r}) + [F(\xi^r) - \hat{F}(\xi^r)]\ \vec{E}^r(\vec{r})$$

$$+ g(k\rho)\ [\hat{z}(\chi^i - \chi^r)\ E_z^i(0,\phi^i) + \hat{\phi}(\chi^i + \chi^r)\ E_\phi^i(0,\phi^i)] \quad (4.27c)$$

$$\vec{H}^t(\rho,\phi) = [F(\xi^i) - \hat{F}(\xi^i)]\ \vec{H}^i(\vec{r}) + [F(\xi^r) - \hat{F}(\xi^r)]\ \vec{H}^r(\vec{r})$$

$$+ g(k\rho)\ [\hat{z}(\chi^i + \chi^r)\ H_z^i(0,\phi^i) + \hat{\phi}(\chi^i - \chi^r)\ H_\phi^i(0,\phi^i)]. \quad (4.27d)$$

The induced surface current on the half plane may be calculated from (4.25b) for the E-wave and from (4.18) for the H-wave. The result is, for $x > 0$,

E-wave:

$$J_z(x) = \left\{1 - 2\left[F(\sqrt{2kx}\ \sin\frac{\phi^i}{2}) - \hat{F}(\sqrt{2kx}\ \sin\frac{\phi^i}{2})\right]\right\} 2\ H_x^i(\rho = x, \phi = 2\pi-) \quad (4.28)$$

H-wave:

$$J_x(x) = \left\{1 - 2\ F(\sqrt{2kx}\ \sin\frac{\phi^i}{2})\right\} (-2)\ H_z^i(\rho = x, \phi = 2\pi-) \quad . \quad (4.29)$$

<u>Near Field</u>. When the observation point is close to the edge ($k\rho \to 0$), the series expansion of the Fresnel function in (3.4) may be used in (4.18), then the total field becomes

$$u^t(\rho,\phi) = \left[\frac{1}{2} - e^{-i\pi/4}\sqrt{\frac{2}{\pi}}\ (\chi^i)^{-1}\ (k\rho)^{1/2}\right] u^i(\rho = 0)$$

$$+ \{i \to r\} + O(k\rho),\ k\rho \to 0 \quad . \quad (4.30)$$

Since $u^r(\rho = 0) = R\, u^i(\rho = 0)$, (4.30) may be written more explicitly

E-wave: As $k\rho \to 0$,

$$E_z^t(\rho,\phi) = -e^{-i\pi/4}\sqrt{\frac{2}{\pi}}\,[(\chi^i)^{-1} - (\chi^r)^{-1}]\,(k\rho)^{1/2}\,E_z^i(\rho = 0) + O(k\rho) \qquad (4.31)$$

H-wave: As $k\rho \to 0$,

$$H_z^t(\rho,\phi) = H_z^i(\rho = 0) - e^{-i\pi/4}\sqrt{\frac{2}{\pi}}\,[(\chi^i)^{-1} + (\chi^r)^{-1}]\,(k\rho)^{1/2}\,H_z^i(\rho = 0)$$
$$+ O(k\rho) \quad . \qquad (4.32)$$

When (4.31) and (4.32) are used in (4.1) and (4.2), the transverse field components as $k\rho \to 0$ can be determined. It is found that all the latter components have a $k\rho$-dependence as $(k\rho)^{-1/2}$, which satisfies the edge condition for singular field behavior specified in (4.10).

Detour Parameters and Shadow Indicators. Away from the edge ($k\rho >> 1$), the field behavior is governed by the detour parameters $\xi^{i,r}$. Let us concentrate on ξ^i. At an observation point $\vec{r} = (\rho,\phi)$, ξ^i is defined in (4.16) or

$$\xi^i(\vec{r}) = \sqrt{2k\rho}\,\sin\frac{1}{2}(\phi^i - \phi) \quad . \qquad (4.33)$$

Note that the sign of ξ^i denoted by

$$\varepsilon^i(\vec{r}) = \mathrm{sgn}\,\xi^i(\vec{r}) \qquad (4.34)$$

is +1 if $\vec{r}$ is in the shaded region in Figure 1-4a ($\phi < \phi^i$), and is -1 if $\vec{r}$ is in the unshaded region ($\phi > \phi^i$). These two regions correspond to exactly the shadow and lit regions of the incident field according to the (classical) geometrical optics theory. For this reason, ε^i is called the shadow indicator of the incident field, thus,

$$\varepsilon^i(\vec{r}) = \begin{cases} +1, & \text{if } \vec{r} \text{ is in the shadow region of the incident field} \\ -1, & \text{if } r \text{ is in the lit region of the incident field} \end{cases} \qquad (4.35)$$

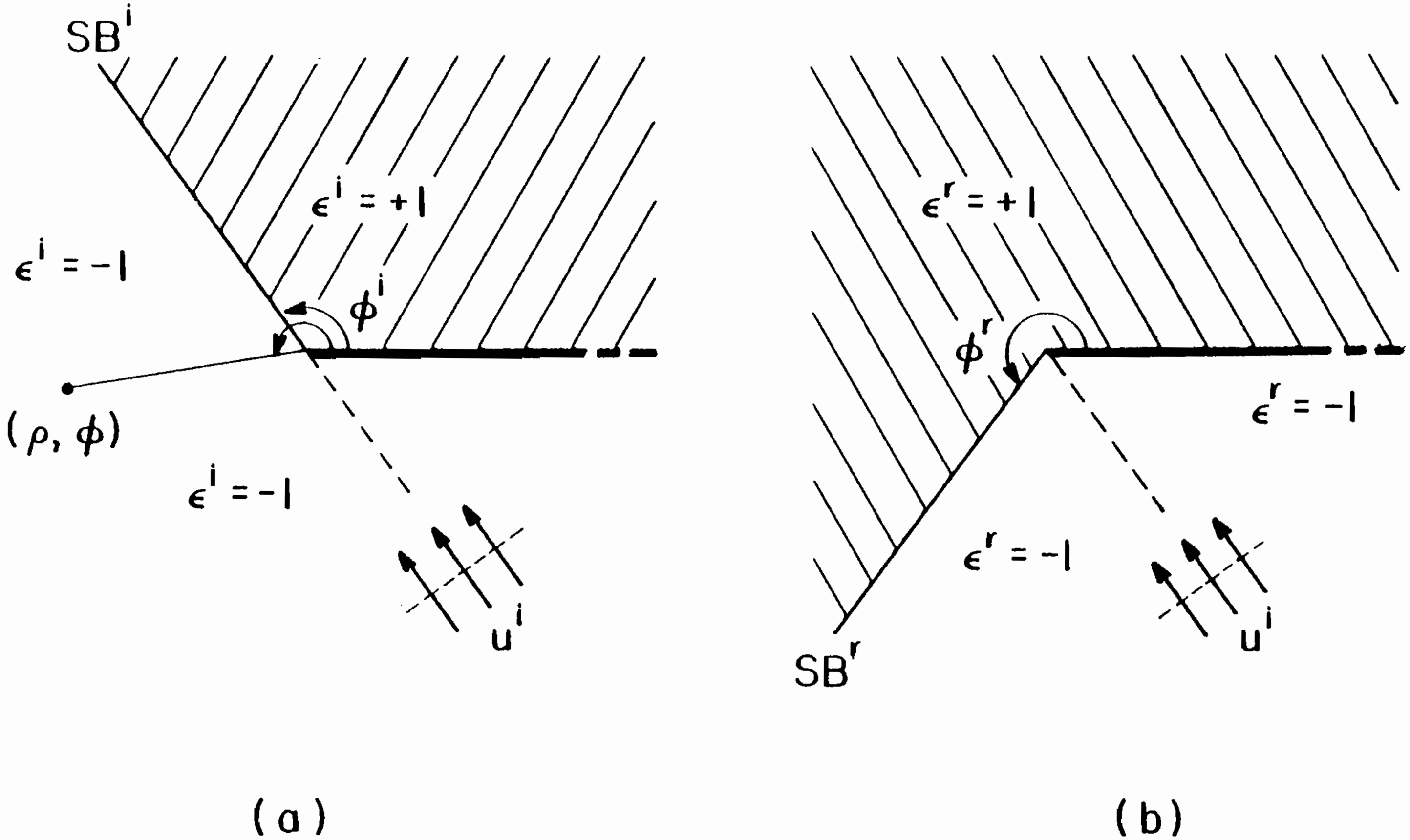

Figure 1-4. Shadow and lit regions of the incident field, and those of the reflected field.

The shadow and lit regions are separated by an incident shadow boundary SB^i at $\phi = \phi^i$, where $\xi^i = 0$. In exactly the same manner, we define the shadow indicator of the reflected field ε^r and the reflected shadow boundary SB^r (Figure 1-4b).

An equimagnitude contour of ξ^i is defined by $|\xi^i| = C$, or

$$k\rho = \frac{1}{2} C^2[\csc \frac{1}{2}(\phi^i - \phi)]^2 \tag{4.36}$$

which describes a parabola with its axis at $SB^i(\phi = \phi^i)$ and focus at the edge ($\rho = 0$) (Figure 1-5a). As usual, a similar equimagnitude parabola exists for ξ^r (Figure 1-5b).

Far Field. Return to the exact solution in (4.18). For a given accuracy requirement, a sufficiently large constant C can be always found such that, provided $|\xi^{i,r}| > C$, $F(\xi^{i,r})$ can be approximated by the first two terms of its asymptotic expansion in (3.5), i.e.,

$$F(\xi^{i,r}) \simeq \theta(-\varepsilon^{i,r}) + \hat{F}(\xi^{i,r}), \ |\xi^{i,r}| > C \tag{4.37}$$

where we have made use of the fact

$$\theta(-\xi^{i,r}) = \theta(-\varepsilon^{i,r}) \quad . \tag{4.38}$$

Therefore, when the observation point is outside of the two dotted transition regions indicated in Figure 1-5, the total field in (4.18) may be asymptotically approximated by

$$u^t(\rho,\phi) = [\theta(-\varepsilon^i) + \hat{F}(\xi^i)]\ u^i(\rho,\phi) + \{i \to r\} + O(k^{-3/2}) \quad . \tag{4.39}$$

We will separate u^t in (4.39) into two components:

$$u^t(\rho,\phi) = u^g(\rho,\phi) + u^d(\rho,\phi) \quad . \tag{4.40}$$

Here u^g, called geometrical optics field, is given by

$$u^g(\rho,\phi) = \theta(-\varepsilon^i)\ u^i(\rho,\phi) + \{i \to r\} \tag{4.41}$$

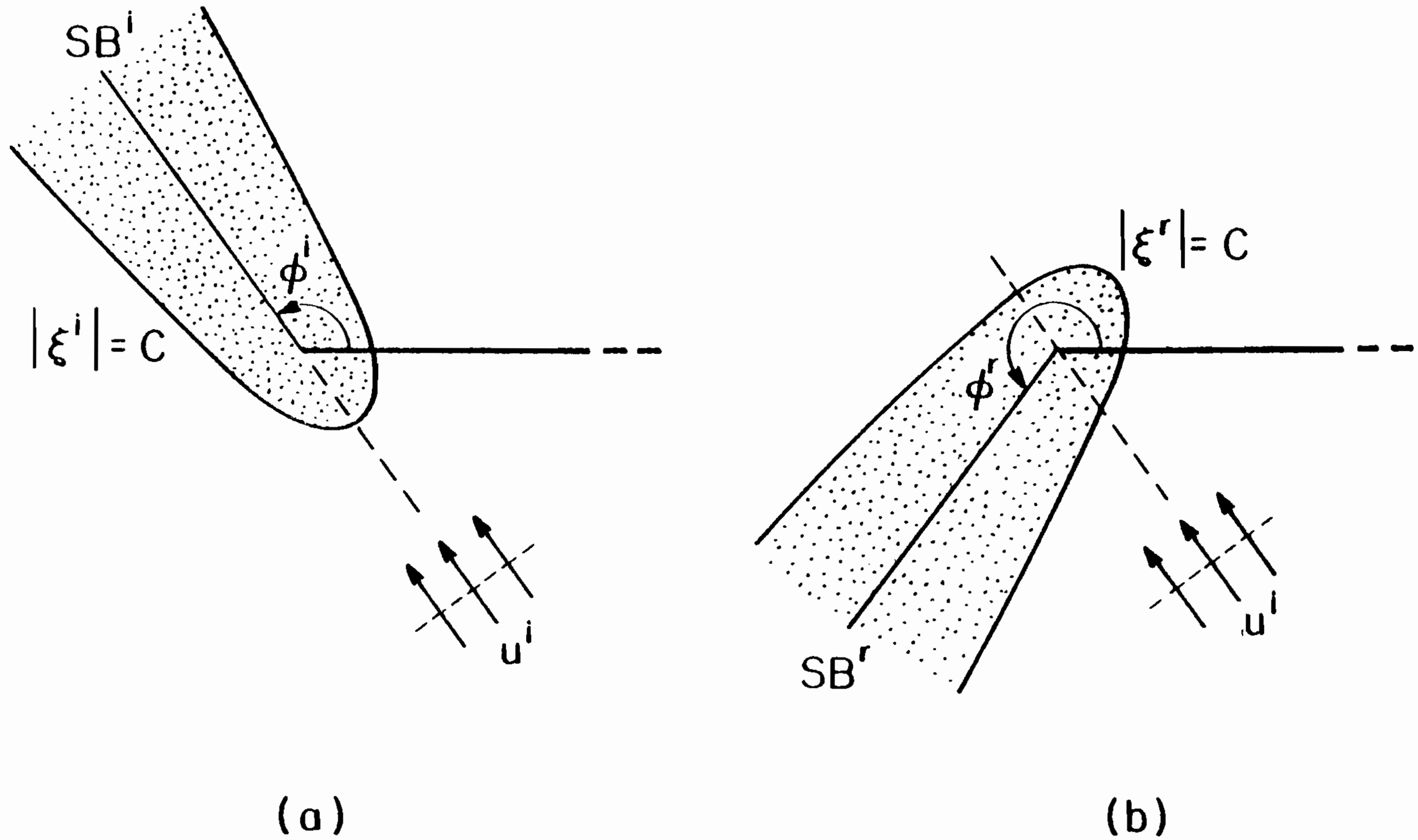

Figure 1-5. Two parabolic transition regions (dotted areas) around shadow boundaries $SB^{i,r}$. Outside these two regions, the exact total field (4.18) can be asymptotically approximated by (4.40).

which is the field predicted by the geometrical optics theory for the half-plane problem. The second component u^d in (4.40), called (Keller's) diffracted field, is given by

$$u^d(\rho,\phi) = g(k\rho)\ \chi^i\ u^i(\rho = 0) + \{i \rightarrow r\} + O(k^{-3/2}) \quad . \tag{4.42}$$

In deriving (4.42) from (4.39), we have made use of (4.22), (4.23), and (4.24). Since $u^r(\rho = 0) = R\ u^i(\rho = 0)$, (4.42) can be written more explicitly as

E-wave:

$$E_z^d(\rho,\phi) = g(k\rho)\ [\chi^i - \chi^r]\ u^i(\rho = 0) + O(k^{-3/2}) \tag{4.43}$$

H-wave:

$$H_z^d(\rho,\phi) = g(k\rho)\ [\chi^i + \chi^r]\ u^i(\rho = 0) + O(k^{-3/2}) \quad . \tag{4.44}$$

Several remarks are in order: (i) u^g is of order k^0 and u^d of order $k^{-1/2}$. Hence, u^g is the dominant term. u^d may be regarded as the correction term for the geometrical optics theory, accounting for the diffraction phenomenon. (ii) u^d is a cylindrical wave emanating from the edge. (iii) At the shadow boundaries $SB^{i,r}$, u^g becomes discontinuous and u^d infinite. This, of course, should not disturb us as the field representation in (4.40) is valid only outside of the two parabolic transition regions shown in Figure 1-5.

Field on Shadow Boundaries. When the observation point $\vec{r} = (\rho,\phi)$ is inside the two transition regions, we have to use the exact representation in (4.18). For a fixed C, the transition regions become narrower as k increases. They collapse into $SB^{i,r}$ as $k \rightarrow \infty$. Therefore, for large k, the total field varies very rapidly in the transition regions and no further simplification from (4.18) is possible. When $\vec{r}$ is exactly on

SB^i and away from the edge, we have $\xi^i = 0$ and $\xi^r >> 1$; then the total field in (4.18) becomes

$$u^t(\rho,\phi^i) = \frac{1}{2} u^i(\rho,\phi^i) + g(k\rho) \; |\csc \phi^i| \; u^r(\rho = 0) + O(k^{-3/2})$$

$$= \frac{1}{2} e^{ik\rho} + R \; g(k\rho) \; |\csc \phi^i| + O(k^{-3/2}) \quad . \tag{4.45}$$

When $\vec{r}$ is exactly on SB^r and away from the edge, we have $\xi^i >> 1$ and $\xi^r = 0$; then the total field becomes

$$u^t(\rho,\phi^r) = u^i(\rho,\phi^r) + \frac{1}{2} u^r(\rho,\phi) - g(k\rho) \; |\csc \phi^r| \; u^i(\rho = 0) + O(k^{-3/2})$$

$$= u^i(\rho,\phi^r) + \frac{1}{2} Re^{ik\rho} - g(k\rho) \; |\csc \phi^i| + O(k^{-3/2}) \quad . \tag{4.46}$$

The values of other field components on shadow boundaries are given in Problems 1-5 and 1-6.

Removal of the Restriction on ϕ^i. As a final remark, we have so far restricted the incident direction by $0 \leq \phi^i < \pi$ in (4.5). For the other case $\pi < \phi^i \leq 2\pi$, it may be shown that all the equations given in this section remain valid only if the definitions in (4.16) and (4.24) are replaced by

$$\xi^{i,r}(\vec{r}) = -\sqrt{2k\rho} \; \sin \frac{1}{2}(\phi^{i,r} - \phi) \tag{4.47}$$

$$\chi^{i,r} = -\csc \frac{1}{2}(\phi^{i,r} - \phi) \quad . \tag{4.48}$$

This change in sign may be explained by the following fact. When $0 \leq \phi^i < \pi$, the increasing direction of ϕ or ϕ^i is from the shadow region toward the lit region of the incident field; whereas in the case of $\pi < \phi^i \leq 2\pi$, the increasing direction of ϕ or ϕ^i is from the lit region toward the shadow region. Combining the definitions in (4.16), (4.24), (4.47), and (4.48), we may write them in two single expressions:

$$\xi^{i,r}(\vec{r}) = \varepsilon^{i,r} \; \left|\sqrt{2k\rho} \; \sin \frac{1}{2}(\phi^{i,r} - \phi)\right| \tag{4.49}$$

$$\chi^{i,r} = \varepsilon^{i,r} \; \left|\csc \frac{1}{2}(\phi^{i,r} - \phi)\right| \quad . \tag{4.50}$$

Using the new definitions in (4.49) and (4.50), all the equations given in this section are valid for any value of ϕ^i in the range of $(0,2\pi)$.

Numerical results. For a normally incident plane wave given by (4.4) with $\phi^i = \pi/2$, the magnitude of the total field u^t has been calculated from the exact solution in (4.11). The results for the observation point in a 2λ by 2λ square region around the edge are presented in Figure 1-6 for an E-wave, and in Figure 1-7 for an H-wave. As an aid to visualizing the three-dimensional plots, some remarks about the approximate values of $|u^t|$ are in order. (i) In Region A, the total field is a standing wave such that $|u^t| \sim 2|\cos ky|$ for the H-wave and $|u^t| \sim 2|\sin ky|$ for the E-wave. (ii) In region B, which is the geometrical shadow, $|u^t|$ is very small. (iii) As the observation point moves away from the edge in Region C, u^t approaches u^i in the limit. The rate of approach generally is faster for the H-wave than that for the E-wave. (iv) At the edge point $(x = y = 0)$, $u^t = 1$ for the H-wave, and $u^t = 0$ for the E-wave. (v) At $x = 0$, $|u^t| \sim 1/2$ as $ky \to +\infty$, and $|u^t|$ oscillates between (1/2) and (3/2), as $ky \to -\infty$. (vi) At the lit side of the half plane $(x > 0,\ y = 0-)$, $|u^t|$ roughly equals 2 for the H-wave, and is identically zero for the E-wave. (vii) At the shadow side of the half plane $(x > 0,\ y = 0+)$, $|u^t|$ decreased gradually from 1 to 0 for the H-wave, and is identically zero for the E-wave.

In summary, for an incident plane wave, the exact solution of the half-plane problem is given in (4.18) and (4.27) with $\xi^{i,r}$ and $\chi^{i,r}$ defined in (4.49) and (4.50). When the observation point is outside of the two parabolic transition regions in Figure 1-5, the asymptotic expansion of the exact solution in (4.40) consists of two terms: the geometrical optics field in (4.41) and the (Keller's) diffracted field in (4.42). Exactly on

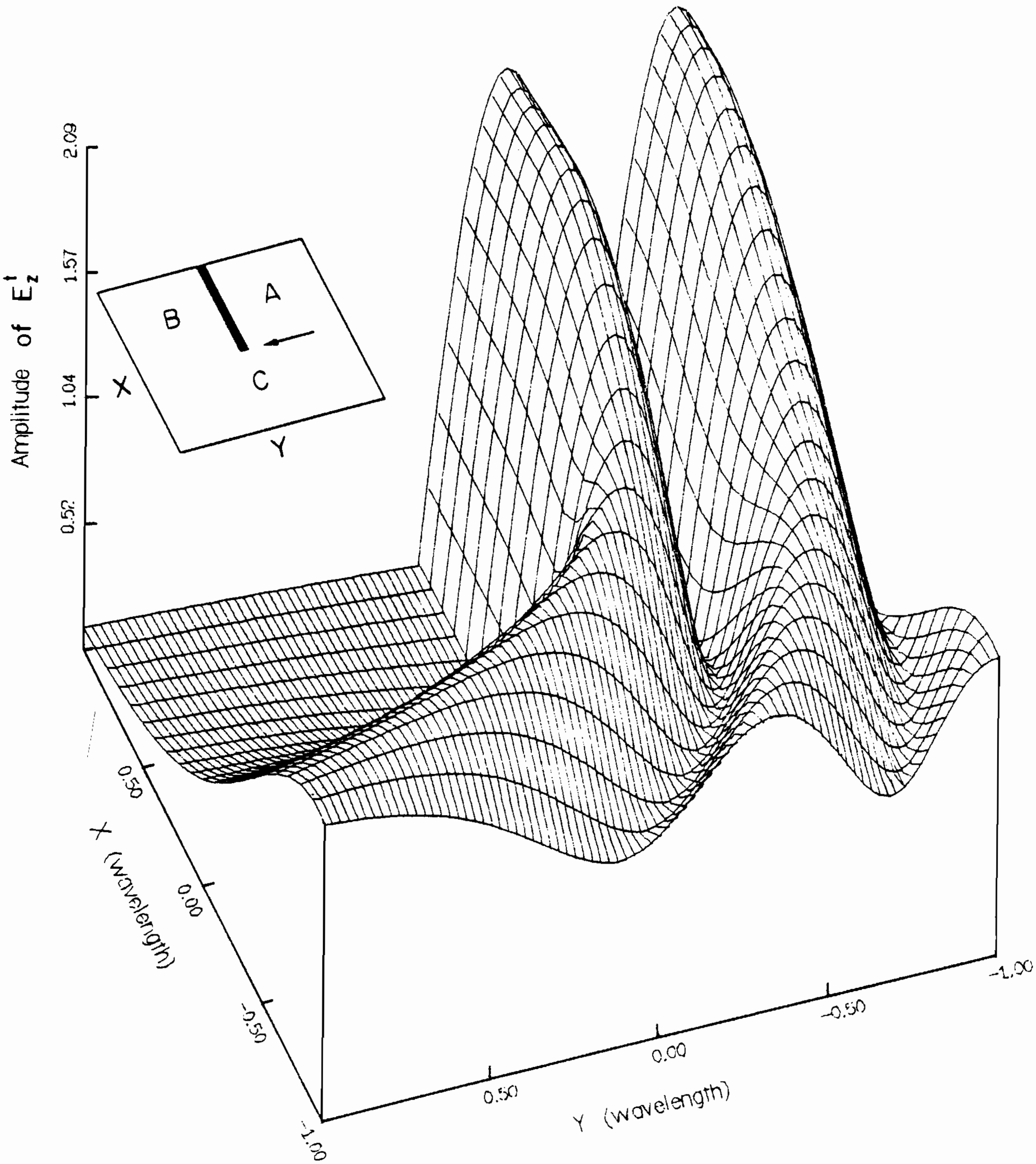

Figure 1-6. Magnitude of the total field E_z^t calculated from the exact solution in (4.11) for the half-plane diffraction. The incident field is given in (4.4) with $\phi^i = \pi/2$ (normal incidence).

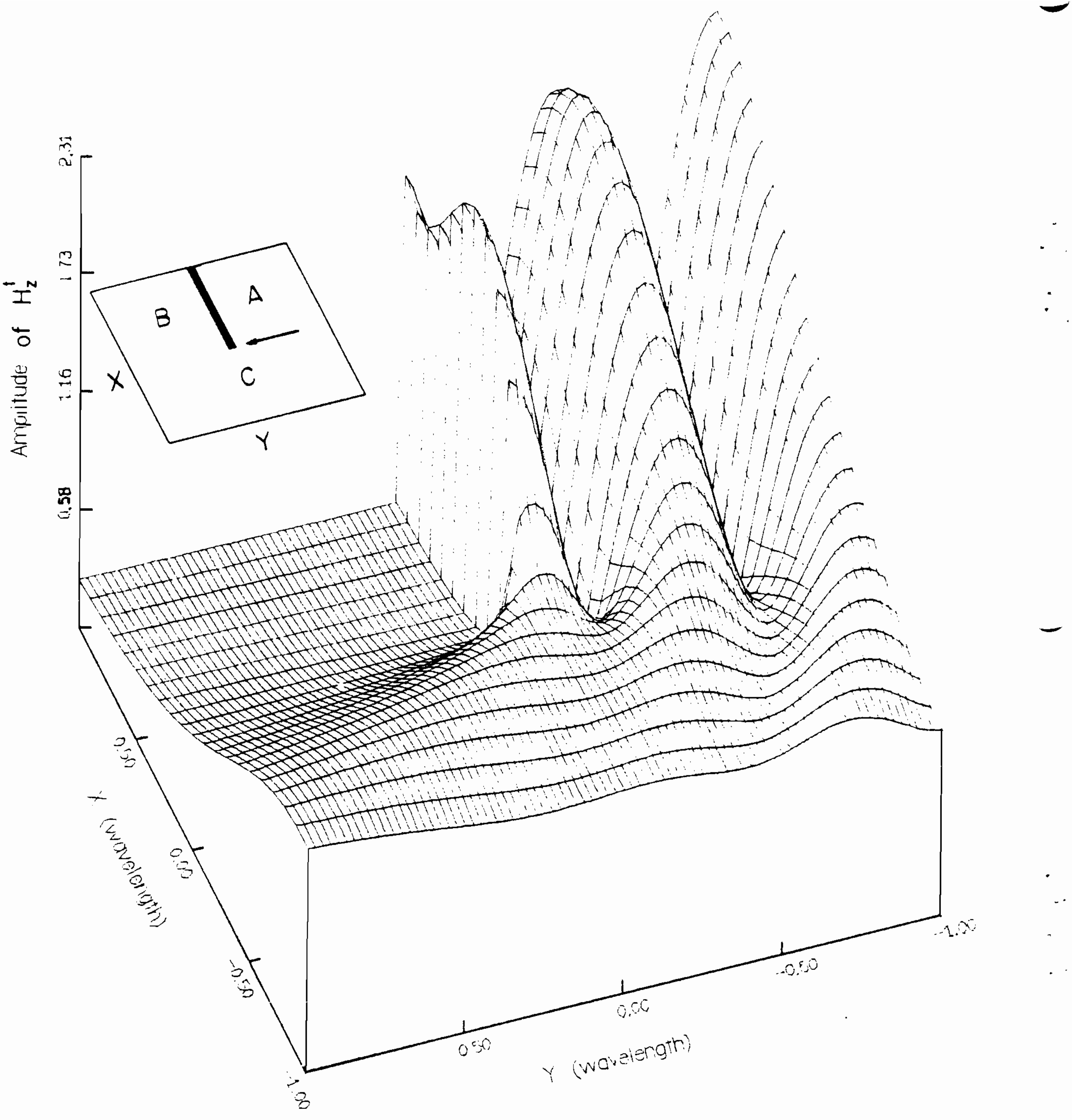

Figure 1-7. Magnitude of the total field H_z^t calculated from the exact solution in (4.11) for the half-plane diffraction. The incident field is given in (4.4) with $\phi^i = \pi/2$ (normal incidence).

the shadow boundaries but away from the edge, the total field is given in (4.45), (4.46), Problems 1-5 and 1-6. In the transition regions next to shadow boundaries, the total field varies rapidly according to the exact solution.

1.5 Sommerfeld Half-Plane Problem II. Three-Dimensional Case

In the two-dimensional half-plane problem studied in the previous section, the incident plane wave propagates in the plane transverse to the edge. Now let us consider the same problem with an obliquely incident plane wave given by

$$\vec{E}^i(\vec{r}) = e^{iks^i(\vec{r})} \vec{e}^i_0 \tag{5.1a}$$

where

$$s^i(\vec{r}) = x \sin \theta^i \cos \phi^i + y \sin \theta^i \sin \phi^i + z \cos \theta^i \quad . \tag{5.1b}$$

The spherical angles (θ^i,ϕ^i) describe the direction where the plane wave travels into (not comes from) as shown in Figure 1-8, and their values are restricted in the range

$$0 < \theta^i < \pi, \ 0 \leq \phi^i < \pi \quad . \tag{5.2}$$

(The restriction on ϕ^i will be removed later.) The problem is to determine the exact total field everywhere.* Since the half plane is uniform in z, the total field shall have the same z-variation $\exp(ikz \cos \theta^i)$ as the incident one. In this sense, the present problem is quasi-two-dimensional. In fact, it turns out that its solution can be deduced from that of the two-dimensional problem discussed in the previous section. As a preparation for this deduction, let us consider the following three points:

(i) Incident Field. With respect to the incident direction (θ^i,ϕ^i), we introduce three constant unit vectors

$$\hat{r}^i = \hat{x} \sin \theta^i \cos \phi^i + \hat{y} \sin \theta^i \sin \phi^i + \hat{z} \cos \theta^i \tag{5.3a}$$

$$\hat{\theta}^i = \hat{x} \cos \theta^i \cos \phi^i + \hat{y} \cos \theta^i \sin \phi^i - \hat{z} \sin \theta^i \tag{5.3b}$$

$$\hat{\phi}^i = -\hat{x} \sin \phi^i + \hat{y} \cos \phi^i \quad . \tag{5.3c}$$

*The two-dimensional case studied in Section 1.4 is a special case of the present problem with $\theta^i = \pi/2$.

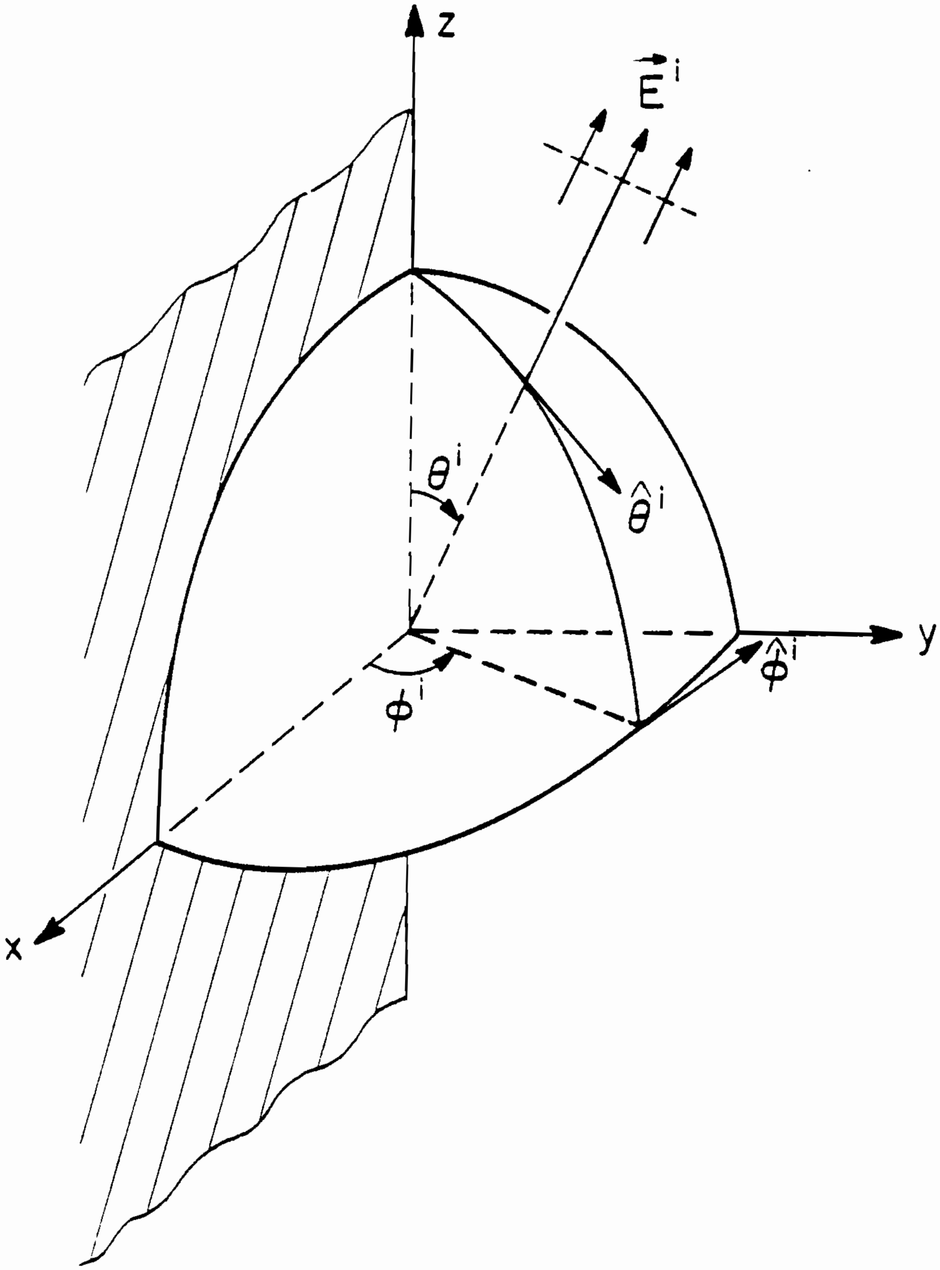

Figure 1-8. Diffraction of an incident plane wave propagating in the direction (θ^i,ϕ^i) by a half plane at $(x > 0,\ y = 0)$.

We emphasize that $(\hat{r}^i, \hat{\theta}^i, \hat{\phi}^i)$ are constant vectors, independent of observation point $\vec{r}$. At points where $(\theta = \theta^i, \phi = \phi^i)$, $(\hat{r}^i, \hat{\theta}^i, \hat{\phi}^i)$ coincide with the spherical unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$, but this relation does not hold elsewhere. Furthermore, we define a propagation vector $\vec{k}^i$ for the incident field such that

$$\vec{k}^i = k \nabla s^i \tag{5.4}$$

which in the present case becomes

$$\vec{k}^i = k \hat{r}^i \quad . \tag{5.5}$$

Then (5.1a) can be rewritten as

$$\vec{E}^i(\vec{r}) = (\hat{\theta}^i A^i + \hat{\phi}^i B^i) e^{i\vec{k}^i \cdot \vec{r}} \tag{5.6}$$

The fact that $\vec{e}_0^{\,i}$ in (5.1a) must be transverse to $\vec{k}^i$ is made use of in (5.6).

(ii) Reflected Field. According to the geometrical optics theory, the incidence of $\vec{E}^i$ in (5.6) on the half plane gives rise to a reflected field

$$\vec{E}^r(\vec{r}) = (\hat{\theta}^r A^r + \hat{\phi}^r B^r) e^{i\vec{k}^r \cdot \vec{r}} \tag{5.7}$$

which travels in the direction (θ^r, ϕ^r). Here $(\vec{k}^r, \hat{\theta}^r, \hat{\phi}^r)$ are again given by (5.3) and (5.5) after replacing superscript "i" by "r" in the latter equations. From Snell's law, we have

$$\theta^r = \theta^i, \ \phi^r = 2\pi - \phi^i \quad . \tag{5.8}$$

From the boundary conditions of the half plane, we have

$$A^r = -A^i, \ B^r = +B^i \quad . \tag{5.9}$$

(iii) TM and TE Waves. When a scatterer is uniform in the z-direction and is illuminated by an incident plane wave, the field $(\vec{E}, \vec{H})$, representing incident, scattered, or total fields, can be decomposed into TM and TE waves with respect to z. These waves are derivable from two scalar potentials as follows:*

TM:

$$\vec{H} = \sqrt{\frac{\varepsilon}{\mu}}\ \nabla \times (\hat{z}\psi) \tag{5.10a}$$

$$\vec{E} = \sqrt{\frac{\mu}{\varepsilon}}\ \frac{i}{k}\ \nabla \times \vec{H} = \frac{i}{k}\ \nabla \times \nabla \times (\hat{z}\psi) \tag{5.10b}$$

TE:

$$\vec{E} = \nabla \times (\hat{z}\bar{\psi}) \tag{5.11a}$$

$$\vec{H} = \sqrt{\frac{\varepsilon}{\mu}}\ \frac{1}{ik}\ \nabla \times \vec{E} = \sqrt{\frac{\varepsilon}{\mu}}\ \frac{1}{ik}\ \nabla \times \nabla \times (z\bar{\psi}) \quad . \tag{5.11b}$$

Both potentials satisfy the scalar wave equation

$$(\nabla^2 + k^2) \begin{bmatrix} \psi(\vec{r}) \\ \bar{\psi}(\vec{r}) \end{bmatrix} = 0 \quad . \tag{5.12}$$

Now, let us return to the half-plane diffraction problem sketched in Figure 1-6. Concentrate on the case when $B^i = 0$ in (5.6), i.e., the incident field is given by

$$\vec{E}^i(\vec{r}) = \hat{\theta}^i\ A^i\ e^{i\vec{k}^i \cdot \vec{r}} \tag{5.13a}$$

$$\vec{H}^i(\vec{r}) = \hat{\phi}^i \sqrt{\frac{\varepsilon}{\mu}}\ A^i\ e^{i\vec{k}^i \cdot \vec{r}} \quad . \tag{5.13b}$$

I can be simply verified that (5.13) is derivable from (5.10) with

$$\psi^i(\vec{r}) = \frac{i}{k \sin \theta^i}\ A^i\ e^{i\vec{k}^i \cdot \vec{r}} \quad . \tag{5.14}$$

Since the incident field is TM and the half plane is uniform in z, the scattered (total minus incident) field is also TM, derivable from a scalar potential $\psi(\vec{r})$. The conditions on $\psi(\vec{r})$ are studied below:

---

*In the two-dimensional case, TM becomes E-wave and TE becomes H-wave.

(a) Since the half plane is uniform in z, the scattered field has the same z-variation as the incident one. Thus, $\psi(\vec{r})$ can be written as

$$\psi(\vec{r}) = e^{ikz\cos\theta^i} \psi_0(x,y) \quad . \tag{5.15}$$

The use of (5.15) in (5.12) leads to

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + (k \sin \theta^i)^2\right] \psi_0(x,y) = 0 \tag{5.16}$$

which is a two-dimensional wave equation.

(b) The boundary conditions for the total field to be satisfied on the half plane are

$$E_x^t = E_z^t = 0, \text{ for } x > 0,\ y = 0 \quad . \tag{5.17}$$

Note that (5.10b) can be explicitly written as

$$\vec{E}(\vec{r}) = \left[-\cos \theta^i(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}) + \hat{z}\ ik(\sin \theta^i)^2\right] \psi(\vec{r}) \quad . \tag{5.18}$$

It follows that (5.17) is equivalent to the condition*

$$\psi_0 + \left(\frac{i}{k \sin \theta^i} A^i\right) e^{ik\rho\sin \theta^i \cos(\phi^i-\phi)} = 0, \text{ for } x > 0,\ y = 0 \ . \tag{5.19}$$

(c) The scattered field $\psi_0(x,y)$ should satisfy the radiation condition as $(x^2 + y^2) \to \infty$, and the edge condition as $(x^2 + y^2) \to 0$.

In view of (a), (b), and (c) above, it is clear that $\psi_0(x,y)$ is precisely the E-wave solution of the two-dimensional half-plane problem discussed in Section 1.4 except for a multiplicative constant and the replacement of k by k sin θ^i. Explicitly, the scalar potential of the total field may be written down from (4.11), namely,

*The fact $f(x,y) = 0$ for $x > 0$, $y = 0$ implies $\partial f/\partial x = 0$ for $x > 0$, $y = 0$.

$$\psi^t(\vec{r}) = \left(\frac{i}{k \sin \theta^i} A^i\right) e^{ikz\cos\theta^i} [F(\xi^i)\, e^{ik\rho\cos(\phi^i-\phi)\sin\theta^i}$$

$$- F(\xi^r)\, e^{ik\rho\cos(\phi^r-\phi)\sin\theta^i}] \tag{5.20}$$

or, in a more symmetrical form,

$$\psi^t(\vec{r}) = \left(\frac{i}{k \sin \theta^i} A^i\right) F(\xi^i)\, e^{i\vec{k}^i\cdot\vec{r}} + \{i \rightarrow r\} \tag{5.21}$$

where $\vec{k}^i$ is given in (5.5),

$$\xi^i(r) = (2kr \sin \theta \sin \theta^i)^{1/2} \sin \frac{1}{2}(\phi^i - \phi) \quad , \tag{5.22}$$

and the symbol $\{i \rightarrow r\}$ means to repeat all the terms after the equal sign and to change the superscript "i" to "r" in those terms. In conclusion, due to the incident field in (5.13), the scalar potential of the total field for the half-plane diffraction problem is given in (5.21). From (5.10), the complete field components may be calculated. With the manipulations outlined in Problems 1-7 and 1-8, we give the final result for the exact total field as follows:

$$\vec{H}^t(\vec{r}) = [F(\xi^i) - \hat{F}(\xi^i)]\, \vec{H}^i(\vec{r}) + \hat{\phi}\, M\, g(kr) \frac{\chi^i}{\sin \theta^i} H^i_\phi(0)$$

$$+ \{i \rightarrow r\} \tag{5.23a}$$

$$\vec{E}^t(\vec{r}) = [F(\xi^i) - \hat{F}(\xi^i)]\, \vec{E}^i(\vec{r}) + M\, g(kr) \frac{\chi^i}{\sin \theta^i} [\hat{r} \sin(\theta - \theta^i)$$

$$+ \hat{\theta} \cos(\theta - \theta^i)]\, E^i_\theta(0) + \{i \rightarrow r\} \tag{5.23b}$$

where

$$\chi^i = \csc \frac{1}{2}(\phi^i - \phi) \tag{5.23c}$$

$$g(kr) = \frac{1}{2\sqrt{2\pi kr}} e^{i(kr+\pi/4)} \tag{5.23d}$$

$$M = \left(\frac{\sin\theta^i}{\sin\theta}\right)^{1/2} \exp\{ikr[\cos(\theta - \theta^i) - 1]\} \ . \tag{5.23e}$$

We emphasize that $E^i_\theta(0)$ in (5.23b), for example, is the θ-component of $\vec{E}^i$ at $(r = 0, \theta = \theta^i, \phi = \phi^i)$, and equals to A^i according to (5.6). It is not the θ-component of $\vec{E}^i$ at $(r = 0, \theta, \phi)$. The result in (5.23) is valid only for $0 \leq \phi^i < \pi$. To make it valid for all values of ϕ^i in the range of $(0, 2\pi)$, we only have to replace the definitions of ξ^i in (5.22) and χ^i in (5.23c) by

$$\xi^i(\vec{r}) = \varepsilon^i \left|2kr \sin\theta \sin\theta^i\right|^{1/2} \left|\sin\tfrac{1}{2}(\phi^i - \phi)\right| \tag{5.24a}$$

$$\chi^i(\vec{r}) = \varepsilon^i \left|\csc\tfrac{1}{2}(\phi^i - \phi)\right| \tag{5.24b}$$

where $\varepsilon^i(\vec{r})$ is the shadow indicator defined in (4.35).

Next consider the incident field in (5.6) when $A^i = 0$. The solution due to this incident field can be determined in exactly the same manner. We combine this result and that in (5.23), and present them below in a single equation. When the half plane is illuminated by a general incident plane wave in (5.6), the exact solution for the total field is

$$\vec{E}^t(\vec{r}) = [F(\xi^i) - \hat{F}(\xi^i)]\,\vec{E}^i(\vec{r}) + M\,g(kr)\,\frac{\chi^i}{\sin\theta^i}\{[\hat{r}\sin(\theta - \theta^i)$$

$$+ \hat{\theta}\cos(\theta - \theta^i)]\,E^i_\theta(0) + \hat{\phi}\,E^i_\phi(0)\} + \{i \rightarrow r\} \tag{5.25}$$

The same equation (5.25) holds for $\vec{H}^t(\vec{r})$ after replacing $\vec{E}^{t,i}$ by $\vec{H}^{t,i}$ This solution can be put in a simpler form. Note that, for a given incident field and observation point, we can always choose an origin such that

$$\theta = \theta^i \quad , \tag{5.26}$$

which implies that the three vectors $\vec{k}^i$, $\vec{k}^r$, and $\vec{r}$ lie on the surface of a cone whose axis is the edge of the half plane (Figure 1-9). Then (5.25) becomes

$$\vec{E}^t(\vec{r}) = [F(\xi^i) - \hat{F}(\xi^i)]\ \vec{E}^i(\vec{r}) + g(kr)\ \frac{\chi^i}{\sin\theta^i}\ [\hat{\theta}\ E^i_\theta(0) + \hat{\phi}\ E^i_\phi(0)]$$

$$+ \{i \rightarrow r\},\ \text{if}\ \theta = \theta^i \quad . \tag{5.27}$$

The same equation (5.27) holds for $\vec{H}^t$ after replacing $\vec{E}^{t,i}$ by $\vec{H}^{t,i}$. It should be remarked that the half-plane solution appears in the literature in several different (but of course equivalent) forms. None of them is as simple as the one in (5.27). The latter was suggested by the uniform asymptotic theory to be studied in Chapter 5.

When the observation point $\vec{r} = (r, \theta = \theta^i, \phi)$ is away from the edge ($kr >> 1$) and away from the shadow boundaries $SB^{i,r}$ defined by $\phi = \phi^{i,r}$, i.e., $\vec{r}$ is outside the transition regions in Figure 1-5, the relation [cf. (3.5)]

$$F(\xi^{i,r}) - \hat{F}(\xi^{i,r}) \sim \theta(-\xi^{i,r}) = \theta(-\varepsilon^{i,r}) \tag{5.28}$$

may be used in (5.27). The manipulation leads to the result

$$\vec{E}^t(r, \theta^i, \phi) = \vec{E}^g + \vec{E}^d \quad . \tag{5.29}$$

Here $\vec{E}^g$ is the geometrical optics field given by

$$\vec{E}^g(\vec{r}) = \theta(-\varepsilon^i)\ \vec{E}^i(\vec{r}) + \{i \rightarrow r\} \quad . \tag{5.30}$$

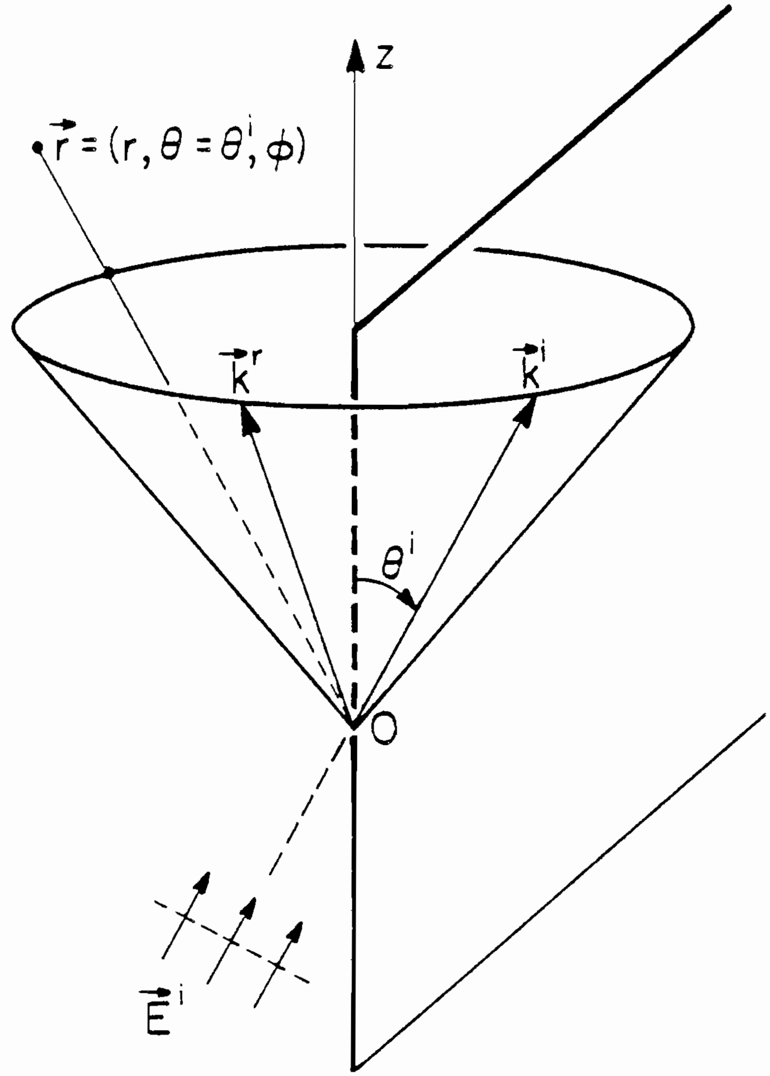

Figure 1-9. Diffraction of a plane wave by a half plane when the origin O is chosen so that $\theta = \theta^i$.

$\vec{E}^d$ is the (Keller's) diffracted field given by

$$\vec{E}^d(\vec{r}) = g(kr) \frac{\chi^i}{\sin \theta^i} [\hat{\theta} E^i_\theta(0) + \hat{\phi} E^i_\phi(0)] + \{i \to r\} + O(k^{-3/2}) \quad . \quad (5.31a)$$

Since $\vec{E}^i(0)$ and $\vec{E}^r(0)$ are simply related by (5.9), the diffracted field may be rewritten as

$$\vec{E}^d(\vec{r}) = g(kr) \frac{1}{\sin \theta^i} [\hat{\theta} (\chi^i - \chi^r) E^i_\theta(0) + \hat{\phi}(\chi^i + \chi^r) E^i_\phi(0)]$$

$$+ O(k^{-3/2}) \quad . \quad (5.31b)$$

A detailed discussion of (5.29) in terms of rays will be given in Chapter 4.

1.6 Some Scattering Parameters and Theorems

For convenient reference, we summarize here some scattering parameters and theorems. Detailed discussion of them can be found in standard books on electromagnetic theory*.

(i) Radar cross section. Consider a three-dimensional perfectly conducting scatterer Σ illuminated by an incident linearly polarized plane wave (Figure 1-10)

$$\vec{E}^i(\vec{r}) = \hat{e}^i A e^{ikx} \tag{6.1}$$

where $\hat{e}^i$ is a unit constant real vector normal to $\hat{x}$. At an observation point $\vec{r} = (r,\theta,\phi)$ at a large distance from Σ, let us assume that the scattered (total minus incident) field $\vec{E}$ is a linearly polarized spherical wave which has the form

$$\vec{E}(\vec{r}) \sim \frac{e^{ikr}}{kr}\,\hat{e}\,AS(\theta,\phi), \quad r \to \infty \tag{6.2}$$

Here $S(\theta,\phi)$ is commonly known as the far-field scattering pattern and is dimensionless. We define the (back-scattered) radar cross section RCS of Σ by

$$\mathrm{RCS} = \lim_{r\to\infty} 4\pi r^2 \frac{|\vec{E}(\vec{r} = -\hat{x}\,r)|^2}{|\vec{E}^i(\vec{r})|^2} \tag{6.3}$$

which has a dimension of square meters. In terms of scattering pattern $S(\theta,\phi)$, (6.3) becomes

$$\mathrm{RCS} = \frac{4\pi}{k^2}\left|S(\theta = \frac{\pi}{2},\ \phi = \pi)\right|^2 \quad . \tag{6.4}$$

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*M. Born and E. Wolf, Principles of Optics, 2nd Ed., Pergamon Press, New York, 1964. J. J. Bowman, T. B. A. Senior and P. L. E. Uslenghi, Electromagnetic and Acoustic Scattering by Simple Shapes. North-Holland Publishing Company, Amsterdam, Netherlands, 1969.

In case that Σ is an infinite scatterer and is uniform in the z-direction, the above definition should be slightly modified. Let the incident field be a linearly polarized plane wave

$$u^i(x,y) = Ae^{ikx} \tag{6.5}$$

where $u^i = E_z^i$ for E-wave and $u^i = H_z^i$ for H-wave. At an observation point $\vec{\rho} = (\rho,\phi)$ at a large distance from Σ, we assume that the scattered field is a cylindrical wave

$$u(x,y) \sim \frac{e^{i(k\rho+\pi/4)}}{2\sqrt{2\pi k\rho}} AS(\phi), \quad \rho \to \infty \quad . \tag{6.6}$$

The (back-scattered) radar cross section per unit length along z defined by

$$\text{RCS} = \lim_{\rho\to\infty} 2\pi\rho \frac{|u(x = -|x|, y = 0)|^2}{|u^i(x,y)|^2} \tag{6.7}$$

is found to be

$$\text{RCS} = \frac{1}{4k} |S(\phi = \pi)|^2 \quad . \tag{6.8}$$

The RCS in (6.7) or (6.8) has the dimension of meters.

(ii) Scattering cross section. Consider again the scattering problem sketched in Figure 1-10 with the incident field given in (6.1) and scattered field in (6.2). We note that the time-averaged incident power density is given by

$$\text{Re}(\vec{E}^i \times \vec{H}^{i*}) \quad ,$$

and the time-averaged total scattered power in the far field is given by

$$\text{Re} \oiint (\vec{E} \times \vec{H}^*) \cdot d\vec{a}$$

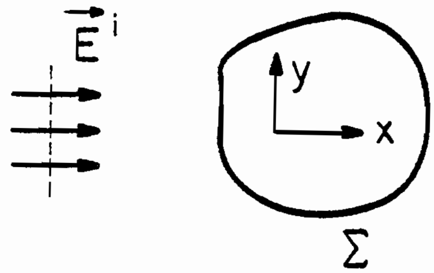

Figure 1-10. A conducting scatterer illuminated by an incident plane wave.

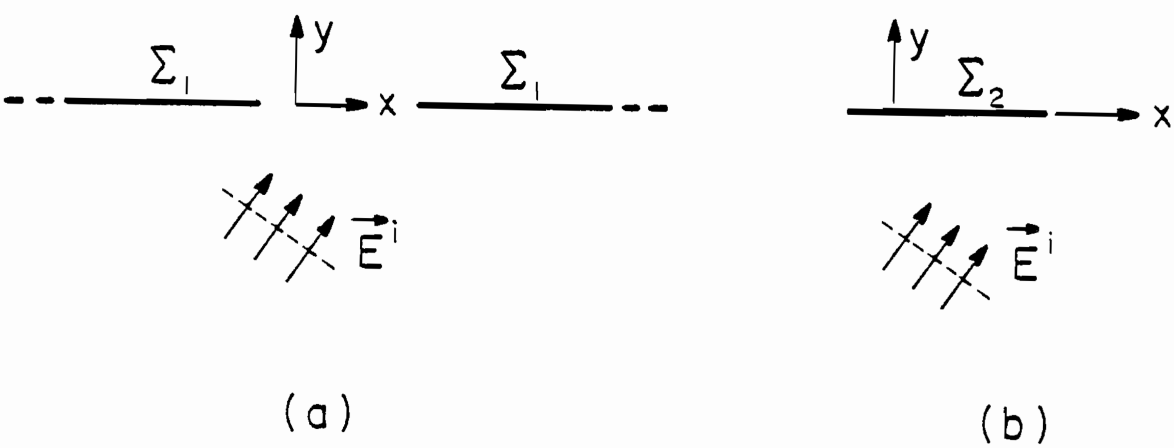

Figure 1-11. A planar aperture, or its complementary plate illuminated by an incident plane wave.

where the surface integration is carried over a sphere $r = R$ with $R \to \infty$. Then the scattering cross section SCS of Σ is defined by the ratio of the above two quantities, namely,

$$\mathrm{SCS} = \frac{\mathrm{Re} \iint (\vec{E} \times \vec{H}^*) \cdot d\vec{a}}{|\mathrm{Re}(\vec{E}^i \times \vec{H}^{i*})|} \tag{6.9}$$

which has a dimension of square meters. In terms of the scattering pattern $S(\theta,\phi)$ in (6.2), it can be shown that

$$\mathrm{SCS} = \frac{4\pi}{k^2} (\hat{e}^i \cdot \hat{e}) \ \mathrm{Im}\ S(\theta = \frac{\pi}{2}, \phi = 0) \quad . \tag{6.10}$$

This relation states that the SCS of a lossless scatterer is proportional to the imaginary part of the scattering pattern in the forward direction of the incident plane wave. Next consider the case when Σ is an infinite scatterer and is uniform in the z-direction. With respect to the incident field in (6.5) and scattered field in (6.6), the scattering cross section per unit length along z is found to be

$$\mathrm{SCS} = \frac{1}{k} \ \mathrm{Im}\ S(\phi = 0) \tag{6.11}$$

which has a dimension of meters.

(iii) Transmission cross section. At $y = 0$, there is an infinitely large perfectly conducting plane Σ_1 with an aperture as shown in Figure 1-11a. It is illuminated by an incident linearly polarized plane wave

$$\vec{E}^i(\vec{r}) = \hat{e}^i A e^{i\vec{k}^i \cdot \vec{r}} \tag{6.12}$$

where the vectors $\hat{e}^i$ and $\vec{k}^i$ describe, respectively, the polarization and propagation direction of $\vec{E}^i$. In the half space $y > 0$, let us assume that the total field in the far zone is also linearly polarized and has the form

$$\vec{E}^t(\vec{r}) \sim \frac{e^{ikr}}{kr} \hat{e} AS(\hat{r}), \quad y > 0 \text{ and } r \to \infty \tag{6.13}$$

at an observation point $\vec{r} = \hat{r}\, r$. We define a transmission cross section TCS of the planar aperture as the ratio of time-averaged power transmitted through the aperture and the time-averaged incident power density. It may be shown that

$$\text{TCS} = -\frac{2\pi}{k^2} (\hat{e}^i \cdot \hat{e}) \text{ Im } S(\hat{r} = \hat{k}^i) \tag{6.14}$$

which has a dimension of square meters. TCS normalized with respect to the area of the aperture is called the transmission coefficient of the aperture. In a corresponding two-dimensional problem (no z-variation), the incident field and total field transmitted through the aperture at a point $\vec{\rho} = \hat{\rho}\, \rho$ are represented by

$$u^i(x,y) = Ae^{i\vec{k}^i \cdot \rho} \tag{6.15}$$

$$u^t(x,y) \sim \frac{e^{i(k\rho+\pi/4)}}{2\sqrt{2\pi k\rho}} AS(\hat{\rho}), \quad y > 0 \text{ and } \rho \to \infty \quad . \tag{6.16}$$

Then TCS per unit length along the z-direction is given by

$$\text{TCS} = -\frac{1}{2k} \text{ Im } S(\hat{\rho} = \hat{k}^i) \tag{6.17}$$

which has a dimension of meters.

(iv) Scattered field from a planar aperture or plate. Consider the infinitely large perfectly conducting plane Σ_1 with an aperture, and the plate Σ_2 at $y = 0$ shown in Figure 1-11. For an arbitrary incident field, show that the following symmetry exists for the scattered (total minus incident) fields:

$$E_x(x,y,z) = E_x(x,-y,z), \; H_x(x,y,z) = -H_x(x,-y,z) \tag{6.18a}$$

$$E_y(x,y,z) = -E_y(x,-y,z), \; H_y(x,y,z) = H_y(x,-y,z) \tag{6.18b}$$

$$E_z(x,y,z) = E_z(x,-y,z), \; H_z(x,y,z) = -H_z(x,-y,z) \quad . \tag{6.18c}$$

The above result is a consequence of the fact that the scattered field is due to the radiation of induced currents on Σ_1 or Σ_2, and those currents radiate symmetrically into the two half spaces $y > 0$ and $y < 0$.

(v) Babinet's Principle. Consider the following two scattering problems:

(1) An infinitely large perfectly conducting plane Σ_1 with an aperture is illuminated by an incident electrical field $\vec{E}_1^i = \vec{F}^i$ as shown in Figure 1-11a. The total electrical field everywhere is the sum of the incident field $\vec{E}_1^i$ and scattered field $\vec{E}_1$.

(2) A perfectly conducting plate Σ_2, which is complementary to the plane Σ_1, is illuminated by an incident magnetic field $\vec{H}_2^i = \vec{F}^i$ as shown in Figure 1-11b. The total magnetic field everywhere is the sum of incident field $\vec{H}_2^i$ and scattered field $\vec{H}_2$.

Suppose that the incident fields in both problems come from the half space $y < 0$. It may be shown that,

$$\vec{E}_1^s + \vec{H}_2^s + \vec{F}^i = 0, \; y > 0 \tag{6.19a}$$

in the transmitted half space; and

$$\begin{cases} \hat{z} \times (\vec{E}_1^s - \vec{H}_2^s + \vec{F}^r) = 0 \\ \hat{z} \cdot (-\vec{E}_1^s + \vec{H}_2^s + \vec{F}^r) = 0 \end{cases} , \ y < 0 \qquad (6.19b)$$

in the incident half space, where $\vec{F}^r(x,y,z) = \vec{F}^i(x,-y,z)$. The relation in (6.19) remains valid if the roles of Σ_1 and Σ_2 are interchanged (i.e., if Σ_1 is a plate and Σ_2 is an infinite plane).

1-1. The symbols O and o used in this book have the following meanings:

(a) $f(k) = O[g(k)]$ as $k \to \infty$ means that $f(k)$ does not grow faster than $g(k)$ as $k \to \infty$; or there is a constant A such that

$$|f(k)| \leq A|g(k)| \quad , \quad k \to \infty \; .$$

(b) $f(k) = o[g(k)]$ as $k \to \infty$ means that $f(k)$ grows slower than $g(k)$ as $k \to \infty$; or

$$\lim_{k\to\infty} |f(k)/g(x)| = 0 \quad .$$

Now show that, as $k \to \infty$,

$$h(k) = o(k^{-M}) \text{ implies } h(k) = O(k^{-M})$$

and

$$h(k) = O(k^{-M}) \text{ implies } h(k) = o(k^{-M+1}) \; .$$

1-2. Show that the Fresnel function

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^\infty e^{it^2} \, dt, \text{ for real } x$$

has its Taylor series expansion given by

$$F(x) = \frac{1}{2} - \frac{1}{\sqrt{\pi}} e^{-i\pi/4} \, x \sum_{n=0}^{\infty} \frac{(ix^2)^n}{n!(2n+1)}$$

$$= \frac{1}{2} - \frac{1}{\sqrt{\pi}} e^{-i\pi/4} \, x\left[1 + \frac{ix^2}{3} - \frac{x^4}{10} + \ldots\right] \quad .$$

Hint: Rewrite the integral in two: one from t = 0 to ∞ and the other from 0 to x. Expand the integrand of the second integral as

$$e^{it^2} = \sum_{n=0}^{\infty} \frac{(it^2)^n}{n!}$$

and carry out the integration term by term.

1-3. Show that the Fresnel integral defined in Problem 1-2 has its asymptotic expansion given by, as $|x| \to \infty$,

$$F(x) \sim \theta(-x) + \frac{1}{2\pi x} e^{i(x^2+\frac{\pi}{4})} \sum_{n=0}^{\infty} \Gamma(n + \tfrac{1}{2}) (ix^2)^{-n}$$

$$= \theta(-x) + \frac{1}{2\sqrt{\pi}x} e^{i(x^2+\frac{\pi}{4})} \left[1 - \frac{i}{2x^2} - \frac{3}{4x^4} + \ldots\right] .$$

Hint: First consider the case x > 0. Integration by parts gives

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^{\infty} \frac{1}{(2it)} de^{it^2}$$

$$= \frac{e^{-i\pi/4}}{\sqrt{\pi}} \left[\frac{ie^{ix^2}}{2x} + \int_x^{\infty} \frac{e^{it^2}}{2it^2} dt\right]$$

$$= \frac{1}{2\sqrt{\pi}x} e^{i(x^2+\frac{\pi}{4})} \left[1 - (2ix) e^{-ix^2} \int_x^{\infty} \frac{1}{(2i)^2 t^3} de^{it^2}\right]$$

Repeating integration by parts, the desired asymptotic series is obtained. For x < 0, the above procedure cannot be used directly, as the integrand would contain (1/t) which is infinite at t = 0. This difficulty can be circumvented by using the identity F(x) = 1 - F(-x).

1-4. Derive the relation in (4.21).

Hint:

$$\frac{d}{dx} F(x) = i2x \hat{F}(x)$$

$$\frac{\partial}{\partial \rho} \xi^i = \frac{k}{2\xi^i} [1 - \cos(\phi^i - \phi)]$$

$$\frac{1}{\rho} \frac{\partial}{\partial \phi} \xi^i = - \frac{k}{2\xi^i} \sin(\phi^i - \phi) \quad .$$

1-5. From the exact solution given in (4.26) show that the total fields on the incident shadow boundary $\phi = \phi^i$ (where $0 < \phi^i < 2\pi$) and away from the edge are given by

E-wave:

$$E_z^t = \frac{1}{2} e^{ik\rho} - g(k\rho) \, |\csc \phi^i| + O(k^{-3/2})$$

$$H_\rho^t = \mp \sqrt{\frac{\varepsilon}{\mu}} \, 2g(k\rho) + O(k^{-3/2})$$

$$H_\phi^t = - \sqrt{\frac{\varepsilon}{\mu}} \, E_z^t + O(k^{-3/2})$$

H-wave:

$$H_z^t = \frac{1}{2} e^{ik\rho} + g(k\rho) \, |\csc \phi^i| + O(k^{-3/2})$$

$$E_\rho^t = \pm \sqrt{\frac{\varepsilon}{\mu}} \, 2g(k\rho) + O(k^{-3/2})$$

$$E_\phi^t = \sqrt{\frac{\varepsilon}{\mu}} \, H_z^t + O(k^{-3/2})$$

Here the upper sig in H_ρ^t or E_ρ^t applies when $0 < \phi^i < \pi$, and the lower sign applies when $< \phi^i < 2\pi$. This change in sign may be attributed to the fact that, en $0 < \phi^i < \pi$, H_ϕ^t or E_ϕ^t points to the lit region of the incident field and when $\pi < \phi^i < 2\pi$, it points to the shadow region.

1-6. From the exa solution given in (4.26) show that the total fields on th reflected shadow boundary $\phi = \phi^r = 2\pi - \phi^i$ (where $0 < \phi^i < 2\pi$) and a y from the edge are given by

E-wave:

$$E_z^t - E_z^i = -\frac{1}{2} \quad ^{ik\rho} - g(k\rho) \; |\csc \phi^i| + O(k^{-3/2})$$

$$H_\rho^t - H_\rho^i = \pm \sqrt{\ } \; 2g(k\rho) + O(k^{-3/2})$$

$$H_\phi^t - H_\phi^i = -\sqrt{\ } \; (E_z^t - E_z^i) + O(k^{-3/2})$$

H-wave:

$$H_z^t - H_z^i = \frac{1}{2} e \quad - g(k\rho) \; |\csc \phi^i| + O(k^{-3/2}$$

$$E_\rho^t - E_\rho^i = \mp \sqrt{\ } \; 2g(k\rho) + O(k^{-3/2})$$

$$E_\phi^t - E_\phi^i = \sqrt{\frac{\mu}{\epsilon}} \quad {}^t_z - H_z^i) + O(k^{-3/2})$$

where the upper (l er) sign in H_ρ^t or E_ρ^t applies when $0 < \phi^i < \pi$ $(\pi < \phi^i < 2\pi)$.

1-7. Given

$$\xi^i = (2kr \sin\theta \sin\theta^i)^{1/2} \sin\frac{1}{2}(\phi^i - \phi)$$

$$F(x) = \frac{e^{-i\pi/4}}{\sqrt{\pi}} \int_x^\infty e^{it^2}\, dt$$

$$\hat{F}(x) = \frac{1}{2(\pi)^{1/2}\, x} \exp i(x^2 + \frac{\pi}{4}) \quad .$$

Show the following identities

$$\nabla\xi^i = (2\xi^i)^{-1} (k \sin\theta^i)\ (\hat{\rho} - \hat{\rho}^i)$$

$$\nabla F(\xi^i) = (ik \sin\theta^i)\ \hat{F}(\xi^i)\ (\hat{\rho} - \hat{\rho}^i)$$

$$\nabla\hat{F}(\xi^i) = (ik \sin\theta^i)\ [1 + \frac{i}{2}\ (\xi^i)^{-2}]\ \hat{F}(\xi^i)\ (\hat{\rho} - \hat{\rho}^i)$$

$$\nabla[F(\xi^i) - \hat{F}(\xi^i)] = \frac{1}{2}(k \sin\theta^i)\ (\xi^i)^{-2}\ \hat{F}(\xi^i)\ (\hat{\rho} - \hat{\rho}^i) \quad .$$

Hint:

$$\frac{\partial}{\partial r} F(\xi^i) = i\ \hat{F}(\xi^i)\ [1 - \cos(\phi^i - \phi)]\ k \sin\theta \sin\theta^i$$

$$\frac{1}{r}\frac{\partial}{\partial\theta} F(\xi^i) = i\ \hat{F}(\xi^i)\ [1 - \cos(\phi^i - \phi)]\ k \cos\theta \sin\theta^i$$

$$\frac{1}{r \sin\theta}\frac{\partial}{\partial\phi} F(\xi^i) = i\ \hat{F}(\xi^i)\ (-k \sin\theta^i) \sin(\phi^i - \phi) \quad .$$

1-8. Using the results in Problem 1-7, derive the exact total field in (5.23) for the half plane from (5.21) and (5.10).

Hint: To calculate $\vec{H}^t$, note the vector identity

$$\nabla \times (U\vec{A}) = \nabla U \times \vec{A} + U\nabla \times \vec{A} \quad .$$

To calculate $\vec{E}^t$, note that

$$\sqrt{\frac{\mu}{\varepsilon}}\,\frac{i}{k}\,\nabla \times [F(\xi^i) - \hat{F}(\xi^i)]\,\vec{H}^i(\vec{r}) = [F(\xi^i) - \hat{F}(\xi^i)]\,\vec{E}^i(\vec{r})$$

$$+ \hat{z}\,M\,g(kr)\,\frac{\chi^i}{\sin\theta^i}\,(2ikr\sin\theta)^{-1}\,E^i_\theta(0,\phi^i)$$

$$\sqrt{\frac{\mu}{\varepsilon}}\,\frac{i}{k}\,\nabla \times [\hat{\phi}\,M\,g(kr)\,\frac{\chi^i}{\sin\theta^i}\,H^i_\phi(0,\phi^i)] = M\,g(kr)\,\frac{\chi^i}{\sin\theta^i}\,[\hat{r}\sin(\theta - \theta^i)$$

$$+ \hat{\theta}\cos(\theta - \theta^i) - \hat{z}(2ikr\sin\theta)^{-1}]\,E^i_\theta(0,\phi^i) \quad .$$

Chapter 2. ASYMPTOTIC SOLUTION OF MAXWELL'S EQUATIONS

Chapter 2. ASYMPTOTIC SOLUTION OF MAXWELL'S EQUATIONS

2.1. Introduction

For a prescribed source and boundary conditions, only a few electromagnetic problems can be exactly and explicitly solved. Thus from a practical viewpoint, it is extremely important to develop approximate (analytical or numerical) techniques that can be used for different occasions. Throughout this work we are concerned with high frequency electromagnetic fields, that is, fields with small wavelengths compared to either a significant dimension of the object that the fields interact, or the distance between the source and the observation point. At high frequencies, several well-developed analytical asymptotic methods in mathematics can be applied to the solution of Maxwell's equations. These applications generally can be grouped into two types: the direct application and the indirect application, according to the stage in the solution process when the asymptotic method is introduced.

In the indirect application, we work with the Maxwell equations in their exact forms (as given in Section 1.1.). For a small class of problems, e.g., problems with separable geometry, we may be able to derive the exact solution in integral representations; a typical form is given below:

$$\vec{E}(\vec{r}) = \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \; \vec{A}(\alpha,\beta,kz) e^{i(\alpha kx+\beta ky)} \tag{1.1}$$

where $k = \omega\sqrt{\mu\varepsilon}$ is the wavenumber of the free space. Frequently, either the above integral cannot be carried out explicitly, or when it does, the result is too complex to be useful. At a high frequency (or more explicitly, $k\sqrt{x^2 + y^2}$ is large), the well-known saddle point integration method can be applied to (1.1) to derive an asymptotic series

$$\vec{E}(\vec{r}) \sim \sum_{\nu} (ik)^{-\nu} \vec{E}_{\nu}(\vec{r},k) \quad , \quad k \to \infty \tag{1.2}$$

where $\{\nu\}$ is a set of integers or fractional numbers, and $\{\vec{E}_{\nu}\}$ are, in general, functions of k and are bounded as $k \to \infty$. We call (1.2) the asymptotic expansion of the exact solution in (1.1). Studies on such an indirect application of asymptotic methods are well-documented in books on electromagnetic theory.*

In the direct application, we apply asymptotic methods directly to Maxwell's equations at the beginning of the problem, instead of to its solutions. Since it is no longer necessary to derive (if possible at all) a representation such as (1.1), the direct application is invariably simpler, and, more importantly, can be adopted to a much broader class of problems. It is the direct application of asymptotic methods to electromagnetic edge diffraction problems that will be studied in this book.

Based on our experience with edge diffraction problems, an "educated" conjecture is that an asymptotic solution of Maxwell's equations in the free space may take the following form,† as $k \to \infty$,

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*e.g., D. S. Jones, The Theory of Electromagnetism, Macmillan, New York, 1964. L. B. Felsen and N. Marcuvitz, Radiation and Scattering of Waves, Prentice-Hall, New Jersey, 1973.

†The use of the asymptotic series in the form of (1.3) for solving Maxwell's equations was first suggested by R. K. Luneburg in his mimeographed notes on Mathematical Theory of Optics issued by Brown University in 1944, and also in M. Kline, "An asymptotic solution of Maxwell's equations," Comm. Pure Appl. Math., 4, 225-263, 1951. This series is sometimes called the Luneburg-Kline expansion.

$$\vec{E}(\vec{r}) \sim k^{\tau} e^{iks(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m} \vec{e}_m(\vec{r}) \qquad (1.3a)$$

$$\vec{H}(\vec{r}) \sim k^{\tau} e^{iks(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m} \vec{h}_m(\vec{r}) \qquad (1.3b)$$

where the amplitudes $\{\vec{e}_m, \vec{h}_m\}$ and the phase function $s(\vec{r})$ are functions of the space variable $\vec{r}$, and are independent of k. When the incident field is assumed to be of order k^0, the parameter τ in (1.3) takes a value between -1 and 0, and determines the nature of the field represented by the series, e.g., geometrical optics field ($\tau = 0$) or edge diffracted field ($\tau = -1/2$). In a given problem, a complete solution of the total field may be a superposition of several asymptotic series with possibly different τ. We call (1.3) an asymptotic solution of the problem under consideration. No general proof that the asymptotic solution (1.3) is identical with the asymptotic expansion of the exact solution (1.2) exists. Nevertheless, the agreement found at various special problems provides strong evidence of the validity of the present direct application of asymptotic methods.

The purpose of this chapter is to study some general properties of the series (1.3) as imposed by Maxwell's equations. Those properties will be used throughout the remainder of this book.

2.2. Equations Governing Asymptotic Solutions

An asymptotic solution in the form of (1.3) must satisfy Maxwell's equations. Based on this fact we will now derive explicit equations that govern the variation of the phase function $s(\vec{r})$ and the amplitude vectors $\{\vec{e}_m, \vec{h}_m\}$. In the free space, Maxwell's equations in a source-free region for a time-harmonic field take the following form:

$$\nabla \times \vec{E} = i\omega\mu\vec{H} \quad , \qquad \nabla \times \vec{H} = -i\omega\varepsilon\vec{E} \quad , \tag{2.1a}$$

$$\nabla \cdot \vec{E} = 0 \quad , \qquad \nabla \cdot \vec{H} = 0 \quad , \tag{2.1b}$$

where the time dependence $\exp(-i\omega t)$ as usual has been dropped. Eliminating $\vec{H}(\vec{r})$ in (2.1) leads to two equations that govern $\vec{E}(\vec{r})$, namely,

$$\begin{cases} (\nabla^2 + k^2)\vec{E}(\vec{r}) = 0 \quad , & (2.2) \\ \nabla \cdot \vec{E}(\vec{r}) = 0 \quad . & (2.3) \end{cases}$$

Once $\vec{E}(\vec{r})$ is determined, $\vec{H}(\vec{r})$ may be calculated from (2.1a) or

$$\vec{H}(\vec{r}) = \frac{1}{i\omega\mu} \nabla \times \vec{E}(\vec{r}) \quad . \tag{2.4}$$

Let us now concentrate on (2.2) and (2.3).

When an asymptotic solution in the form of (1.3) is substituted into (2.2) and (2.3), the results are obviously independent of the multiplication factor k^τ. Thus, for simplicity, let us set $\tau = 0$ in (1.3a) such that

$$\vec{E}(\vec{r}) \sim e^{iks(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m} \vec{e}_m(\vec{r}) \quad , \qquad k \to \infty \quad . \tag{2.5}$$

This is the form of $\vec{E}$ that will be considered in the remainder of this chapter. A word about the meaning of $k \to \infty$ is in order. Although there is nothing inappropriate mathematically, it is desirable that the parameter of the asymptotic expansion be dimensionless. As a built-in property of Maxwell's equations, the parameter k always appears together with one (or more) parameter of distance, say d, in the solution (except perhaps for a proportional constant in the solution which is independent of d). As kd is dimensionless, our asymptotic expansion can be also regarded as one for $kd \to \infty$. The precise definition of d becomes apparent only after the problem is explicitly specified. Thus, for the time being, we are satisfied with an asymptotic expansion with respect to large k and we understand that, in a specific problem, the same expansion often can be interpreted as the one with respect to large kd, a dimensionless parameter.

Return to the asymptotic series in (2.5). We desire to determine the conditions imposed by Maxwell's equations (not including boundary and source conditions yet) on the phase function $s(\vec{r})$ and amplitudes $\{\vec{e}_m(\vec{r})\}$. Substituting (2.5) into the source-free wave equation in (2.2) and collecting the terms of the same power of $(ik)^{-1}$, we have

$$\sum_{m=0}^{\infty} (ik)^{-m+2}\left\{[(\nabla s)^2 - 1]\,\vec{e}_m + \frac{1}{ik}\,[2(\nabla s \cdot \nabla)\,\vec{e}_m + \nabla^2 s\,\vec{e}_m] + \frac{1}{(ik)^2}\,\nabla^2\,\vec{e}_m\right\} = 0 \tag{2.6}$$

where $(\nabla s)^2$ means $\nabla s \cdot \nabla s$. As $k \to \infty$, the coefficients of each power of $(ik)^{-1}$ must be zero, cf. Section 1.2, namely,

$$[\nabla s(\vec{r})]^2 = 1 \tag{2.7}$$

which is known as the eikonal* equation for the phase function $s(\vec{r})$, and

$$2(\nabla s \cdot \nabla)\,\vec{e}_m(\vec{r}) + \nabla^2 s\,\vec{e}_m(\vec{r}) = -\nabla^2\,\vec{e}_{m-1}(\vec{r}) \quad , \tag{2.8}$$

$$m = 0,\ 1,\ 2,\ \ldots\ ; \qquad \vec{e}_{-1} = 0$$

*The word eikonal is derived from the Greek εικών meaning image. The term was used first by H. Bruns in 1895.

which are known as transport equations for amplitudes $\{\vec{e}_m(\vec{r})\}$. Furthermore, the use of (2.5) in (2.3) leads to

$$\sum_{m=0}^{\infty} (ik)^{-m+1} (\nabla s \cdot \vec{e}_m + \frac{1}{ik} \nabla \cdot \vec{e}_m) = 0$$

and it yields in turn

$$\nabla s \cdot \vec{e}_m(\vec{r}) = -\nabla \cdot \vec{e}_{m-1}(\vec{r}) , \qquad (2.9)$$

$$m = 0, 1, 2, \ldots ; \quad \vec{e}_{-1} = 0$$

which is a relation imposed by the Gauss law. Finally, the use of (2.5) in (2.4) gives

$$\vec{h}_m(\vec{r}) = \sqrt{\frac{\varepsilon}{\mu}} [\nabla s \times \vec{e}_m(\vec{r}) + \nabla \times \vec{e}_{m-1}(\vec{r})] , \qquad (2.10)$$

$$m = 0, 1, 2, \ldots ; \quad \vec{e}_{-1} = 0$$

which may be used to calculate the magnetic field in (1.3b) once $\{\vec{e}_m\}$ are known.

In summary, an asymptotic solution in the form of (1.3) satisfies the source-free Maxwell equations provided that the four conditions in (2.7) through (2.10) are satisfied. In the next four sections, we will study the implications of the four conditions.

2.3. Wavefront, Ray, and Pencil

We will now study the solution of the eikonal equation in the free space as given in (2.7) or

$$[\nabla s(\vec{r})]^2 = 1 \tag{3.1}$$

by the concept of rays. The surfaces of constant phase defined by

$$s(\vec{r}) = \text{constant} \tag{3.2}$$

are called wavefronts. The curves everywhere orthogonal to wavefronts are called rays. They are tangent to the unit vector ∇s. In optics, a set of curves filling a portion of space in such a way that, in general, a single curve passes through any given point is called a congruence. If there exists an infinite family of surfaces cut orthogonally by the curves of a congruence, the congruence is said to be normal; if no such family of surfaces exists, the congruence is skew. Obviously, rays form a normal congruence.

Along each ray the partial differential equation in (3.1) can be reduced to an ordinary differential equation, which, of course, is more manageable. This will be demonstrated below. Since a ray is a curve in space, it can be represented by a parametric equation

$$x = x(\sigma) \quad , \qquad y = y(\sigma) \quad , \qquad z = z(\sigma) \quad , \tag{3.3}$$

or in abbreviated form,

$$\vec{r}(\sigma) = (x(\sigma), y(\sigma), z(\sigma)) \tag{3.4}$$

where σ is a parameter. For studying rays in an isotropic, homogeneous medium, the parameter σ is invariably taken as the arc length of a ray. The positive direction of σ defined by the direction of increasing $s(\vec{r})$, i.e., the direction of wave propagation. Then the unit tangent of a ray is

$$\frac{d\vec{r}}{d\sigma} = \left(\frac{dx}{d\sigma}, \frac{dy}{d\sigma}, \frac{dz}{d\sigma}\right) \tag{3.5}$$

which is equal to ∇s,

$$\frac{d\vec{r}}{d\sigma} = \nabla s \quad . \tag{3.6}$$

Differentiate (3.6) with respect to σ,

$$\frac{d^2\vec{r}}{d\sigma^2} = \frac{d}{d\sigma} \nabla s = \left(\frac{d}{d\sigma}\frac{\partial s}{\partial x}, \frac{d}{d\sigma}\frac{\partial s}{\partial y}, \frac{d}{d\sigma}\frac{\partial s}{\partial z}\right) \quad . \tag{3.7}$$

Let us concentrate on the x-component of (3.7),

$$\begin{aligned} \frac{d}{d\sigma}\frac{dx}{d\sigma} &= \frac{d}{d\sigma}\frac{\partial s}{\partial x} \\ &= \left(\frac{dx}{d\sigma}\frac{\partial}{\partial x} + \frac{dy}{d\sigma}\frac{\partial}{\partial y} + \frac{dz}{d\sigma}\frac{\partial}{\partial z}\right)\frac{\partial s}{\partial x} \\ &= \left(\frac{\partial s}{\partial x}\frac{\partial}{\partial x} + \frac{\partial s}{\partial y}\frac{\partial}{\partial y} + \frac{\partial s}{\partial z}\frac{\partial}{\partial z}\right)\frac{\partial s}{\partial x} \\ &= \frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2 + \left(\frac{\partial s}{\partial z}\right)^2\right] \\ &= \frac{1}{2}\frac{\partial}{\partial x}(\nabla s)^2 \end{aligned} \tag{3.8}$$

where the third identity follows from (3.6). Similar equations to (3.8) can be derived for y and z components of (3.7). Combining these results with (3.1), we obtain the desired ordinary differential equation governing a ray in the free space,

$$\frac{d^2\vec{r}}{d\sigma^2} = 0 \quad . \tag{3.9}$$

The solution of (3.9) is

$$\vec{r}(\sigma) = \sigma\nabla s + \vec{b} \tag{3.10}$$

where ∇s and $\vec{b}$ are constant vectors independent of $\vec{r}$. Equation (3.10) describes a straight line with tangent ∇s. Thus rays in the free space are straight lines. The variation of the phase* along a ray from point $\vec{r}_0$ to point $\vec{r}$ can be calculated in the following manner:

$$s(\vec{r}) - s(\vec{r}_0) = \int_{\vec{r}_0}^{\vec{r}} \frac{ds}{d\sigma}\, d\sigma$$

$$= \int_{\vec{r}_0}^{\vec{r}} \left(\nabla s \cdot \frac{d\vec{r}}{d\sigma}\right) d\sigma$$

$$= \int_{\vec{r}_0}^{\vec{r}} \nabla s \cdot d\vec{r} \quad . \tag{3.11}$$

Using (3.10) in (3.11) leads to

$$s(\vec{r}) = s(\vec{r}_0) + (\sigma - \sigma_0) \quad . \tag{3.12}$$

This relates the phase of an asymptotic solution (2.5) at one point $\vec{r}$ on a ray to that at another point $\vec{r}_0$ on the same ray. Thus, tracing along a ray we can determine the phase at any point once an initial value is given. Since σ is the length of a ray, (3.12) reveals that the distance between a wavefront defined by the value $s(\vec{r})$ and another wavefront by $s(\vec{r}_0)$ is $(\sigma - \sigma_0)$. This distance is the same for all rays; therefore, the wavefronts form a family of parallel surfaces.

Ray and Pencil Coordinates. To label rays, we may introduce two parameters (β,α). Along a given ray (fixed β and α), points on the ray are identified by the arc length σ measured positively in the direction

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*The "phase," according to (2.5), is given by ks. For simplicity, we will refer to s as the "phase" while the common factor k is understood.

of wave propagation and from a reference point $\vec{r}_0$ (where $\sigma = 0$). Thus, (β,α,σ) form ray coordinates and, in general, they are curvilinear coordinates. For example, consider the family of radial rays emanating from a point source at the origin (point $\vec{r}_0$). We may choose the spherical coordinates (θ,ϕ,r) as the ray coordinates such that $(\theta = \beta, \phi = \alpha, r = \sigma)$. A fixed set of (β,α) specifies a ray, while a fixed value of σ defines a wavefront.

Let us concentrate on a typical ray passing through $\vec{r}_0$ and labeled by (β_0,α_0) (Figure 2-1). Our later study will show that the variation of the field amplitudes $\{\vec{e}_m\}$ along ray (β_0,α_0) depends on the geometrical properties of neighboring rays. For this reason, it is convenient to consider a small tube of rays centered around ray (β_0,α_0) called a pencil, rather than a single ray. With respect to this pencil, ray (β_0,α_0) is the axial ray and the others are paraxial rays. To describe the position of a point in a pencil, we introduce a rectangular coordinate system (x,y,z), or alternatively written as (x_1,x_2,z), such that its origin is at $\vec{r}_0$ and its z-axis coincides with the axial ray (β_0,α_0). The coordinates (x_1,x_2,z) are called pencil coordinates. On the axial ray, we note that

$$z = \sigma \qquad \text{(on an axial ray)} \quad . \tag{3.13}$$

Since any ray may be considered as the axial ray of a certain pencil, all the formulas developed later for an axial ray with the variable z are also valid for any ray with the variable σ [compare (4.2) and (4.3)], and vice versa.

Curvature Matrix. The wavefronts of a pencil are parallel surfaces. Let us concentrate on the particular wavefront passing through a reference point $\vec{r}_0$ where $(x_1 = 0, x_2 = 0, z = 0)$ and $(\beta = \beta_0, \alpha = \alpha_0, \sigma = 0)$ (Figure 2-2). In the neighborhood of $\vec{r}_0$, the wavefront may be approximated by a

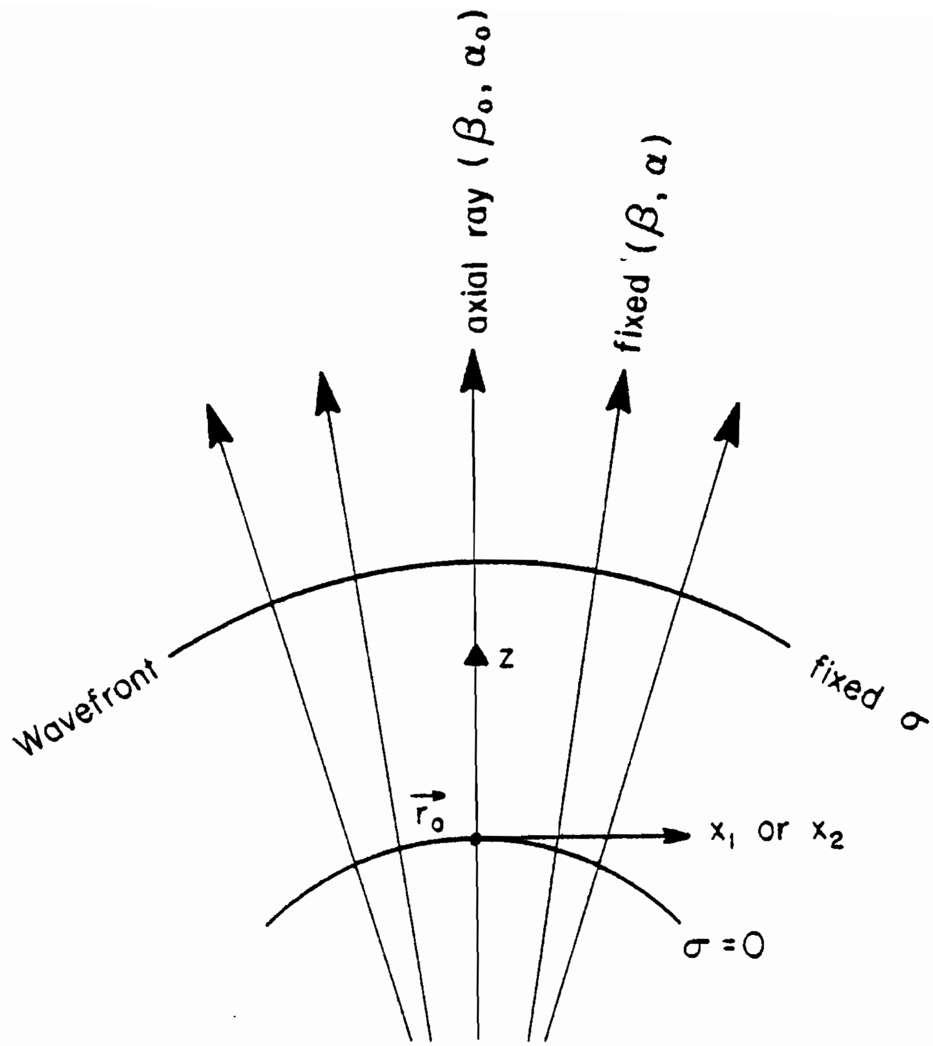

Figure 2-1. Ray coordinates (β,α,σ) and pencil coordinates (x_1,x_2,z).

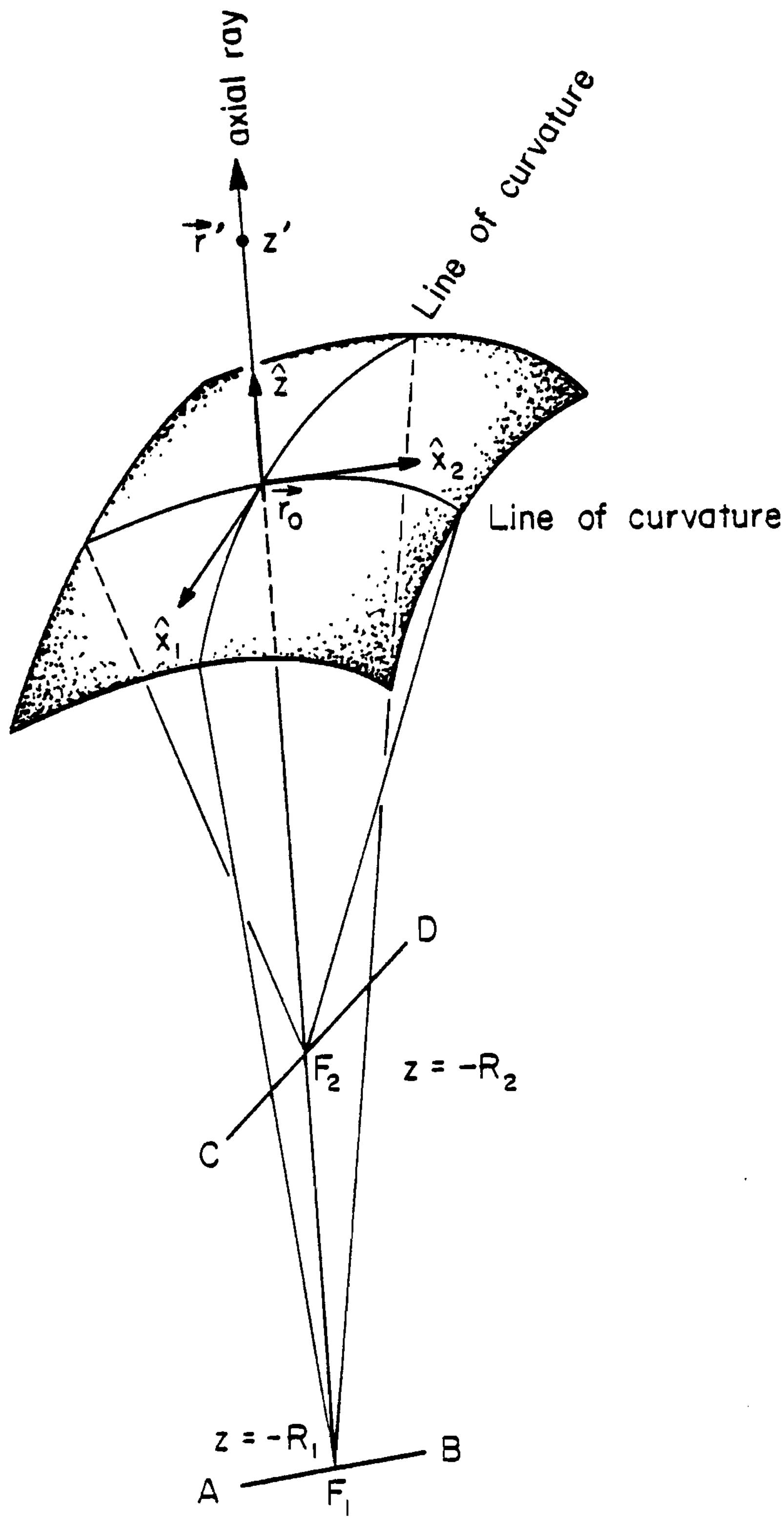

Figure 2-2. Wavefront of a pencil with $(\hat{x}_1, \hat{x}_2)$ in the principal directions. In the sketch, both R_1 and R_2 are positive (diverging pencil).

second-degree surface (Appendix A). Thus, a typical point (x_1, x_2, z) on the wavefront satisfies the following equation

$$\text{Wavefront:} \quad z = -\frac{1}{2}\left(Q_{11}x_1^2 + 2Q_{12}x_1x_2 + Q_{22}x_2^2\right) + O(x_{1,2}^3) \qquad (3.14a)$$

where $O(x_{1,2}^3)$ means that terms of order $x_1^\nu x_2^\mu$ with $\nu + \mu = 3$ and higher have been neglected. In matrix notation, (3.14a) can be rewritten as

$$z = -\frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \bar{\bar{Q}}(z = 0) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + O(x_{12}^3) \qquad (3.14b)$$

where $\bar{\bar{Q}}$ is a symmetrical 2×2 matrix

$$\bar{\bar{Q}}(z = 0) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} . \qquad (3.15)$$

As discussed in Appendix A, if $(\hat{x}_1, \hat{x}_2)$ are the principal directions of the surface (Figure 2-2), $\bar{\bar{Q}}$ is diagonalized such that

$$\bar{\bar{Q}}(z = 0) = \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix} \qquad (3.16)$$

where (R_1, R_2) are the principal radii of curvature of the wavefront passing through $\vec{r}_0$. The sign convention of (R_1, R_2) is chosen to be (Figure 2-3):

$$R = \begin{cases} +|R|, & \text{for a divergent pencil ,} \\ -|R|, & \text{for a convergent pencil .} \end{cases} \qquad (3.17)$$

In (3.17) R stands for either R_1 and R_2, and the terms "divergent" and "convergent" are, of course, referred to rays passing through the respective

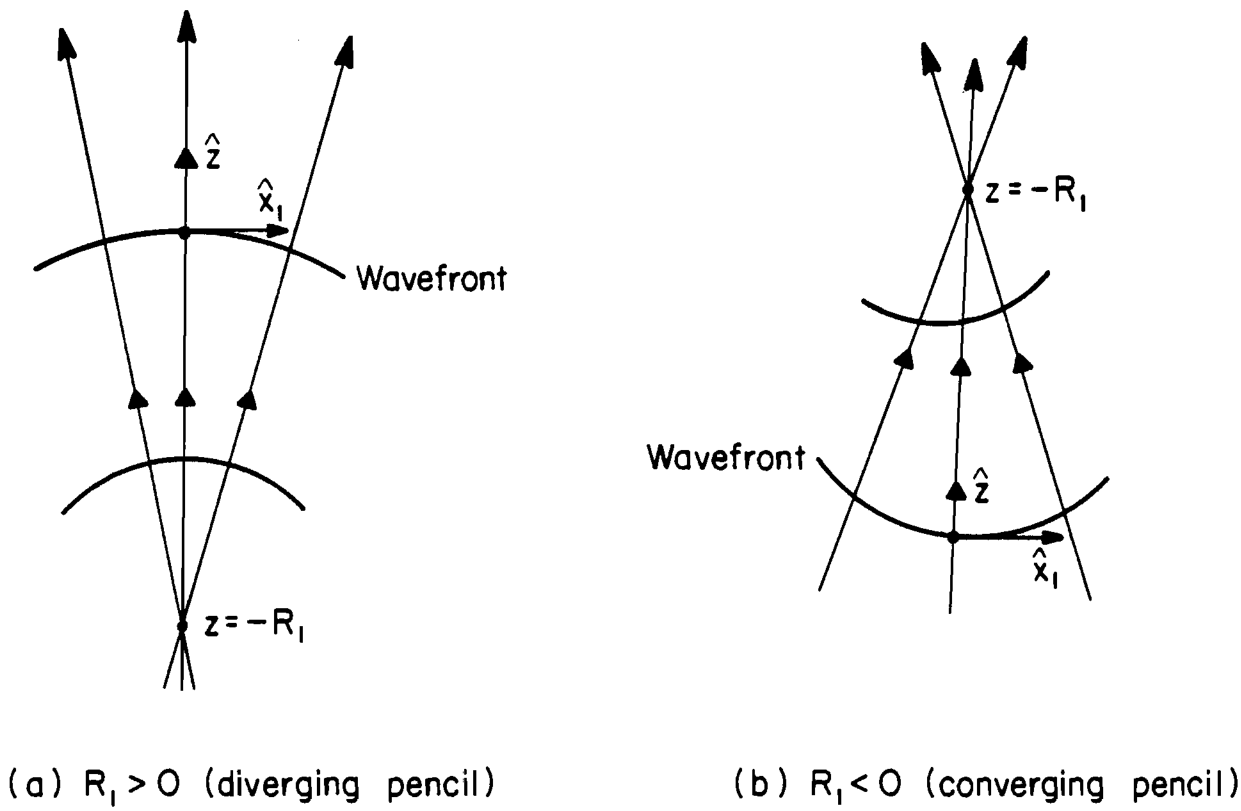

Figure 2-3. Sign convention for principal radii of curvature of a wavefront.

normal section of the pencil. If the base vectors $(\hat{x}_1, \hat{x}_2)$ do not coincide with the principal directions of the wavefront but make an angle ψ with respect to them (Figure 2-4), it is a simple matter to verify that $\bar{\bar{Q}}(z = 0)$ then becomes

$$\bar{\bar{Q}}(z = 0) = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}^T \begin{bmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \tag{3.18a}$$

$$= \begin{bmatrix} \frac{\cos^2\psi}{R_1} + \frac{\sin^2\psi}{R_2} & \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\sin 2\psi \\ \frac{1}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\sin 2\psi & \frac{\sin^2\psi}{R_1} + \frac{\cos^2\psi}{R_2} \end{bmatrix} \tag{3.18b}$$

where T is the transpose operator. Since $\bar{\bar{Q}}(z = 0)$ is determined by the principal curvatures, it is called the curvature matrix of the wavefront passing through $z = 0$, i.e., point $\vec{r}_0$. Two interesting properties of $\bar{\bar{Q}}(z = 0)$ are

$$\det \bar{\bar{Q}}(z = 0) = \frac{1}{R_1 R_2} = \text{Gaussian curvature} \tag{3.19}$$

$$\frac{1}{2} \cdot \text{trace } \bar{\bar{Q}}(z = 0) = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{mean curvature} \quad . \tag{3.20}$$

These two formulas are valid when $\bar{\bar{Q}}(z = 0)$ is given by either (3.16) or (3.18). More discussion on the curvature matrix is given in Appendix A.

According to differential geometry, paraxial rays in the plane spanned by the principal directions $\hat{x}_1$ and $\hat{z}$ intersect at focus F_1 defined by $(x_1 = 0, x_2 = 0, z = -R_1)$ (Figure 2-2), and those in the plane $(\hat{x}_2, \hat{z})$ intersect at focus F_2 defined by $(x_1 = 0, x_2 = 0, z = -R_2)$. Other rays in the pencil

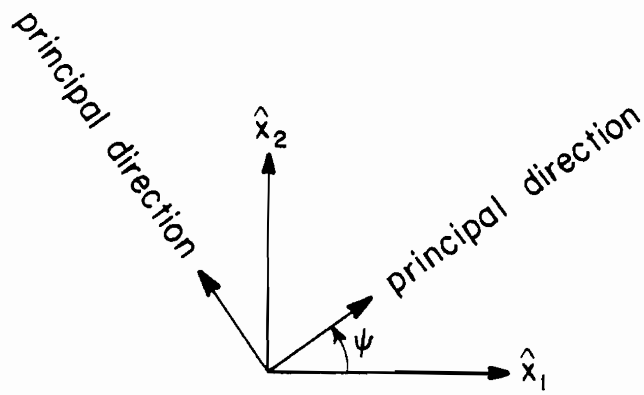

Figure 2-4. Case when $(\hat{x}_1, \hat{x}_2)$ do not coincide with the principal directions of a wavefront.

may be considered to intersect approximately at two focal lines AF_1B, which is parallel to $\hat{x}_2$, and CF_2D, which is parallel to $\hat{x}_1$. This property can be seen better from Figure 2-5. If we do not restrict our consideration to a small tube of rays, the focal lines of a family of wavefronts generate two surfaces known as the caustic surfaces. (In Figure 2-6, only one of them is shown.) Note that all rays are tangent to caustic surfaces. Sometimes a caustic may degenerate to a curve (e.g., in the case of edge diffraction) or a point (e.g., the focus of a paraboloidal reflector).

Phase Variation in a Pencil. Referring to Figure 2-2 let us consider another wavefront passing through $\vec{r}'$ on the axial ray. Since all wavefronts are parallel surfaces, the principal directions $(\hat{x}_1,\hat{x}_2)$ remain unchanged. Corresponding to (3.14) the second-degree surface that describes the wavefront passing through $\vec{r} = \vec{r}'$ or $(x_1 = 0, x_2 = 0, z = z')$ is given by

$$z - z' = -\frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \bar{\bar{Q}}(z = z')\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + O(x_{1,2}^3) \quad . \tag{3.21}$$

When $(\hat{x}_1,\hat{x}_2)$ are the principal directions, it is easily seen from Figure 2-2 that

$$\bar{\bar{Q}}(z) = \begin{bmatrix} \frac{1}{R_1 + z} & 0 \\ 0 & \frac{1}{R_2 + z} \end{bmatrix} \tag{3.22}$$

which is the curvature matrix of the wavefront passing through a general point $(x_1 = 0, x_2 = 0, z)$ on the axial ray. In terms of $\bar{\bar{Q}}(z = 0)$ in (3.16), we may rewrite (3.22) as

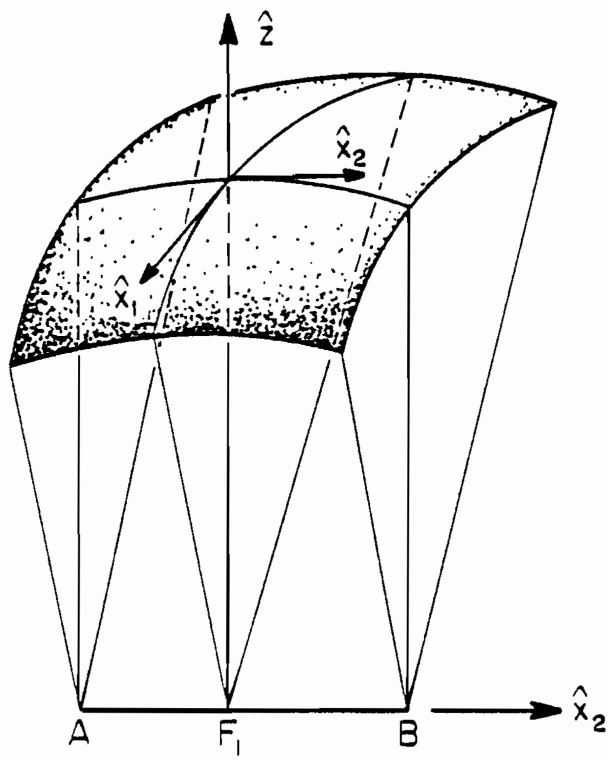

Figure 2-5. All rays in a pencil intersect at the focal line AF_1B.

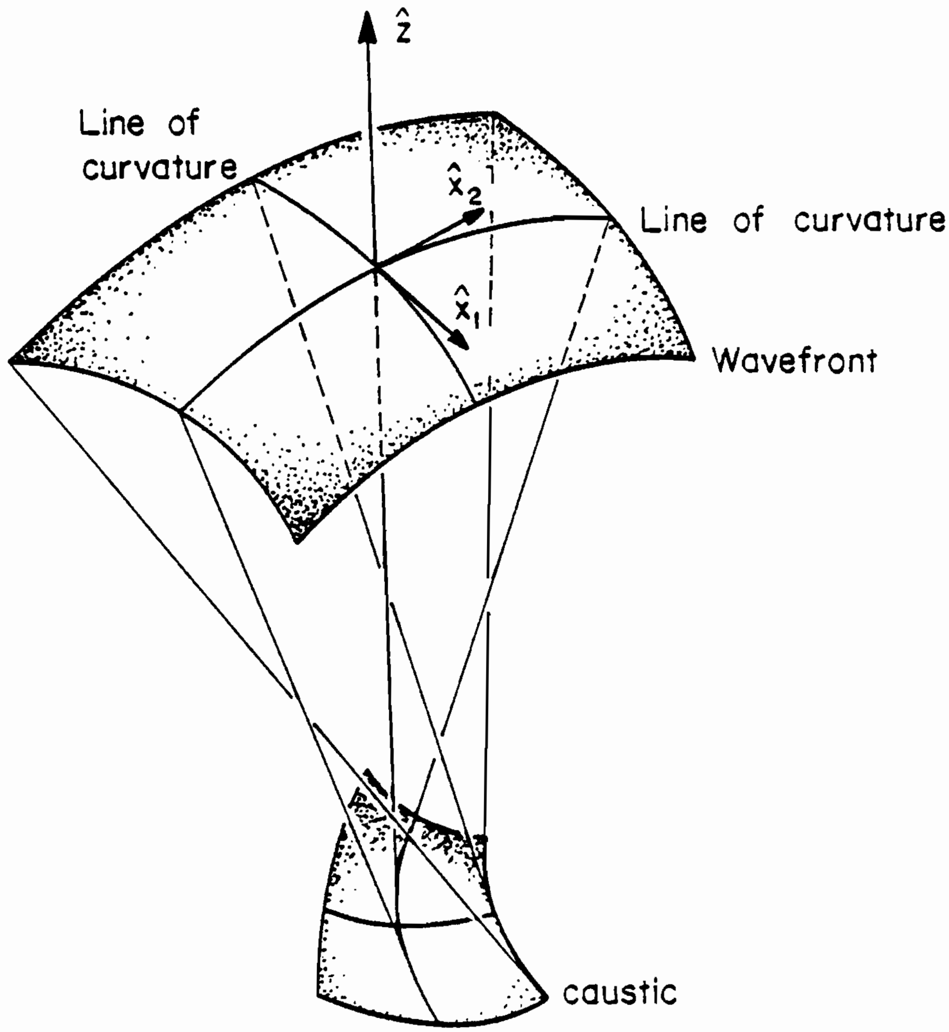

Figure 2-6. One of the two caustic surfaces of a family of wavefronts.

$$[\bar{\bar{Q}}(z)]^{-1} = [\bar{\bar{Q}}(z = 0)]^{-1} + z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.23}$$

It can be shown (Problem 2-5) that the formula in (3.23) is also valid when $(\hat{x}_1, \hat{x}_2)$ are not the principal directions and $\bar{\bar{Q}}(z = 0)$ is given by (3.18). For a given pencil, (3.23) relates the curvature matrix at one point on the axial ray to that of another. Thus, once an initial value is given, the curvature matrix can be continued to any other point along the same ray via (3.23).

The variation of the phase function $s(\vec{r})$ along a fixed ray is given by (3.12) ($\vec{r}$ and $\vec{r}_0$ are on the <u>same</u> ray). Now we will derive a more general formula that governs the variation of $s(\vec{r})$ within a pencil. Suppose an initial value of the phase function at $(x_1 = 0, x_2 = 0, z = 0)$ is known, then phase variation along the axial ray is determined from (3.12), namely,

$$s(0,0,z) = s(0,0,0) + z \quad . \tag{3.24}$$

The question of interest then is to determine $s(x_1, x_2, z)$ when $|x_1|$ and $|x_2|$ are small (on a paraxial ray). Since point $(0,0,z)$ and point c are on the same wavefront (Figure 2-7), the difference between the phase function at $(0,0,z)$ and that at (x_1, x_2, z) is given by ε, the distance from point c to the tangent plane of the wavefront at $(0,0,z)$. From elementary differential geometry ε is found to be (Appendix A):

$$\varepsilon = \frac{1}{2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \bar{\bar{Q}}(z) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + O(x_{1,2}^3) \quad . \tag{3.25}$$

Then the desired formula is

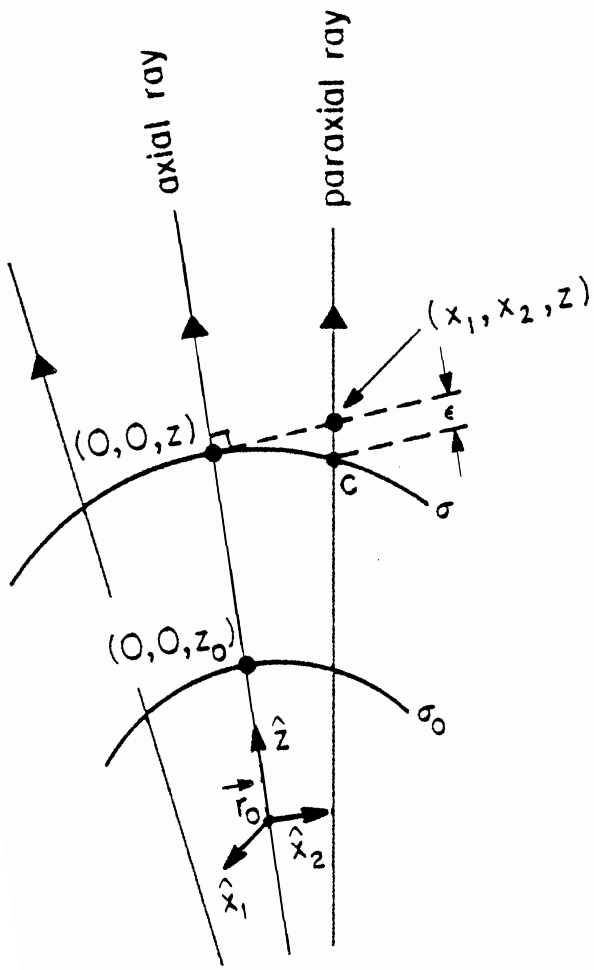

Figure 2-7. Phase of a pencil.

$$s(x_1,x_2,z) = s(0,0,0) + z + \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \bar{\bar{Q}}(z)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + O(x_{1,2}^3) \qquad (3.26)$$

which includes (3.24) as a special case.

In summary, the wavefronts in the free space are parallel surfaces, and rays are straight lines. The phase variation of a pencil is described by the quadratic approximation in (3.26). The curvature matrix $\bar{\bar{Q}}(z)$ is given by (3.22) when $(\hat{x}_1,\hat{x}_2)$ are the principal directions of the wavefront, or by (3.23) and (3.18), when $(\hat{x}_1,\hat{x}_2)$ are not.

2.4 Expansion Ratio and Divergence Factor

As may be seen from Figure 2-7, the cross section of a pencil may expand (diverging pencil) or contract (converging pencil) as it propagates. This fact will later play an important role in the solution of the transport equations for the field amplitudes $\{\vec{e}_m\}$. We will now examine the detailed computation procedures for such a variation of cross section.

Because rays in the free space travel along straight lines, the cross section of a pencil can be expressed simply in terms of the principal curvatures of the wavefront. Referring to Figure 2-8, consider two differential cross sections at $z = 0$, and $z = z_0$, denoted by $da(0)$, and $da(z_0)$, respectively. (We have drawn the cross sections as rectangles and will calculate their areas as so. However, it is not difficult to show that our conclusion is not restricted by this assumption.) From the geometrical construction in Figure 2-8, we have

$$\frac{da(0)}{da(z_0)} = \frac{\widehat{ab}}{\widehat{a'b'}} \cdot \frac{\widehat{ad}}{\widehat{a'd'}} = \frac{R_2}{z_0 + R_2} \cdot \frac{R_1}{z_0 + R_1} = \frac{R_1 R_2}{(R_1 + z_0)(R_2 + z_0)} \qquad (4.1)$$

where (R_1, R_2) are the principal radii of curvature of the wavefront passing through $x_1 = x_2 = z = 0$. Written in a more general form, the expansion ratio of the cross section at z_0 and that at z is

$$\frac{da(z_0)}{da(z)} = \frac{(R_1 + z_0)(R_2 + z_0)}{(R_1 + z)(R_2 + z)} \qquad (4.2)$$

$$= \frac{\det \bar{\bar{Q}}(z)}{\det \bar{\bar{Q}}(z_0)}$$

$$= \frac{\text{Gaussian curvature at } z}{\text{Gaussian curvature at } z_0} .$$

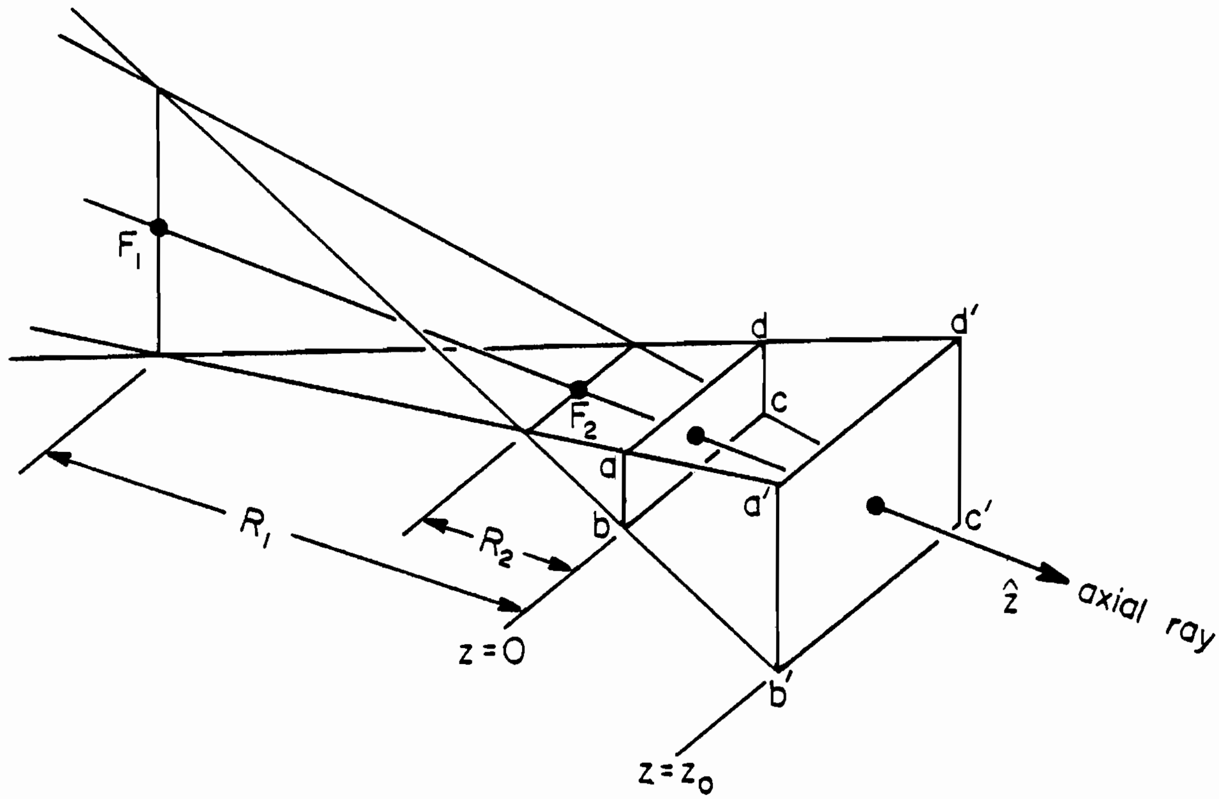

Figure 2-8. Variation of cross section of a pencil. In the sketch, both R_1 and R_2 are positive (diverging pencil).

In (4.2), z may be identified with the arc length along the axial ray measured from the reference point $\vec{r}_0$ (Figure 2-1). Since any ray may be regarded as an axial ray, we can replace z in (4.2) by the arc length σ, namely,

$$\frac{da(\sigma_0)}{da(\sigma)} = \frac{(R_1 + \sigma_0)\ (R_2 + \sigma_0)}{(R_1 + \sigma)\ (R_2 + \sigma)}$$

$$= \frac{\det \bar{\bar{Q}}(\sigma)}{\det \bar{\bar{Q}}(\sigma_0)}$$

$$= \frac{\text{Gaussian curvature at } \sigma}{\text{Gaussian curvature at } \sigma_0} \tag{4.3}$$

where (R_1,R_2) are the principal radii of curvature of the local wavefront passing through $\sigma = 0$.

An alternative formula for the expansion ratio is expressed in terms of the Jacobian of the transformation from the ray coordinates to the pencil coordinates (Section 2.3). This will be discussed below.

The position of a point in a pencil may be described by (rectangular) pencil coordinates $(x_1 = x, x_2 = y, z)$, or by (curvilinear) ray coordinates (β,α,σ), as indicated in Figure 2-1. For a fixed σ, the variation of (β,α) defines a wavefront:

$$\text{Wavefront:}\quad \vec{r}(x,y,z) = (x(\beta,\alpha),y(\beta,\alpha),z(\beta,\alpha)) \quad , \quad \text{with fixed } \sigma \quad . \tag{4.4}$$

By a standard formula (7.8) in Appendix A, the differential area on a wavefront at σ is given by

$$da(\sigma) = \left|\frac{\partial\vec{r}}{\partial\beta} \times \frac{\partial\vec{r}}{\partial\alpha}\right|\ d\beta\ d\alpha \quad .$$

On the other hand, for a fixed (β,α), the variation of σ defines a ray:

$$\text{Ray:} \quad \vec{r}(x,y,z) = (x(\sigma),y(\sigma),z(\sigma)) \quad , \qquad \text{with fixed } (\beta,\alpha) \tag{4.5}$$

whose unit tangent is given by

$$\hat{t} = \frac{d\vec{r}}{d\sigma} = \frac{\partial}{\partial\sigma}\,\vec{r}(x,y,z) \quad . \tag{4.6}$$

Since the vector

$$\frac{\partial\vec{r}}{\partial\beta} \times \frac{\partial\vec{r}}{\partial\alpha}$$

is normal to the wavefront and, therefore, parallel to $\hat{t}$, the differential surface in (4.4) can be rewritten as

$$da(\sigma) = \left|\frac{\partial\vec{r}}{\partial\beta} \times \frac{\partial\vec{r}}{\partial\alpha} \cdot \frac{\partial\vec{r}}{\partial\sigma}\right| \, d\beta \; d\alpha \quad . \tag{4.7}$$

The factor in the absolute value sign is identified as the Jacobian of the transform from the ray coordinates (β,α,σ) to the pencil coordinates (x,y,z), namely,

$$j(\beta,\alpha,\sigma) = \frac{\partial(x,y,z)}{\partial(\beta,\alpha,\sigma)} = \frac{\partial\vec{r}}{\partial\beta} \times \frac{\partial\vec{r}}{\partial\alpha} \cdot \frac{\partial\vec{r}}{\partial\sigma} \tag{4.8}$$

$$= \det \begin{bmatrix} \frac{\partial x}{\partial\beta} & \frac{\partial x}{\partial\alpha} & \frac{\partial x}{\partial\sigma} \\ \frac{\partial y}{\partial\beta} & \frac{\partial y}{\partial\alpha} & \frac{\partial y}{\partial\sigma} \\ \frac{\partial z}{\partial\beta} & \frac{\partial z}{\partial\alpha} & \frac{\partial z}{\partial\sigma} \end{bmatrix} .$$

Usually, we are interested in the value of the Jacobian along a given ray (fixed β and α). Then, $j(\beta,\alpha,\sigma)$ in abbreviated form is written as $j(\sigma)$. Making use of (4.7) and (4.8) in (4.3) we have a new formula for the expansion ratio

$$\frac{da(\sigma_0)}{da(\sigma)} = \frac{(R_1 + \sigma_0)(R_2 + \sigma_0)}{(R_1 + \sigma)(R_2 + \sigma)} = \frac{\det \bar{\bar{Q}}(\sigma)}{\det \bar{\bar{Q}}(\sigma_0)} = \frac{j(\sigma_0)}{j(\sigma)} \quad . \tag{4.9}$$

Thus, at a given point $\vec{r} = (x,y,z)$ or (β,α,σ) on a ray, (4.8) and (4.9) may be used as an alternative way to calculate the expansion ratio.

Along a given ray, the divergence factor DF from a reference point σ_0 to a general point σ is defined by

$$DF = \left(\frac{da(\sigma_0)}{da(\sigma)}\right)^{1/2} = \left(\frac{j(\sigma_0)}{j(\sigma)}\right)^{1/2} \tag{4.10a}$$

which is equal to the square root of the expansion ratio. With the help of (4.9), we find

$$DF = \frac{1}{\sqrt{1 + [(\sigma - \sigma_0)/(R_1 + \sigma_0)]}\ \sqrt{1 + [(\sigma - \sigma_0)/(R_2 + \sigma_0)]}} \ . \tag{4.10b}$$

The factors in (4.10b) have the following meaning: $(R_1 + \sigma_0)$ and $(R_2 + \sigma_0)$ are the two radii of curvature of the wavefront passing through the reference point σ_0; and $(\sigma - \sigma_0)$ is the signed arclength along the ray measured from σ_0. In the remainder of the section, let us choose $\sigma_0 = 0$ (without loss of generality) and study

$$DF = \frac{1}{\sqrt{1 + (\sigma/R_1)}\ \sqrt{1 + (\sigma/R_2)}} \ . \tag{4.11}$$

The proper value of the square root function in (4.11) must be defined. As discussed in the next section, DF describes the variation of field amplitudes $\{\vec{e}_m\}$ along a ray. Guided by rigorous solutions of some canonical problems. it is found that the proper value of the square root function in (4.11) must be chosen such that

$$f = \sqrt{1 + \frac{\sigma}{R_n}} = \begin{cases} +|f| \ , & \text{if } f \text{ is real} \qquad (4.12a) \\ +i|f| \ , & \text{if } f \text{ is imaginary and } \sigma > 0 \qquad (4.12b) \\ -i|f| \ , & \text{if } f \text{ is imaginary and } \sigma < 0 \qquad (4.12c) \end{cases}$$

where $n = 1$ or 2. Note that, with respect to the wave propagation, $\sigma > 0$

corresponds to a point in the forward direction of the reference point $\sigma = 0$; whereas $\sigma < 0$ corresponds to a point in the backward direction. Because of the choice in (4.12), we conclude that, along the direction of wave propagation, DF has a phase jump $\exp(-i\pi/2)$ whenever a focus is crossed. If the number of foci between $\sigma = 0$ and a positive σ is M (Morse index), it is readily verified that

$$\mathrm{DF} = \begin{cases} +|\mathrm{DF}| & , \text{ if } M = 0 \\ -i|\mathrm{DF}| & , \text{ if } M = 1 \\ -|\mathrm{DF}| & , \text{ if } M = 2 \end{cases} \quad . \tag{4.13}$$

Consider the divergent cylindrical pencil in Figure 2-9a as an example. At the reference point O_1 where $\sigma_1 = 0$, the two radii of curvature of the wavefront are $R_1 = +5$ and $R_2 = \infty$. We measure σ_1 positively from O_1 in the direction of wave propagation. Then the divergence factor relative to O_1 is given by

$$\mathrm{DF} = \frac{1}{\sqrt{1 + (\sigma_1/5)}} \quad , \qquad \text{for } -\infty < \sigma_1 < \infty \quad . \tag{4.14}$$

According to the convention in (4.12), DF in (4.14) is positive real for $(-5) < \sigma_1$, and positive imaginary for $\sigma_1 < (-5)$. At the focus where $\sigma_1 = -5$, DF in (4.14) becomes infinite, as the cylindrical wavefront degenerates into a line. For the same pencil, we next choose a different reference point at O_2 where $\sigma_2 = 0$ (Figure 2-9b). The cylindrical wavefront that passes through O_2 is convergent, with $R_1 = -7$ and $R_2 = \infty$. Then the divergent factor relative to O_2 is

$$\mathrm{DF} = \frac{1}{\sqrt{1 - (\sigma_2/7)}} \quad , \qquad \text{for } -\infty < \sigma_2 < \infty \tag{4.15}$$

which is positive real for $\sigma_2 < 7$, and negative imaginary for $7 < \sigma_2$.

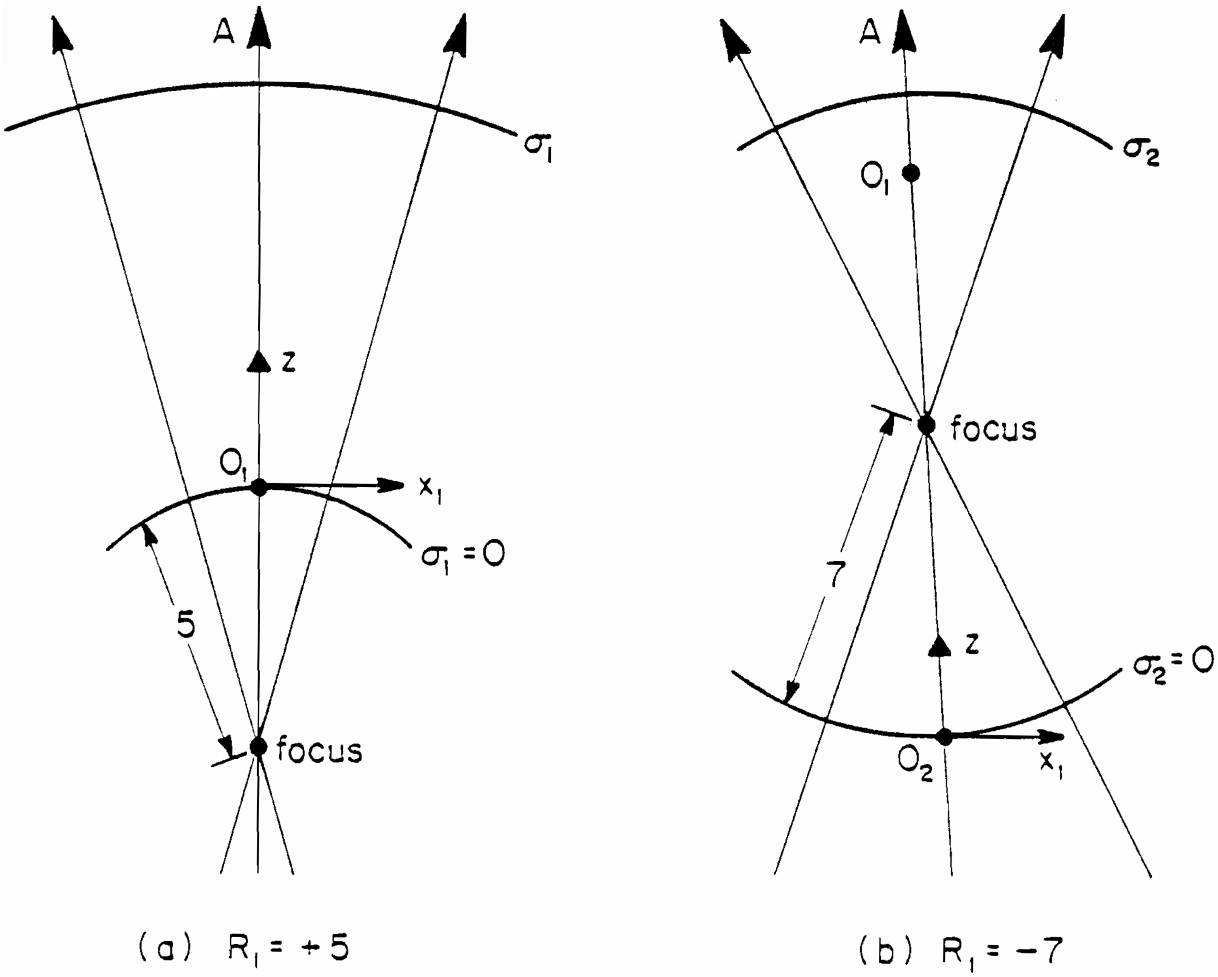

Figure 2-9. Examples for calculating divergence factor of a cylindrical pencil.

2.5 Continuation of Field Amplitudes

We now turn to the solution of the transport equations in (2.8), which is repeated below:

$$2(\nabla s \cdot \nabla)\ \vec{e}_m + \nabla^2 s\ \vec{e}_m = -\nabla^2\ \vec{e}_{m-1}, \qquad m = 0,1,2,\ldots;\vec{e}_{-1} = 0 \quad . \tag{5.1}$$

Here $\vec{e}_m(\vec{r})$ is a function of position, e.g., a function of ray coordinates (β,α,σ). We will now fix $(\beta = \beta_0, \alpha = \alpha_0)$ and study the variation of $\vec{e}_m(\vec{r})$ with respect to σ. If one makes use of the identity

$$\nabla s \cdot \nabla = \frac{d\vec{r}}{d\sigma} \cdot \nabla = \frac{\partial x}{\partial \sigma}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \sigma}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \sigma}\frac{\partial}{\partial z} = \frac{d}{d\sigma} \tag{5.2}$$

(5.1) becomes

$$2\frac{d}{d\sigma}\vec{e}_m(\sigma) + \nabla^2 s\ \vec{e}_m(\sigma) = -\nabla^2\ \vec{e}_{m-1}(\sigma) \quad , \qquad m = 0,1,2,\ldots \quad . \tag{5.3}$$

Thus, along a given ray, the partial differential equation in (5.1) is simplified to an ordinary differential equation (5.3). We emphasize that the symbols $\vec{e}_m(\sigma)$ and $\nabla^2\vec{e}_m(\sigma)$ in (5.3) have the meaning

$$\vec{e}_m(\sigma) = \vec{e}_m(\beta_0,\alpha_0,\sigma) \tag{5.4a}$$

$$\nabla^2\vec{e}_m(\sigma) = \nabla^2\vec{e}_m(\beta,\alpha,\sigma)\Big|_{\beta=\beta_0,\alpha=\alpha_0} \quad . \tag{5.4b}$$

The solution of (5.3) can be derived by a standard method in differential equations (Problem 2-6) and is given by

$$\vec{e}_m(\sigma) = \vec{e}_m(\sigma_0)\kappa(\sigma,\sigma_0) - \frac{1}{2}\int_{\sigma_0}^{\sigma}\frac{\kappa(\sigma,\sigma_0)}{\kappa(\sigma',\sigma_0)}\nabla^2\ \vec{e}_{m-1}(\sigma')\ d\sigma' \quad , \qquad m = 0,1,2,\ldots \tag{5.5a}$$

where $\kappa(\sigma,\sigma_0)$ is defined by

$$\kappa(\sigma,\sigma_0) = \exp\left[-\frac{1}{2}\int_{\sigma_0}^{\sigma} \nabla^2 s(\sigma')\, d\sigma'\right]. \tag{5.5b}$$

To calculate the integrand in (5.5b), we use (3.26) and (3.22):

$$s(x_1,x_2,z) = s(0,0,0) + z + \frac{1}{2}\left(\frac{x_1^2}{R_1 + z} + \frac{x_2^2}{R_2 + z}\right) + O(x_{1,2}^3) \tag{5.6}$$

$$\nabla^2 s(\sigma) = \nabla^2 s(x_1,x_2,z)\Big|_{x_1=x_2=0,\, z=\sigma}$$

$$= \frac{1}{R_1 + \sigma} + \frac{1}{R_2 + \sigma} \tag{5.7}$$

where (R_1,R_2) are the principal radii of curvature of the local wavefront passing through $\sigma = 0$. The use of (5.7) in (5.5b) leads immediately to

$$\kappa(\sigma,\sigma_0) = \left[\frac{(R_1 + \sigma_0)(R_2 + \sigma_0)}{(R_1 + \sigma)(R_2 + \sigma)}\right]^{1/2} = \left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} = DF \quad . \tag{5.8}$$

The divergence factor DF was studied at the end of Section 2-4. Then the solution for the transport equation in (5.5) can be written as

$$\vec{e}_0(\sigma) = \vec{e}_0(\sigma_0)\left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} = \vec{e}_0(\sigma_0)(DF) \tag{5.9}$$

$$\vec{e}_m(\sigma) = \vec{e}_m(\sigma_0)\left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} - \frac{1}{2}\int_{\sigma_0}^{\sigma}\left[\frac{j(\sigma')}{j(\sigma)}\right]^{1/2} \nabla^2\, \vec{e}_{m-1}(\sigma')\, d\sigma' \ , \ m = 1,2,... \tag{5.10}$$

The above results can be also derived by an alternative way using the Jacobian ratio directly (Problems 2-7 and 2-8).

Consider the zeroth order solution $\vec{e}_0(\sigma)$. Once an initial value is known at one point σ_0 on a ray, it is determined along the entire ray by (5.9). The direction of $\vec{e}_0(\sigma)$ is maintained constant along a given ray. Using (4.9), we have an interesting interpretation of (5.9), namely,

$$\vec{e}_0(\sigma) = \vec{e}_0(\sigma_0) \left(\frac{da(\sigma_0)}{da(\sigma)} \right)^{1/2} . \tag{5.11}$$

Thus, the magnitude of $\vec{e}_0(\sigma)$ at a point on a ray is inversely proportional to the square root of the cross section of a pencil centered around that ray. This is a well-known fact in optics. An alternative way for deriving this conclusion is given in Problems 2-9 and 2-10.

The solutions for the higher-order $\{\vec{e}_m\}$, $m = 1,2,\ldots$, given in (5.10), are not as simple. In addition to the initial value $\vec{e}_m(\sigma_0)$ on a ray, it is necessary to know $\nabla^2 \vec{e}_{m-1}(\sigma')$ for all σ' in the range $\sigma_0 < \sigma' < \sigma$ before the value of $\vec{e}_m(\sigma)$ can be determined. The simple geometrical interpretation in (5.11) no longer applies to higher-order $\{\vec{e}_m\}$.

From (5.9), (5.10) and (4.9), we note that $\{\vec{e}_m(\sigma)\}$ become infinite whenever $\sigma = -R_1$ or $\sigma = -R_2$. These points lie on caustic surfaces (Figures 2-2 and 2-6). The ray techniques discussed in this book, in general, cannot predict the field correctly in the neighborhood of caustic surfaces* and, therefore, we have to avoid these points in using (5.9) and (5.10).

Another complication in association with a caustic point $\sigma_c = -R_1$ or $-R_2$ is that, whenever $\sigma_0 < \sigma_c < \sigma$, the integral in (5.10) may diverge. Thus, it is necessary to deduce an alternative representation which remains valid when a caustic point appears on the path of integration†. For this purpose, let us introduce the definition of the "finite part" of a divergent integral.

*One exception is when the caustic is the edge of a screen. This case will be studied in Chapter 5.

†R. M. Lewis and J. Boersma, "Uniform asymptotic theory of edge diffraction," J. Math. Phys. 10, 2291-2305, 1969.

Assume $f(\varepsilon)$ has an asymptotic expansion in powers of ε as $\varepsilon \to 0$

$$f(\varepsilon) = \int_{\varepsilon}^{b} g(x)\, dx \sim a_0 + \sum_n a_n \varepsilon^{\alpha_n} \quad , \quad \varepsilon \to 0 \tag{5.12}$$

where α_n's are nonzero real numbers (positive or negative; integer or fractional). Then, the finite part of the integral is defined by

$$⨍_0^b g(x)\, dx = \text{finite part of} \int_0^b g(x)\, dx = a_0 \tag{5.13}$$

where the slash on the integral sign denotes this special operation. As may be seen from (5.12), this definition is simply equivalent to that we evaluate the result of the integration at the upper limit $x = b$ only and ignore the lower limit $x = 0$. Now return to (5.10). Without loss of generality, let us choose the origin of σ at the caustic point, $\sigma_c = 0$, and rewrite the integral in (5.10):

$$\int_{\sigma_0}^{\sigma} = ⨍_0^{\sigma} - ⨍_0^{\sigma_0} \quad . \tag{5.14}$$

Then (5.10) becomes

$$[j(\sigma)]^{1/2}\, \vec{e}_m(\sigma) + \frac{1}{2} ⨍_0^{\sigma} = [j(\sigma_0)]^{1/2}\, \vec{e}_m(\sigma_0) + \frac{1}{2} ⨍_0^{\sigma_0} \quad . \tag{5.15}$$

The right-hand side of (5.15) is independent of σ. If we denote its value by $\vec{\delta}'_m$, then we obtain

$$\vec{e}_m(\sigma) = \frac{\vec{\delta}'_m}{[j(\sigma)]^{1/2}} - \frac{1}{2} ⨍_0^{\sigma} \left\{ \frac{j(\sigma')}{j(\sigma)} \right\}^{1/2} \nabla^2 \vec{e}_{m-1}(\sigma')\, d\sigma' \quad , \quad m = 0,1,2,\ldots \quad . \tag{5.16}$$

This is a modified version of (5.9) and (5.10), and it is valid even when

$\sigma = 0$ is a caustic point. Setting $\sigma = 0$ in (5.16), we have

$$\vec{\delta}'_m = \lim_{\sigma \to 0} \vec{e}_m(\sigma)[j(\sigma)]^{1/2} \quad . \tag{5.17}$$

At the caustic point $\sigma = 0$, $j(\sigma = 0)$ goes to zero but $\vec{e}_m(\sigma = 0)$ goes to infinity in such a way that $\vec{\delta}'_m$ is finite. This may be considered an interpretation of the initial values $\{\vec{\delta}'_m\}$.

In summary, along a given ray (fixed β and α), the continuation of the field amplitudes $\{\vec{e}_m\}$ as a function of σ is governed by (5.9) and (5.10). If the reference point $\sigma_0 = 0$ is a caustic point, the latter equations should be replaced by (5.16).

2.6. Condition Imposed by Gauss' Law

The condition imposed by Gauss' law is given in (2.9), which reads

$$\nabla s \cdot \vec{e}_m(\sigma) = -\nabla \cdot \vec{e}_{m-1}(\sigma) \quad , \quad m = 0,1,2,\ldots;\vec{e}_{-1} = 0 \quad . \tag{6.1}$$

It has an interesting property which is stated as follows*: If (i) s and $\{\vec{e}_m\}$ satisfy the eikonal equation (2.7) and the transport equation (2.8), and (ii) the relation in (6.1) is satisfied at one point $\sigma = \sigma_0$ on a ray, then (6.1) is satisfied at all other points on the same ray. This property will be found useful later on. We will now establish it by induction. Starting with the m = 0 case, (6.1) becomes

$$\nabla s \cdot \vec{e}_0(\sigma) = 0 \quad . \tag{6.2}$$

Because of (5.9), it is obvious that if (6.2) holds for one point $\sigma = \sigma_0$ on a ray, it holds for any other point on the same ray. The relation in (6.2) indicates that the zeroth-order solution $\vec{e}_0$ always lies in a plane perpendicular to the direction of propagation and, therefore, represents a transverse wave. Next, given the conditions that

$$\text{(i)} \quad \nabla s \cdot \vec{e}_{m-1}(\sigma) = -\nabla \cdot \vec{e}_{m-2}(\sigma) \quad , \text{ and} \tag{6.3}$$

$$\text{(ii)} \quad \nabla s \cdot \vec{e}_m(\sigma_0) = -\nabla \cdot \vec{e}_{m-1}(\sigma_0) \tag{6.4}$$

we would like to show that

$$\nabla s \cdot \vec{e}_m(\sigma) = -\nabla \cdot \vec{e}_{m-1}(\sigma) \tag{6.5}$$

*J. Boersma and P. H. M. Kersten, "Uniform asymptotic theory of electromagnetic diffraction by a plane screen," Technical Report, Department of Mathematics, Tech. University of Eindhoven, Eindhoven, Netherlands (in Dutch), 1967.

for all σ. To this end, let us multiply (6.5) by $[j(\sigma)]^{1/2}$ and differentiate the resultant equation with respect to σ. Then, the right-hand side becomes

$$F_1 = \frac{d}{d\sigma}\{[j(\sigma)]^{1/2}(-1)\ \nabla \cdot \vec{e}_{m-1}(\sigma)\} \tag{6.6}$$

$$= -\frac{1}{2}\frac{j'(\sigma)}{[j(\sigma)]^{1/2}}\nabla \cdot \vec{e}_{m-1} - [j(\sigma)]^{1/2}\frac{d}{d\sigma}(\nabla \cdot \vec{e}_{m-1})$$

$$= -\frac{1}{2}[j(\sigma)]^{1/2}[\nabla^2 s(\nabla \cdot \vec{e}_{m-1}) + 2(\nabla s \cdot \nabla)(\nabla \cdot \vec{e}_{m-1})]$$

where the last identity follows from (5.3) and the relation (Problem 2-11)

$$\nabla^2 s = \frac{j'(\sigma)}{j(\sigma)} = \frac{1}{j(\sigma)}\frac{d}{d\sigma}j(\sigma) \quad ; \tag{6.7}$$

while the left-hand side becomes

$$F_2 = \frac{d}{d\sigma}\{[j(\sigma)]^{1/2}\ \nabla s \cdot \vec{e}_m(\sigma)\} \tag{6.8}$$

$$= [j(\sigma)]^{1/2}\ \vec{e}_m \cdot \frac{d}{d\sigma}\nabla s + \nabla s \cdot \frac{d}{d\sigma}[j(\sigma)]^{1/2}\ \vec{e}_m$$

$$= 0 - \frac{1}{2}[j(\sigma)]^{1/2}[\nabla s \cdot \nabla^2\ \vec{e}_{m-1}]$$

$$= -\frac{1}{2}[j(\sigma)]^{1/2}[\nabla^2 s(\nabla \cdot \vec{e}_{m-1}) + 2(\nabla s \cdot \nabla)(\nabla \cdot \vec{e}_{m-1})]$$

where the third identity follows from (3.28) and (5.10), and the fourth identity is established in Problem 2-12. Compare F_1 in (6.6) and F_2 in (6.8). We note that they are equal, or

$$\frac{d}{d\sigma}\{[j(\sigma)]^{1/2}[\nabla s \cdot \vec{e}_m(\sigma) + \nabla \cdot \vec{e}_{m-1}(\sigma)]\} = 0$$

or

$$[j(\sigma)]^{1/2}[\nabla s \cdot \vec{e}_m(\sigma) + \nabla \cdot \vec{e}_{m-1}(\sigma)] = \text{a constant independent of } \sigma \quad . \qquad (6.9)$$

Because of (6.4), the constant in (6.9) is zero. Thus, we have established the desired relation in (6.5), and completed the proof by induction.

2.7. Summary

(1) In this book we are concerned with the high-frequency asymptotic solutions of electromagnetic edge diffraction in the free space. The total field solution in a given problem often can be conveniently identified as a superposition of several partial fields, such as the incident field, the geometrical optics field, and the diffraction field. For each partial field, we conjecture that it can be represented (at least in some regions) by a formal asymptotic series:

$$\vec{E}(\vec{r}) \sim k^{\tau}\, e^{iks(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m}\, \vec{e}_m(\vec{r}) \quad , \quad k \to \infty \tag{7.1a}$$

$$\vec{H}(\vec{r}) \sim k^{\tau}\, e^{iks(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m}\, \vec{h}_m(\vec{r}) \quad , \quad k \to \infty \tag{7.1b}$$

which is called an asymptotic solution. There is no general proof that the asymptotic solution always agrees with the asymptotic expansion of the exact solution, even though there exist many affirmative examples. In order for (7.1) to satisfy the source-free Maxwell equations, it is subject to the following constraints:

$$(\nabla s)^2 = 1 \qquad \text{(eikonal equation)} \tag{7.2}$$

$$2(\nabla s \cdot \nabla)\, \vec{e}_m + \nabla^2 s\, \vec{e}_m = -\nabla^2\, \vec{e}_{m-1} \qquad \text{(transport equations)} \tag{7.3}$$

$$\nabla s \cdot \vec{e}_m = -\nabla \cdot \vec{e}_{m-1} \qquad \text{(Gauss' law)} \tag{7.4}$$

$$\vec{h}_m = \sqrt{\frac{\varepsilon}{\mu}}\, [\nabla s \times \vec{e}_m + \nabla \times \vec{e}_{m-1}] \tag{7.5}$$

where $m = 0,1,2,3,\ldots$, and $\vec{e}_{-1} = 0$. In addition, (7.1) has to satisfy the source boundary and edge conditions in a given problem, which have yet to be enforced.

(2) Ray and pencils: Rays in the free space are a family of straight lines described by two parameters (β,α). Along a given ray, its arc length is denoted by σ which is measured positively in the direction of increasing s (direction of wave propagation). The curvilinear coordinates (β,α,σ) form ray coordinates. A pencil is a small tube of rays (an axial ray and surrounding paraxial rays). To describe a point in a pencil, we use the (rectangular) pencil coordinates (x_1,x_2,z), whose base vectors $(\hat{x}_1,\hat{x}_2,\hat{z})$ are right-handed and orthonormal. The axis $\hat{z}$ coincides with the axial ray, and $(\hat{x}_1,\hat{x}_2)$ may or may not coincide with the principal directions of the wavefront of the pencil.

(3) Phase of a pencil: At any point $\vec{r} = (x_1,x_2,z)$ within a pencil (Figure 2-7), the phase function $s(\vec{r})$ can be approximated by a second-degree equation:

$$s(x_1,x_2,z) = s(0,0,0) + z + \frac{1}{2}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \bar{\bar{Q}}(z) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0(x^3_{1,2}) \quad . \tag{7.6}$$

When $(\hat{x}_1,\hat{x}_2)$ are the principal directions, we have

$$\bar{\bar{Q}}(z = 0) = \begin{pmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{pmatrix} \quad . \tag{7.7}$$

When $(\hat{x}_1,\hat{x}_2)$ make an angle ψ with respect to the principal directions (Figure 2-4), we have

$$\bar{\bar{Q}}(z = 0) = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}^T \begin{pmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} . \tag{7.8}$$

In either case, the continuation of $\bar{\bar{Q}}(z)$ is described by

$$[\bar{\bar{Q}}(z)]^{-1} = [\bar{\bar{Q}}(0)]^{-1} + z \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{7.9}$$

The sign convention of (R_1,R_2) is that they are positive (negative) for diverging (converging) normal sections of the wavefront (Figure 2-3).

(4) Divergence factor: Along a given ray, DF at a general point σ with respect to a reference point σ_0 is defined by

$$\begin{aligned} DF &= \left(\frac{j(\sigma_0)}{j(\sigma)} \right)^{1/2} = [\text{Jacobian ratio}]^{1/2} \\ &= \left(\frac{da(\sigma_0)}{da(\sigma)} \right)^{1/2} = [\text{pencil cross-section ratio}]^{1/2} \\ &= \left(\frac{\det \bar{\bar{Q}}(\sigma)}{\det \bar{\bar{Q}}(\sigma_0)} \right)^{1/2} = [\text{Gaussian curvature ratio}]^{1/2} \\ &= \frac{1}{\sqrt{1 + \left(\frac{\sigma - \sigma_0}{R_1 + \sigma_0}\right)} \sqrt{1 + \left(\frac{\sigma - \sigma_0}{R_2 + \sigma_0}\right)}} \end{aligned} \tag{7.10}$$

where (R_1,R_2) are the radii of curvature of the wavefront passing through the origin $\sigma = 0$. The square root function in the last line of (7.10) is defined such that

$$f = \frac{1}{\sqrt{1 + \left(\frac{\sigma - \sigma_0}{R_n + \sigma_0}\right)}} = \begin{cases} +\,|f| \quad , & \text{if } f \text{ is real} \\ -i\,|f| \quad , & \text{if } f \text{ is imaginary and } (\sigma - \sigma_0) > 0 \\ +i\,|f| \quad , & \text{if } f \text{ is imaginary and } (\sigma - \sigma_0) < 0 \end{cases} \tag{7.11}$$

(5) Zeroth-order solution: Along a given ray, we have

$$\vec{e}_0(\sigma) = \vec{e}_0(\sigma_0) \left(\frac{j(\sigma_0)}{j(\sigma)} \right)^{1/2} = \vec{e}_0(\sigma_0)(DF) \tag{7.12}$$

$$\nabla s \cdot \vec{e}_0(\sigma) = 0 \tag{7.13}$$

$$\vec{h}_0(\sigma) = \sqrt{\frac{\varepsilon}{\mu}}\, \nabla s \times \vec{e}_0(\sigma) \tag{7.14}$$

which indicates that the zeroth-order solution is locally a plane wave.

(6) Higher-order solution: Along a given ray, the propagation of the amplitude vectors is governed by the relation

$$\vec{e}_m(\sigma) = \vec{e}_m(\sigma_0) \left(\frac{j(\sigma_0)}{j(\sigma)} \right)^{1/2} - \frac{1}{2} \int_{\sigma_0}^{\sigma} \left(\frac{j(\sigma')}{j(\sigma)} \right)^{1/2} \nabla^2 \vec{e}_{m-1}(\sigma')\, d\sigma',$$

$$m = 0,1,2,\ldots \quad . \tag{7.15}$$

(7) If $\sigma_0 = 0$ is a caustic point, alternative formulas for (7.12) and (7.15) are

$$\vec{e}_m(\sigma) = \frac{\vec{\delta}'_m}{[j(\sigma)]^{1/2}} - \frac{1}{2} ⨍_0^{\sigma} \left(\frac{j(\sigma')}{j(\sigma)} \right)^{1/2} \nabla^2 \vec{e}_{m-1}(\sigma') \, d\sigma' \quad ,$$

$$m = 0,1,2, \ldots \tag{7.16}$$

where the slash on the integral sign indicates the "finite part" of a (possibly divergent) integral.

(8) A useful property of (7.4) is that if it is satisfied at one point on a ray, it is satisfied at all other points along the same ray.

2.8. Propagation of Cylindrical Waves

Consider a field represented by the asymptotic series

$$\vec{E}(\vec{r}) \sim e^{iks(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m} \vec{e}_m(\vec{r}) \quad , \quad k \to \infty \quad . \tag{8.1}$$

The continuation of the amplitude vectors $\{e_m\}$ along a given ray is governed by the relation in (7.15) or

$$\vec{e}_m(\sigma) = \vec{e}_m(\sigma_0) \left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} - \frac{1}{2} \int_{\sigma_0}^{\sigma} \left[\frac{j(\sigma')}{j(\sigma)}\right]^{1/2} \nabla^2 \vec{e}_{m-1}(\sigma')\, d\sigma' \quad ,$$

$$m = 0,1,2, \ldots \quad , \quad \text{and } \vec{e}_{-1} = 0 \quad . \tag{8.2}$$

In this section we will give a simple example* to illustrate the application of (8.2). Consider the two-dimensional problem (no z-variation) sketched in Figure 2-10:an infinitely long line source at the origin 0 radiating an E-wave (with nonzero field components E_z, H_x, and H_y) or an H-wave (H_z, E_x, and E_y). Let u stand for E_z or H_z. Away from point 0, we assume that $\hat{z}u$ can be asymptotically represented by a series (8.1), or

$$u(\rho,\phi) \sim e^{ik\rho} \sum_{m=0}^{\infty} (ik)^{-m} z_m(\rho,\phi) \quad , \quad k \to \infty \quad . \tag{8.3}$$

(Do not confuse the amplitudes $\{z_m\}$ with the coordinate z.) Over an arbitrarily closed surface Σ enclosing 0 and defined by the equation

$$\Sigma: \quad \rho = a(\phi) \quad , \tag{8.4}$$

we assume that the values of $\{z_m\}$, i.e., $\{z_m(a(\phi),\phi)\}$, are known. The problem is to determine $\{z_m\}$ away from Σ from (8.2).

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*J. B. Keller, R. M. Lewis and B. D. Seckler, "Asymptotic solution of some diffraction problem," Comm. Pure Appl. Math., vol. 9, pp. 207-265, 1956.

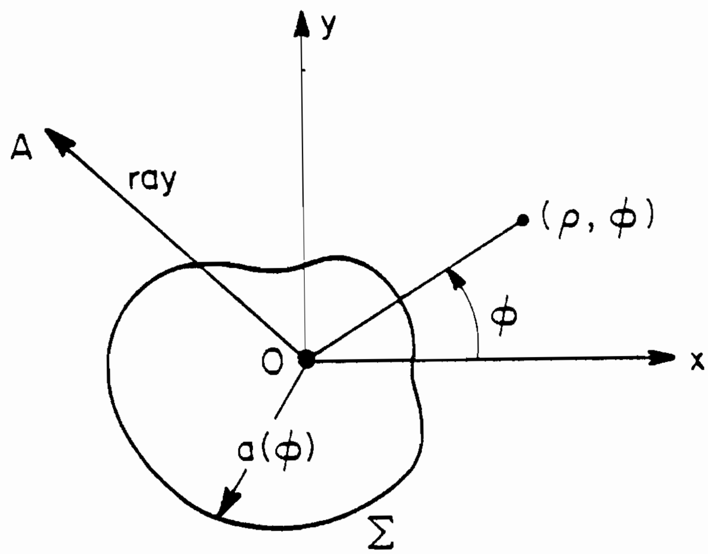


Figure 2-10. A cylindrical wave emanating from the line source at O.

For the present system of rays emanating from the focal line passing through 0, it is convenient to introduce ray coordinates

$$\beta = \phi, \ \alpha = z, \ \sigma = \rho \quad . \tag{8.5}$$

Then a fixed ρ defines a wavefront, and a fixed (ϕ, z) defines a ray. Along a given ray, e.g., ray OA in Figure 2-10, ρ represents the arc length measured from the focal line. The variation of the asymptotic field solution with respect to ρ is governed by (8.2) or

$$z_m(\rho) = z_m(\rho_0)\left[\frac{j(\rho_0)}{j(\rho)}\right]^{1/2} - \frac{1}{2}\int_{\rho_0}^{\rho}\left[\frac{j(\rho')}{j(\rho)}\right]^{1/2} \nabla^2 \, z_{m-1}(\rho')\, d\rho'$$

$$m = 0,1,2,\ldots \quad , \quad \text{and } z_{-1} = 0 \tag{8.6}$$

where ρ_0 is a reference point on ray OA. In the present problem, ρ_0 is taken to be the point on Σ according to (8.4), i.e., $\rho_0 = a$.

To apply (8.6), the Jacobian ratio must be calculated first. There are two ways of doing this. With the ray coordinates given in (8.5), the determinant in (4.8) is readily calculated with the result

$$j(\rho) = -\rho \quad . \tag{8.7}$$

Then the Jacobian ratio is

$$\frac{j(\rho_0)}{j(\rho)} = \frac{a}{\rho} \quad . \tag{8.8}$$

Alternatively, we note that the radii of the present cylindrical wave at a point ρ on ray OA are ρ and ∞. Thus, the curvature matrix may be written as

$$\overline{\overline{Q}}(\rho) = \begin{bmatrix} \rho^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix} \quad , \text{ where } b \to \infty \quad . \tag{8.9}$$

The use of (8.9) in (7.10) leads again to (8.8).

Setting m = 0 in (8.6), we obtain the formula for continuing the zeroth-order field:

$$z_0(\rho,\phi) = (a/\rho)^{1/2}\, z_0(a,\phi) \quad . \tag{8.10}$$

To calculate z_1, the Laplacian of z_0 is needed;

$$\nabla^2 z_0(\rho,\phi) = \left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right] (a/\rho)^{1/2}\, z_0(a,\phi)$$

$$= \rho^{-3/2}\left[\left(\frac{1}{2}\right)^2 + \rho^{-1}\frac{\partial^2}{\partial\phi^2}\right] a^{1/2}\, z_0(a,\phi) \quad . \tag{8.11}$$

Using (8.8) and (8.11) in (8.6) with m = 1, we obtain after a simple integration

$$z_1(\rho,\phi) = \rho^{-1/2}\, f_{01}(\phi) + \rho^{-3/2}\, f_{11}(\phi) \tag{8.12}$$

where

$$f_{01}(\phi) = a^{1/2}\, z_1(a,\phi) - a^{-1}\, f_{11}(\phi)$$

$$f_{11}(\phi) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right] a^{1/2}\, z_0(a,\phi) \quad .$$

Unlike z_0 in (8.10), we note that z_1 depends not only on its own initial value on Σ, but also on the initial values of z_0 and $(\partial^2 z_0/\partial\phi^2)$ on Σ. Following the same procedure, z_2, z_3, ... , etc., may be determined in succession. However, this is not necessary as there exists a better way where all $\{z_m\}$ are determined recursively.

Guided by the solutions of z_0 in (8.10) and z_1 in (8.12), it is conjectured that the general form of z_m is

$$z_m(\rho,\phi) = \rho^{-1/2}\sum_{n=0}^{m} \rho^{-n}\, f_{nm}(\phi) \quad , \qquad m = 0,1,2, \ldots \quad . \tag{8.13}$$

Inserting (8.13) into (8.6) and carrying out the integration, we obtain

$$z_m(\rho,\phi) = \rho^{-1/2} \left\{ a^{1/2} z_m(a,\phi) - \sum_{n=1}^{m} \frac{1}{2n} a^{-n} \left[(n-\tfrac{1}{2})^2 + \frac{\partial^2}{\partial\phi^2} \right] f_{n-1,m-1} \right.$$

$$\left. + \sum_{n=1}^{m} \frac{1}{2n} \rho^{-n} \left[(n-\tfrac{1}{2})^2 + \frac{\partial^2}{\partial\phi^2} \right] f_{n-1,m-1} \right\} . \qquad (8.14)$$

A comparison of (8.13) and (8.14) gives immediately

$$f_{00}(\phi) = a^{1/2} z_0(a,\phi) \qquad (8.15a)$$

$$f_{0m}(\phi) = a^{1/2} z_m(a,\phi) - \sum_{n=1}^{m} a^{-n} f_{nm}(\phi) \quad , \quad m \geq 1 \qquad (8.15b)$$

$$f_{nm}(\phi) = \frac{1}{2n} \left[(n-\tfrac{1}{2})^2 + \frac{\partial^2}{\partial\phi^2} \right] f_{n-1,m-1}(\phi) \quad , \quad m \neq 0 \text{ and } m \geq 1 \ . \qquad (8.15c)$$

It is a simple matter to check that the known results in (8.10) and (8.12) are special cases of (8.15).

In summary, in terms of the known initial values of $\{z_m\}$ on Σ, the asymptotic field solution radiated from a line source at 0 (Figure 2-9) is given by

$$u(\rho,\phi) \sim e^{ik\rho} \rho^{-1/2} \sum_{m=0}^{\infty} (ik)^{-m} \left[\sum_{n=0}^{m} \rho^{-n} f_{nm}(\phi) \right] \quad , \quad k \to \infty \qquad (8.16)$$

where $\{f_{nm}\}$ are given by (8.15). Alternatively, u can be written as (8.3) with the first three amplitudes given by

$$z_0(\rho,\phi) = \rho^{-1/2} f_{00} \qquad (8.17a)$$

$$z_1(\rho,\phi) = \rho^{-1/2} f_{01} + \rho^{-3/2} (\tfrac{1}{8} f_{00} + \tfrac{1}{2} f''_{00}) \qquad (8.17b)$$

$$z_2(\rho,\phi) = \rho^{-1/2} f_{02} + \rho^{-3/2} \left(\frac{1}{8} f_{01} + \frac{1}{2} f''_{01}\right)$$

$$+ \rho^{-5/2} \frac{1}{8} \left(\frac{9}{16} f_{00} + \frac{5}{2} f''_{00} + f''''_{00}\right) \quad (8.17c)$$

where the primes on f_{nm} indicate the derivative with respect to ϕ. As may be seen from (8.15a) and (8.15b), $\{f_{0m}\}$ depends on the initial values on Σ, and therefore are arbitrary functions of ϕ.

Next, let us consider a special case of the solution in (8.16). In a problem which involves no characteristic dimension (i.e., whose geometrical configuration is defined by angles only), k and ρ must appear in the solution only in the combination of $k\rho$ [Problem 2-13]. Inserting an additional factor $k^{-1/2}$ on the right-hand-side of (8.16), the requirement that k and ρ appear as $k\rho$ is satisfied if

$$f_{nm}(\phi) = 0 \quad , \quad \text{for } n \neq m \quad . \quad (8.18)$$

In this case, the recurrence formula (8.15c) becomes

$$f_{mm}(\phi) = \frac{1}{2m} \left[\left(m - \frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right] f_{m-1,m-1} \quad (8.19$$

$$= \frac{1}{2m} \left[\left(m - \frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right] \frac{1}{2(m-1)} \left[\left(m - 1 - \frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right] f_{m-2,m-2}$$

$$= \frac{1}{2^m m!} \prod_{n=1}^{m} \left[\left(n - \frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right] f_{00} \quad . \quad (8.19)$$

Hence, in problems containing no characteristic dimension, the asymptotic expansion of a cylindrical wave emanating from O (Figure 2-10) is given by

$$u(\rho,\phi) \sim e^{ik\rho} (k\rho)^{-1/2} \left\{1 + \sum_{m=1}^{\infty} \frac{1}{m!} (2ik\rho)^{-m} \prod_{n=1}^{m} \left[\left(n - \frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right]\right\} f_{00}(\phi) \quad , \quad k \to \infty \quad (8.20)$$

where f_{00} is an arbitrary function of ϕ, and may be identified with the initial value of z_0 on Σ according to (8.15a). For this special case in (8.20), we note that only the initial value of z_0, not those of higher order $\{z_m\}$, is needed in the determination of u away from Σ. If we choose f_{00} to be

$$f_{00}(\phi) = (\frac{2}{\pi})^{1/2} \exp[-i(\nu + \frac{1}{2}) \frac{\pi}{2} + i \nu\phi] \tag{8.21}$$

then u in (8.20) becomes

$$u(\rho,\phi) \sim (\frac{2}{\pi k\rho})^{1/2} \left\{1 + \sum_{m=1}^{\infty} \frac{1}{m!} (2ik\rho)^{-m} \prod_{n=1}^{m} [(n - \frac{1}{2})^2 - \nu^2]\right\}$$

$$\cdot \exp i[k\rho + \nu\phi - (\nu + \frac{1}{2}) \frac{\pi}{2}] \quad , \quad k \to \infty \tag{8.22}$$

which coincides exactly with the known asymptotic expansion of $H_\nu^{(1)}(k\rho) \exp i\nu\phi$.

2-1. Consider the asymptotic solution of an electromagnetic diffraction problem given in (2.5), and denote its partial sum by

$$\vec{E}_M(\vec{r}) = e^{iks(\vec{r})} \sum_{m=0}^{M} (ik)^{-m} \vec{e}_m(\vec{r}) \quad , \quad k \to \infty \quad .$$

Show that

$$\begin{cases} (\nabla^2 + k^2)\, \vec{E}_M(\vec{r}) = O(k^{-M}) & \text{(1a)} \\ \nabla \cdot \vec{E}_M(\vec{r}) = O(k^{-M}) & \text{(1b)} \end{cases}$$

for any positive integer M. Alternatively, (1) may serve as a definition of the asymptotic solution of Maxwell's equations. The error in using $\vec{E}_M$ to approximate the exact solution $\vec{E}$ asymptotically satisfies the following relations:

$$\begin{cases} (\nabla^2 + k^2)(\vec{E} - \vec{E}_M) = O(k^{-M}) \\ \nabla \cdot (\vec{E} - \vec{E}_M) = O(k^{-M}) \end{cases}$$

Next, consider the asymptotic expansion of the exact solution $\vec{E}$ given in (1.2). Let the partial sum up to and including k^{-M} term be denoted by $\vec{\mathcal{E}}_M(\vec{r})$. Then, in contrast to (1), show that

$$\vec{E} - \vec{\mathcal{E}}_M = O(k^{-M-1}) \qquad (2)$$

2-2. In an isotropic inhomogeneous medium, with $\mu = \mu(\vec{r})$ and $\varepsilon = \varepsilon(\vec{r})$, show that the source-free wave equations for the electric and magnetic fields are

$$\nabla^2\vec{E} + n^2k^2\vec{E} + (\nabla\ \ln\ \mu) \times (\nabla \times \vec{E}) + \nabla(\vec{E} \cdot \nabla\ \ln\ \varepsilon) = 0 \tag{3a}$$

$$\nabla^2\vec{H} + n^2k^2\vec{H} + (\nabla\ \ln\ \varepsilon) \times (\nabla \times \vec{H}) + \nabla(\vec{H} \cdot \nabla\ \ln\ \mu) = 0 \tag{3b}$$

where $n(\vec{r}) = (\varepsilon\mu/\varepsilon_0\mu_0)^{1/2}$, $k = \omega(\mu_0\varepsilon_0)^{1/2}$ and (μ_0,ε_0) are the permeability and permittivity of free space. As $k \to \infty$, let us write the leading term in an asymptotic solution of the fields as

$$\vec{E}(\vec{r}) \sim \vec{e}_0(\vec{r})\ \exp[iks(\vec{r})] \tag{4a}$$

$$\vec{H}(\vec{r}) \sim \vec{h}_0(\vec{r})\ \exp[iks(\vec{r})] \tag{4b}$$

By substituting (4a) into (3a), show the eikonal equation in an isotropic inhomogeneous medium:

$$(\nabla s)^2 = n^2 \tag{5}$$

and the transport equation for $\vec{e}_0$:

$$\mu\vec{e}_0\nabla \cdot \left(\frac{1}{\mu}\nabla s\right) + 2(\vec{e}_0 \cdot \nabla\ \ln\ n)\ \nabla s + 2(\nabla s \cdot \nabla)\ \vec{e}_0 = 0\ . \tag{6}$$

By substituting (4b) into (3b), show the transport equation for $\vec{h}_0$:

$$\varepsilon\vec{h}_0\nabla \cdot \left(\frac{1}{\varepsilon}\nabla s\right) + 2(\vec{h}_0 \cdot \nabla\ \ln\ n)\ \nabla s + 2(\nabla s \cdot \nabla)\ \vec{h}_0 = 0\ . \tag{7}$$

2-3. Study the solution of the eikonal equation in (5) by carrying out the following steps. Corresponding to (3.9), show that the equation that governs the ray $\vec{r} = \vec{r}(\sigma)$ in an isotropic inhomogeneous medium is

$$n\frac{d}{d\sigma}\left(n\frac{d\vec{r}}{d\sigma}\right) = \nabla\left(\frac{n^2}{2}\right) \tag{8}$$

where σ is the arc length parameter of a ray. Corresponding to (3.12), show that the propagation of the phase function along a given ray is described by the relation

$$s(\vec{r}) = s(\vec{r}_0) + \int_{\vec{r}_0}^{\vec{r}} n \, d\sigma \qquad (9)$$

where $\vec{r}$ and $\vec{r}_0$ are on the same ray, and the integration is to be carried out along that ray. (Hint: let $d\vec{r}/d\sigma = n^{-1}\nabla s$.)

2-4. The solutions of (6) and (7) are quite involved. They were first obtained by R. K. Luneburg in 1944 (R. K. Luneburg, Mathematical Theory of Optics. University of California Press, Berkeley, Calif., 1964. M. Kline and I. W. Kay, Electromagnetic Theory and Geometrical Optics. Interscience Publishers, New York, 1965). We summarize the main results below: (i) Along a given ray, the propagation of the field magnitudes is given by

$$|\vec{e}_0(\sigma)| = |\vec{e}_0(\sigma_0)| \left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} \left[\frac{\mu(\sigma)}{\mu(\sigma_0)}\right]^{1/2} \qquad (10a)$$

$$|\vec{h}_0(\sigma)| = |\vec{h}_0(\sigma_0)| \left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} \left[\frac{\varepsilon(\sigma)}{\varepsilon(\sigma_0)}\right]^{1/2} \qquad (10b)$$

where the magnitude of a complex vector a is defined to be

$$|\vec{a}| = +\sqrt{\vec{a} \cdot \vec{a}^*} \quad .$$

Results in (10) should be compared with those in (5.9). Note that in an inhomogeneous medium, the Jacobian $j(\sigma)$ can no longer be expressed in terms of curvatures of the wavefront in a simple manner. (ii) In general, the electric vector $\vec{e}_0(\vec{r})$ of a linearly polarized wave rotates along a ray. The amount of rotation in the right-hand sense with respect to the tangent of the ray in the normal plane (a plane transverse to the tangent) is determined by the relation

$$\Omega(\vec{r}) = \Omega(\vec{r}_0) - \int_{\vec{r}_0}^{\vec{r}} \tau \, d\sigma$$

where $\vec{r}$ and $\vec{r}_0$ are points on the same ray, τ is the torsion of the ray, and Ω is the angle of $\vec{e}_0$ measured from the normal of the ray. For the special case in which the ray is a planar curve ($\tau \equiv 0$), the angle $\Omega(\vec{r})$ is a constant along that ray. (This of course does not necessarily mean that $\vec{e}$ points to a constant direction in space. The latter statement is generally true only when the ray is a straight line.)

2-5. Show that (3.23) is true even if $(\hat{x}_1, \hat{x}_2)$ makes an angle ψ with the principal directions. Hint: $\bar{\bar{R}}^T \, \bar{\bar{Q}} \, \bar{\bar{R}}$ is a diagonal matrix and satisfies (3.23), where $\bar{\bar{R}}$ is a rotation matrix given in (3.18a). Furthermore, from (3.18a) show that

$$[\bar{\bar{Q}}(\sigma = 0)]^{-1} = \begin{pmatrix} R_1 \cos^2 \psi + R_2 \sin^2 \psi & \frac{1}{2}(R_1 - R_2) \sin 2\psi \\ \frac{1}{2}(R_1 - R_2) \sin 2\psi & R_1 \sin^2 \psi + R_2 \cos^2 \psi \end{pmatrix} .$$

This is useful for the computation in (3.23).

2-6. Consider a linear ordinary differential equation of first order

$$\frac{dy}{dx} + a(x) \, y = b(x) \quad .$$

Show that its solution is given by

$$y(x) = e^{-A(x)} \left[c - \int b(x)\, e^{A(x)}\, dx \right]$$

where

$$A(x) = \int a(x)\, dx \quad , \quad c = \text{an arbitrary constant.}$$

Hint: Consider first the special case $b(x) = 0$ and show that its solution is

$$y(x) = ce^{-A(x)} \quad .$$

Next, consider the general case $b(x) \neq 0$. Let

$$y(x) = u(x)\, e^{-A(x)} \quad .$$

Determine $u(x)$. See, e.g., R. Courant, Differential and Integral Calculus, Vol. II. London: Blackie and Son, 1936, pp. 429-430.

2-7. Show that the Jacobian of the transformation from rectangular coordinates ($x = x_1$, $y = x_2$, $z = x_3$) to ray coordinates ($\beta = \sigma_1, \alpha = \sigma_2, \sigma = \sigma_3$) can be written as

$$j = \det \left(\frac{\partial x_i}{\partial \sigma_v} \right) = \sum_{v=1}^{3} \frac{\partial x_1}{\partial \sigma_v} \operatorname{cof} \frac{\partial x_1}{\partial \sigma_v}$$

where cof $(\partial x_i / \partial \sigma_v)$ is the cofactor of the ith row and the vth column of the determinant.

Hint: Since a determinant vanishes if two rows are identical, we have

$$\sum_{v=1}^{3} \frac{\partial x_k}{\partial \sigma_v} \operatorname{cof} \frac{\partial x_i}{\partial \sigma_v} = j \delta_{ik}$$

where δ_{ik} is the Kronecker symbol: $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ if $i \neq k$.

2-8. Making use of the results in Problem 2-6 and the transport equation in (5.1), first establish the relation

$$\frac{d}{d\sigma}\left(j^{1/2}\,\vec{e}_m\right) = -\frac{j^{1/2}}{2}\,\nabla^2\vec{e}_{m-1}$$

and, next, by integration along a ray, show that the solution of the transport equation is given by

$$\vec{e}_m(\sigma) = \left[\frac{j(\sigma_0)}{j(\sigma)}\right]^{1/2} \vec{e}_m(\sigma_0) - (1/2)\int_{\sigma_0}^{\sigma} \left[\frac{j(\sigma')}{j(\sigma)}\right]^{1/2} \nabla^2\vec{e}_{m-1}(\sigma')\, d\sigma' \quad ,$$

$$m = 0,1,2,\ldots \quad .$$

Hint: Note the following manipulation:

$$\frac{dj}{d\sigma} = \frac{\partial j}{\partial\sigma_3} = \sum_{i,v} \frac{\partial^2 x_i}{\partial\sigma_3\partial\sigma_v}\,\mathrm{cof}\,\frac{\partial x_i}{\partial\sigma_v} = \sum_{i,v,k} \frac{\partial}{\partial x_k}\left(\frac{\partial x_i}{\partial\sigma_3}\right)\left[\frac{\partial x_k}{\partial\sigma_v}\,\mathrm{cof}\,\frac{\partial x_i}{\partial\sigma_v}\right]$$

$$= j\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial\sigma_3}\right) = j\nabla\cdot\frac{d\vec{r}}{d\sigma} = j\nabla\cdot\nabla s = j\nabla^2 s$$

$$\frac{d}{d\sigma}(j^{1/2}\,\vec{e}_m) = j^{1/2}\left[\frac{d\vec{e}_m}{d\sigma} + \frac{\vec{e}_m}{2j}\frac{dj}{d\sigma}\right] = \frac{j^{1/2}}{2}\,[2(\nabla s\cdot\nabla)\,\vec{e}_m + \vec{e}_m\nabla^2 s]$$

$$= -\frac{j^{1/2}}{2}\,\nabla^2\vec{e}_{m-1} \quad .$$

The above derivation was first given in D. S. Ahluwalia, R. M. Lewis and J. Boersma, "Uniform asymptotic theory of diffraction by a plane screen," <u>SIAM J. Appl. Math.</u>, <u>16</u>, pp. 783-807, 1968.

2-9. The zeroth-order transport equation in (5.1) reads

$$2(\nabla s \cdot \nabla)\vec{e}_0 + \nabla^2 s\vec{e}_0 = 0 \quad . \tag{11}$$

Since $\vec{e}_0$, in general, is a complex vector, let us define its magnitude and "direction" by

$$\vec{e}_0 = |\vec{e}_0|\hat{e}_0 \tag{12}$$

where $|\vec{e}_0| = \sqrt{\vec{e}_0 \cdot \vec{e}_0^*}$, $\vec{e}_0^*$ = complex conjugate of $\vec{e}_0$, and

$$\hat{e}_0 = \vec{e}_0/|\vec{e}_0| = \text{a complex vector.}$$

Show that the magnitude of $\vec{e}_0$ satisfies the relation

$$\nabla \cdot (|\vec{e}_0|^2 \nabla s) = 0 \tag{13}$$

and by integrating it over a volume formed by a pencil and two segments of wavefronts show that

$$|\vec{e}_0(\sigma)|^2 = |\vec{e}_0(\sigma_o)|^2 \frac{da(\sigma_0)}{da(\sigma)} \quad . \tag{14}$$

Hint: Multiplying (11) by $\vec{e}_0^*$ and adding to the resulting equation its complex conjugate, we obtain

$$(\nabla s \cdot \nabla)|\vec{e}_0|^2 + \nabla^2 s|\vec{e}_0|^2 = \nabla \cdot (|\vec{e}_0|^2 \nabla s) = 0 \quad .$$

When integrating (13) over the volume, we note that there is no contribution from the side surface because ∇s is tangent to rays everywhere.

2-10. Substituting (12) into (11) shows that

$$(\nabla s \cdot \nabla)\, \hat{e}_0 = \frac{d}{d\sigma} \hat{e}_0 = 0$$

which implies the direction of $\vec{e}_0$ is a constant along a ray. When this result is combined with (14), we have the complete solution of the zeroth-order transport equation in (5.11).

2-11. Starting with (5.10), show the relation given in (6.7).

Hint: $$\frac{d}{d\sigma} \vec{e}_m(\sigma) = \vec{e}_m(\sigma_0)[j(\sigma_0)]^{1/2} \frac{-j'(\sigma)}{2[j(\sigma)]^{3/2}} - \frac{1}{2} \nabla^2 \vec{e}_{m-1}(\sigma)$$

$$= \vec{e}_m(\sigma_0)[j(\sigma_0)]^{1/2} \frac{-j'(\sigma)}{2[j(\sigma)]^{3/2}} + \frac{1}{2} \left(2 \frac{d}{d\sigma} \vec{e}_m(\sigma) + \nabla^2 s \vec{e}_m(\sigma) \right)$$

or

$$\vec{e}_m(\sigma_0)[j(\sigma_0)]^{1/2} \frac{j'(\sigma)}{[j(\sigma)]^{3/2}} = \nabla^2 s \vec{e}_m(\sigma) \quad .$$

2-12. From (2.8) and (6.3), show that

$$\nabla s \cdot \nabla^2 \vec{e}_{m-1}(\sigma) = \nabla^2 s (\nabla \cdot \vec{e}_{m-1}) + 2(\nabla s \cdot \nabla)(\nabla \cdot \vec{e}_{m-1}) \quad .$$

Hint: With (m - 1) replacing m in (2.8), we have

$$2(\nabla s \cdot \nabla)\, \vec{e}_{m-1} + \nabla^2 s \vec{e}_{m-1} = -\nabla^2 \vec{e}_{m-2} \quad .$$

The gradient of (6.3) gives

$$(\vec{e}_{m-1} \cdot \nabla)\, \nabla s + (\nabla s \cdot \nabla)\, \vec{e}_{m-1} + \nabla s \times (\nabla \times \vec{e}_{m-1}) = -\nabla \nabla \cdot \vec{e}_{m-2} \quad .$$

The difference of the above two equations gives

$$\nabla \times (\vec{e}_{m-1} \times \nabla s) + \nabla s(\nabla \cdot \vec{e}_{m-1}) - \nabla s \times (\nabla \times \vec{e}_{m-1}) = \nabla \times \nabla \times \vec{e}_{m-2} \quad .$$

Taking the divergence of it yields the desired relation.

2-13. Show that in a problem with no characteristic dimension, the wave number k and the space variables (x,y,z) appear only in the combination (kx,ky,kz) in the field solution.

Hint: In a problem with no characteristic dimension, the geometrical configuration can be defined by angles (θ,ϕ). Consequently, it remains the same if the scale of (x,y,z) is changed such that

$$x \to x' = kx \quad , \quad y \to y' = ky \quad , \quad z \to z' = kz'$$

Correspondingly, the source-free Maxwell equations in the free space

$$k^{-1} \nabla \times \vec{E}(x,y,z) = i\eta \, \vec{H}(x,y,z)$$

$$k^{-1} \nabla \times \vec{H}(x,y,z) = -i\eta^{-1} \vec{E}(x,y,z)$$

where $\eta = (\mu/\varepsilon)^{1/2}$, become

$$\nabla' \times \vec{E}(x',y',z') = i\eta \, \vec{H}(x',y',z')$$

$$\nabla' \times \vec{H}(x',y',z') = -i\eta^{-1} \vec{E}(x',y',z')$$

Thus, when the new scale is used, k no longer appears in the Maxwell equations, nor in a description of the boundary conditions.

Chapter 3. GEOMETRICAL OPTICS THEORY

3.1. Introduction

The basic problem studied in this part of the book is the diffraction of electromagnetic waves by a perfectly conducting, infinitely thin screen Σ. We assume the screen to be a portion of a curved surface, which is sufficiently smooth to make all of our mathematical manipulations meaningful. For a given incident field $\vec{E}^i$, the total field $\vec{E}^t$ is to be determined everywhere. The total field is defined as a sum of the incident field $\vec{E}^i$ and the scattered field $\vec{E}$:

$$\vec{E}^t(\vec{r}) = \vec{E}^i(\vec{r}) + \vec{E}(\vec{r}) \quad . \tag{1.1}$$

The incident field may be regarded as the field produced by the source in the absence of screen Σ. We assume that it can be represented, at high frequencies, by an asymptotic series:

$$\vec{E}^i(\vec{r}) \sim e^{iks^i(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m} \vec{e}^{\,i}_m(\vec{r}) \quad , \quad k \to \infty \quad . \tag{1.2}$$

Since the incident field can exist independently, the series in (1.2) must satisfy the Maxwell's equations, and therefore all the conditions summarized in Section 2.7. A field which can be asymptotically represented by (1.2) is called a <u>ray field</u>. Generally speaking, all the ray techniques are applicable only when the incident fields are ray fields. Otherwise, an arbitrary incident field, if possible, should be decomposed into a superposition of ray fields, and the ray techniques be applied to each ray-field constituent. Some examples of the latter case are given in Chapter 5.

For a given screen and an incident ray field in (1.2), our problem at hand is to determine $\vec{E}^t$. Except for the few special cases, e.g., the

Sommerfeld half-plane problem studied in Chapter 1, $\vec{E}^t$ cannot be found exactly. Since the frequency is high, we settle for a (formally) asymptotic solution of $\vec{E}^t$. According to the ascending degrees of sophistication, this step can be carried out according to any one of the following theories:

(i) Geometrical optics theory (GO)

(ii) Keller's geometrical theory of diffraction (GTD)

(iii) Uniform asymptotic theory (UAT), or other uniform theories.

In this chapter we will start out with the simplest one: GO. In fact, the results obtained here are necessary for the development of the more elaborate theories listed in (ii) and (iii) above.

3.2 Incident and Reflected Fields

Referrring to Figure 3-1, let us consider a point source at $\vec{r}_0$, which in the absence of screen Σ radiates a field $\vec{E}^i$ given in (1.2). The problem is to determine the total field $\vec{E}^t$ produced by the same source when Σ is present. According to the (classical) geometrical optics theory, the incident field is blocked by Σ and casts a shadow behind it. An observation point $\vec{r}$, whether it is in the shadow or lit regions of the incident field, can be readily determined by the following test. Drawing a straight line from the source $\vec{r}_0$ to the observation point $\vec{r}$, i.e., a ray, then $\vec{r}$ is in the shadow region if the line intercepts Σ; otherwise, $\vec{r}$ is in the lit region. To state this fact mathematically, let us introduce a <u>shadow indicator of the incident field</u> $\varepsilon^i(\vec{r})$ such that

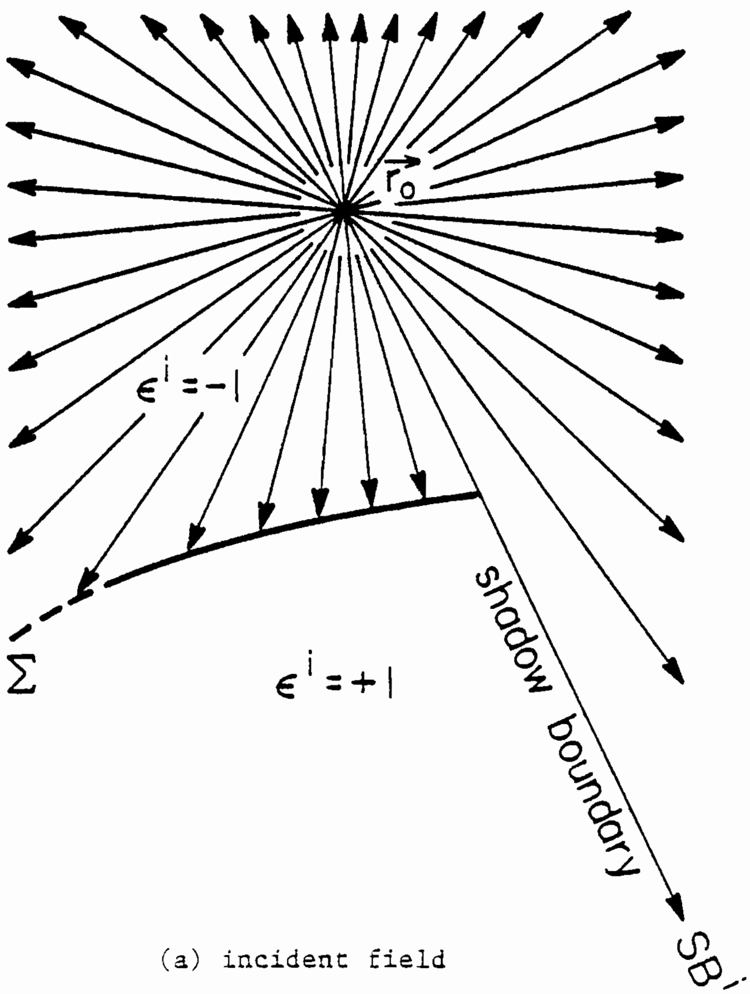
$$\varepsilon^i(\vec{r}) = \begin{cases} +1 \quad , \quad \text{if } \vec{r} \text{ is in the shadow region of the incident field} \\ \\ -1 \quad , \quad \text{if } \vec{r} \text{ is in the lit region of the incident field} \end{cases} \tag{2.1}$$

Then, in the presence of Σ, the incident field $\vec{E}^i$ in (1.2) is modified to become

$$\theta(-\,\varepsilon^i)\;\vec{E}^i(\vec{r}) \tag{2.2}$$

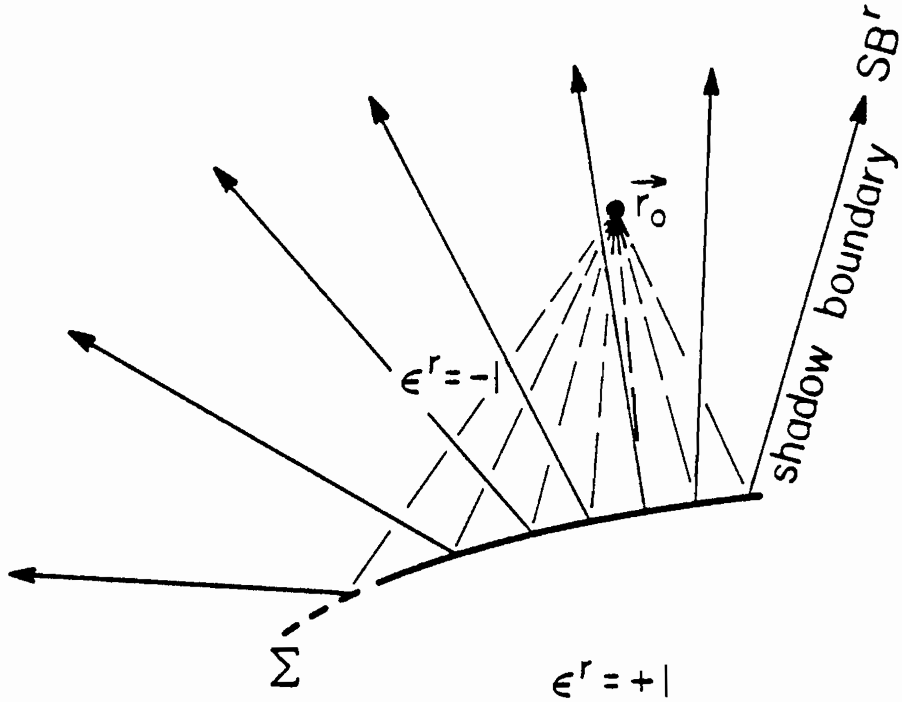
where $\theta(x)$ is the unit step function

$$\theta(x) = \begin{cases} +1 \quad , \quad x > 0 \\ \\ 0 \quad , \quad x < 0 \end{cases} \;. \tag{2.3}$$



(a) incident field

Figure 3-1. Illuminated and shadow regions of the incident (reflected) field from a point source.



(b) reflected field

Figure 3-1. Illuminated and shadow regions of the incident (reflected) field from a point source.

In Figure 3-1a, we give a picturesque description of this fact by launching several typical rays from the source at $\vec{r}_0$ and terminating the rays only when Σ is encountered. Note that there is no incident ray that reaches the shadow region, indicating the fact that the asymptotic field is zero there. The surface formed by the extension of the straight lines, drawing from $\vec{r}_0$ to a point on the edge of the screen, separates the shadow and the lit regions of the incident field. This surface is called the incident shadow boundary, denoted by SB^i.

In addition to the incident field, the classical geometrical optics theory also predicts the existence of a reflected field $\vec{E}^r$ such that

$$\vec{E}^t(\vec{r}) \sim \theta(-\varepsilon^i)\ \vec{E}^i(\vec{r}) + \theta(-\varepsilon^r)\ \vec{E}^r(\vec{r}) \quad , \quad k \to \infty \quad . \tag{2.4}$$

The two terms on the right-hand side of (2.4) are also known as the geometrical optics field. When $\vec{E}^i$ is given in (1.2), $\vec{E}^r$ can be represented by a similar asymptotic expansion, namely,

$$\vec{E}^r(\vec{r}) \sim e^{iks^r(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m}\ \vec{e}^r(\vec{r}) \quad , \quad k \to \infty \quad . \tag{2.5}$$

The shadow indicator of the reflected wave ε^r in (2.4) is defined exactly the same as ε^i in (2.1) except that the "incident field" in (2.1) should be replaced by "reflected field." The reflected field $\vec{E}^r$ in (2.5), as it stands, is mathematically defined for all $\vec{r}$ (in the lit as well as in the shadow regions of the reflected field), despite the fact that only $\vec{E}^r$ in its lit region (where $\varepsilon^r = -1$) is physically meaningful and contributes to $\vec{E}^t$ in accordance with (2.4). The precise definition of $\vec{E}^r$ in its shadow region is of no concern at this moment. The subject is discussed further in Chapter 5.

There exist two conditions that are to be enforced for the determination of $\vec{E}^r$ in its lit region. First, the usual boundary condition on the perfectly conducting screen Σ requires that the tangential components of the total electric field be zero:

$$\hat{N} \times (\vec{E}^i + \vec{E}^r) = 0 \quad , \quad \vec{r} \text{ on } \Sigma \tag{2.6}$$

where $\hat{N}$ is the outward normal* of Σ. The second condition is that the total field satisfy Gauss' law:

$$\nabla \cdot (\vec{E}^i + \vec{E}^r) = 0 \quad , \tag{2.7}$$

for all $\vec{r}$ in the common lit region of $\vec{E}^i$ and $\vec{E}^r$. It has been established in Section 2.6 that the satisfaction of Gauss' law at one point on a ray implies its satisfaction at all other points on the same ray. Because of this property, it is only necessary to enforce (2.7) for points on screen Σ, as they are common points on incident and reflected rays. Thus, (2.7) may be replaced by

$$\nabla \cdot (\vec{E}^i + \vec{E}^r) = 0 \quad , \quad \vec{r} \text{ on } \Sigma \quad . \tag{2.8}$$

The two conditions in (2.6) and (2.8) are enforced for the fields in (1.2) and (2.5) for the solution of the reflected field.

Consider the reflection from a typical point O on screen Σ (point of reflection). There are three quantities involved: the incident pencil, the reflected pencil, and the screen Σ. We describe each of them below.

(i) <u>Incident pencil</u> (Figure 3-2): The rectangular pencil coordinates $(x_1^i, x_2^i, z^i = x_3^i)$ describe the position in the incident pencil. $\hat{z}^i$ coincides with the axial ray which passes through O, and z^i is measured positively in the direction of wave propagation from O. According to (7.6) in Section 2.7,

*"Outward" normal means the normal pointing toward the source of $\vec{E}^i$.

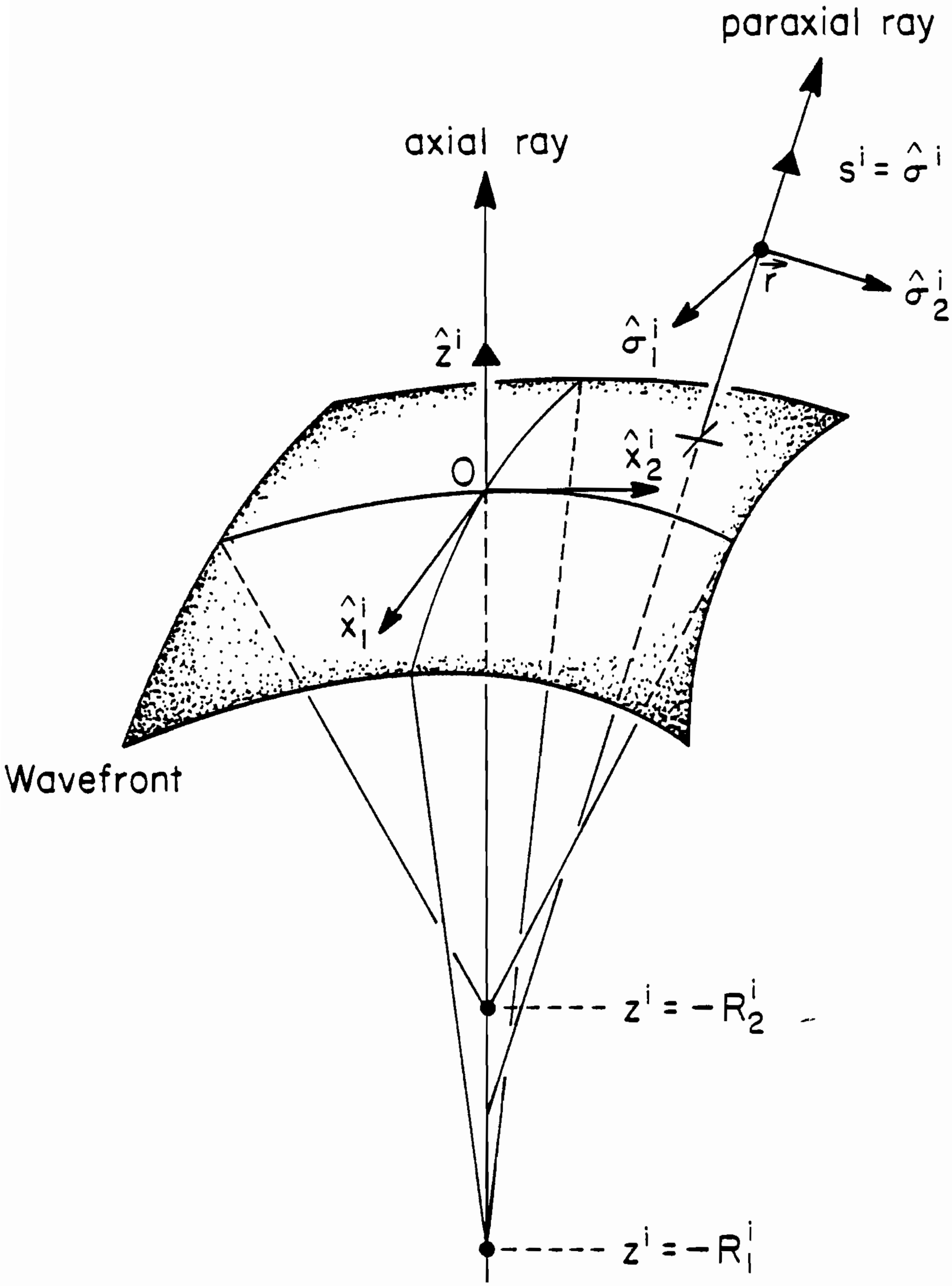


Figure 3-2. Incident pencil. In this sketch, $\left(\hat{x}_1^i, \hat{x}_2^i\right)$ are principal directions, and $\left(R_1^i, R_2^i\right)$ are positive.

the phase function $s^i(\vec{r})$ at an arbitrary point $\vec{r} = (x_1^i, x_2^i, z^i)$ in the incident pencil is given by

$$s^i(\vec{r}) = s^i(0,0,0) + z^i + \frac{1}{2} \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix} \cdot \bar{\bar{Q}}^i(z^i) \begin{pmatrix} x_1^i \\ x_2^i \end{pmatrix} + O[(x^i)^3] \quad , \qquad (2.9)$$

where $O[(x^i)^3]$ means terms of order $(x_1^i)^\mu \, (x_2^i)^\nu$ with $\mu + \nu = 3$. The curvature matrix $\bar{\bar{Q}}^i(z^i)$ in (2.9) is as usual determined by the principal radii (R_1^i, R_2^i) of the incident pencil and the angle ψ between $(\hat{x}_1^i, \hat{x}_2^i)$ and the principal directions of the wavefront. Throughout this chapter, we <u>always</u> choose $(\hat{x}_1^i, \hat{x}_2^i)$ in the principal directions ($\psi = 0$) unless it is explicitly stated otherwise. Then, $\bar{\bar{Q}}(z^i)$ has the simple form

$$\bar{\bar{Q}}^i(z^i) = \begin{pmatrix} \dfrac{1}{R_1^i + z^i} & 0 \\ 0 & \dfrac{1}{R_2^i + z^i} \end{pmatrix} \qquad (2.10)$$

as given in (7.7) and (7.9) in Section 2.7. Next, consider the vector components of the amplitude vectors $\{\vec{e}_m^{\,i}\}$ at $\vec{r}$ in the incident pencil. For later applications, it is convenient to resolve the amplitude vectors into longitudinal and transverse components with respect to <u>each</u> individual ray (axial or paraxial), rather than into components in the directions of $(\hat{x}_1^i, \hat{x}_2^i, \hat{z}^i)$. For this purpose, we introduce three <u>local</u> orthonormal base vectors at a point $\vec{r} = (x_1^i, x_2^i, z^i)$:

$$\hat{\sigma}_1^i = \hat{x}_1^i - \hat{z}^i \frac{x_1^i}{R_1^i + z^i} + O[(x^i)^2] \tag{2.11a}$$

$$\hat{\sigma}_2^i = \hat{x}_2^i - \hat{z}^i \frac{x_2^i}{R_2^i + z^i} + O[(x^i)^2] \tag{2.11b}$$

$$\nabla s^i = \hat{\sigma}^i = \hat{x}_1^i \frac{x_1^i}{R_1^i + z^i} + \hat{x}_2^i \frac{x_2^i}{R_2^i + z^i} + \hat{z}^i + O[(x^i)^2] \quad . \tag{2.11c}$$

Note that in (2.11) terms of order $(x_1^i)^2$, $(x_2^i)^2$ or $(x_1^i x_2^i)$ have been neglected. Within this approximation, ∇s^i in (2.11c), which was calculated from (2.9) and (2.10), is in the direction of the paraxial ray passing through $\vec{r}$; $(\hat{\sigma}_1^i, \hat{\sigma}_2^i)$ are two orthogonal directions transverse to ∇s^i, and $\hat{\sigma}_1^i(\sigma_2^i)$ is also orthogonal to the principal direction $\hat{x}_2^i(\hat{x}_1^i)$. For the special case when $\vec{r}$ is on the axial ray $(x_1^i = 0, x_2^i = 0)$, the local base vectors $(\hat{\sigma}_1^i, \hat{\sigma}_2^i, \nabla s^i)$ coincide with the pencil base vectors $(\hat{x}_1^i, \hat{x}_2^i, \hat{z}^i)$. At any point in the pencil, we represent $\{\vec{e}_m^i\}$ by

$$\vec{e}_m^i(\vec{r}) = \hat{\sigma}_1^i e_{m1}^i + \hat{\sigma}_2^i e_{m2}^i + \nabla s^i e_{m3}^i \quad , \quad m = 0,1,2,\dots \quad . \tag{2.12}$$

Thus,

$$(e_{m1}^i, e_{m2}^i) = \text{transverse components} \quad ,$$

and

$$e_{m3}^i = \text{longitudinal component}$$

of $\vec{e}_m^i$ with respect to the direction of wave propagation. Substituting (2.11) into (2.12), we have

$$\vec{e}^{\,i}_m(\vec{r}) = \hat{x}^i_1 \left(e^i_{m1} + \frac{x^i_1}{R^i_1 + z^i} e^i_{m3} \right) + \hat{x}^i_2 \left(e^i_{m2} + \frac{x^i_2}{R^i_2 + z^i} e^i_{m3} \right)$$

$$+ \hat{z}^i \left(e^i_{m3} - \frac{x^i_1}{R^i_1 + z^i} e^i_{m1} - \frac{x^i_2}{R^i_2 + z^i} e^i_{m2} \right) + O[(x^i)^2] \quad . \qquad (2.13)$$

This is the representation of $\vec{e}^{\,i}_m$ in terms of the rectangular pencil base vectors $(\hat{x}^i_1, \hat{x}^i_2, \hat{z}^i)$ at a point $\vec{r}$ in a pencil.

(ii) Reflected pencil: After replacing the superscript "i" by "r," the same formulas and conventions for the incident pencil are used for the reflected pencil.

(iii) Screen Σ(Figure 3-3): At point of reflection O where the axial ray of the incident pencil meets Σ, let us introduce three orthonormal base vectors $\left(\hat{x}^\Sigma_1, \hat{x}^\Sigma_2, \hat{z}^\Sigma = \hat{x}^\Sigma_3\right)$. Here $\hat{z}^\Sigma$ coincides with the outward normal of Σ at O; $\left(x^\Sigma_1, x^\Sigma_2\right)$ lie in the tangent plane and are in the principal directions of Σ at O. Then, in the neighborhood of O, Σ can be approximated by a quadratic surface:

$$z^\Sigma = \frac{1}{2} \begin{pmatrix} x^\Sigma_1 \\ x^\Sigma_2 \end{pmatrix} \cdot \bar{\bar{Q}}^\Sigma \begin{pmatrix} x^\Sigma_1 \\ x^\Sigma_2 \end{pmatrix} + O[(x^\Sigma)^3] \qquad (2.14)$$

where the curvature matrix $\bar{\bar{Q}}^\Sigma$ of Σ is given by

$$\bar{\bar{Q}}^\Sigma = \begin{pmatrix} \frac{1}{R^\Sigma_1} & 0 \\ 0 & \frac{1}{R^\Sigma_2} \end{pmatrix} \quad . \qquad (2.15)$$

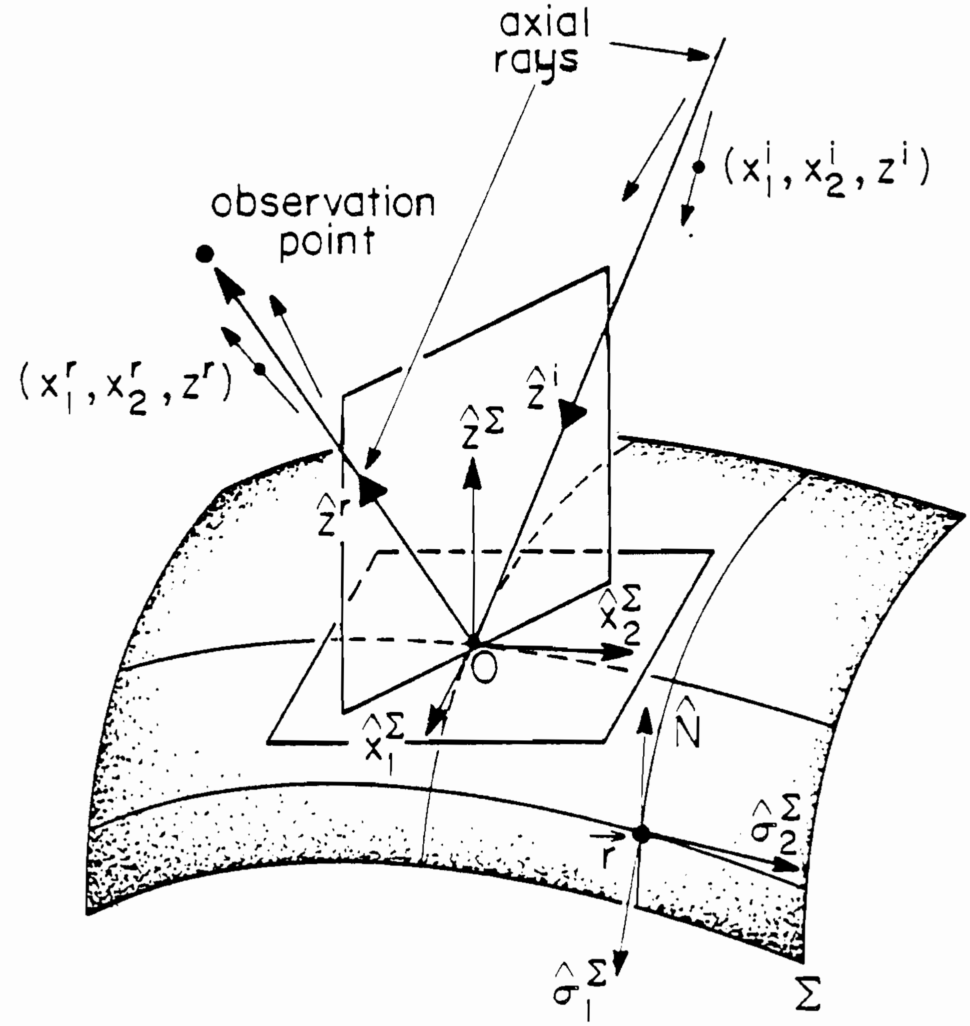


Figure 3-3. Reflection from a smooth conducting surface Σ.

The sign of R_1^Σ or R_2^Σ is positive (negative) if the normal section of Σ bends toward (away from) $\hat{z}^\Sigma$. For example, when Σ is a sphere of radius a, $\hat{z}^\Sigma$ points away from the center, and $R_1^\Sigma = R_2^\Sigma = -a$.

At an arbitrary point $\vec{r} = (x_1^\Sigma, x_2^\Sigma, z^\Sigma)$ on Σ and in the neighborhood of O, we define three local orthonormal base vectors:

$$\hat{\sigma}_1^\Sigma = \hat{x}_1^\Sigma + \hat{z}^\Sigma \frac{x_1^\Sigma}{R_1^\Sigma} + O[(x^\Sigma)^2] \tag{2.16a}$$

$$\hat{\sigma}_2^\Sigma = \hat{x}_2^\Sigma + \hat{z}^\Sigma \frac{x_2^\Sigma}{R_2^\Sigma} + O[(x^\Sigma)^2] \tag{2.16b}$$

$$\hat{N} = \hat{\sigma}^\Sigma = -\hat{x}_1^\Sigma \frac{x_1^\Sigma}{R_1^\Sigma} - \hat{x}_2^\Sigma \frac{x_2^\Sigma}{R_2^\Sigma} + \hat{z}^\Sigma + O[(x^\Sigma)^2] \tag{2.16c}$$

Note that $\left(\hat{\sigma}_1^\Sigma, \hat{\sigma}_2^\Sigma\right)$ lie in the tangent plane, and $\hat{N} = \hat{\sigma}^\Sigma$ is the outward normal of Σ. At O, $\left(\hat{\sigma}_1^\Sigma, \hat{\sigma}_2^\Sigma, \hat{N}\right)$ coincide with $\left(\hat{x}_1^\Sigma, \hat{x}_2^\Sigma, \hat{z}^\Sigma\right)$.

In summary, to describe a position in space, we have introduced six coordinate systems. Three of them are rectangular coordinates with a common origin at O (the point of reflection on Σ), namely,

(i) Incident pencil coordinates with base vectors $(\hat{x}_1^i, \hat{x}_2^i, \hat{z}^i = \hat{x}_3^i)$.

(ii) Reflected pencil coordinates with base vectors $(\hat{x}_1^r, \hat{x}_2^r, \hat{z}^r = \hat{x}_3^r)$.

(iii) Screen coordinates with base vectors $\left(\hat{x}_1^\Sigma, \hat{x}_2^\Sigma, \hat{z}^\Sigma = \hat{x}_3^\Sigma\right)$.

At an arbitrary point $\vec{r}$ on Σ, three additional coordinates are introduced, namely,

(iv) Incident ray coordinates with base vectors $(\hat{\sigma}_1^i, \hat{\sigma}_2^i, \hat{\sigma}^i = \hat{\sigma}_3^i)$.

(v) Reflected ray coordinates with base vectors $(\hat{\sigma}_1^r, \hat{\sigma}_2^r, \hat{\sigma}^r = \hat{\sigma}_3^r)$.

(vi) Local screen coordinates with base vectors $\left(\hat{\sigma}_1^\Sigma, \hat{\sigma}_2^\Sigma, \hat{\sigma}^\Sigma = \hat{\sigma}^\Sigma\right)$.

Note that $z^{i,r}$ measures the arc length along an axial ray, while $\sigma^{i,r}$ measures that along an arbitrary ray. Since any ray can be regarded as the axial ray in a certain pencil, the roles of $z^{i,r}$ and $\sigma^{i,r}$ are interchangeable, as described in connection with (3.13) in Section 2.3.

With the descriptions of the incident pencil, reflected pencil, and screen Σ above, we are ready to determine the reflected field $\vec{E}^r$ in (2.5) for a given incident field $\vec{E}^i$ in (1.2). As discussed in the previous chapter, to determine $\vec{E}^r$ on a reflected ray it is necessary to know:

(i) the curvature matrix $\bar{\bar{Q}}(z^r)$ at a reference point on the same ray, and

(ii) the initial values of $\{\vec{e}_m^{\,r}(\vec{r})\}$, $m = 0,1,2,\ldots$, at a reference point on the same ray.

The continuation of $\bar{\bar{Q}}^r$ follows from (7.9), and that of $\{\vec{e}_m^{\,r}\}$ from (7.15) in Section 2.7. The reference point in the present problem is taken to be the point of reflection O. The initial values of $\bar{\bar{Q}}^r$ and $\{\vec{e}_m^{\,r}\}$ at O are determined from the conditions in (2.6) and (2.8) in the next two sections.

3.3. Phase Matching

In this section, we determine the curvature matrix $\bar{\bar{Q}}^r(z^r)$.* When (1.2) and (2.5) are used in (2.6) or (2.8), it is clear that the boundary conditions can be satisfied only if the phases of the incident and reflected fields are matched over Σ, i.e.,

$$s^i(\vec{r}) = s^r(\vec{r}) \quad , \quad \vec{r} \text{ on } \Sigma \quad . \tag{3.1}$$

We will carry out the matching in (3.1) at a general point $\vec{r}$ in the neighborhood of the point of reflection 0. The coordinates of $\vec{r}$ are $(x_1^\Sigma, x_2^\Sigma, z^\Sigma)$. Alternatively, it can be described by a position vector drawn from 0:

$$\vec{r} = \hat{x}_1^\Sigma x_1^\Sigma + \hat{x}_2^\Sigma x_2^\Sigma + \hat{z}^\Sigma z^\Sigma \quad . \tag{3.2}$$

These coordinates, of course, satisfy (2.14), which describes screen Σ:

$$z^\Sigma = \frac{1}{2} \begin{pmatrix} x_1^\Sigma \\ \\ x_2^\Sigma \end{pmatrix} \cdot \bar{\bar{Q}}^\Sigma \begin{pmatrix} x_1^\Sigma \\ \\ x_2^\Sigma \end{pmatrix} + O[(x^\Sigma)^3] \quad . \tag{3.3}$$

To calculate s^i at $\vec{r}$ from (2.9), it is necessary to know the corresponding coordinates (x_1^i, x_2^i, z^i) of $\vec{r}$ expressed in terms of $(\hat{x}_1^i, \hat{x}_2^i, \hat{z})$. They can be readily found from the relations

$$x_n^i = \vec{r} \cdot \hat{x}_n^i = (\hat{x}_1^\Sigma x_1^\Sigma + \hat{x}_2^\Sigma x_2^\Sigma + \hat{z}^\Sigma z^\Sigma) \cdot \hat{x}_n^i \quad , \quad n = 1,2 \tag{3.4a}$$

$$z^i = \vec{r} \cdot \hat{z}^i = (\hat{x}_1^\Sigma x_1^\Sigma + \hat{x}_2^\Sigma x_2^\Sigma + \hat{z}^\Sigma z^\Sigma) \cdot \hat{z}^i \quad . \tag{3.4b}$$

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*G. A. Deschamps, "Ray techniques in electromagnetics," Proc. IEEE, vol. 60, pp. 1022-1035, 1972.

Let us introduce the notations*

$$p^i_{mn} = \hat{x}^i_m \cdot \hat{x}^\Sigma_n \quad , \quad m,n = 1,2,3 \tag{3.5}$$

where, we emphasize, $(\hat{x}^i_1, \hat{x}^i_2)$ are the principal directions of the incident pencil, and $(\hat{x}^\Sigma_1, \hat{x}^\Sigma_2)$ are the principal directions of screen Σ at O. Then, (3.4) can be written as

$$\begin{pmatrix} x^i_1 \\ \\ x^i_2 \end{pmatrix} = \bar{\bar{P}}^i \begin{pmatrix} x^\Sigma_1 \\ \\ x^\Sigma_2 \end{pmatrix} + O[(x^\Sigma)^2] \tag{3.6a}$$

$$z^i = \left[x^\Sigma_1(\hat{z}^i \cdot \hat{x}^\Sigma_1) + x^\Sigma_2(\hat{z}^i \cdot \hat{x}^\Sigma_2)\right] + \frac{1}{2}\, p^i_{33} \begin{pmatrix} x^\Sigma_1 \\ \\ x^\Sigma_2 \end{pmatrix} \cdot \bar{\bar{Q}}^\Sigma \begin{pmatrix} x^\Sigma_1 \\ \\ x^\Sigma_2 \end{pmatrix} + O[(x^\Sigma)^3] \tag{3.6b}$$

where the 2 x 2 matrix $\bar{\bar{P}}^i$ is defined by

$$\bar{\bar{P}}^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ \\ p^i_{21} & p^i_{22} \end{pmatrix} . \tag{3.7}$$

Substituting (3.6) into (2.9), we obtain the phase function of the incident pencil evaluated at a typical $\vec{r}$ on Σ, namely,

$$s^i(r) = s^i(0) + \left[x^\Sigma_1(\hat{z}^i \cdot \hat{x}^\Sigma_1) + x^\Sigma_2(\hat{z}^i \cdot \hat{x}^\Sigma_2)\right]$$
$$+ \frac{1}{2} \begin{pmatrix} x^\Sigma_1 \\ \\ x^\Sigma_2 \end{pmatrix} \cdot \left[(\bar{\bar{P}}^i)^T\, \bar{\bar{Q}}^i(z^i = 0)\, \bar{\bar{P}}^i + p^i_{33}\, \bar{\bar{Q}}^\Sigma\right] \begin{pmatrix} x^\Sigma_1 \\ \\ x^\Sigma_2 \end{pmatrix} + O[(x^\Sigma)^3] \quad . \tag{3.8}$$

*Remember $\hat{x}^i_3 = \hat{z}^i$ and $\hat{x}^\Sigma_3 = \hat{z}^\Sigma$.

In exactly the same manner, the phase function of the reflected pencil can be evaluated at $\vec{r}$, and the result is

$$s^r(\vec{r}) = s^r(0) + [x_1^\Sigma(\hat{z}^r \cdot \hat{x}_1^\Sigma) + x_2^\Sigma(\hat{z}^r \cdot \hat{x}_2^\Sigma)]$$

$$+ \frac{1}{2} \begin{pmatrix} x_1^\Sigma \\ x_2^\Sigma \end{pmatrix} \cdot [(\bar{\bar{P}}^r)^T \bar{\bar{Q}}^r(z^r = 0)\, \bar{\bar{P}}^r + p_{33}^r\, \bar{\bar{Q}}^\Sigma] \begin{pmatrix} x_1^\Sigma \\ x_2^\Sigma \end{pmatrix} + O[(x^\Sigma)^3] \tag{3.9}$$

where p_{mn}^r are defined by

$$p_{mn}^r = \hat{x}_m^r \cdot \hat{x}_n^\Sigma \quad , \quad m,n = 1,2,3 \quad , \tag{3.10}$$

and $(\hat{x}_1^r, \hat{x}_2^r)$ are the principal directions of the reflected pencil. A slight difficulty may arise here. Except for simple problems, $(\hat{x}_1^r, \hat{x}_2^r)$ may not be known at this moment and, consequently, we may not be able to calculate $\{p_{mn}^r\}$ in (3.10). This difficulty can be readily overcome, as will be explained later.

Now, with s^i and s^r evaluated at $\vec{r}$ and given respectively in (3.8) and (3.9), we can match them in accordance with (3.1). Note that as a function of (x_1^Σ, x_2^Σ), $\vec{r}$ represents a general point on Σ in the neighborhood of O. The condition in (3.1) requires that the phase functions be matched for all values of (x_1^Σ, x_2^Σ). In other words, s^i and s^r must be equal for each order of (x_1^Σ, x_2^Σ). Matching the linear terms of (x_1^Σ, x_2^Σ) in (3.8) and (3.9), we obtain

$$\hat{z}^i \cdot \hat{x}_n^\Sigma = \hat{z}^r \cdot \hat{x}_n^\Sigma \quad , \quad n = 1,2 \tag{3.11a}$$

which states that the projections of $\hat{z}^i$ and $\hat{z}^r$ on the tangent plane of Σ at O must be equal. This is the well-known <u>Snell's law</u> of reflection. The consequences of Snell's law are

$$\hat{z}^r = \hat{z}^i - 2(\hat{z}^\Sigma \cdot \hat{z}^i)\,\hat{z}^\Sigma \tag{3.11b}$$

$$p_{33}^i = -p_{33}^r \quad . \tag{3.12}$$

Matching the quadratic terms of (3.8) and (3.9) yields

$$(\bar{\bar{P}}^i)^T\,\bar{\bar{Q}}^i(z^i = 0)\,\bar{\bar{P}}^i + p_{33}^i\,\bar{\bar{Q}}^\Sigma = (\bar{\bar{P}}^r)^T\,\bar{\bar{Q}}^r(z^r = 0)\,\bar{\bar{P}}^r + p_{33}^r\,\bar{\bar{Q}}^\Sigma \quad . \tag{3.13a}$$

In an abbreviated form, (3.13a) is written as

$$(\bar{\bar{P}}^i)^T\,\bar{\bar{Q}}^i(z^i = 0)\,\bar{\bar{P}}^i + p_{33}^i\,\bar{\bar{Q}}^\Sigma = \{i \to r\} \tag{3.13b}$$

where $\{i \to r\}$ means that all terms on the left of the equal sign are repeated and that the superscript "i" is changed to "r." From (3.13), the curvature matrix $\bar{\bar{Q}}^r$ at point of reflection O can be solved. The continuation of $\bar{\bar{Q}}^r$ to other points on the reflected ray is governed by (7.9) in Section 2.7. Several discussions about the relation in (3.13) are given below.

(i) In (3.5), (3.7) and (3.10), we have specified that $(\hat{x}_1^{i,r,\Sigma}, \hat{x}_2^{i,r,\Sigma})$ are principal directions. This specification, however, is not necessary. As can be easily seen from its derivation, the final result (3.13) is valid for <u>arbitrary</u> directions $(\hat{x}_1^{i,r,\Sigma}, \hat{x}_2^{i,r,\Sigma})$ which are mutually orthogonal and are transverse to $\hat{z}^{i,r,\Sigma}$, respectively.

(ii) If $(\hat{x}_1^r, \hat{x}_2^r)$ are the principal directions of the reflected pencil, $\bar{\bar{Q}}^r$ calculated from (3.13) is diagonal, i.e., in the form of (7.7) in Section 2.7. In some problems, however, the principal directions of the reflected pencil cannot be readily recognized before $\bar{\bar{Q}}^r$ is found. Then, for the purpose of calculating $\bar{\bar{Q}}^r$, two arbitrary orthogonal directions $(\hat{x}_1^r, \hat{x}_2^r)$ can be used in (3.10). A good choice of the two arbitrary directions is

$$\hat{x}_n^r = \hat{x}_n^i - 2(\hat{z}^\Sigma \cdot \hat{x}_n^i)\,\hat{z}^\Sigma \quad , \quad n = 1,2 \tag{3.14}$$

where $(\hat{x}_1^r, \hat{x}_2^r)$ are obtained by "reflecting" $(\hat{x}_1^i, \hat{x}_2^i)$ from the surface Σ at O, and are no longer of unit length. With such a choice, $(\hat{x}_1^r, \hat{x}_2^r, \hat{z}^r)$ form a left-hand system (which is perfectly acceptable), and

$$\bar{\bar{P}}^i = \bar{\bar{P}}^r \quad . \tag{3.15}$$

Making use of (3.15) and $p_{33}^i = -p_{33}^r$, (3.13) is simplified to become

$$\bar{\bar{Q}}^r(z^r = 0) = \bar{\bar{Q}}^i(z^i = 0) + 2p_{33}^i((\bar{\bar{P}}^i)^{-1})^T \, \bar{\bar{Q}}^\Sigma (\bar{\bar{P}}^i)^{-1} \quad . \tag{3.16}$$

Generally, $\bar{\bar{Q}}^r$ calculated from (3.16) is not diagonal. The principal curvature and directions of the reflected pencil nevertheless can be readily calculated from the general $\bar{\bar{Q}}^r$ by using (13.29) in Appendix A.

(iii) In Section 3.4 (next section), $(\hat{x}^{i,r,\Sigma}, \hat{x}^{i,r,\Sigma})$ are always taken as principal directions unless they are specified otherwise.

3.4. Amplitude Matching

In this section we determine the amplitude vectors $\{\vec{e}_m^r\}$ of the reflected field*. At point of reflection O, the boundary condition in (2.6) is equivalent to

$$(\vec{E}^i + \vec{E}^r) \cdot \hat{x}_n^\Sigma = 0 \quad , \quad n = 1,2 \quad . \tag{4.1}$$

Substituting (1.2) and (2.5) into (4.1) and (2.8), we obtain two sets of equations for determining $\{\vec{e}_m^r\}$ at O, namely,

$$\begin{cases} \vec{e}_m^i \cdot \hat{x}_n^\Sigma = -\vec{e}_m^r \cdot \hat{x}_n^\Sigma \quad , \quad n = 1,2 & (4.2a) \\ \left(\vec{e}_m^i \cdot \hat{z}^\Sigma\right) \dfrac{\partial s^i}{\partial z^\Sigma} + \nabla \cdot \vec{e}_{m-1}^i = -\left(\vec{e}_m^r \cdot \hat{z}^\Sigma\right) \dfrac{\partial s^r}{\partial z^\Sigma} - \nabla \cdot \vec{e}_{m-1}^r \quad , & (4.2b) \end{cases}$$

$$m = 0,1,2, \ldots, \text{ and } \vec{r} \text{ at O.}$$

In deriving (4.2b) we have made use of (4.2a). We emphasize that (4.2) is valid only at O, not in the neighboring points of O [cf. Problem 3-1]. Since O is a typical point on screen Σ, the solution of (4.2) is sufficient to give us the desired initial values of $\{\vec{e}_m^r\}$ over Σ.

Let us study (4.2) in some detail. Since $\vec{e}_{-1}^{i,r} = 0$, the zeroth-order solution of the amplitude vector at O can be easily found from (4.2), namely,

$$e_{03}^r = 0 \tag{4.3a}$$

$$\begin{pmatrix} p_{11}^i & p_{21}^i \\ p_{12}^i & p_{22}^i \end{pmatrix} \begin{pmatrix} e_{01}^i \\ e_{02}^i \end{pmatrix} = - \begin{pmatrix} p_{11}^r & p_{21}^r \\ p_{12}^r & p_{22}^r \end{pmatrix} \begin{pmatrix} e_{01}^r \\ e_{02}^r \end{pmatrix} \tag{4.3b}$$

where we have made use of the fact that

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*S. W. Lee, "Electromagnetic reflection from a conducting surface: geometrical optics solution," IEEE Trans. Antennas Propagat., Vol. AP-23, pp. 184-191, 1975.

$$\vec{e}_0^{\,i,r}(\vec{r} \text{ at } 0) = \hat{x}_1^{i,r} e_{01}^{i,r} + \hat{x}_2^{i,r} e_{02}^{i,r} + \hat{z}^{i,r} e_{03}^{i,r} \quad , \tag{4.3c}$$

a fact derived from (2.13) with $x_1^{i,r} = x_2^{i,r} = z^{i,r} = 0$. It can be readily shown that (4.3) remains valid when $(\hat{x}_1^{i,r}, \hat{x}_2^{i,r})$ are not principal directions. The solution of $\vec{e}_0^{\,r}$ determined from (4.3) is a transverse wave, the same as $\vec{e}_0^{\,i}$. The solution in (4.3) may be expressed in a different form, namely,

$$\vec{e}_0^{\,r} = -\vec{e}_0^{\,i} + 2(\vec{e}_0^{\,i} \cdot \hat{z}^{\Sigma})\, \hat{z}^{\Sigma} \tag{4.3d}$$

where $\vec{e}_0^{\,i,r}$ are evaluated at the reflection point 0. Once the initial value of $\vec{e}_0$ at 0 is known, it may be continued by using (7.12) in Section 2.7, namely,

$$\vec{e}_0^{\,r}(z^r) = \frac{\vec{e}_0^{\,r}(z^r = 0)}{\sqrt{1 + (z^r/R_1^r)}\ \sqrt{1 + (z^r/R_2^r)}} \tag{4.4}$$

where $z^r = \sigma^r$ is the arclength measured from 0 along a reflected ray. The square roots in (4.4) must be calculated by following the convention specified in (7.11) in Section 2.7.

The higher-order solutions of $\{\vec{e}_m^{\,r}\}$ are more complex. The two derivative quantities in (4.2b) have to be calculated first. From (2.9), (2.10), and (3.5) it follows that

$$\left.\frac{\partial s^{i,r}}{\partial z^{\Sigma}}\right|_{\vec{r} \text{ at } 0} = \sum_{n=1}^{3} \frac{\partial x_n^{i,r}}{\partial z^{\Sigma}} \frac{\partial s^{i,r}}{\partial x_n^{i,r}} = p_{33}^{i,r} \quad . \tag{4.5}$$

From (2.13), we have

$$\left.\nabla \cdot \vec{e}_{m-1}^{\,i,r}\right|_{\vec{r} \text{ at } 0} = \sum_{q=1}^{3} \frac{\partial}{\partial x_q^{i,r}} \left(\vec{e}_{m-1}^{\,i,r} \cdot \hat{x}_q^{i,r}\right)$$

$$= \left(\frac{1}{R_1^{i,r}} + \frac{1}{R_2^{i,r}}\right) e_{m-1,3}^{i,r} + \sum_{q=1}^{3} \frac{\partial e_{m-1,q}^{i,r}}{\partial x_q^{i,r}} \quad . \tag{4.6}$$

If (4.5) is substituted into (4.2), the latter becomes

$$\begin{pmatrix} p^i_{11} & p^i_{21} & p^i_{31} \\ p^i_{12} & p^i_{22} & p^i_{32} \\ p^i_{13}p^i_{33} & p^i_{23}p^i_{33} & p^i_{33}p^i_{33} \end{pmatrix} \begin{pmatrix} e^i_{m,1} \\ e^i_{m,2} \\ e^i_{m,3} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \nabla \cdot \vec{e}^{\,i}_{m-1} \end{pmatrix} = (-1)\ \{i \to r\} \quad ,$$

$$m = 1,2,3,\ \ldots \qquad (4.7)$$

where $\nabla \cdot \vec{e}^{\,i,r}_{m-1}$ was given in (4.6). This is the desired equation from which one can solve for the value of $\vec{e}^{\,r}_m$ at 0, in terms of $\vec{e}^{\,i}_m$, $\vec{e}^{\,i,r}_{m-1}$, and the derivatives of $\vec{e}^{\,i,r}_{m-1}$ at 0. For a given incident field $\vec{E}^i$ in (1.2), $\vec{e}^{\,i}_m$ and its derivatives are known. $\vec{e}^{\,r}_{m-1}$ is found from the previous calculation. The only less explicit parts in (4.7) are the derivatives of $\vec{e}^{\,r}_{m-1}$ of the reflected field that appear in $\nabla \cdot \vec{e}^{\,r}_{m-1}$. For m = 1, the derivatives of $\vec{e}^{\,r}_0$ at 0 can be solved from the following equations:

$$\begin{bmatrix} \dfrac{p^i_{13}}{R^{\Sigma}_1} - \dfrac{p^i_{31}p^i_{11}}{R^i_1} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{31}p^i_{11} \\ - \dfrac{p^i_{32}p^i_{11}}{R^i_1} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{31}p^i_{12} \\ - \dfrac{p^i_{31}p^i_{12}}{R^i_1} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{32}p^i_{11} \\ \dfrac{p^i_{13}}{R^{\Sigma}_2} - \dfrac{p^i_{32}p^i_{12}}{R^i_1} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{32}p^i_{12} \end{bmatrix} e^i_{01} + \begin{bmatrix} \dfrac{p^i_{23}}{R^{\Sigma}_1} - \dfrac{p^i_{31}p^i_{21}}{R^i_2} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{31}p^i_{21} \\ - \dfrac{p^i_{32}p^i_{21}}{R^i_2} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{31}p^i_{22} \\ - \dfrac{p^i_{31}p^i_{22}}{R^i_2} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{32}p^i_{21} \\ \dfrac{p^i_{23}}{R^{\Sigma}_2} - \dfrac{p^i_{32}p^i_{22}}{R^i_2} - \dfrac{1}{2}\left(\dfrac{1}{R^i_1} + \dfrac{1}{R^i_2}\right) p^i_{32}p^i_{22} \end{bmatrix} e^i_{02}$$

$$
+\begin{bmatrix}
p^i_{11}p^i_{11} & p^i_{11}p^i_{21} & p^i_{21}p^i_{11} & p^i_{21}p^i_{21} \\
p^i_{11}p^i_{12} & p^i_{11}p^i_{22} & p^i_{21}p^i_{12} & p^i_{21}p^i_{22} \\
p^i_{12}p^i_{11} & p^i_{12}p^i_{21} & p^i_{22}p^i_{11} & p^i_{22}p^i_{21} \\
p^i_{12}p^i_{12} & p^i_{12}p^i_{22} & p^i_{22}p^i_{12} & p^i_{22}p^i_{22}
\end{bmatrix}
\begin{bmatrix}
\dfrac{\partial e^i_{01}}{\partial x^i_1} \\[2mm]
\dfrac{\partial e^i_{02}}{\partial x^i_1} \\[2mm]
\dfrac{\partial e^i_{01}}{\partial x^i_2} \\[2mm]
\dfrac{\partial e^i_{02}}{\partial x^i_2}
\end{bmatrix}
= (-1)\ \{i \to r\} \tag{4.8}
$$

where all the values of $e^{i,r}_{0m}$ and their derivatives are those at O. Note that (4.8) represents four linear equations for four unknowns

$$
\frac{\partial e^r_{01}}{\partial x^r_1}\ ,\quad \frac{\partial e^r_{02}}{\partial x^r_1}\ ,\quad \frac{\partial e^r_{01}}{\partial x^r_2}\ ,\quad \text{and}\ \frac{\partial e^r_{02}}{\partial x^r_2}\ . \tag{4.9}
$$

The derivation of (4.8) is given in Problem 3-1.

To calculate the higher-order $\vec{e}^{\,r}_m$ beyond m = 1 requires the knowledge of the derivatives of $\vec{e}^{\,r}_{m-1}$. General explicit formulas for calculating the derivatives of $\vec{e}^{\,r}_{m-1}$ are too cumbersome to be included here. It is often simpler to derive them for each individual problem. Furthermore, in practical calculations, one seldom takes more than two terms in the reflected field in (2.5), therefore, (4.8) suffices in those situations.

Let us summarize the steps in determining the reflected field in (2.5) for a given incident field in (1.2) and a reflecting surface Σ described in (2.14) and (2.15).

(i) Determine the orthonormal base vectors $(\hat{\sigma}^\Sigma_1, \hat{\sigma}^\Sigma_2, \hat{\sigma}^\Sigma)$, with $(\hat{\sigma}^\Sigma_1, \hat{\sigma}^\Sigma_2)$ in the principal directions of Σ. Then using Snell's law in (3.11), locate the point of reflection O for a given observation point.

(ii) Arbitrarily assuming two orthogonal transverse directions $\{\hat{x}_1^r, \hat{x}_2^r\}$ in the reflected pencil, calculate $\bar{\bar{Q}}^r(z^r = 0)$ from (3.13). In the case that $\bar{\bar{Q}}^r$ so obtained is diagonal, $\{\hat{x}_1^r, \hat{x}_2^r\}$ are in the principal directions of the reflected wavefront. Otherwise, one has to diagonalize $\bar{\bar{Q}}^r$ and determine principal directions $\{\hat{x}_1^r, \hat{x}_2^r\}$ by using (13.29) in Appendix A.

(iii) The initial value of s^r at O is determined from (3.1), and its continuation is governed by (7.6) in Section 2.7.

(iv) The initial values of $\{\vec{e}_m^{\,r}\}$ at O are determined in succession. For $m = 0$, $\vec{e}_0^{\,r}$ can be solved from (4.3). For $m > 0$, $\vec{e}_m^{\,r}$ can be solved from (4.7). In the latter calculation, (4.8) is useful in the determination of the derivatives of $\vec{e}_0^{\,r}$.

(v) The continuation of $\{\vec{e}_m^{\,r}\}$ along a ray is governed by (7.15) in Section 2.7.

Corresponding to the electric fields given in (1.2) and (2.5), the magnetic fields are

$$\vec{H}^{i,r}(\vec{r}) \sim e^{iks^{i,r}(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m}\, \vec{h}_m^{\,i,r}(\vec{r}) \quad , \quad k \to \infty \tag{4.10}$$

$$\vec{h}_m^{\,i,r} = \sqrt{\frac{\varepsilon}{\mu}} \left[\nabla s^{i,r} \times \vec{e}_m^{\,i,r} + \nabla \times \vec{e}_{m-1}^{\,i,r}\right] \quad . \tag{4.11}$$

In the calculation of $\nabla \times \vec{e}_{m-1}^{\,i,r}$ in (4.11), it is important that the expression in (2.13) be used for $\vec{e}_{m-1}^{\,i,r}$. For the first two orders, the amplitude vectors of the incident and reflected fields at O determined from (4.11) are found to be

$$\vec{h}_0^{\,i} = \sqrt{\frac{\varepsilon}{\mu}} \left[-\hat{x}_1^i e_{02}^i + \hat{x}_2^i e_{01}^i\right] \tag{4.12}$$

$$\vec{h}_1^i = \sqrt{\frac{\varepsilon}{\mu}} \left\{ \hat{x}_1^i \left[-e_{12}^i + \frac{1}{2} e_{02}^i \left(\frac{1}{R_1^i} - \frac{1}{R_2^i} \right) \right] + \hat{x}_2^i \left[e_{11}^i + \frac{1}{2} e_{01}^i \left(\frac{1}{R_1^i} - \frac{1}{R_2^i} \right) \right] \right.$$

$$\left. + \hat{z}^i \left[\frac{\partial e_{02}^i}{\partial x_1^i} - \frac{\partial e_{01}^i}{\partial x_2^i} \right] \right\} \quad . \qquad (4.13)$$

As usual, (4.12) and (4.13) remain valid when superscript "i"'s are changed to "r"'s. The derivatives of $\vec{e}_0^r$ of the reflected field have been already found from the solutions of (4.8).

The surface current density at O is related to the magnetic field through the familiar equation:

$$\vec{J}_s = e^{iks^i} \hat{z}^\Sigma \times \left[\left(\vec{h}_0^i + \vec{h}_0^r \right) + \frac{1}{ik} \left(\vec{h}_1^i + \vec{h}_1^r \right) + O(k^{-2}) \right] \quad . \qquad (4.14)$$

Because of the fact that

$$\hat{z}^\Sigma \times \left(\vec{h}_0^i + \vec{h}_0^r \right) = 2\hat{z}^\Sigma \times \vec{h}_0^i \quad , \qquad (4.15)$$

(4.14) can be rewritten as

$$\vec{J}_s = e^{iks^i} \hat{z}^\Sigma \times \left[2\vec{h}_0^i + \frac{1}{ik} \left(\vec{h}_1^i + \vec{h}_1^r \right) + O(k^{-2}) \right] \quad . \qquad (4.16)$$

The first term in (4.16) is often known as the physical-optics current. Note that since $\vec{J}_s$ depends only on the initial values of $\{\vec{h}_m^{i,r}\}$ (and therefore $\{\vec{e}_m^{i,r}\}$) on Σ, its computation is simpler than those of fields at points away from Σ.

3.5. Summary

(1) We are interested in the scattering of a given incident field $\vec{E}^i$ by a perfectly conducting screen Σ (Figure 3-3). According to the (classical) geometrical optics theory, the total field $\vec{E}^t$ at an observation point $\vec{r}$ is

$$\vec{E}^t(\vec{r}) = \theta(-\varepsilon^i)\ \vec{E}^i(\vec{r}) + \theta(-\varepsilon^r)\ \vec{E}^r(\vec{r}) \quad . \tag{5.1}$$

Here $\theta(x)$ is the unit step function. The incident field $\vec{E}^i$ is assumed to be a ray field, which is asymptotically given by

$$\vec{E}^i(\vec{r}) \sim e^{iks^i(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m}\ \vec{e}^{\,i}_m(\vec{r}) \quad , \quad k \to \infty \quad . \tag{5.2}$$

The shadow indicator $\varepsilon^i(\vec{r})$ of the incident field is defined by

$$\varepsilon^i(\vec{r}) = \begin{cases} +1 \quad , & \text{if } \vec{r} \text{ is in the shadow region of the incident field,} \\ -1 \quad , & \text{if } \vec{r} \text{ is in the lit region of the incident field} \end{cases} \quad . \tag{5.3}$$

The lit and shadow regions of the incident field are separated by the incident shadow boundary SB^i. In exactly the same manner, $\vec{E}^r$, ε^r, and SB^r of the reflected field are defined.

(2) The solution given in (5.1), known as the geometrical optics field $\vec{E}^g$, is only approximately valid for high frequency. It does not include the contribution from diffraction. Furthermore, in the discussion of this chapter, possible multiple reflections between different portions of Σ are ignored, as they can be accounted for separately.

(3) Snell's law of reflection (Figure 3-3). Consider an incident ray propagating in the direction $\hat{z}^i$ and meeting Σ at O (point of reflection).

The direction $\hat{z}^r$ of the reflected ray can be determined from the relation

$$\hat{z}^r = \hat{z}^i - 2(\hat{z}^\Sigma \cdot \hat{z}^i)\,\hat{z}^\Sigma \tag{5.4}$$

where $\hat{z}^\Sigma$ is the outward normal of Σ at O.

(4) Incident and reflected pencils. The incident pencil is characterized by its direction of propagation $\hat{z}^i$, the principal directions $(\hat{x}_1^i, \hat{x}_2^i)$ of its wavefronts, and the principal radii of curvature (R_1^i, R_2^i) at O. Its phase function at a point $\vec{r} = (x_1^i, x_2^i, z^i)$ is given by

$$s^i(\vec{r}) = s^i(x_1^i = x_2^i = z^i = 0) + z^i + \frac{1}{2}\begin{pmatrix} x_1^i \\ \\ x_2^i \end{pmatrix} \cdot \bar{\bar{Q}}^i(z^i)\begin{pmatrix} x_1^i \\ \\ x_2^i \end{pmatrix} + O[(x^i)^3] \tag{5.5}$$

where the curvature matrix $\bar{\bar{Q}}^i(z^i)$ was defined in (7.7) and (7.9) of Section 2.7. The amplitude vector $\vec{e}_m^{\,i}$ can be resolved into three components:

$$\vec{e}_m^{\,i}(\vec{r}) = \hat{\sigma}_1^i e_{m1}^i + \hat{\sigma}_2^i e_{m2}^i + \nabla s^i e_{m3}^i \tag{5.6}$$

where $(\hat{\sigma}_1^i, \hat{\sigma}_2^i, \nabla s^i)$ are defined in (2.11). Thus, with respect to the direction of propagation of each ray (axial or paraxial), (e_{m1}^i, e_{m2}^i) are the transverse components, and e_{m3}^i the longitudinal component. The same formula and convention hold for the reflected pencil.

(5) Screen Σ. At O, the principal directions of Σ are $(\hat{x}_1^\Sigma, \hat{x}_2^\Sigma)$ and its outward normal is $\hat{z}^\Sigma$. The principal radii of Σ, denoted by (R_1^Σ, R_2^Σ), are positive (negative) if the respective normal section bends toward (away from) $\hat{z}^\Sigma$.

(6) Curvature matrix of reflected field. The initial value of the curvature matrix $\bar{\bar{Q}}^r(z^r = 0)$ at 0 can be calculated from the matrix equation:

$$(\bar{\bar{P}}^i)^T \bar{\bar{Q}}^i(z^i = 0) \bar{\bar{P}}^i + p_{33}^i \bar{\bar{Q}}^\Sigma = (\bar{\bar{P}}^r)^T \bar{\bar{Q}}^r(z^r = 0) \bar{\bar{P}}^r + p_{33}^r \bar{\bar{Q}}^\Sigma \tag{5.7}$$

where

$$\bar{\bar{P}}^{i,r} = \begin{bmatrix} p_{11}^{i,r} & p_{12}^{i,r} \\ p_{21}^{i,r} & p_{22}^{i,r} \end{bmatrix} \tag{5.8}$$

$$p_{mn}^{i,r} = \hat{x}_m^{i,r} \cdot \hat{x}_n^\Sigma \quad , \quad m,n = 1,2,3 \quad . \tag{5.9}$$

The three equations in (5.7) through (5.9) remain valid even if $(\hat{x}_1^{i,r}, \hat{x}_2^{i,r})$ are not principal directions but two arbitrary orthogonal directions. In the later case, $\bar{\bar{Q}}^r$ determined from (5.7) is not diagonal. Its principal curvatures and directions can be found by using (13.29) in Appendix A. Once $\bar{\bar{Q}}^r(z^r = 0)$ is found from (5.7), its continuation to other points on the reflected (axial) ray follows from (7.9) in Section 2.7. For a clever choice of $(\hat{x}_1^r, \hat{x}_2^r)$, the matrix equation in (5.7) may be simplified to become (3.16).

(7) Phase of reflected field. Along a reflected axial ray, the phase of the reflected field is

$$s^r(0,0,z^r) = s^i(0,0,0) + z^r \tag{5.10a}$$

where $(x_1^r = 0,\ x_2^r = 0,\ z^r = 0)$ is the reflection point 0 (Figure 3-3). Since any ray can be considered as an axial ray, (5.10a) may be rewritten as

$$s^r(\sigma^r) = s^i(\sigma^r = 0) + \sigma^r \tag{5.10b}$$

where σ^r is the arclength of the reflected ray measured from its reflection point.

(8) Zeroth-order solution of reflected field. At reflection point 0, the initial values of the zeroth-order amplitude vectors are given by

$$\vec{e}_0^{\,r} = -\vec{e}_0^{\,i} + 2(\hat{z}^{\Sigma} \cdot \vec{e}_0^{\,i})\, \hat{z}^{\Sigma} \tag{5.11a}$$

$$\vec{h}_0^{\,r} = +\vec{h}_0^{\,i} - 2(\hat{z}^{\Sigma} \cdot \vec{h}_0^{\,i})\, \hat{z}^{\Sigma} \tag{5.11b}$$

where $\vec{e}_0^{\,i,r}$, $\vec{h}_0^{\,i,r}$ and $\hat{z}^{\Sigma}$ are all evaluated at 0 (Figure 3-3). Along a reflected ray passing through 0, the zeroth-order reflected field is given by

$$\vec{E}^r(\sigma^r) = e^{iks^r} \frac{\vec{e}_0^{\,r}(\sigma^r = 0)}{\sqrt{1 + (\sigma^r/R_1^r)}\ \sqrt{1 + (\sigma^r/R_2^r)}} + O(k^{-1}) \tag{5.12}$$

where s^r is given in (5.10b) and $\vec{e}_0^{\,r}(\sigma^r = 0)$ in (5.11). The two radii of curvature (R_1^r, R_2^r) can be calculated from $\bar{\bar{Q}}^r$ determined from (5.7). An expression similar to (5.12) holds for $\vec{H}^r(\sigma^r)$. The zeroth-order solution is locally a plane wave, namely,

$$\nabla s^r \cdot \vec{e}_0^{\,r}(\sigma^r) = 0 \tag{5.13a}$$

$$\vec{h}_0^{\,r}(\sigma^r) = \sqrt{\frac{\varepsilon}{\mu}}\, \nabla s^r \times \vec{e}_0^{\,r}(\sigma^r) \quad . \tag{5.13b}$$

(9) Higher-order solution of reflected field. At reflection point 0, the initial value of a higher-order amplitude vector $\vec{e}_m$ with $m = 1,2,\ldots,$ can be solved from (4.6). In such a determination, the knowledge of $\nabla \cdot \vec{e}_{m-1}$ at 0 is needed. For the case $m = 1$, this knowledge can be obtained from (4.5) and (4.7). For cases with $m > 1$, no simple explicit formulas have been deduced. Once the initial values of $\{e_m^r\}$ at 0 are known, their continuation to other points on the reflected ray follows from (7.15) in Section 2.7.

3.6. Reflection From a Two-Dimensional Parabolic Cylinder

For application of formulas developed in this chapter, we begin by considering a simple two-dimensional problem: reflection of a normally incident plane wave by a parabolic cylinder.* Referring to Figure 3-4, the incident field is given by

$$\vec{E}^i(\vec{r}) = \hat{z}e^{-ikx} \quad . \tag{6.1}$$

At a typical point $(\rho = \rho_0, \phi)$ on the surface Σ of the parabolic cylinder, the following relation holds:

$$\Sigma: \quad \rho_0(\phi) = a \sec^2 \frac{\phi}{2} \quad , \quad -\pi < \phi < \pi \tag{6.2}$$

where a is the focal length of the parabolic cylinder. Since there is no variation along z, the reflected field $\vec{E}^r$ will also be polarized in the z-direction, and we have equivalently a scalar problem. As a consequence of this, the amplitude matching becomes very simple, and in fact none of the formulas presented in Section 3.4 is needed.

First, we have to expand the incident field in an asymptotic series as in (5.2). A comparison of (6.1) and (5.2) yields immediately, at an observation point $\vec{r}$ with cylindrical coordinates (ρ, ϕ),

$$s^i(\rho,\phi) = -x \quad , \quad \vec{e}_0^{\,i}(\rho,\phi) = \hat{z} \quad , \quad \vec{e}_m^{\,i}(\rho,\phi) = 0 \quad , \quad m = 1,2,\ldots \tag{6.3}$$

Let us express the reflected field in a similar asymptotic series

$$\vec{E}^r(\vec{r}) \sim e^{iks^r(\vec{r})} \sum_{m=0}^{\infty} (ik)^{-m} \vec{e}_m^{\,r}(\vec{r}) \quad , \quad k \to \infty \quad . \tag{6.4}$$

Matching the boundary condition of the tangential electric field on Σ gives

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*J. B. Keller, R. M. Lewis, and B. D. Seckler, "Asymptotic solution of some diffraction problems," Comm. Pure Appl. Math. 9, 207-265, 1956.

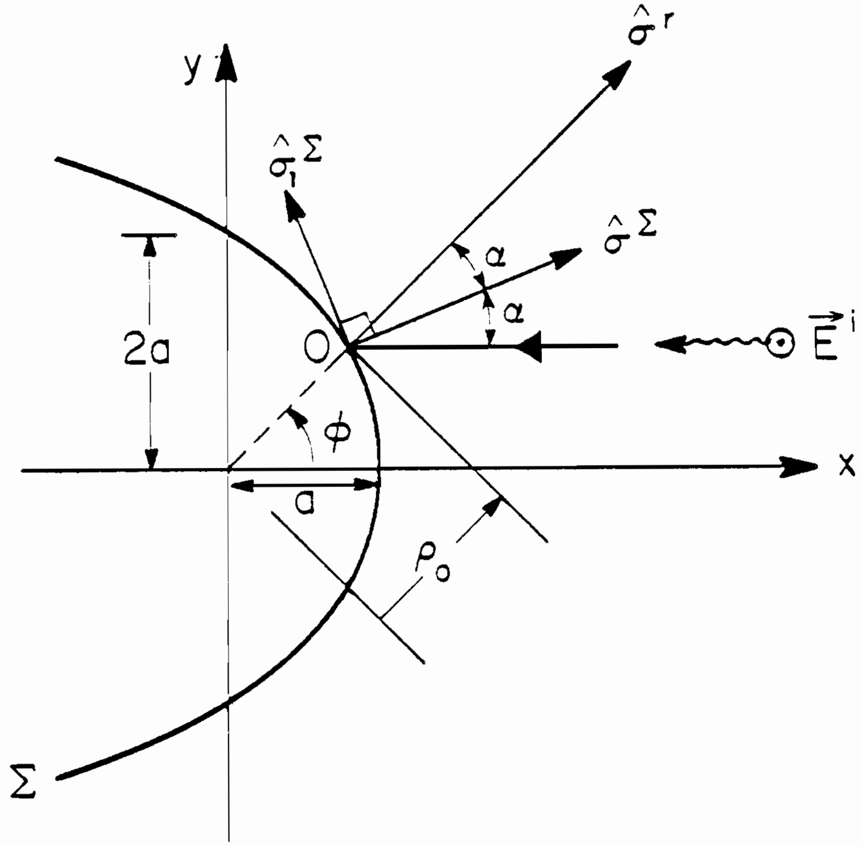

Figure 3-4. Reflection of a normally incident plane wave from a parabolic cylinder.

$$s^r(\rho_0,\phi) = s^i(\rho_0,\phi) = -\rho_0 \cos\phi \quad , \tag{6.5a}$$

$$\vec{e}^{\,r}_0(\rho_0,\phi) = \hat{z}(-1) \quad , \tag{6.5b}$$

$$\vec{e}^{\,r}_m(\rho_0,\phi) = 0 \quad , \quad m = 1,2,\ldots \quad . \tag{6.5c}$$

We emphasize that (6.5) gives only the initial waves of the reflected field on Σ. They have to be continued in order to obtain the reflected field at an arbitrary observation point (ρ,ϕ) away from Σ.

As shown in Problem 3-3, the reflected field is a cylindrical wave with "phase center" at the focal line of the cylinder $(x = 0, y = 0)$. This is a well-known property of a parabolic cylinder. Written explicity, we have

$$\hat{\sigma}^r = \hat{\rho} \tag{6.6}$$

$$\bar{\bar{Q}}^r(\sigma^r = 0) = \begin{pmatrix} (\rho_0)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \quad . \tag{6.7}$$

The arclength σ^r along a reflected ray is measured from the point (ρ_0,ϕ) on Σ.

The continuation of the phase along a reflected ray is governed by (5.10). For the present problem we obtain

$$s^r(\rho,\phi) = -\rho_0 \cos\phi + (\rho - \rho_0) = \rho - 2a \tag{6.8}$$

where we have made use of (6.5a).

As a last step, consider the continuation of the amplitude vectors $\{\vec{e}^{\,r}_m\}$. The appropriate formula for this purpose is (7.15) in Section 2.7. We have for the present problem

$$\sigma^r = \rho - \rho_0 \quad , \quad \sigma^r_0 = 0 \tag{6.9a}$$

$$\frac{j(0)}{j(\sigma^r)} = \frac{\det \bar{\bar{Q}}^r(\sigma^r)}{\det \bar{\bar{Q}}^r(0)} = \frac{\rho_0}{\rho} \quad . \tag{6.9b}$$

Then (7.15) in Section 2.7 becomes

$$\vec{e}^{\,r}_m(\rho,\phi) = \vec{e}^{\,r}_m(\rho_0,\phi)\left(\frac{\rho_0}{\rho}\right)^{1/2} - \frac{1}{2}\int_{\rho_0}^{\rho}\left(\frac{\rho'}{\rho}\right)^{1/2} \nabla^2\vec{e}^{\,r}_{m-1}(\rho',\phi)\, d\rho' \quad . \tag{6.10}$$

We have to calculate (6.10) consecutively in m. Starting with $m = 0$, we have, with the help of (6.5b),

$$\vec{e}^{\,r}_0(\rho,\phi) = \hat{z}(-1)\left(\frac{\rho_0}{\rho}\right)^{1/2} = \hat{z}(-1)\left(\frac{a}{\rho}\sec^2\frac{\phi}{2}\right)^{1/2} \quad . \tag{6.11}$$

For $m = 1$, the Laplacian of $\vec{e}^{\,r}_0$ is given by

$$\begin{aligned}
\nabla^2\vec{e}_0(\rho,\phi) &= \hat{z}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right](-1)\left(\frac{\rho_0}{\rho}\right)^{1/2} \\
&= \hat{z}\rho^{-5/2}\left\{\left[\left(\frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2}\right](-1)\,\rho_0^{1/2}\right\} \\
&= \hat{z}\left(-\frac{1}{2}\right)\left(\frac{a}{\rho}\right)^{1/2}\rho^{-2}\sec^3\frac{\phi}{2} \quad .
\end{aligned} \tag{6.12}$$

Substitute (6.5c) and (6.12) into (6.10) and perform the integration. The result is found to be

$$\vec{e}^{\,r}_1(\rho,\phi) = \hat{z}\,\frac{1}{a}\left[\frac{1}{4}\left(\frac{a}{\rho}\sec^2\frac{\phi}{2}\right)^{1/2} + \left(-\frac{1}{4}\right)\left(\frac{a}{\rho}\sec^2\frac{\phi}{2}\right)^{3/2}\right] \quad . \tag{6.13}$$

Guided by the solutions in (6.11) for $m = 0$ and in (6.13) for $m = 1$, we can determine the higher-order amplitude vectors recursively instead of successively, as done below. Assume that $\vec{e}^{\,r}_m$ can be written in the form

$$\vec{e}_m^{\,r}(\rho,\phi) = \hat{z}a^{-m} \sum_{n=0}^{m} d_{nm} \left(\frac{a}{\rho} \sec^2 \frac{\phi}{2} \right)^{n+(1/2)} \quad , \quad m = 0,1,2,\ldots \quad . \tag{6.14}$$

Here, $\{d_{nm}\}$ are undetermined constants except for the first one

$$d_{00} = -1 \quad . \tag{6.15}$$

This inital value was obtained by a comparison of (6.14) and (6.11). From the expression in (6.14), we can calculate the Laplacian of $\vec{e}_{m-1}^{\,r}$:

$$\nabla^2 \vec{e}_{m-1}^{\,r}(\rho,\phi) = \hat{z}a^{-(m+1)} \sum_{n=0}^{m-1} d_{n,m-1} \left(\frac{a}{\rho}\right)^{n+(5/2)} \left[\left(n + \frac{1}{2}\right)^2 + \frac{\partial^2}{\partial\phi^2} \right] \left(\sec \frac{\phi}{2}\right)^{2n+1}$$

$$= \hat{z}a^{-(m+1)} \sum_{n=0}^{m-1} d_{n,m-1} \left(\frac{a}{\rho}\right)^{n+(5/2)} \left(n + \frac{1}{2}\right) (n+1) \left(\sec \frac{\phi}{2}\right)^{2n+3} \quad . \tag{6.16}$$

Substitute (6.3c) and (6.16) into (6.10) and perform the integration. This operation gives the result

$$\vec{e}_m^{\,r}(\rho,\phi) = \hat{z}a^{-m} \left\{ -\left[\sum_{n=0}^{m-1} \frac{1}{2}\left(n + \frac{1}{2}\right) d_{n,m-1} \right] \left(\frac{a}{\rho} \sec^2 \frac{\phi}{2} \right)^{1/2} + \sum_{n=0}^{m-1} \left[\frac{1}{2} \left(n + \frac{1}{2}\right) d_{n,m-1} \right] \left(\frac{a}{\rho} \sec^2 \frac{\phi}{2} \right)^{n+(3/2)} \right\} \quad . \tag{6.17}$$

Replacing n by n - 1 for the index of summations in (6.17), we obtain

$$\vec{e}_m^{\,r}(\rho,\phi) = \hat{z}a^{-m} \left\{ -\left[\sum_{n=1}^{m} \frac{1}{2}\left(n - \frac{1}{2}\right) d_{n-1,m-1} \right] \left(\frac{a}{\rho} \sec^2 \frac{\phi}{2} \right)^{1/2} + \sum_{n=1}^{m} \left[\frac{1}{2} \left(n - \frac{1}{2}\right) d_{n-1,m-1} \right] \left(\frac{a}{\rho} \sec^2 \frac{\phi}{2} \right)^{n+(1/2)} \right\} \quad . \tag{6.18}$$

A comparison of (6.18) with (6.14) yields immediately the following recursive formulas for $\{d_{nm}\}$:

$$d_{nm} = \frac{1}{2}\left(n - \frac{1}{2}\right) d_{n-1,m-1} \quad , \quad n,m = 0,1,2,\ldots \tag{6.19a}$$

$$d_{0m} = -\sum_{n=1}^{m} d_{nm} \quad , \quad m = 1,2,\ldots \quad . \tag{6.19b}$$

With the initial value in (6.15), the recursive formulas enable us to determine all $\{d_{nm}\}$. The first few of them are

$$d_{01} = \frac{1}{4} \quad , \quad d_{11} = -\frac{1}{4}$$

$$d_{02} = \frac{1}{8} \quad , \quad d_{12} = \frac{1}{16} \quad , \quad d_{22} = \frac{3}{16} \quad .$$

Summarizing the results in (6.8) and (6.14), we obtain the final solution for the reflected field in (6.4), namely,

$$\vec{E}^r(\rho,\phi) \sim e^{ik(\rho-2a)} \sum_{m=0}^{\infty} (ika)^{-m} \sum_{n=0}^{m} d_{nm} \left(\frac{a}{\rho} \sec^2 \frac{\phi}{2}\right)^{n+(1/2)} \quad , \quad k \to \infty \quad . \tag{6.20}$$

The first two orders of (6.20) read

$$\vec{E}^r(\rho,\phi) = -\sqrt{\frac{a}{\rho}} \sec \frac{\phi}{2}\, e^{ik(\rho-2a)} \left\{1 + \frac{i}{4ka}\left[1 - \frac{a}{\rho} \sec^2 \frac{\phi}{2}\right] + O(k^{-2})\right\} \quad . \tag{6.21}$$

An exact solution for the problem treated in this section was obtained by H. Lamb in 1906 by using parabolic cylinder coordinates.* Problem 3-4 shows that when the exact solution is expanded asymptotically in k, it yields precisely (6.20). Thus, our ray method has recovered the exact asymptotic

*H. Lamb, "On Sommerfeld's diffraction problem and on reflection by a parabolic mirror," Proc. London Math. Soc., Series 2, 4, 190-203, 1906.

solution. The dominant terms of the asymptotic solution in (6.21) were compared numerically with the exact solution by Keller et al. (op. cit.). The two results agree very well for $ka \geq 2$.

3.7. Reflection from a Surface of Revolution

The parabolic cylinder problem considered in the previous section is relatively simple because it is equivalently a scalar problem. For a more general problem, the amplitude matching on the reflecting surface becomes complex. The formulas developed in Section 3.4 can be used to carry out the amplitude matching in a systematic fashion for the fields of the first two orders. This is illustrated by an example below. In this example, two conditons are assumed:

(i) The reflecting surface Σ is a surface of revolution about the z-axis (Figure 3-5), and is described by the equation

$$\Sigma: \quad z = f(\rho) \tag{7.1}$$

where (ρ,ϕ,z) are the usual cylindrical coordinates.

(ii) The incident field is a plane wave propagating in the axial direction of Σ:

$$\vec{E}^i(\vec{r}) = \hat{x}e^{ikz} \quad . \tag{7.2}$$

The problem is to determine the reflected field everywhere and the surface current on Σ. In the following manipulations we give sufficient details to demonstrate that the procedure given in Section 3.4, despite its cumbersome appearance, is rather systematic and straightforward.

Calculation of $\{\hat{x}_n\}$ and $\{p_{mn}\}$: A comparison of (7.2) with (5.2) leads immediately to

$$s^i = z \quad , \quad \vec{e}^{\,i}_0 = \hat{x} \quad , \quad \vec{e}^{\,i}_m = 0 \text{ for } m = 1,2,\ldots \quad . \tag{7.3}$$

We choose the base vectors of the incident pencil coordinate as

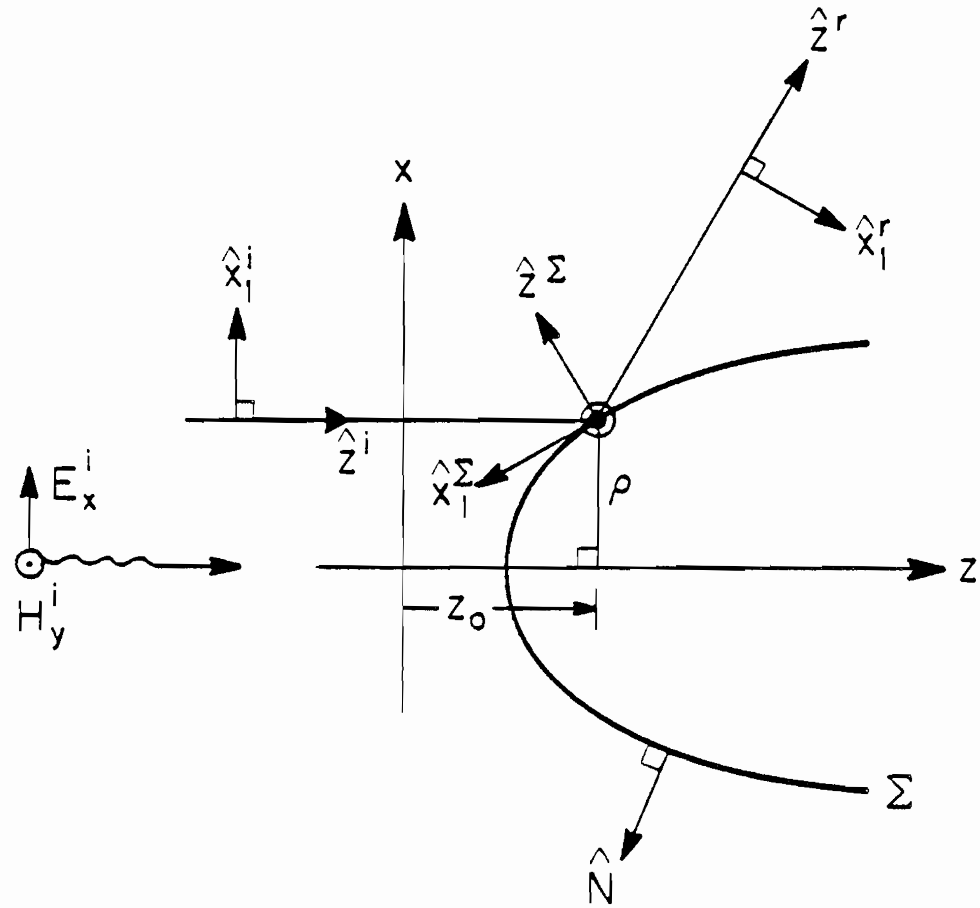

Figure 3-5. Reflection of an incident plane wave along the axial direction of Σ which is a surface of revolution.

$$\hat{x}_1^i = \hat{x} \quad , \quad \hat{x}_2^i = \hat{y} \quad , \quad \hat{z}^i = \hat{z} \quad . \tag{7.4}$$

Consider a point of reflection O on Σ, described by

$$[\rho, \phi, z_0 = f(\rho)] \quad .$$

Note that through the variation of (ρ, ϕ), O may represent any point on Σ. The principal directions and the outward normal of Σ at O are found to be [Problem 3-5]:

$$\hat{x}_1^\Sigma = -g^{-1} \left(\hat{x} \cos \phi + \hat{y} \sin \phi + \hat{z} f^{(1)} \right) \tag{7.5a}$$

$$\hat{x}_2^\Sigma = \hat{x}(- \sin \phi) + \hat{y} \cos \phi = \hat{\phi} \Big|_{\vec{r} \text{ at } 0} \tag{7.5b}$$

$$\hat{z}^\Sigma = g^{-1} \left(\hat{x} f^{(1)} \cos \phi + \hat{y} f^{(1)} \sin \phi - \hat{z} \right) \tag{7.5c}$$

where $f^{(n)}(\rho)$ means the nth derivative of $f(\rho)$, and

$$g(\rho) = \sqrt{1 + \left(f^{(1)}(\rho) \right)^2} \quad . \tag{7.5d}$$

Application of Snell's law determines $\hat{z}^r$ of the reflected pencil. We choose $\hat{x}_2^r = \hat{x}_2^i$. The base vectors of the reflected pencil are

$$\hat{x}_1^r = \frac{g^2 - 2}{g^2} (\hat{x} \cos \phi + \hat{y} \sin \phi) - \hat{z} \frac{2f^{(1)}}{g^2} \tag{7.6a}$$

$$\hat{x}_2^r = \hat{x}(- \sin \phi) + \hat{y} \cos \phi = \hat{x}_2^\Sigma \tag{7.6b}$$

$$\hat{z}^r = \frac{2f^{(1)}}{g^2} (\hat{x} \cos \phi + \hat{y} \sin \phi) + \hat{z} \frac{g^2 - 2}{g^2} \quad . \tag{7.6c}$$

As verified later, $\bar{\bar{Q}}^r$ calculated in using (7.6) is diagonal. Hence, $\hat{x}_1^r$ and $\hat{x}_2^r$ above are indeed principal directions of the reflected wavefront.

Next calculate $\{p_{mn}^{i,r}\}$ according to (5.9) and the results are

$$p_{11}^{i} = -g^{-1} \cos \phi \qquad p_{21}^{i} = -g^{-1} \sin \phi \qquad p_{31}^{i} = -g^{-1}f^{(1)}$$

$$p_{12}^{i} = -\sin \phi \qquad p_{22}^{i} = \cos \phi \qquad p_{32}^{i} = 0$$

$$p_{13}^{i} = g^{-1}f^{(1)} \cos \phi \qquad p_{23}^{i} = g^{-1}f^{(1)} \sin \phi \qquad p_{33}^{i} = -g^{-1} \qquad (7.7)$$

$$p_{11}^{r} = g^{-1} \qquad p_{21}^{r} = 0 \qquad p_{31}^{r} = -g^{-1}f^{(1)}$$

$$p_{12}^{r} = 0 \qquad p_{22}^{r} = 1 \qquad p_{32}^{r} = 0$$

$$p_{13}^{r} = g^{-1}f^{(1)} \qquad p_{23}^{r} = 0 \qquad p_{33}^{r} = g^{-1} \quad . \qquad (7.8)$$

Calculation of $\bar{\bar{Q}}^{r}$: The radii of curvature of the incident pencil are $R_1^i = R_2^i \to \infty$, while those of Σ are found to be [Problem 3-5]

$$R_1^{\Sigma} = -\frac{g^3}{f^{(2)}} , \quad R_2^{\Sigma} = -\frac{\rho g}{f^{(1)}} \quad . \qquad (7.9)$$

From (5.7) we can calculate $\bar{\bar{Q}}^{r}(\sigma^{r} = 0)$, which turns out to be diagonal and yields

$$R_1^{r} = \frac{g^2}{2f^{(2)}} , \quad R_2^{r} = \frac{\rho g^2}{2f^{(1)}} \quad . \qquad (7.10)$$

The continuation of $\bar{\bar{Q}}^{r}(\sigma^{r})$ follows (7.9) in Section 2.7.

Solution of $\vec{e}_0^{\,r}$: From (5.11a), the initial value of the zeroth-order reflected electrical field can be found with the result

$$\vec{e}_0^{\,r}(x_1^r = x_2^r = z^r = 0) = \hat{x}_1^r \cos \phi + \hat{x}_2^r \sin \phi \quad . \qquad (7.11)$$

The continuation of $\vec{e}_0(z^r)$ along a reflected axial ray follows from (5.12), namely,

$$\vec{e}_0(x_1^r = 0, x_2^r = 0, z^r) = \frac{1}{\sqrt{1 + (z^r/r_1^r)}\ \sqrt{1 + (z^r/R_2^r)}} \left\{ \hat{x}_1^r \cos\phi + \hat{x}_2^r \sin\phi \right\} . \tag{7.12}$$

Thus, the zeroth-order solution of the reflected field is

$$\vec{E}^r(\vec{r}) = e^{ik[f(\rho)+z^r]}\ \vec{e}_0(x_1^r = 0, x_2^r = 0, z^r) \left[1 + O(k^{-1}) \right] \quad . \tag{7.13}$$

We emphasize that $[\rho, \phi, f(\rho)]$ are the cylindrical coordinates of the point of reflection O, while observation point $\vec{r}$ is at a distance z^r away from O in the direction of $\hat{z}^r$ defined in (7.6c).

<u>Initial value of $\vec{e}_1^{\,r}$:</u> As a first step, we use (4.7) to calculate the derivatives of $\vec{e}_0^{\,r}$ at O. The results are

$$\frac{1}{\cos\phi}\ \frac{\partial e_{01}^r}{\partial x_1^r} = -\frac{1}{2}\left\{ \frac{1}{R_1^r} + \frac{1}{R_2^r} \right\} f^{(1)} = \frac{1}{\sin\phi}\ \frac{\partial e_{02}^r}{\partial x_1^r} \tag{7.14a}$$

$$\frac{1}{\cos\phi}\ \frac{\partial e_{02}^r}{\partial x_2^r} = \frac{2(g^2 - 1)}{g^2 \rho} = \frac{(-1)}{\sin\phi}\ \frac{\partial e_{01}^r}{\partial x_2^r} \quad . \tag{7.14b}$$

Then the initial value of $\vec{e}_1^{\,r}$ at O is found after solving (4.6):

$$\vec{e}_1^{\,r}(x_1^r = x_2^r = z^r = 0) = \hat{x}_1^r e_{11} + \hat{z}^r e_{13} \tag{7.15a}$$

$$= \hat{z}^{\Sigma}\ \frac{g}{f^{(1)}}\ e_{11} \tag{7.15b}$$

where

$$e_{11} = f^{(1)}\ e_{13} = \frac{1 - g^2}{g^2}\left\{ \frac{f^{(1)}}{\rho} - f^{(2)} \right\} \cos\phi \quad . \tag{7.15c}$$

Note from (7.15b) that the tangential components of $\vec{e}_1^r$ on Σ are zero as expected. With the initial value of $\vec{e}_1^r$ at 0 given in (7.15), the continuation of $\vec{e}_1^r$ to an arbitrary point on the same ray is governed by (7.15) in Section 2.7. In the evaluation of the integral, it is necessary to know $\nabla^2\vec{e}_0^r$ at every point between 0 and the observation point along the ray. It is possible to derive general explicit formulas for it, but they are very cumbersome.* We leave this step to each individual problem. An illustrating example is given in the next section.

Initial value of $\vec{h}_0^r$ and $\vec{h}_1^r$: From (4.11) and (4.12), we can calculate the first two orders of the reflected magnetic field at 0. The results are

$$\vec{h}_0^r(z^r = 0) = \sqrt{\frac{\varepsilon}{\mu}} \left(-x_1^r \sin\phi + \hat{x}_2^r \cos\phi\right) \tag{7.16}$$

$$\vec{h}_1^r(z^r = 0) = \sqrt{\frac{\varepsilon}{\mu}} \left(\hat{x}_1^r h_{11} + \hat{x}_2^r h_{12} + \hat{z}^r h_{13}\right) \tag{7.17}$$

where

$$h_{11} = \left(\frac{f^{(2)}}{g^2} - \frac{f^{(1)}}{\rho g^2}\right) \sin\phi$$

$$h_{12} = \left(f^{(2)} - \frac{f^{(1)}}{\rho}\right) \cos\phi$$

$$h_{13} = \left(\frac{g^2 - 1}{g^2 \rho} - \frac{f^{(1)} f^{(2)}}{g^2}\right) \sin\phi \quad .$$

*An explicit formula for $\vec{e}_1^r$ at points away from Σ was given in C. E. Schensted, "Electromagnetic and acoustical scattering by a semi-infinite body of revolution," J. Appl. Phys. 26, 306-308, 1955. However, it is believed that that formula is not completely correct as it gives an erroneous solution for reflection from a sphere (Section 3.8, and Problem 3-6). Schensted's formula does yield the correct solution of $\vec{e}_0^r$ in (7.12) and the correct initial value of $\vec{e}_1^r$ in (7.15).

Solution of surface current: The surface current density at O can be calculated from (4.15). It is convenient to express the surface current in terms of two principal directions of Σ at O. The result is

$$\vec{J}_s = (\hat{x}_1^{\Sigma} J_1 + \hat{x}_2^{\Sigma} J_2) + O(k^{-2}) \tag{7.18}$$

where

$$J_1 = -\sqrt{\frac{\varepsilon}{\mu}}\, 2 \cos\phi\, e^{iks^i} \left\{ 1 + \frac{i}{2k} \left[\frac{f^{(1)}}{\rho} - f^{(2)} \right] + O(k^{-2}) \right\}$$

$$J_2 = -\sqrt{\frac{\varepsilon}{\mu}}\, \frac{2}{g} \sin\phi\, e^{iks^i} \left\{ 1 + \frac{i}{2k} \left[f^{(2)} - \frac{f^{(1)}}{\rho} \right] + O(k^{-2}) \right\} \quad .$$

Reflection from a paraboloid: As an application, let the reflecting surface Σ in (7.1) be a paraboloid of revolution, namely,

$$\Sigma: \quad z = f(\rho) = \rho^2/2a \quad , \tag{7.19}$$

where (a/2) is the focal length. The derivatives of f are found to be

$$f^{(1)} = \frac{\rho}{a} \quad , \quad f^{(2)} = \frac{1}{a} \quad , \quad g = \left[\left(\frac{\rho}{a}\right)^2 + 1 \right]^{1/2} \quad . \tag{7.20}$$

At a point of reflection O with cylindrical coordinates $[\rho, \phi, z_0 = f(\rho)]$, the base vectors of the reflected pencil are calculated from (7.6), namely,

$$\hat{x}_1^r = \frac{(\rho/a)^2 - 1}{(\rho/a)^2 + 1} (\hat{x}\cos\phi + \hat{y}\sin\phi) - \hat{z}\, \frac{2(\rho/a)}{(\rho/a)^2 + 1} \tag{7.21a}$$

$$\hat{x}_2^r = \hat{x}(-\sin\phi) + \hat{y}\cos\phi \tag{7.21b}$$

$$\hat{z}^r = \frac{2(\rho/a)}{(\rho/a)^2 + 1} (\hat{x}\cos\phi + \hat{y}\sin\phi) + \hat{z}\, \frac{(\rho/a)^2 - 1}{(\rho/a)^2 + 1} \quad . \tag{7.21c}$$

The radii of curvature of the reflected pencil at O calculated from (7.10) are found to be

$$R_1^r = R_2^r = \frac{a}{2} \left[\left(\frac{\rho}{a} \right)^2 + 1 \right] \quad .$$

The solution of $\vec{e}_0^{\,r}$ on the axial ray of the reflected pencil is found from (7.12) with the result

$$\vec{e}_0(x_1^r = 0, x_2^r = 0, z^r) = \frac{\rho^2 + a^2}{\rho^2 + a^2 + 2az^r} (\hat{x}_1^r \cos\phi + \hat{x}_2^r \sin\phi) \qquad (7.22)$$

where z^r is the distance between the observation point and O. The initial value of $\vec{e}_1^{\,r}$ at O is given (7.15) or

$$\vec{e}_1(x_1^r = 0, x_2^r = 0, z^r = 0) = 0 \quad . \qquad (7.23)$$

To determine $\vec{e}_1$ at points away from O, we may make use of (7.22) and (7.23) above in (7.15) of Section 2.7. After some tedious calculations, it is found that

$$\vec{e}_1(x_1^r = 0, x_2 = 0, z^r) = 0 \quad . \qquad (7.24)$$

Since $\vec{e}_1$ is identically zero everywhere in space, it follows from (7.15) in Section 2.7 that

$$\vec{e}_m = 0 \quad , \quad m = 2,3,\ldots \quad . \qquad (7.25)$$

In summary, when a paraboloid in (7.19) is illuminated by a normally incident plane wave in (7.2), the reflected field at an observation point $\vec{r}$ is found to be

$$\vec{E}^r(\vec{r}) = e^{ik(z_0+z^r)} \frac{\rho^2 + a^2}{\rho^2 + a^2 + 2az^r} (\hat{x}_1^r \cos\phi + \hat{x}_2^r \sin\phi) \qquad (7.26)$$

where (ρ,ϕ,z_0) are the cylindrical coordinates of the reflection point O on the paraboloid (Figure 3-5), and z^r is the distance between $\vec{r}$ and O. Since (7.26) satisfies the wave equation, the Gauss' law, the boundary condition, and the radiation condition, we recognize that (7.26) is not only an asymptotic solution, but also an exact solution. This remarkable fact was first established by C. E. Schensted in 1955 (op. cit.). The surface current on the paraboloid may be calculated from (7.18) and (7.5). The result is

$$\vec{J}(\rho,\phi) = \sqrt{\frac{\varepsilon}{\mu}}\, 2e^{ik(\rho^2/2a)} \left[\left(\frac{\rho}{a}\right)^2 + 1\right]^{1/2} [(\hat{\rho} + \hat{z}\,\frac{\rho}{a}) \cos\phi - \hat{\phi}\sin\phi] \quad . \tag{7.27}$$

This current is recognized as the physical optics current, given by the first term of (4.6). It is also the exact current on the paraboloid.

3.8. Reflection from a Sphere

Let us consider an explicit example which makes use of the results derived in Section 3.7. The incident field is a plane wave given in (7.2), and the surface Σ is a perfectly conducting sphere with a radius a (Figure 3-6). For a point of reflection 0 on Σ with cylindrical coordinates (ρ,ϕ,z_0) and spherical coordinates (r_0,θ,ϕ), we have

$$\Sigma: \quad z_0 = f(\rho) = -\sqrt{a^2 - \rho^2} = a \cos \theta \quad . \tag{8.1}$$

We are only interested in the determination of the reflected field from the illuminated half of the sphere. Hence, $z_0 < 0$ and $\pi \leq \theta < \pi/2$ as indicated in (8.1) (the square root takes a non-negative value). The diffracted field due to creeping-wave contribution is not considered here. From (7.5), the rectangular base vectors $(\hat{x}_1^\Sigma,\hat{x}_2^\Sigma,\hat{z}^\Sigma)$ are found to coincide with $(\hat{\theta},\hat{\phi},\hat{r})$ at point 0. From (7.6), we have

$$\hat{x}_1^r = (-\cos 2\theta)(\hat{x}\cos\phi + \hat{y}\sin\phi) + \hat{z}\sin 2\theta \tag{8.2a}$$

$$\hat{x}_2^r = \hat{x}(-\sin\phi) + \hat{y}\cos\phi = \hat{x}_2^\Sigma \tag{8.2b}$$

$$\hat{z}^r = -\sin 2\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) - \hat{z}\cos 2\theta \tag{8.2c}$$

where, we emphasize, (θ,ϕ) are coordinates of 0 and are not those of the observation point. At an observation point A at a distance z^r from 0 in the direction of $\hat{z}^r$, the zeroth-order solution $\vec{e}_0$ is determined from (7.12), namely,

$$\vec{e}_0(x_1^r = 0, x_2^r = 0, z^r) = \left[1 + z^r\left(\frac{-2}{a\cos\phi}\right)\right]^{-1/2}\left[1 + z^r\left(\frac{-2\cos\theta}{a}\right)\right]^{-1/2}$$

$$\cdot\ (\hat{x}_1^r\cos\phi + \hat{x}_2^r\sin\phi) \tag{8.3}$$

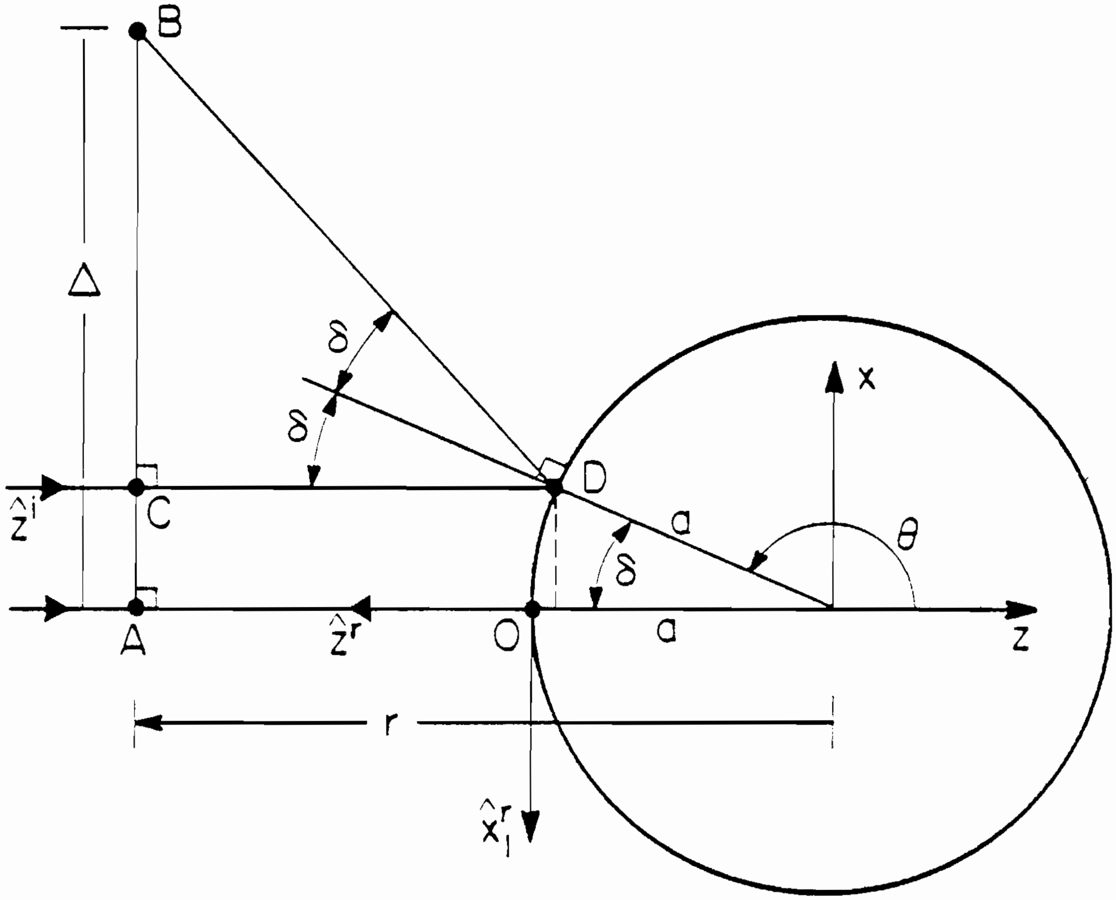

Figure 3-6. Reflection from a sphere of an incident plane wave.

where the square roots always take positive real values. The initial value of $\vec{e}_1$ at 0 is found from (7.15) with the result

$$\vec{e}_1^{\,r}(x_1^r = x_2^r = z^r = 0) = \left(\frac{\sin^4 \theta \cos \phi}{a \cos^3 \theta} \right) (- \hat{x}_1^r + \cos \theta \hat{x}_2^r) \quad . \tag{8.4}$$

To determine the value of $\vec{e}_1^{\,r}$ at observation point A, it is necessary to evaluate (7.15) in Section 2.7. Let us now concentrate on a special case when A is on the negative z-axis (Figure 3-6).

A crucial step is to calculate $\nabla^2 \vec{e}_0^{\,r}$ as a function of z^r along ray OA. Since $\hat{x}_1^r = -\hat{x}, \hat{x}_2^r = \hat{y}, \hat{z}^r = -\hat{z}$, we have

$$\begin{aligned} \nabla^2 \vec{e}_0^{\,r} \Big|_{\vec{r} \text{ at A}} &= \hat{x}_1^r \left(\frac{\partial^2}{\partial x_1^{r2}} + \frac{\partial^2}{\partial x_2^{r2}} + \frac{\partial^2}{\partial z^{r2}} \right) (\vec{e}_0^{\,r} \cdot \hat{x}_1^r) \Big|_{\vec{r} \text{ at A}} \\ &= (- \hat{x}) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left[\vec{e}_0^{\,r} \cdot (- \hat{x}) \right] \Big|_{\vec{r} \text{ at A}} \quad . \end{aligned} \tag{8.5}$$

We now calculate terms on the right-hand side of (8.5). Consider $\vec{e}_0^{\,r}$ at another observation point B, which is close to A and has coordinates

$$\text{B:} \quad x = \Delta \to 0 \quad , \quad y = 0 \quad , \quad z = -(r - a) \quad . \tag{8.6}$$

The reflected ray passing B is ray DB. The point D on Σ has coordinates $(r = a, \theta = \pi - \delta, \phi = 0)$. Simple geometrical consideration leads to

$$\delta = \frac{\Delta}{2r - a} \, [1 + 0(\Delta)] \tag{8.7}$$

$$DB = (r - a\cos\delta)/\cos 2\delta = (r - a)\left\{1 + \delta^2\left[2 + \frac{a}{2(r - a)}\right] + O(\Delta^3)\right\} . \tag{8.8}$$

Then, according to (8.3), $\vec{e}_0^{\,r}$ at B can be calculated with $z^r = DB$, $\theta = \pi - \delta$, and $\phi = 0$. The result is

$$(-\hat{x}) \cdot \vec{e}_0^{\,r}(B) = \frac{a}{2r - a}\left[1 + \frac{5a - 8r}{(2r - a)^3}\Delta^2 + O(\Delta^3)\right] . \tag{8.9}$$

From the Taylor expansion, it is known that

$$(-\hat{x}) \cdot \vec{e}_0^{\,r}(B) = (-\hat{x}) \cdot \vec{e}_0^{\,r}(A) + \Delta\frac{\partial}{\partial x}(-\hat{x} \cdot \vec{e}_0^{\,r}) + \frac{\Delta^2}{2}\frac{\partial^2}{\partial x^2}(-\hat{x} \cdot \vec{e}_0^{\,r}) + O(\Delta^3) . \tag{8.10}$$

A comparison of (8.9) and (8.10) gives immediately

$$\left.\frac{\partial^2}{\partial x^2}(-\hat{x} \cdot \vec{e}_0^{\,r})\right|_{\vec{r}\text{ at }A} = \frac{2a(5a - 8r)}{(2r - a)^4} . \tag{8.11}$$

In a similar manner, it can be shown that

$$\left.\frac{\partial^2}{\partial y^2}(-\hat{x} \cdot \vec{e}_0^{\,r})\right|_{\vec{r}\text{ at }A} = \frac{a(6a - 8r)}{(2r - a)^4} . \tag{8.12}$$

From (8.3) it follows that

$$\left.\frac{\partial^2}{\partial z^2}(-\hat{x} \cdot \vec{e}_0^{\,r})\right|_{\vec{r}\text{ at }A} = \frac{\partial^2}{\partial z^{r2}}\frac{1}{\left(1 + \frac{2z^r}{a}\right)} = \frac{8a}{(2r - a)^3} . \tag{8.13}$$

Substitution of (8.11) through (8.13) into (8.5) gives

$$\left.\nabla^2\vec{e}_0^{\,r}\right|_{\vec{r}\text{ at }A} = (-\hat{x})\frac{8a(a - r)}{(2r - a)^4} . \tag{8.14}$$

With the help of (8.14) and (8.4), the integral in (7.15) in Section 2.7 can be explicitly evaluated to yield the value of $\vec{e}_1^{\,r}$ at A, namely,

$$\left. \vec{e}_1^{\,r} \right|_{\vec{r}\text{ at A}} = (-\hat{x})\,\frac{2(r-a)^2}{(2r-a)^3}\,. \tag{8.15}$$

When this result is combined with the zeroth-order solution in (8.3), we obtain the reflected field on the negative z-axis:

$$\vec{E}^r(r,\theta=\pi) = (-\hat{x})\,\frac{a}{2r-a}\,e^{ik(r-2a)}\left\{1 - \frac{i}{ka}\,\frac{2(r-a)^2}{(2r-a)^2} + O(k^{-2})\right\}. \tag{8.16}$$

This result is identical to the first two terms of the asymptotic expansion of the exact solution obtained by Weston.*

From (7.17), we can calculate the surface current on the sphere. At a typical point O on the sphere, $(\hat{x}_1^{\Sigma}, \hat{x}_2^{\Sigma})$ coincide with $(\hat{\theta}, \hat{\phi})$. Hence, (J_1, J_2) become in the present case (J_θ, J_ϕ). Their values are

$$J_\theta = \sqrt{\frac{\varepsilon}{\mu}}\,(-2\cos\phi)\,e^{ika\cos\theta}\left\{1 + \frac{i\sin^2\theta}{2\,ka\cos^3\theta} + O(k^{-2})\right\} \tag{8.17a}$$

$$J_\phi = \sqrt{\frac{\varepsilon}{\mu}}\,(2\cos\theta\sin\phi)\,e^{ika\cos\theta}\left\{1 - \frac{i\sin^2\theta}{2ka\cos^3\theta} + O(k^{-2})\right\}. \tag{8.17b}$$

Again, these results are identical to the first two terms of the asymptotic expansion of the exact current solution.†

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*V. H. Weston, "Near zone back scattering from large spheres," Appl. Sci. Res. B9, 107-116, 1961. See also p. 410 of Bowman et al. cited below.

†J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, Electromagnetic and Acoustic Scattering by Simple Shapes. North-Holland Publishing Company, Amsterdam, Netherlands, 1969, p. 408. Note that the z-axis in this reference is the negative z-axis here.

3.9. Reflection from an Arbitrary Reflector

Consider an arbitrary (concave or convex) reflector Σ, which is perfectly conducting and described by the equation (Figure 3-7)

$$\Sigma: \quad z = f(x,y) \tag{9.1}$$

for (x,y) in a domain D. For a given incident field $\vec{E}^i$ from a point source located at $P_1 = (x_1, y_1, z_1)$, the problem is to determine the dominant reflected field $\vec{E}^r$ of order k^0 at an arbitrary observation point $P_2 = (x_2, y_2, z_2)$.

Reflection point: For a given P_1 and P_2, a reflection point O may exist on the reflector Σ, with its coordinates denoted by $(x, y, z = f(x,y))$. The vectors

$$\vec{d}_1 = \hat{x}(x - x_1) + \hat{y}(y - y) + \hat{z}[f(x,y) - z_1] \tag{9.2a}$$

$$\vec{d}_2 = \hat{x}(x_2 - x) + \hat{y}(y_2 - y) + \hat{z}[z_2 - f(x,y)] \tag{9.2b}$$

are the connecting vectors between P_1 and O, and O and P_2, respectively. A condition on the reflection point is that the distance $(d_1 + d_2)$ must be stationary, i.e.,

$$\frac{\partial}{\partial x}(d_1 + d_2) = 0 \quad , \quad \frac{\partial}{\partial y}(d_1 + d_2) = 0 \tag{9.3}$$

which is explicitly given by

$$\begin{cases} \dfrac{1}{d_1}\left\{(x - x_1) + [f(x,y) - z_1]\dfrac{\partial f}{\partial x}\right\} + \dfrac{1}{d_2}\left\{(x - x_2) + [f(x,y) - z_2]\dfrac{\partial f}{\partial x}\right\} = 0 \\ \dfrac{1}{d_1}\left\{(y - y_1) + [f(x,y) - z_1]\dfrac{\partial f}{\partial y}\right\} + \dfrac{1}{d_2}\left\{(y - y_2) + [f(x,y) - z_2]\dfrac{\partial f}{\partial y}\right\} = 0 \end{cases} \tag{9.4a}$$

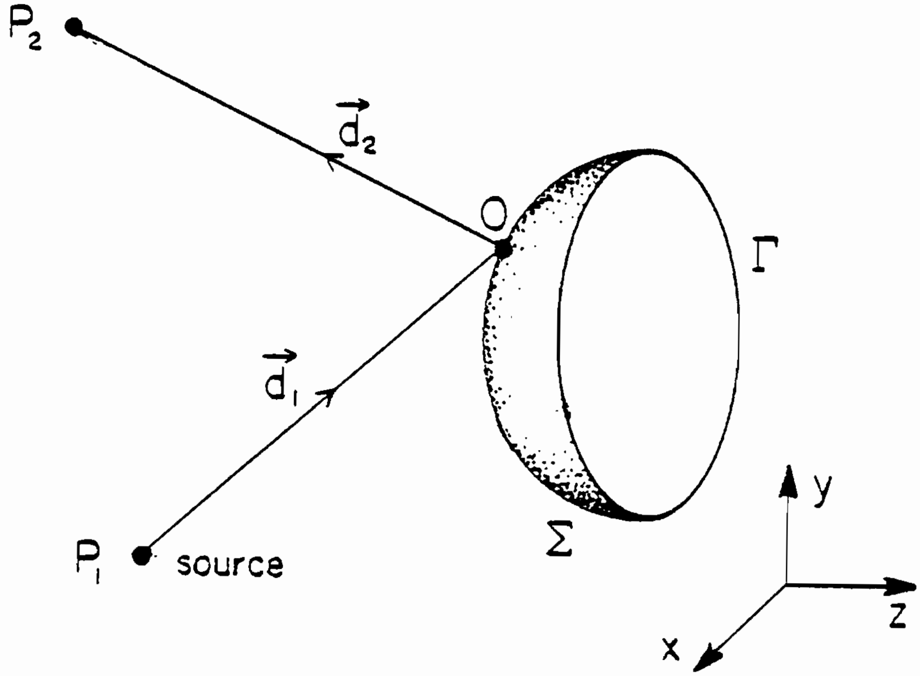

Figure 3-7. A reflector Σ with edge Γ is illuminated by the incident field from a point source at P_1.

A root (x,y,z = f) of the two nonlinear equations in (9.4a) gives the location of a reflection point. For a given P_1 and P_2, there can be none, one, or more than one reflection point. It can be shown that (9.4a) is equivalent to the satisfaction of the Snell's law.

The system of equations (9.4a) can be also satisfied if O is on the line connecting P_1 and P_2. Such a spurious root may be eliminated by an additional conditon

$$\left(\frac{x - x_1}{d_1} + \frac{x - x_2}{d_2}\right)^2 + \left(\frac{y - y_1}{d_1} + \frac{y - y_2}{d_2}\right)^2 + \left(\frac{z - z_1}{d_1} + \frac{z - z_2}{d_2}\right)^2 > 0 \quad . \qquad (9.4b)$$

Zeroth-order reflected field: If only the dominant term of order k^0 is retained, the reflected field is given in (5.12), or

$$\vec{E}^r(P_2) \sim (DF)\, e^{ikd_2} \{- \vec{E}^i(O) + 2[\vec{E}^i(O) \cdot \hat{N}]\, \hat{N}\} \qquad (9.5a)$$

$$\vec{H}^r(P_2) \sim (DF)\, e^{ikd_2} \{+ \vec{H}^i(O) - 2[\vec{H}^i(O) \cdot \hat{N}]\, \hat{N}\} \qquad (9.5b)$$

which is given in terms of the incident field $(\vec{E}^i, \vec{H}^i)$ at O, the normal $\hat{N}$ of the reflector at O, and a divergence factor DF. We choose $\hat{N}$ pointing toward the source. Explicitly, $\hat{N}$ is given by

$$\hat{N} = \Delta(f_x\, \hat{x} + f_y\, \hat{y} - \hat{z}) \qquad (9.6)$$

where $\Delta = +(f_x^2 + f_y^2 + 1)^{-1/2}$ and the subscript x of f_x, for example, means partial derivative with respect to x. The divergence factor in (9.5) is

$$DF = \frac{1}{\sqrt{1 + (d_2/R_1^r)}} \quad \frac{1}{\sqrt{1 + (d_2/R_2^r)}} \qquad (9.7)$$

where the square roots take positive real, positive imaginary, or zero values (so that DF is positive real, negative imaginary or infinite). $(R_1^r,\, R_2^r)$ are the radii of principal curvature of the reflected wavefront passing through O. Their computation is given next.

Curvatures of Reflected Wavefront. We use the formulas given in Section 3.3 for calculating (R_1^r, R_2^r). The three orthonormal base vectors of the incident pencil are chosen to be

$$\hat{x}_1^i = \frac{\hat{y} \times \hat{x}_3^i}{|\hat{y} \times \hat{x}_3^i|} = \frac{\hat{x}(z - z_1) - \hat{z}(x - x_1)}{[(z - z_1)^2 + (x - x_1)^2]^{1/2}} \tag{9.8a}$$

$$\hat{x}_2^i = \frac{\hat{x}_3^i \times \hat{x}_1^i}{|\hat{x}_3^i \times \hat{x}_1^i|} \tag{9.8b}$$

$$\hat{z}^i = \frac{\hat{x}(x - x_1) + \hat{y}(y - y_1) + \hat{z}(z - z_1)}{[(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2]^{1/2}} \tag{9.8c}$$

where (x,y,z) are the coordinates of the reflection point O. Those of the reflected pencil are chosen according to (3.14), namely,

$$\hat{x}_1^r = \hat{x}_1^i - 2(\hat{x}_1^i \cdot \hat{N})\, \hat{N} \tag{9.9a}$$

$$\hat{x}_2^r = \hat{x}_2^i - 2(\hat{x}_2^i \cdot \hat{N})\, \hat{N} \tag{9.9b}$$

$$\hat{z}^r = \frac{\hat{x}(x_2 - x) + \hat{y}(y_2 - y) + \hat{z}(z_2 - z)}{[(x_2 - x)^2 + (y_2 - y)^2 + (z_2 - z)^2]^{1/2}} \cdot \tag{9.9c}$$

Note that (9.9) chosen above is a left-hand system, i.e., $\hat{x}_1^r \times \hat{x}_2^r = \hat{z}^r$, and $(\hat{x}_1^r, \hat{x}_2^r)$ are not unit vectors. This choice, of course, does not affect the final solutions of (R_1^r, R_2^r). The three orthonormal base vectors of subreflector Σ at O are chosen to be

$$\hat{x}_1^{\Sigma} = \frac{\hat{x}\,1 + \hat{z} f_x}{(1 + f_x^2)^{1/2}} \tag{9.10a}$$

$$\hat{x}_2^{\Sigma} = \frac{\hat{y}\,1 + \hat{z} f_y}{(1 + f_y^2)^{1/2}} \tag{9.10b}$$

$$\hat{z}^{\Sigma} = \hat{N} \quad . \tag{9.10c}$$

From (9.8) and (9.10) the elements

$$p^i_{mn} = \hat{x}^i_m \cdot \hat{x}^{\Sigma}_n \quad , \qquad m,n = 1,2,3 \tag{9.11}$$

can be calculated with the results

$$p^i_{11} = \frac{(z - z_1) - f_x(x - x_1)}{[1 + f_x^2]^{1/2}[(x - x_1)^2 + (z - z_1)^2]^{1/2}} \tag{9.12a}$$

$$p^i_{12} = \frac{- f_y(x - x_1)}{[1 + f_y^2]^{1/2}[(x - x_1)^2 + (z - z_1)^2]^{1/2}} \tag{9.12b}$$

$$p^i_{21} = \frac{- (x-x_1)(y-y_1) - f_x(y-y_1)(z-z_1)}{[1+f_x^2]^{1/2}\{(x-x_1)^2(y-y_1)^2 + [(z-z_1)^2 + (x-x_1)^2]^2 + (y-y_1)^2(z-z_1)^2\}^{1/2}} \tag{9.12c}$$

$$p^i_{22} = \frac{(z-z_1)^2 + (x-x_1)^2 - f_y(y-y_1)(z-z_1)}{[1+f_y^2]^{1/2}\{(x-x_1)^2(y-y_1)^2 + [(z-z_1)^2 + (x-x_1)^2]^2 + (y-y_1)^2(z-z_1)^2\}^{1/2}} \tag{9.12d}$$

$$p^i_{33} = \frac{1}{d_1}\,\Delta[f_x(x - x_1) + f_y(y - y_1) - (z - z_1)] \tag{9.12e}$$

The first four elements in (9.12a) through (9.12d) form the 2 x 2 matrix $\bar{\bar{P}}^i$. Because of the particular choice in (9.9), we have $\bar{\bar{P}}^r = \bar{\bar{P}}^i$. The curvature matrix of the incident pencil is

$$\bar{\bar{Q}}^i = \begin{pmatrix} d_1^{-1} & 0 \\ 0 & d_1^{-1} \end{pmatrix} . \tag{9.13}$$

The curvature matrix of reflector Σ at O can be calculated from (13.28) in Appendix A, namely,

$$\bar{\bar{Q}}^\Sigma = \begin{pmatrix} \Delta^2(eG - fF) & , & \Delta^2(fE - eF) \\ \Delta^2(fG - gF) & , & \Delta^2(gE - fF) \end{pmatrix} \tag{9.14}$$

where

$$E = 1 + f_x^2 \quad , \quad F = f_x f_y \quad , \quad G = 1 + f_y^2$$

$$e = -\Delta f_{xx} \quad , \quad f = -\Delta f_{xy} \quad , \quad g = -\Delta f_{yy} .$$

The desired curvature matrix $\bar{\bar{Q}}^r$ at reflection point O can be calculated from the matrix equation in (3.16) or

$$\bar{\bar{Q}}^r = \bar{\bar{Q}}^i + 2p_{33}^i[(\bar{\bar{P}}^i)^{-1}]^T \bar{\bar{Q}}^\Sigma (\bar{\bar{P}}^i)^{-1} . \tag{9.15}$$

Let us denote the four elements of $\bar{\bar{Q}}^r$ by

$$\bar{\bar{Q}}^r = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} . \tag{9.16}$$

Then the desired radii of curvature of the reflected wavefront at O are given by

$$\frac{1}{R_1^r} \; , \; \frac{1}{R_2^r} = \frac{1}{2} \{(Q_{11} + Q_{22})^2 \pm \sqrt{(Q_{11} + Q_{22})^2 - 4(Q_{11}Q_{22} - Q_{12}Q_{21})\}} . \tag{9.17}$$

Both R_1^r and R_2^r are real. Their signs have the following meaning: if R_1^r is positive (negative), the corresponding normal section of the reflected wavefront is divergent (convergent). The same convention applies to R_2^r.

A final remark about the calculation of the reflected field: for a given P_1 and P_2, there may be more than one reflection point on Σ. Then, the total reflected field is the superposition of the contributions from each reflection point. If there is no reflection point on Σ, the reflected field is zero.

Several numerical examples are given below. The reflector Σ is of finite size with its boundary Γ lying on the surface of an elliptical cone. As shown in Figure 3-8, the axis of the cone lies in the y-z plane and its parameters are

$(x = 0, y = 0, z = -p)$ = coordinates of cone tip

θ_3 = inclination angle of the cone axis measured from the z-axis

(θ_1, θ_2) = half-cone angles in the x-z, and y-z planes.

In other words, Γ is the intersection of the surface in (9.1) and the above elliptical cone. The source located at P_1 (Figure 3-7) is assumed to be a linearly polarized (in the y-direction) horn with a simple radiation pattern:

$$\vec{E}^i(x,y,z) \sim \frac{120\pi}{(r/\lambda)} e^{ikr} [v_E(\theta) \sin \phi \, \hat{\theta} + v_H(\theta) \cos \phi \, \hat{\phi}] \qquad (9.18a)$$

where $\lambda = 2\pi/k$ = wavelength, and (r, θ, ϕ) are the spherical coordinates with origin at the source point (x_1, y_1, z_1) such that

$$r = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} \qquad (9.18b)$$

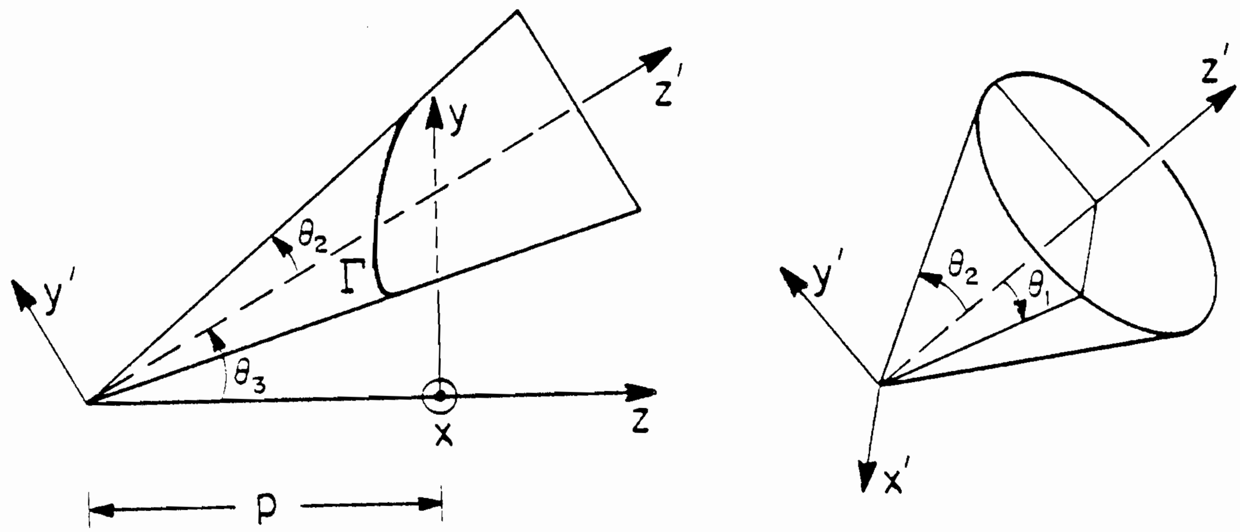

Figure 3-8. Reflector's boundary curve Γ which lies on the surface of an elliptical cone.

$$\theta = \cos^{-1}[(z - z_1)/r] \quad , \quad \phi = \tan^{-1}[(y - y_1)/(x - x_1)] \quad . \tag{9.18c}$$

The E- and H-plane patterns of $\vec{E}^i$ are assumed to be

$$v_E(\theta) = v_H(\theta) = \begin{cases} 1 \text{ volt/m} & , \quad \text{if } |\theta| \leq 51° \\ 0 & , \quad \text{if } |\theta| > 51° \end{cases} \tag{9.18d}$$

which is uniform in a conical sector.

(i) Symmetrical hyperboloid reflector. In the first example, the reflector is a part of a hyperboloid of revolution:

$$\text{Surface A:} \quad \frac{z}{\lambda} = f_0(x,y) = -15 + 6.54 \left\{1 + \frac{x^2 + y^2}{(13.5\lambda)^2}\right\}^{1/2} \quad . \tag{9.19}$$

The source point P_1 is located at one focus of the hyperboloid ($x_1 = 0$, $y_1 = 0$, $z_1 = -30\lambda$). The boundary of the reflector lies on a cone with its tip at P_1 and parameters (Figure 3-8)

$$p = 30\lambda \quad , \quad \theta_1 = \theta_2 = 27.6° \quad , \quad \theta_3 = 0 \quad . \tag{9.20}$$

The pattern of the total field E_y^t in the H-plane [where observation point P_2 is at $x_2 = (10^3 \sin \Omega)\lambda$, $y_2 = 0$, $z_2 = -(10^3 \cos \Omega)\lambda$] is given in Figure 3-9. Two sets of curves are given: the dashed ones are based on the present geometrical optics theory (GO) calculated from (9.5), while the solid ones are based on the uniform asymptotic theory (UAT) to be discussed in Chapter 5. For the time being, let us concentrate on the dashed curves (GO solutions). Several comments are in order: (a) Since the incident field $\vec{E}^i$ is identically zero in the range of Ω shown in the figure, the total field $\vec{E}^t$ consists of the reflected field $\vec{E}^r$ only. (b) The pattern is symmetrical with respect to $\Omega = 0$. (c) The line $\Omega = 64°$ lies on the shadow boundary of $\vec{E}^r$. For Ω beyond this angle, $\vec{E}^r$ is identically zero. (d) The

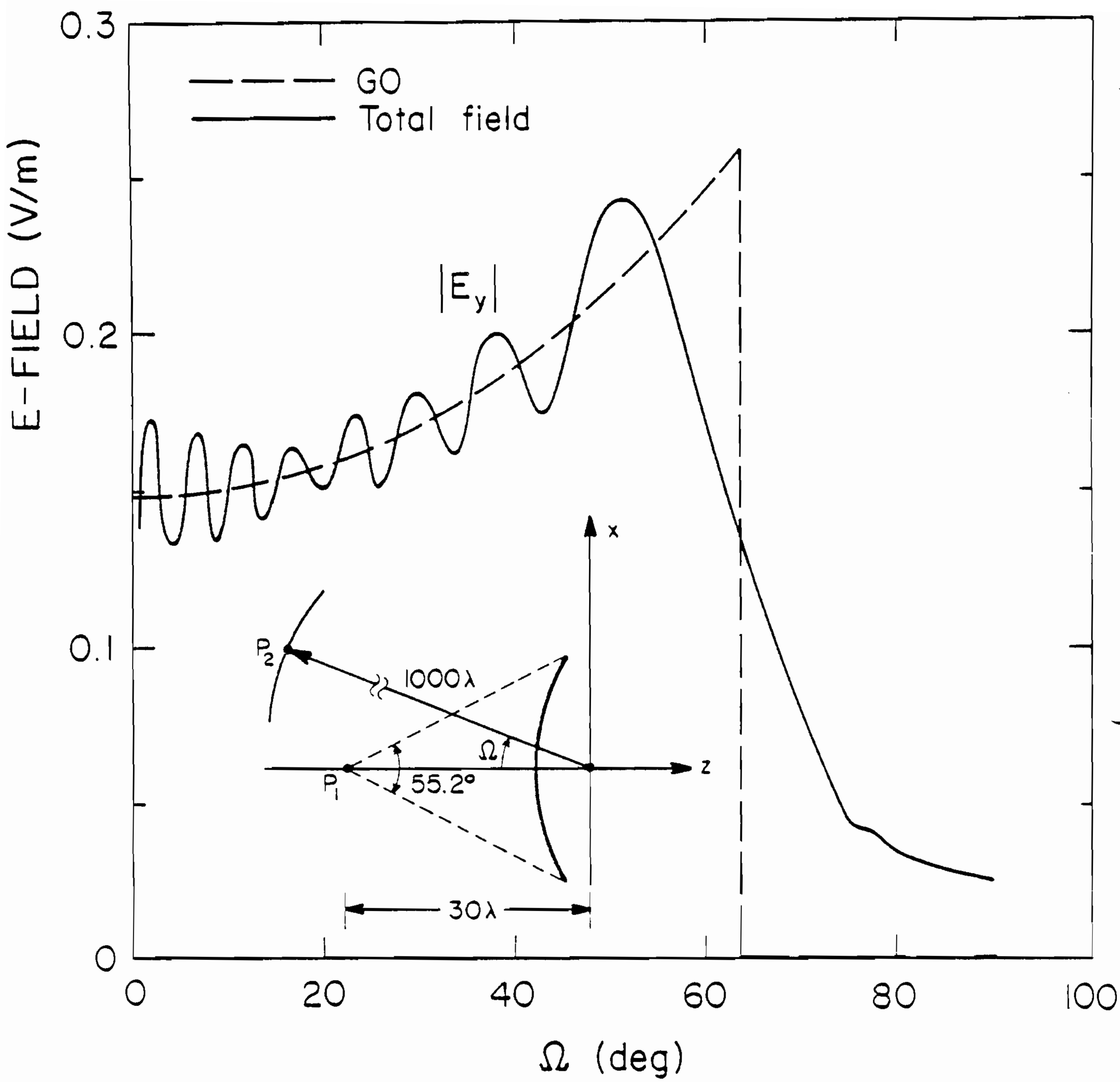

Figure 3-9a. H-plane pattern of the symmetrical hyperbolic reflector described in (9.19) and (9.20), illuminated by a point source located on its focus P_1.

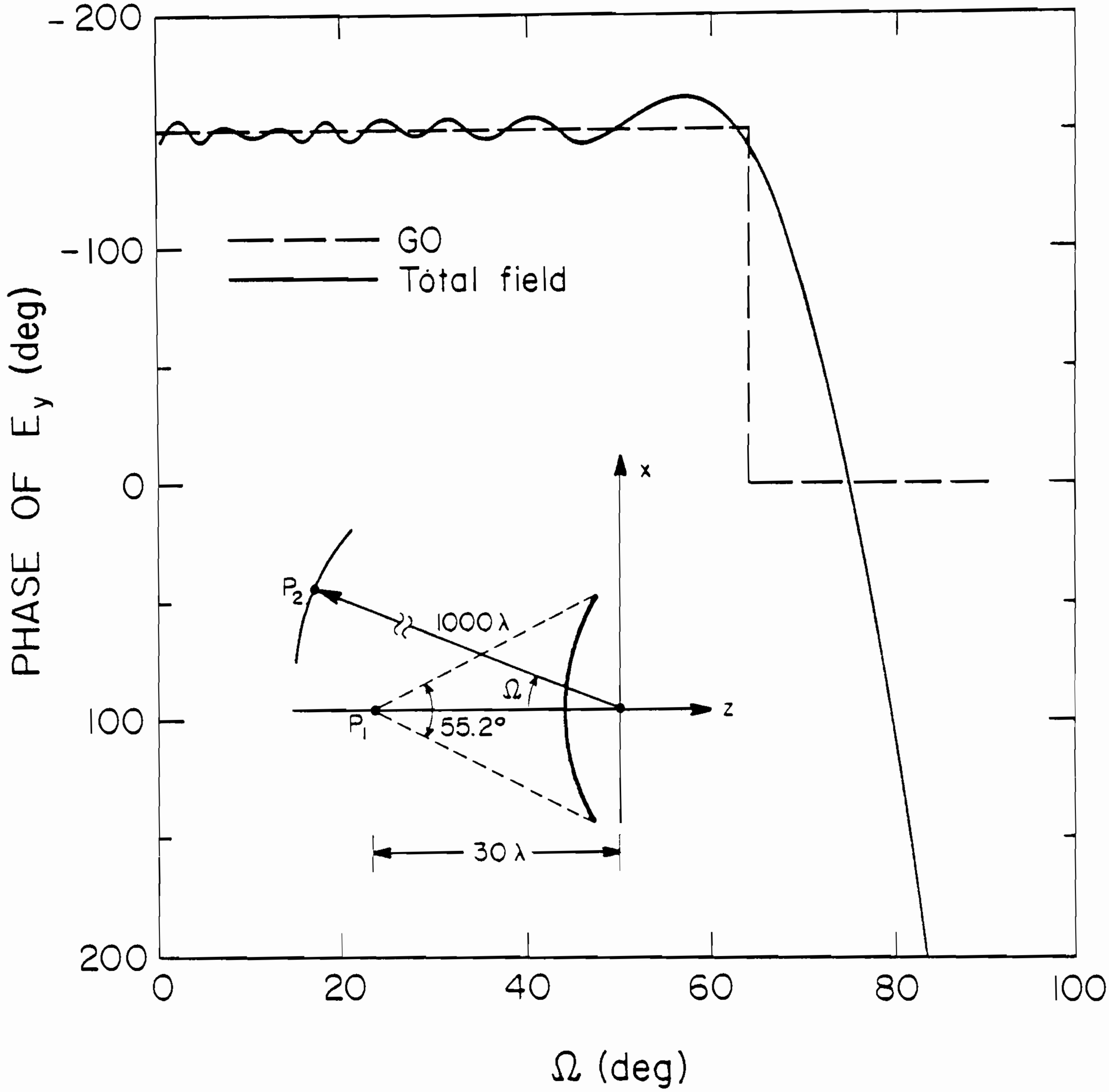

Figure 3-9b. The phase of E_y whose magnitude is displayed in Figure 3-9a.

phase of $\vec{E}^r$ is a constant over a spherical surface centered at the origin which is the other focus of the hyperboloid in (9.19). (e) When the UAT is applied to this problem, the total field $\vec{E}^t$ consists of $\vec{E}^r$ and the diffracted field $\vec{E}^d$ from the rim of the reflector. We note from the solid curves that $\vec{E}^d$ is relatively small, and contributes to the ripples of $\vec{E}^t$.

(ii) Offset hyperboloid reflector. Consider the hyperbolic surface A in (9.19), which is carved out by a cone (Figure 3-8) with parameters

$$p = 30\lambda \quad , \quad \theta_1 = \theta_2 = 27.6^\circ \quad , \quad \theta_3 = 20^\circ \quad . \tag{9.21}$$

A 3-D sketch of the reflector surface is shown in Figure 3-10. The source point P_1 is located at

$$x_1 = 2\lambda \quad , \quad y_1 = 4\lambda \quad , \quad z_1 = -30\lambda \quad . \tag{9.22}$$

The problem has no symmetry. We present the total field E_y^t and E_z^t in the H-plane in Figure 3-11 [where the observation point P_2 is at $x_2 = (10^2 \sin \Omega)\lambda$, $y_2 = 4\lambda$, $z_2 = -(10^2 \cos \Omega)\lambda$], and those in the E-plane in Figure 3-12 [where the observation point P_2 is at $x_2 = 2\lambda$, $y_2 = (10^2 \sin \Omega)\lambda$, $z_2 = -(10^2 \cos \Omega)\lambda$]. Again, let us concentrate on the GO solution for the total field $\vec{E}^t$ (dashed curves). In the H-plane, the total electric field is essentially in the y-direction. In the E-plane, it consists of two main components: E_y^t and E_z^t. The components E_x^t are very small in both planes.

(iii) Perturbed reflector. The Surface A described in (9.19) is perturbed to become

$$\text{Surface B:} \quad \frac{z}{\lambda} = f_0(x,y) - 2 \exp\left\{-\left[\frac{(x/\lambda) - 2}{10}\right]^2 + \left[\frac{(y/\lambda) - 4}{10}\right]^2\right\} \tag{9.23}$$

The reflector is the portion of the Surface B carved out by the cone described in (9.21), as shown in Figure 3-13. For the same source location

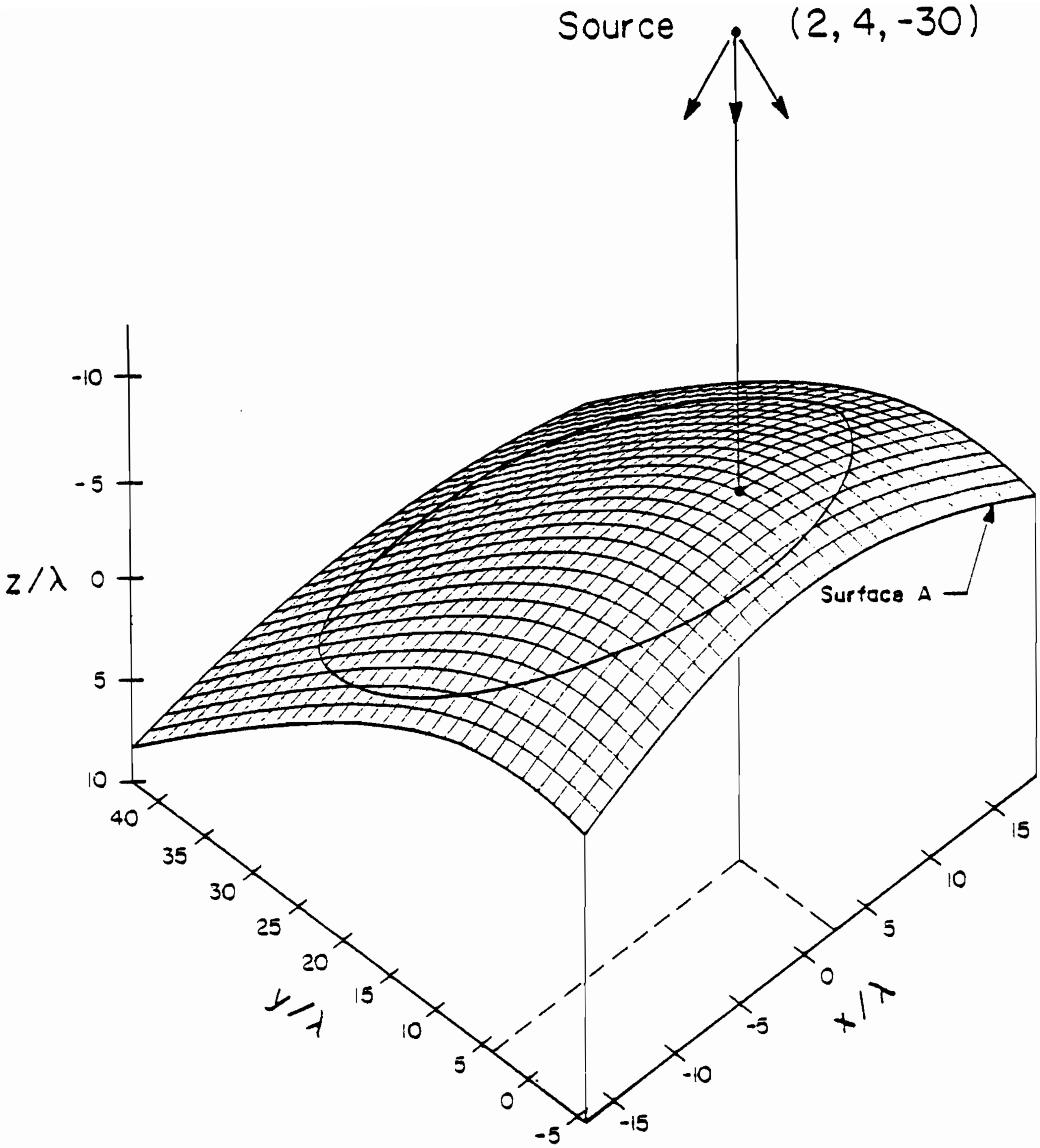

Figure 3-10. An offset hyperbolic reflector, whose boundary lies on a cone, is described by (9.19) and (9.21).

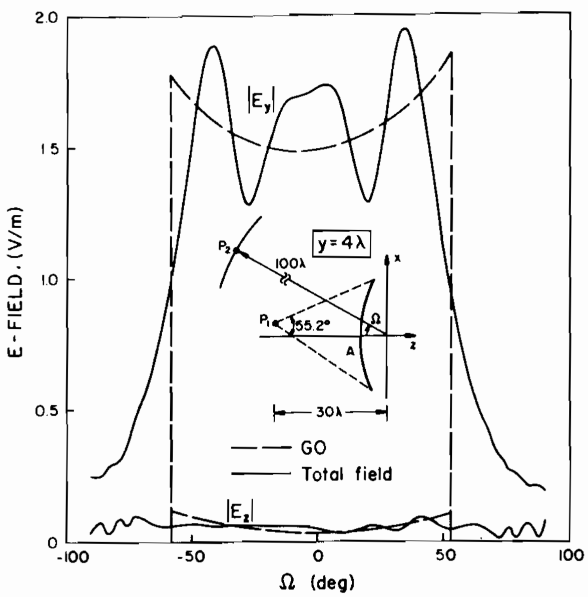

Figure 3-11. H-plane pattern of the reflector in Figure 3-10 illuminated by a source described in (9.22) and (9.13).

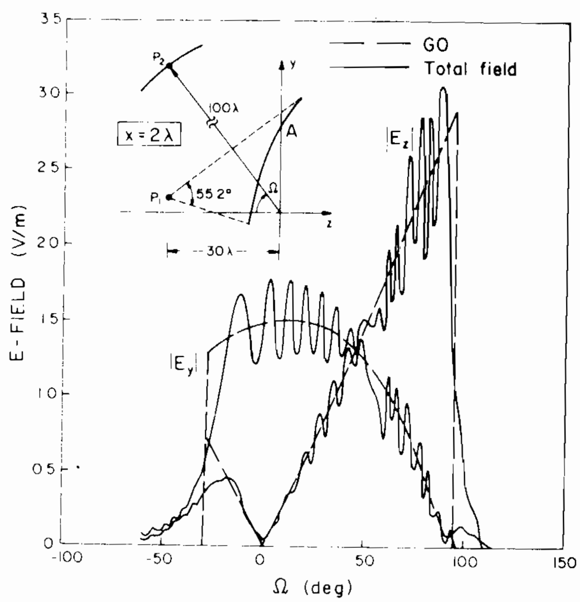

Figure 3-12. E-plane pattern of the reflector in Figure 3-10 illuminated by the source described in (3.22) and (3.13).

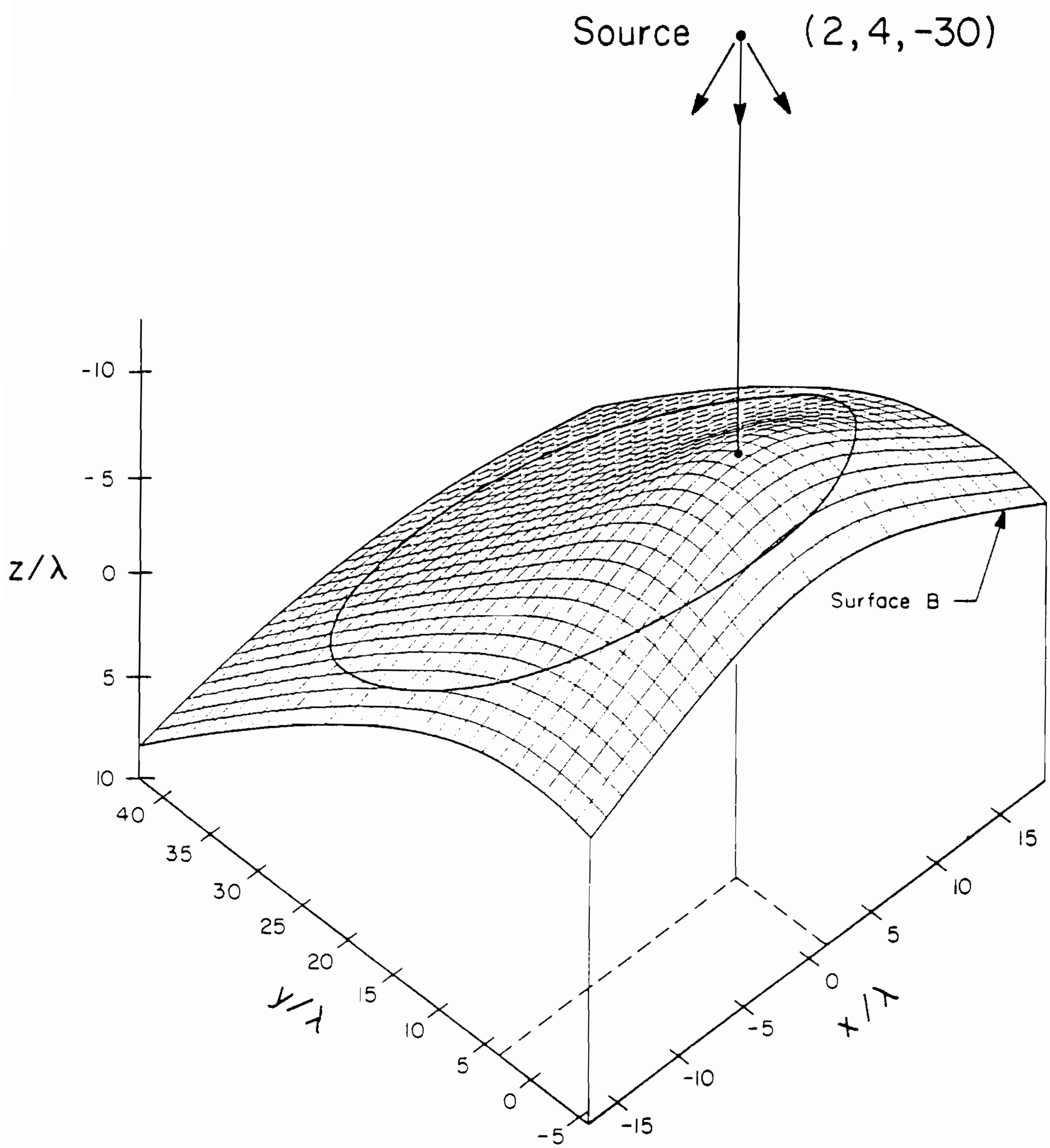

Figure 3-13. A perturbed version of the hyperbolic reflector shown in Figure 3-10.

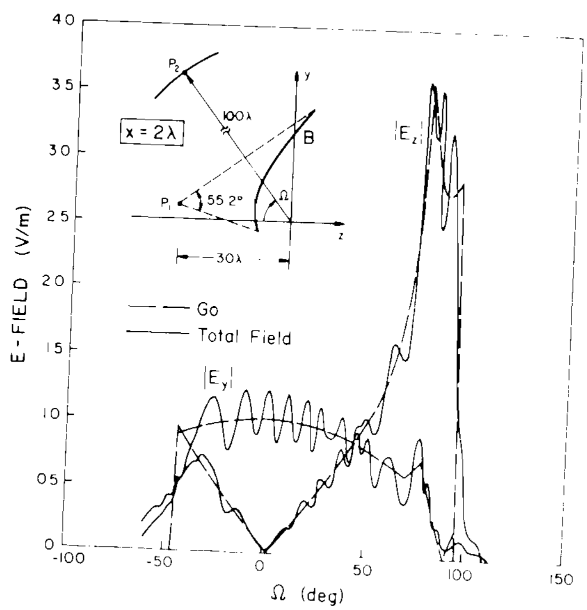

Figure 3-14. Same as Figure 3-12, except that reflector surface A is replaced by surface B in (9.23).

in (9.22), the E-plane patterns of the reflector are plotted in Figure 3-14, which should be compared with Figure 3-12, its counterpart from the unperturbed reflector. We note that, in the present case, $|E_y^t|$ is weaker, and the reflected shadow boundary extended further in the negative Ω direction.

3-1. In the calculation of the initial value of the first-order amplitude vector $\vec{e}_1^{\,r}(\vec{r})$ at 0 on screen Σ, it is necessary to know the derivatives of $\vec{e}_0^{\,r}(\vec{r})$ at 0. The latter quantities can be solved from the linear equations in (4.8). Now derive (4.8).

Hint: For m = 0, a more general version of (4.2a), valid not only at 0 but also in its neighborhood, reads

$$\vec{e}_0^{\,i} \cdot \hat{\sigma}_n^{\Sigma} = -\vec{e}_0^{\,r} \cdot \hat{\sigma}_n^{\Sigma} \quad , \quad n = 1,2 \quad , \text{ and } \vec{r} \text{ on } \Sigma \quad , \tag{1}$$

where the amplitude vectors, according to (2.12), are

$$\vec{e}_0^{\,i,r} = \hat{\sigma}_1^{i,r} e_{01}^{i,r} + \hat{\sigma}_2^{i,r} e_{02}^{i,r} \quad . \tag{2}$$

Substitution of (2) into (1) gives

$$\sum_{m=1}^{2} \left(\hat{\sigma}_m^{i} \cdot \hat{\sigma}_n^{\Sigma}\right) e_{0m}^{i} = - \sum_{m=1}^{2} \left(\hat{\sigma}_m^{r} \cdot \hat{\sigma}_n^{\Sigma}\right) e_{0m}^{r} \quad , \quad n = 1,2 \quad , \text{ and } \vec{r} \text{ on } \Sigma \quad . \tag{3}$$

When (3) is enforced at 0, we recover, of course, (4.3b). In the following equations, however, (3) is enforced at a general point on Σ in the neighborhood of 0. This general point is described by a position vector $\vec{r}$ drawing from 0:

$$\vec{r} = \hat{x}_1^{i,r} x_1^{i,r} + \hat{x}_2^{i,r} x_2^{i,r} + \hat{z}^{i,r} z^{i,r} \tag{4a}$$

$$= \hat{x}_1^{\Sigma} x_1^{\Sigma} + \hat{x}_2^{\Sigma} x_2^{\Sigma} + \hat{z}^{\Sigma} z^{\Sigma} \tag{4b}$$

where

$$z^{\Sigma} = \frac{1}{2} \left[\frac{\left(x_1^{\Sigma}\right)^2}{R_1^{\Sigma}} + \frac{\left(x_2^{\Sigma}\right)^2}{R_2^{\Sigma}} \right] \quad , \tag{4c}$$

$$x_m^{i,r} = p_{m1}^{i,r} x_1^{\Sigma} + p_{m2}^{i,r} x_2^{\Sigma} + O\left[\left(x_1^{\Sigma}\right)^2\right] + O\left[\left(x_2^{\Sigma}\right)^2\right] \quad , \quad m = 1,2,3 \quad . \tag{4d}$$

Note that for $m,n = 1,2$,

$$e_{0m}^{i,r}(\vec{r}) = e_{0m}^{i,r}(\vec{r} = 0) + x_1^{i,r} \left.\frac{\partial e_{0m}^{i,r}}{\partial x_1^{i,r}}\right|_{\vec{r}=0} + x_2^{i,r} \left.\frac{\partial e_{0m}^{i,r}}{\partial x_2^{i,r}}\right|_{\vec{r}=0} + z^{i,r} \left.\frac{\partial e_{0m}}{\partial z^{i,r}}\right|_{\vec{r}=0} + \ldots \tag{5}$$

$$\left.\frac{\partial e_{0m}^{i,r}}{\partial z^{i,r}}\right|_{\vec{r}=0} = -\frac{1}{2}\left(\frac{1}{R_1^{i,r}} + \frac{1}{R_2^{i,r}}\right) e_{0m}(\vec{r} = 0) \tag{6}$$

$$\hat{\sigma}_m^{i,r}(\vec{r}) \cdot \hat{\sigma}_n^{\Sigma}(\vec{r}) = p_{mn}^{i,r} - \frac{x_m^{i,r}}{R_m^{i,r}} p_{3n}^{i,r} + \frac{x_n^{\Sigma}}{R_n^{\Sigma}} p_{m3}^{i,r} + O\left[\left(x_1^{\Sigma}\right)^2\right] + O\left[\left(x_2^{\Sigma}\right)^2\right] \tag{7}$$

where (5) is the Taylor expansion, (6) is derived from (7.12) in Section 2.7, and (7) from (2.11) and (2.16). With the help of (5) through (7), the left-hand side of (3) becomes

$$\begin{aligned}\sum_{m=1}^{2} \left[\hat{\sigma}_m^{i}(\vec{r}) \cdot \hat{\sigma}_n(\vec{r})\right] e_{0m}^{i}(\vec{r}) = & \left\{\sum_{m=1}^{2} p_{mn}^{i} e_{0m}^{i}\right\} + \sum_{q=1}^{2} x_q^{\Sigma} \left\{\sum_{m=1}^{2} \left[\delta_{nq} \frac{p_{m3}^{i}}{R_n^{\Sigma}} e_{0m}^{i}\right.\right. \\ & - \frac{p_{mq}^{i} p_{3n}^{i}}{R_m^{i}} e_{0m}^{i} - \frac{1}{2}\left(\frac{1}{R_1^{i}} + \frac{1}{R_2^{i}}\right) p_{3q}^{i} p_{mn}^{i} e_{0m}^{i} \\ & \left.\left. + p_{1q}^{i} p_{mn}^{i} \frac{\partial e_{0m}^{i}}{\partial x_1^{i}} + p_{2q}^{i} p_{mn}^{i} \frac{\partial e_{0m}^{i}}{\partial x_2^{i}}\right]\right\} \\ & + O\left[\left(x_1^{\Sigma}\right)^2\right] + O\left[\left(x_2^{\Sigma}\right)^2\right] \quad , \quad n,q = 1,2 \end{aligned} \tag{8}$$

where the arguments of e_{0m}^{i} and $\partial e_{0m}^{i}/\partial x_{q}^{i}$ are $(\vec{r} = 0)$; $\delta_{nq} = 1$ if $n = q$; and $\delta_{nq} = 0$ if $n \neq q$. Except for a minus sign, the left-hand side of (3) is also given by (8) after replacing the superscript "i"'s by "r"'s. Since $\left(x_1^{\Sigma}, x_2^{\Sigma}\right)$ are arbitrary, we equate their respective coefficients in (3): the zeroth-order coefficients give (4.3b) which is already known; and the first-order coefficients give the following relations:

$$\sum_{m=1}^{2} \left[\delta_{nq} \frac{p_{m3}^{i}}{R_n^{\Sigma}} e_{0m}^{i} - \frac{p_{mq}^{i} p_{3n}^{i}}{R_m^{i}} e_{0m}^{i} - \frac{1}{2} \left(\frac{1}{R_1^{i}} + \frac{1}{R_2^{i}} \right) p_{3q}^{i} p_{mn}^{i} e_{0m}^{i} + p_{1q}^{i} p_{mn}^{i} \frac{\partial e_{0m}^{i}}{\partial x_1^{i}} \right.$$

$$\left. + p_{2q}^{i} p_{mn}^{i} \frac{\partial e_{0m}^{i}}{\partial x_2^{i}} \right] = (-1) \{ i \to r \} \quad , \quad n,q = 1,2 \qquad (9)$$

which are the desired equations in (4.7).

3-2. Verify that the electric field given in (1.2) and the magnetic field in (4.9), (4.11), and (4.12) satisfy the second Maxwell's equation

$$\nabla \times \vec{H} = -\sqrt{\frac{\varepsilon}{\mu}} \, (ik) \, \vec{E}$$

for the orders of k^0 and k^{-1}.

3-3. Consider a line source radiating in the presence of a perfectly conducting parabolic cylinder Σ described by (Figure 3-15).

$$\Sigma: \quad \rho = \rho_0(\phi) = a \sec^2 \frac{\phi}{2} \quad , \quad -\pi < \phi < \pi \quad .$$

Show that

$$\alpha_2 = \frac{\phi}{2} \quad ,$$

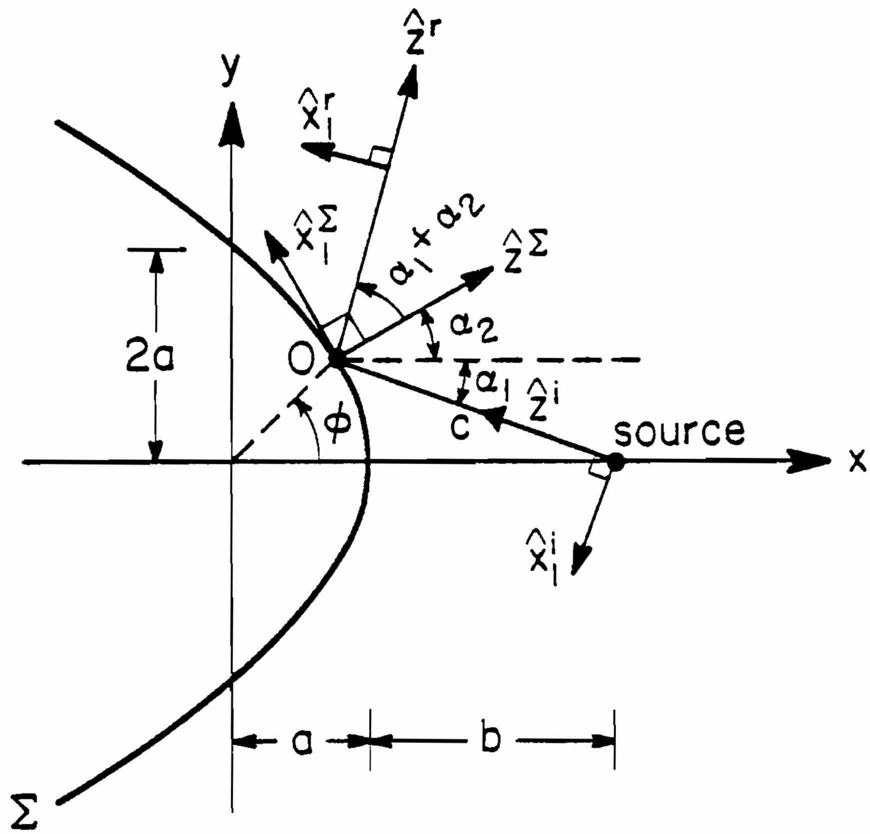

Figure 3-15. Radiation from a line source in the presence of a parabolic cylinder.

$$\bar{\bar{Q}}^r(z^r) = \begin{pmatrix} \frac{1}{R_1^r + z^r} & 0 \\ 0 & 0 \end{pmatrix} ,$$

$$\frac{1}{R_1^r} = \frac{1}{c} + \frac{1}{a} \frac{\left(\cos \frac{\phi}{2}\right)^3}{\cos \left(\alpha_1 + \frac{\phi}{2}\right)} ,$$

$$\alpha_1 = \tan^{-1} \left\{ \frac{2a \sin \phi/(1 + \cos \phi)}{(b + a) - [2a \cos \phi/(1 + \cos \phi)]} \right\} .$$

Also study the special case when the line source is at an infinite distance away ($b \to \infty$) from Σ, and the incident field becomes a plane wave.

Hint: First calculate quantities of Σ by the formulas given in Appendix A:

$$\vec{r}(\phi,z) = (x,y,z) = \left(a \sec^2 \frac{\phi}{2} \cos \phi,\ a \sec^2 \frac{\phi}{2} \sin \phi,\ z\right)$$

$$\vec{r}_\phi = \left(-a \sin \frac{\phi}{2} \sec^3 \frac{\phi}{2} ,\quad a \sec^2 \frac{\phi}{2} ,\ 0\right)$$

$$\vec{r}_z = (0,0,1)$$

$$\vec{r}_{\phi\phi} = \left[- \frac{2a\left(1 + \cos \phi + \sin^2 \phi\right)}{\left(1 + \cos \phi\right)^3} ,\quad \frac{2a \sin \phi}{(1 + \cos \phi)^2} ,\ 0\right]$$

$$\vec{r}_{zz} = \vec{r}_{\phi z} = \vec{r}_{z\phi} = 0$$

$$\hat{z}^\Sigma = \hat{x} \cos \frac{\phi}{2} + \hat{y} \sin \frac{\phi}{2} \text{ , which shows } \alpha_1 = \frac{\phi}{2}$$

$$R_1^\Sigma = - \frac{1}{2a} \cos \frac{\phi}{2}^3 ,\quad R_2^\Sigma \to \infty .$$

Next, calculate the quantities appearing in (5.7) for the determination of $\bar{\bar{Q}}^r$.

$$p_{11}^{i} = -\cos\left(\alpha_1 + \frac{\phi}{2}\right), \quad p_{22}^{i} = 1,$$

$$p_{11}^{r} = \cos\left(\alpha_1 + \frac{\phi}{2}\right), \quad p_{22}^{r} = 1,$$

$$p_{12}^{i,r} = p_{21}^{i,r} = 0, \quad p_{33}^{i} = -p_{33}^{r} = -\cos(\alpha_1 + \alpha_2) = -\cos\left(\alpha_1 + \frac{\phi}{2}\right),$$

$$R_1^{i} = c = \left[\left(a \sin\phi \sec^2\frac{\phi}{2}\right)^2 + \left(a\cos\phi \sec^2\frac{\phi}{2} - a - b\right)^2\right]^{1/2}.$$

Substitution of the above in (5.7) gives $\bar{\bar{Q}}^r(\sigma^r = 0)$. For the special case $b \to \infty$, we have $c \to \infty$, $\alpha_1 \to 0$, and

$$R_1^r = a \sec^2\frac{\phi}{2} = \rho_0(\phi)$$

$$\hat{z}^r = \hat{x}\cos\phi + \hat{y}\sin\phi = \hat{\rho}$$

which means that the reflected field is a cylindrical wave with "phase center" at (x = 0, y = 0).

3-4. The problem of reflection of a normally incident plane wave from a parabolic cylinder studied in Section 3.6 (Figure 3-4) was exactly solved by H. Lamb in 1906 using parabolic cylinder coordinates. (See pp. 467-468 in D. S. Jones, The Theory of Electromagnetism. Macmillan, New York, 1964.) For an incident field given in (6.1), the exact solution of the reflected field reads

$$\vec{E}^r(\rho,\phi) = \hat{z}\,(-1)\,e^{ik\rho\cos\phi}\left(\frac{F(\sqrt{2k\rho}\,\cos\frac{\phi}{2})}{F(\sqrt{2ka})}\right)$$

where the Fresnel integral F(x) is defined in Section 1-3. Using the asymptotic expansion of F(x) given in (3.5) in Section 1-3, show that the asymptotic expansion of the exact $\vec{E}^r$ above agrees with (6.20), the solution obtained by a ray method.

3-5. Consider a surface of revolution about the z-axis (Figure 3-5) described by the equation

$$\Sigma: \quad z = f(\rho) \quad .$$

Show that its principal directions $\left(\hat{x}_1^\Sigma, \hat{x}_2^\Sigma\right)$ and outward normal $\hat{z}^\Sigma$ are given in (7.5), and that its principal radii of curvature $\left(R_1^\Sigma, R_2^\Sigma\right)$ are given in (7.9).

Hint: Start with a parametric equation of Σ

$$\vec{r}(\rho,\phi) = [\rho \cos \phi, \rho \sin \phi, f(\rho)] \quad .$$

3-6. An explicit formula for calculating the reflected field $\vec{E}^r$ of the first two orders for the problem specified in (7.1) and (7.2) (Figure 3-5) was given in Equation (8) in C. Schensted, "Electromagnetic and acoustic scattering by a semi-infinite body of revolution," J. Appl. Phys. 26, 306-308, 1955. Applying that formula to calculate $\vec{E}^r$ in the backward direction for a sphere defined in (8.1) (Figure 3-6) shows that the result is given by

$$\vec{E}^r(r,\theta = \pi)\Big|_{\text{Sch}} = (-\hat{x}) \frac{a}{2r - a} e^{ik(r-2a)} \cdot \left\{1 - \frac{i}{ka}\left[\frac{9(r-a)^2}{4(2r-a)^2} + \frac{1}{8}\frac{a}{(2r-a)}\right] + O(k^{-2})\right\} \quad .$$

This result should be compared with (8.16), which is also the exact asymptotic solution. They do not agree in the term of $O(k^{-1})$. Therefore, it appears that Equation (8) given by Schensted is not completely correct.

Hint: The notations $\{\hat{\rho}, \hat{\phi}, \hat{s}, \vec{E}_0, \vec{E}_1\}$ used by Schensted are identical to $\{-\hat{x}_1^r, \hat{x}_2^r, \hat{z}^r, \vec{e}_0^{\,r}, (-i/2\pi)\, \vec{e}_1^{\,r}\}$ in our notation.