Interaction Notes
Note 361
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EQUIVALENT CIRCUIT REPRESENTATION OF RADIATION SYSTEMS

C. B. Sharpe and C. J. Roussi

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Abstract

In this note the biconical antenna is treated as a representative scattering system. It is shown that at its input terminals the biconical antenna can be modeled by a transmission line terminated in a cannonical LC ladder network. The real and imaginary parts of the input impedance of the biconical antenna serve as useful test functions for studying the approximation of complex functions of frequency by rational functions. An effective algorithm for this purpose was implemented and evaluated. It is also shown that over a limited domain in the complex frequency plane the poles and zeros of the system function can be recovered via the rational approximation.

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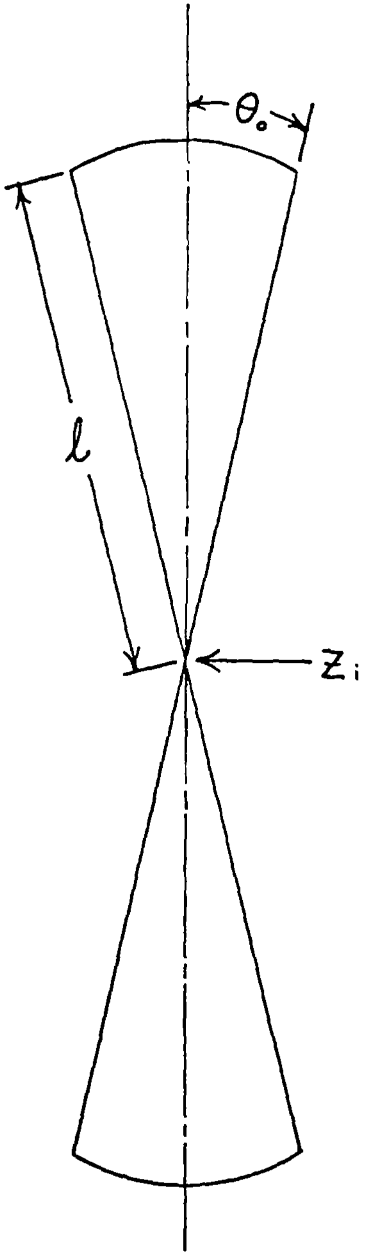
1. Introduction

In developing equivalent circuits for radiating systems we have divided the problem into two basic parts: the development of a rational function approximation technique and the development of lumped network synthesis procedures appropriate to the system in question. Of course, the first part must serve as a basis for the second. Typically the rational function will represent in analytic form the transfer admittance of a system obtained by experimentally measuring the amplitude and phase of the surface current as a function of frequency at some point on the scattering object with reference to the incident electric field at some reference plane. In the general case all we can say about the transfer function is that its poles must all lie in the closed left half-plane. The zeros may lie in either half-plane. Among the parameters in the problem are the polarization and aspect of the incident field, the location and orientation of the current probe on the object and, of course, the shape of the scattering body. An important question which remains to be answered is the nature of the dependence of the poles and zeros of the transfer function on these parameters.

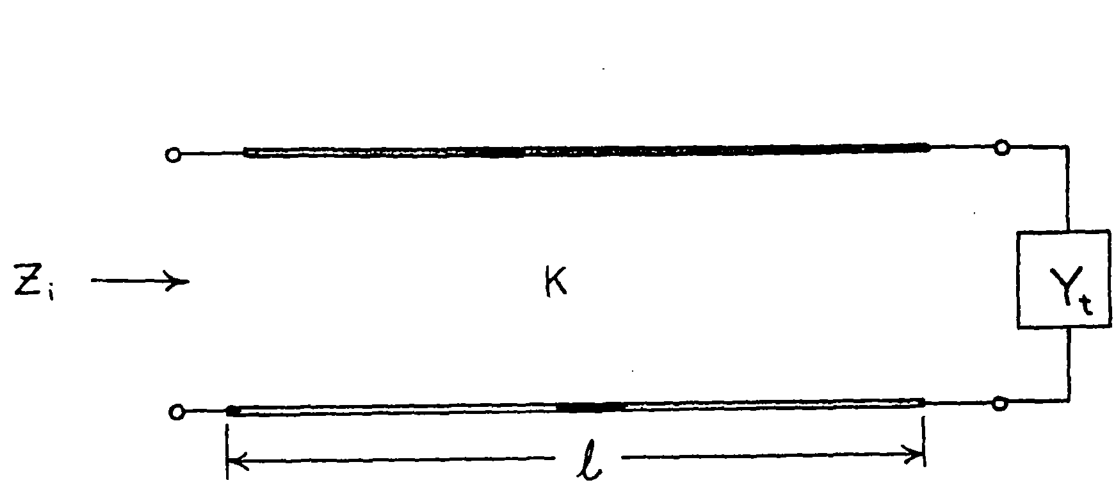
In order to explore the above question, much of our initial effort has been devoted to the development and testing of a numerical approximation technique for rational functions. This technique and its application in several representative approximation problems is described in Section 3. As a preliminary exercise, a network modelling problem leading to a cannonical ladder configuration was also investigated and completed. This work is presented in the next section. Both of the above studies were centered around the biconical antenna. One reason for this approach is that the input impedance of the biconical antenna exhibits many of the frequency response characteristics of more general scattering structures, and is, therefore, a useful vehicle for test purposes.

2. A Network Model for the Biconical Antenna

The biconical antenna offers an interesting example on which to test network modeling techniques because an approximate closed-form expression for the input impedance is available. Tai[1] has shown that the input impedance at the center of the biconical antenna can be represented by a section of uniform line terminated in a frequency-dependent admittance $Y_t(\beta\ell)$. This equivalent circuit is illustrated in Figure 1, where



BICONICAL ANTENNA



EQUIVALENT CIRCUIT FOR THE INPUT IMPEDANCE

$$K = \frac{Z_0}{\pi} \ln\left(\frac{2}{\theta_0}\right)$$

Figure 1

K denotes the characteristic impedance of the line. The following expression was obtained by Tai for Y_t:

$$Y_t = \frac{Z_0}{4\pi K^2} \{2L(2\beta\ell)+e^{2i\beta\ell}[L(2\beta\ell)-L(4\beta\ell)+\ell n\ 2]+e^{-2i\beta\ell}[L^*(2\beta\ell)-\ell n\ 2], \tag{1}$$

where

$$L(x) = \int_0^x \frac{1-\cos t}{t}\, dt + i \int_0^x \frac{\sin t}{t}\, dt$$

and the asterisk denotes the complex conjugate. The real and imaginary parts of $Y_t(\beta\ell)$ are plotted in Figure [2] for the case where the angle of the cone, $\theta_0 = .01$ radians.

The objective of the work described in this section is to construct a lumped network model for the load admittance Y_t of the equivalent circuit shown in Figure 1. Because Y_t represents a positive real driving point admittance, it is possible in principle to synthesize a network model from the real part of Y_t alone. From (1), the asymptotic behavior of the real part of $Y_t(\beta\ell)$ at low frequencies is given by

$$\lim_{\beta\ell\to 0} \mathrm{Re}[Y_t(\beta\ell)] = \frac{Z_0}{6\pi K^2} (\beta\ell)^4 ,$$

where Z_0 is the characteristic impedance of free space. This suggests that we look for a network having an input admittance $Y_n(j\omega)$ such that

$$\mathrm{Re}[Y_n(j\omega)] = \frac{\omega^4}{P(\omega^2)} , \tag{2}$$

where $P(\omega^2)$ is a polynomial of the form

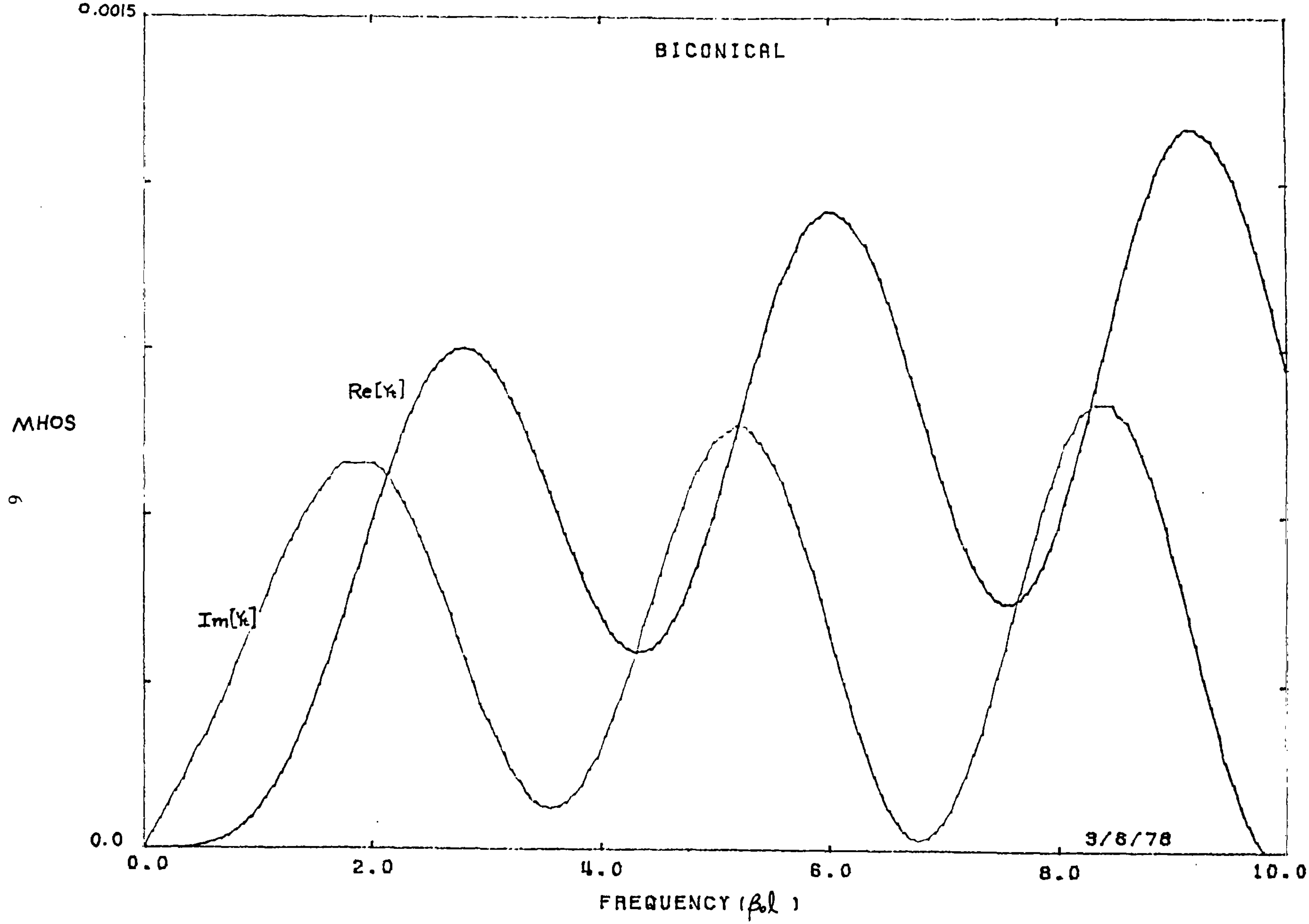
$$P(\omega^2) = p_0 + p_2\omega^2 + \dots + p_{4n}\omega^{4n} .$$

It will now be shown that $Y_n(s)$ can always be realized as an LC ladder terminated in a resistance if $P(\omega^2) > 0$, all ω.

Suppose a lossless network is excited as shown in Figure [3] and assume that the scattering coefficient $S_{21}(s)$ has the form

$$S_{21}(s) = \frac{s^2}{B(s)} , \tag{3}$$

where $B(s)$ is a polynomial of degree $2n$ having all its roots in the LHP. Then if $A(s)$ is another polynomial of degree $2n$, $S_{11}(s)$ will have the form



0.0015
BICONICAL
MHOS
Re[Yt]
Im[Yt]
0.0
3/8/78
0.0
2.0
4.0
6.0
8.0
10.0
FREQUENCY (β₀l)

Figure 2

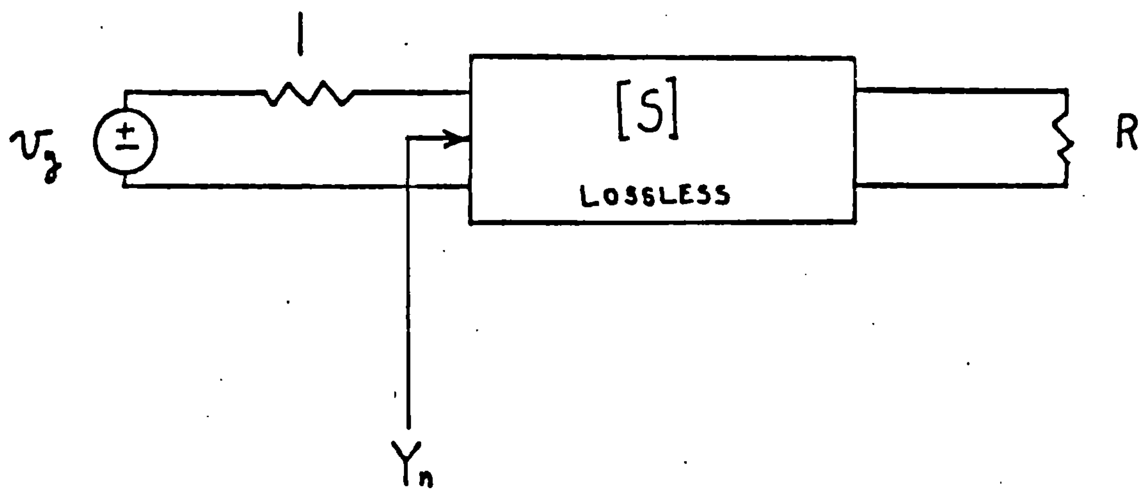


Figure 3

$$S_{11}(s) = \frac{A(s)}{B(s)} .$$

The unitary condition,

$$S_{11}(s)S_{11}(-s) + S_{21}(s)S_{21}(-s) = 1,$$

implies that

$$B(s)B(-s) - A(s)A(-s) = s^4. \tag{4}$$

This is the only condition on A(s). The input admittance can be expressed by

$$Y_n(s) = \frac{1 - S_{11}(s)}{1 + S_{11}(s)} = \frac{W(s)}{U(s)} ,$$

where U(s) = A(s) + B(s) and W(s) = B(s) - A(s). Equation (4) leads to the relation,

$$U(s)W(-s) + U(-s)W(s) = 2s^4 , \tag{5}$$

between U(s) and W(s). Using this result

$$\begin{aligned} \mathrm{Re}[Y_n(j\omega)] &= \frac{1}{2} \left(\frac{W(j\omega)}{U(j\omega)} + \frac{W(-j\omega)}{U(-j\omega)}\right) \\ &= \frac{\omega^4}{U(j\omega)U(-j\omega)} . \end{aligned}$$

Thus, the polynomial $P(\omega^2)$ can be identified with the product

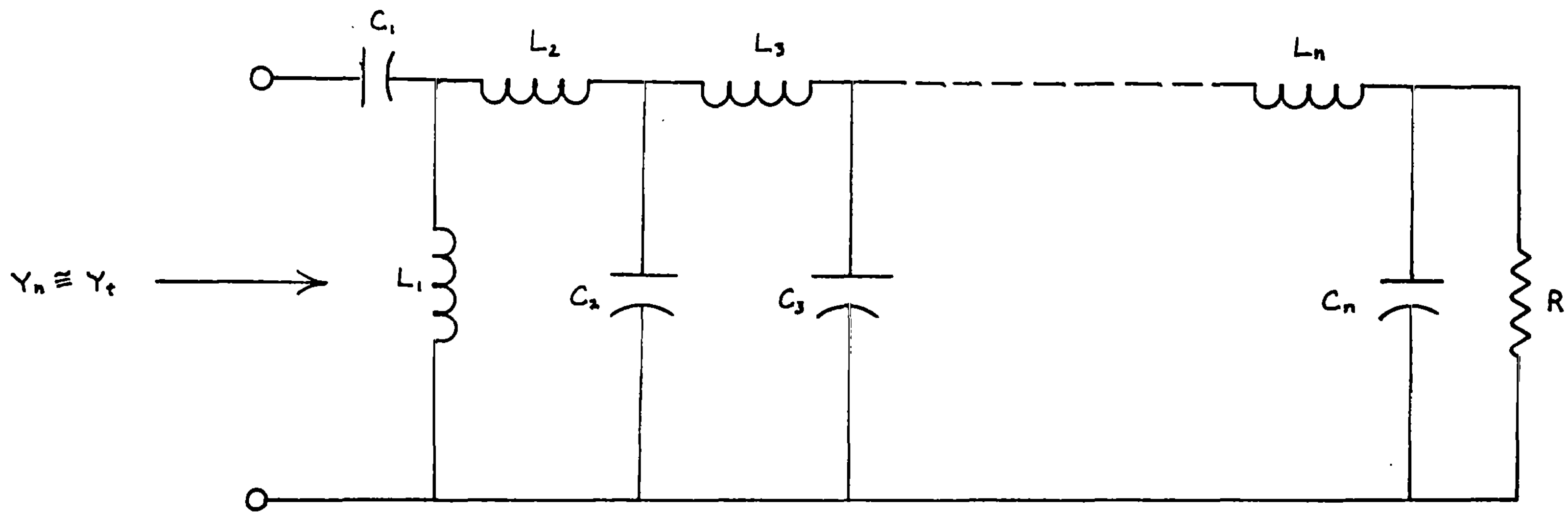
$$P(\omega^2) = U(j\omega)U(-j\omega).$$

Finally, it can be shown that the transmission function given in (3) can be realized by a ladder network of the form shown in Figure [4]. The number of independent energy storage elements, 2n, corresponds to the number of poles of $S_{21}(s)$ and the two zeros of transmission required at s = 0 are provided by C_1 and L_1. It should be noted that this realization of $S_{21}(s)$ is not unique.

The numerical aspect of the modeling problem involves the determination of the coefficients of $P(\omega^2)$ by curve-fitting the function $\omega^4/P(\omega^2)$ to the "data" represented by $\mathrm{Re}[Y_t(\frac{\omega\ell}{c})]$. This can most readily be done by forming the objective function

$$F = \sum_{i=1}^{NFREQ} W_i \left[\frac{\omega_i^4}{\mathrm{Re}[Y_t(\frac{j\omega_i \ell}{c})]} - P(\omega_i^2)\right]^2 , \tag{6}$$

where the W_i are arbitrary weighting coefficients. In the results that follow $W_i = 1$, all i, and the normalization ℓ=c was employed. The optimum



LOW-FREQUENCY LUMPED CIRCUIT MODEL FOR Y_t

Figure 4

frequency range over which the data function was sampled is a function of n and was determined empirically.

Of course the advantage of employing a least-squares objective function as defined in (6) is that the minimization of F with respect to the p_i, i=0, 2, ... , 4n, leads to a system of linear equations for the unknown coefficients. The algorithm is easily implemented and the solution for the coefficients presents no difficulty as long as the corresponding matrix remains well-conditioned. Chebyshev polynomial methods or near minimax approximations such as Lawson's algorithm [2], can be used to avoid the ill-conditioning that frequently occurs for large values of n, although we did not find this to be necessary. Once the polynomial $P(\omega^2)$ is known U(s) can be recovered from

$$P(-s^2) = U(s)U(-s)$$

by factorization if $P(\omega^2) > 0$, all ω. Thus, U(s) contains the LHP roots of $P(-s^2)$.

Suppose U(s) has the form

$$U(s) = d_0 + d_1 s + - - - + d_{2n} s^{2n} .$$

Then W(s) will have the form

$$W(s) = c_1 s + c_2 s^2 + - - - + C_{2n-1} s^{2n-1} .$$

The latter polynomial can be determined uniquely from U(s) by imposing the condition given in (5). For instance, in the case of n=3, the equations for the c_i take the form,

$$\begin{bmatrix} 0 & 0 & 0 & d_6 & -d_5 \\ 0 & d_6 & -d_5 & d_4 & -d_3 \\ -d_5 & d_4 & -d_3 & d_2 & -d_1 \\ -d_3 & d_2 & -d_1 & d_0 & 0 \\ -d_1 & d_0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

The synthesis procedure is completed by expanding $Y_n(s) = W(s)/U(s)$ in a continued fraction and identifying the coefficients with the elements of the ladder network. Thus, for the network of Figure 4, the expansion takes the form,

$$Y_n(s) = \cfrac{1}{\cfrac{1}{sC_1} + \cfrac{1}{\cfrac{1}{sL_1} + \cfrac{1}{sL_2 + \cfrac{1}{sC_2 + \ddots \cfrac{1}{sC_n + \frac{1}{R}}}}}}$$

Figures 5 and 6 illustrate some typical results obtained by the modeling technique described above. In each case $\theta_0 = .01$ radians. The element values for the approximating network are calculated assuming ℓ=1m. It can be seen that the least-squares fit in the real part of the admittance is quite satisfactory and that the band over which the approximation is valid increases, as expected, with the order of the network. It can be seen from Figures 7 and 8 that the approximation in the real and imaginary parts of z_{in}, the input impedance of the antenna, is not as good as that obtained for Y_t. This can be explained by noting that the error in approximating $Im[Y_t]$ is not controlled in the present procedure and, therefore, it contributes to the observed error in Z_{in} when transferred through the transmission line. To avoid this effect, it would be necessary to control both the real and imaginary parts of Y_n. This could be accomplished by using the ladder element values obtained here as initial values in a computer-aided design procedure. In this event a nonlinear function minimization algorithm would be required.

In conclusion, it has been shown that the biconical antenna can be effectively modeled by a transmission line terminated in an LC ladder network with a resistive load. It has been shown that the modeling problem can be reduced to a straightforward numerical approximation procedure followed by a direct synthesis algorithm.

3. Approximation by Rational Functions

3.1 Theory

It is often the case that one would like to express the transfer function of a linear system as the ratio of two polynomials. This form is preferred as it lends itself to linear transform methods of solution. Of the techniques that have been developed to fit experimental data by such rational functions [3]-[4], the one by Levy is the most notable [5]-[6] and forms the basis of the rational approximation method examined in this report.

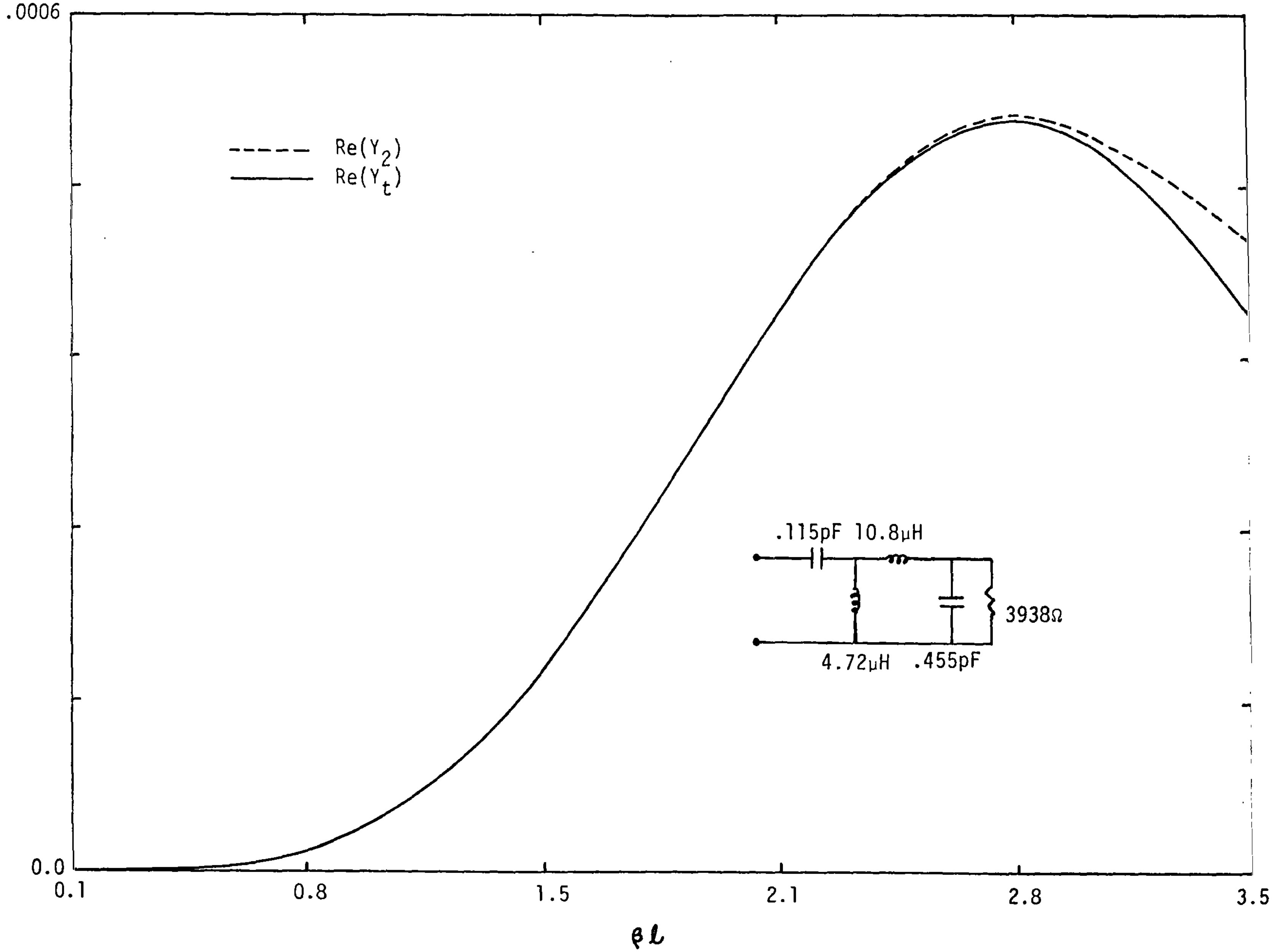

.0006
0.0
0.1
0.8
1.5
2.1
2.8
3.5
βℓ
Re(Y2)
Re(Yt)
.115pF 10.8μH
3938Ω
4.72μH .455pF

Figure 5

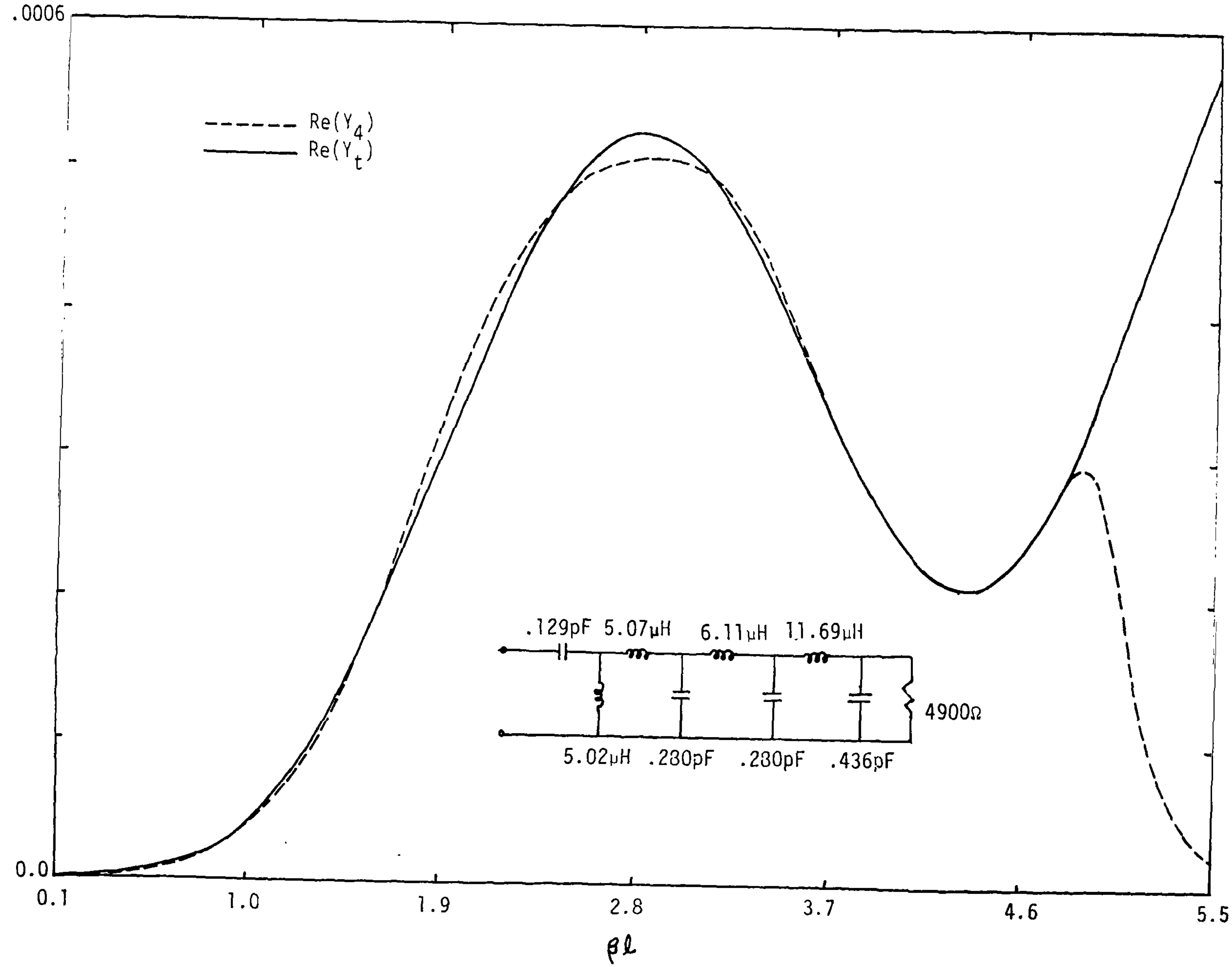
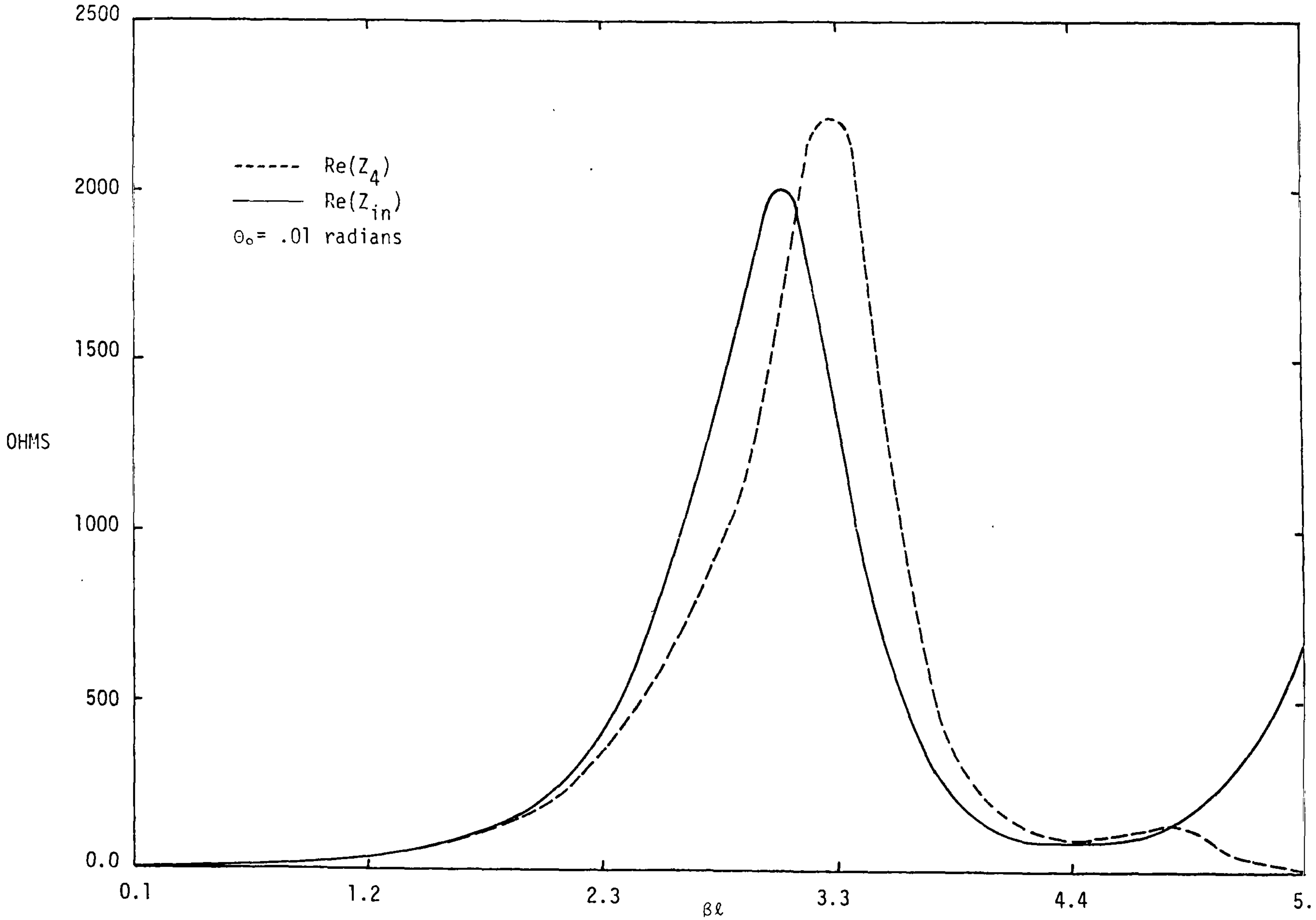

.0006
0.0
Re(Y4)
Re(Yt)
.129pF 5.07μH 6.11μH 11.69μH
4900Ω
5.02μH .280pF .280pF .436pF
0.1
1.0
1.9
2.8
3.7
4.6
5.5
βℓ

Figure 6



Re(Z_4)
Re(Z_{in})
Θ_o= .01 radians
OHMS
2500
2000
1500
1000
500
0.0
0.1
1.2
2.3
3.3
4.4
5.5
$\beta\ell$

Figure 7

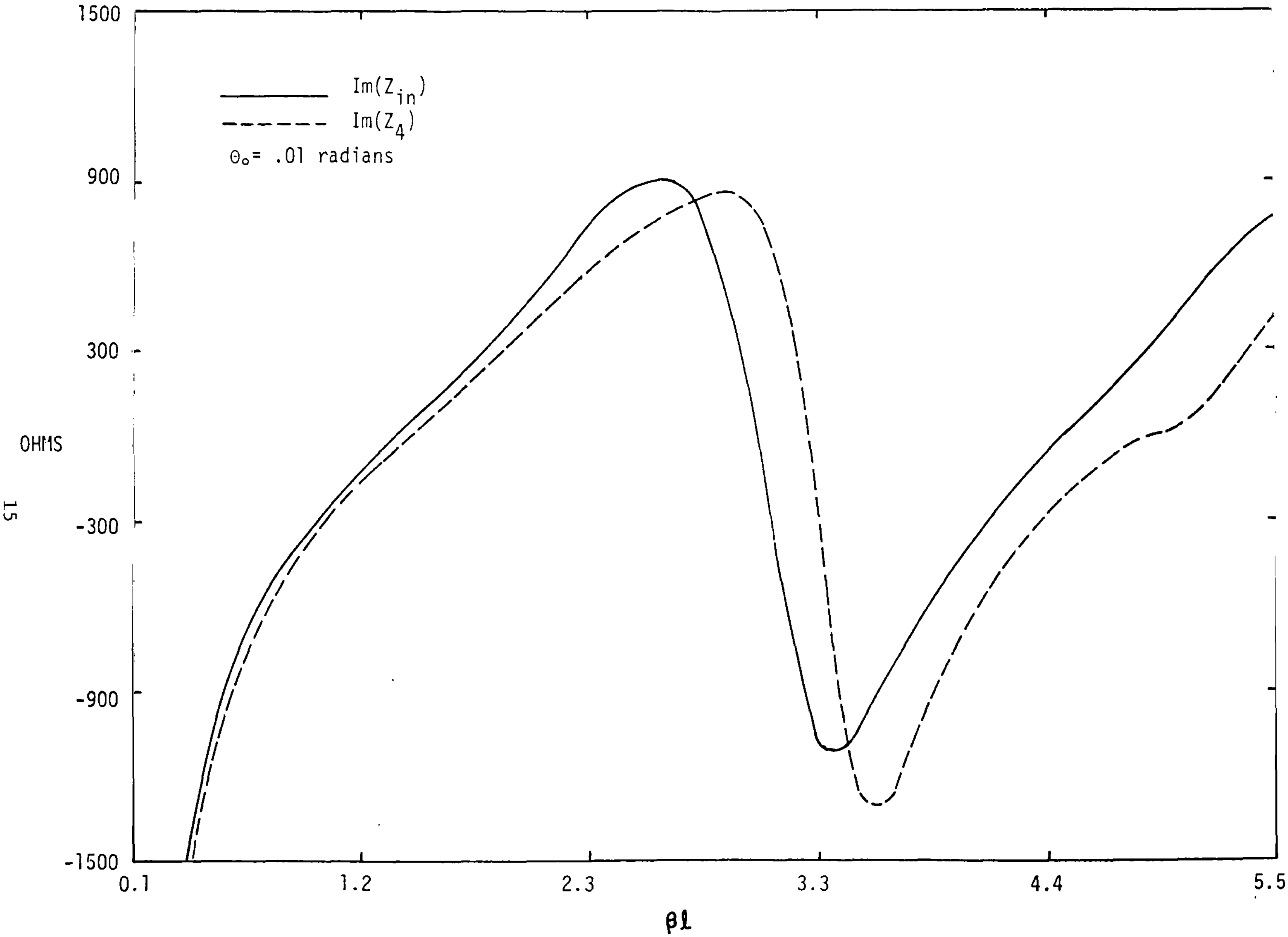

1500
900
300
-300
-900
-1500
OHMS
Im(Z_in)
Im(Z_4)
Θ₀= .01 radians
0.1
1.2
2.3
3.3
4.4
5.5
βl

Figure 8

A function of the form

$$H(j\omega) = \frac{a_0 + a_1(j\omega) + \ldots + a_n(j\omega)^n}{b_0 + b_1(j\omega) + \ldots + b_m(j\omega)^m} = \frac{N(j\omega)}{D(j\omega)}$$

is chosen to approximate (in the least-squares sense) a given complex set of data $F_i = F_{Ri} + jF_{\pm i}$, i=1, ... , N, where $H(j\omega)$ represents, for example, the transfer function of a lumped network and F_i the steady-state data associated, for example, with the current at some point on a scattering object. The a_i and b_i coefficients are found by minimizing

$$E = \sum_{i=1}^{N} |F_i - H(j\omega i)|^2 = \sum_{i=1}^{N} |e_i|^2 .$$

The problem with this formulation is two-fold: E is a nonlinear function of the unknown coefficients and the low frequency data is not weighted sufficiently. As a result, wide swings in the input data will cause large approximating errors at low frequencies. These problems may be remedied by defining a new error,

$$e^*_i = \frac{D^k(j\omega)}{D^{k-1}(j\omega)} e_i ,$$

where the superscript k refers to the iteration number. If, after each iteration, one refines the error estimate in this way and minimizes again, a much better approximation is obtained. Sanathanan and Koerner [7] have shown that $D^k \cong D^{k-1}$ after a sufficient number of linear iterations. With this change, the object function now becomes

$$E^* = \sum_{i=1}^{N} \left| \frac{[D_R(jw_i) + jD_I(w_i)][F_R(jw_i) + jF_I(w_i)] - [N_R(jw_i) + jN_I(w_i)]}{D^{k-1}(jw_i)} \right|^2$$

$$= \sum_{i=1}^{N} \left| [D_R + jD_I][F_R + jF_I] - [N_R + jN_I] \right|^2 W_{ik} ,$$

where $W_{ik} = \frac{1}{|D^{k-1}|^2}$ is a weight function, and the subscripts R and I indicate the real and imaginary parts of the terms. The minimization of E* at each iteration is now a linear problem. To this end E* is partially differentiated with respect to each of the polynomial coefficients and equated to zero. This yields the following matrix equation:

$$
\begin{bmatrix}
\lambda_0 & 0 & -\lambda_2 & 0 & \lambda_4 & \cdots & T_1 & S_2 & -T_3 & S_4 & \cdots \\
0 & \lambda_2 & 0 & -\lambda_4 & 0 & \cdots & -S_2 & T_3 & S_4 & -T_5 & \cdots \\
\lambda_2 & 0 & -\lambda_4 & 0 & \lambda_6 & \cdots & T_3 & S_4 & -T_5 & S_6 & \cdots \\
0 & \lambda_4 & 0 & -\lambda_6 & 0 & \cdots & -S_4 & T_5 & S_6 & -T_7 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \\
T_1 & -S_2 & -T_3 & S_4 & T_5 & \cdots & U_2 & 0 & -U_4 & 0 & \cdots \\
S_2 & T_3 & -S_4 & -T_5 & S_6 & \cdots & 0 & U_4 & 0 & -U_6 & \cdots \\
T_3 & -S_4 & -T_5 & S_6 & -T_7 & \cdots & U_4 & 0 & -U_6 & 0 & \cdots \\
S_4 & T_5 & -S_6 & -T_7 & S_5 & \cdots & 0 & U_6 & 0 & -U_8 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots &
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ \vdots \\ \\ \\ b_0 \\ b_1 \\ \vdots \\ \\ \\
\end{bmatrix}
=
\begin{bmatrix}
S_0 \\ T_1 \\ S_2 \\ T_3 \\ \vdots \\ 0 \\ U_2 \\ 0 \\ U_4 \\ \vdots
\end{bmatrix}
$$

where

$$\lambda_i = \sum_{k=1}^{n} w_k^i \, W_{kL}$$

$$S_i = \sum_{k=1}^{n} w_k^i \, R_k W_{kL}$$

$$T_i = \sum_{k=1}^{n} w_k^i \, I_k W_{kL}$$

$$U_i = \sum_{k=1}^{n} w_k^i \, (R_k^2 + I_k^2) W_{kL}$$

R_k and I_k are the real and imaginary parts of the transfer function at experimental points, and L is the iteration number. The coefficients b_1, b_2, ... evaluated at the L-1 iteration are used to refine the weighting function W_L for the next iteration.

A FORTRAN program has been written implementing the above complex-curve fitting algorithm.

3.2 Applications

The aforementioned method was applied to the data generated by the terminating admittance function Y_t of the biconical-antenna model as given by Tai [1]. The data is shown in Figure 2. A rational function with eighth order numerator and ninth-order denominator was chosen to fit this data over a range of normalized frequency $0 \le \beta\ell \le 15$. This choice of transfer function was based upon the results of tests using the same program to fit the input impedance of an ideal transmission-line

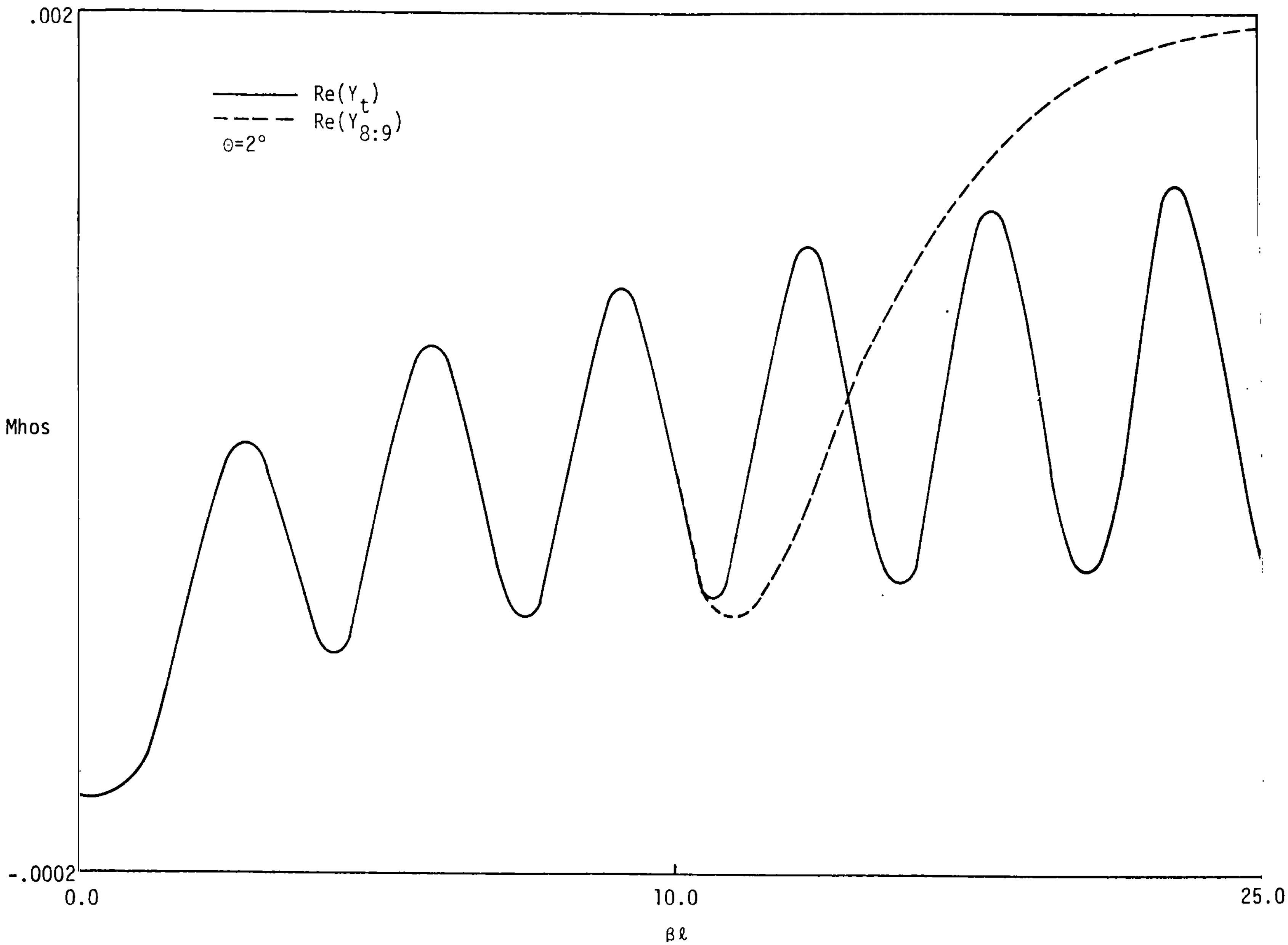


Figure 9

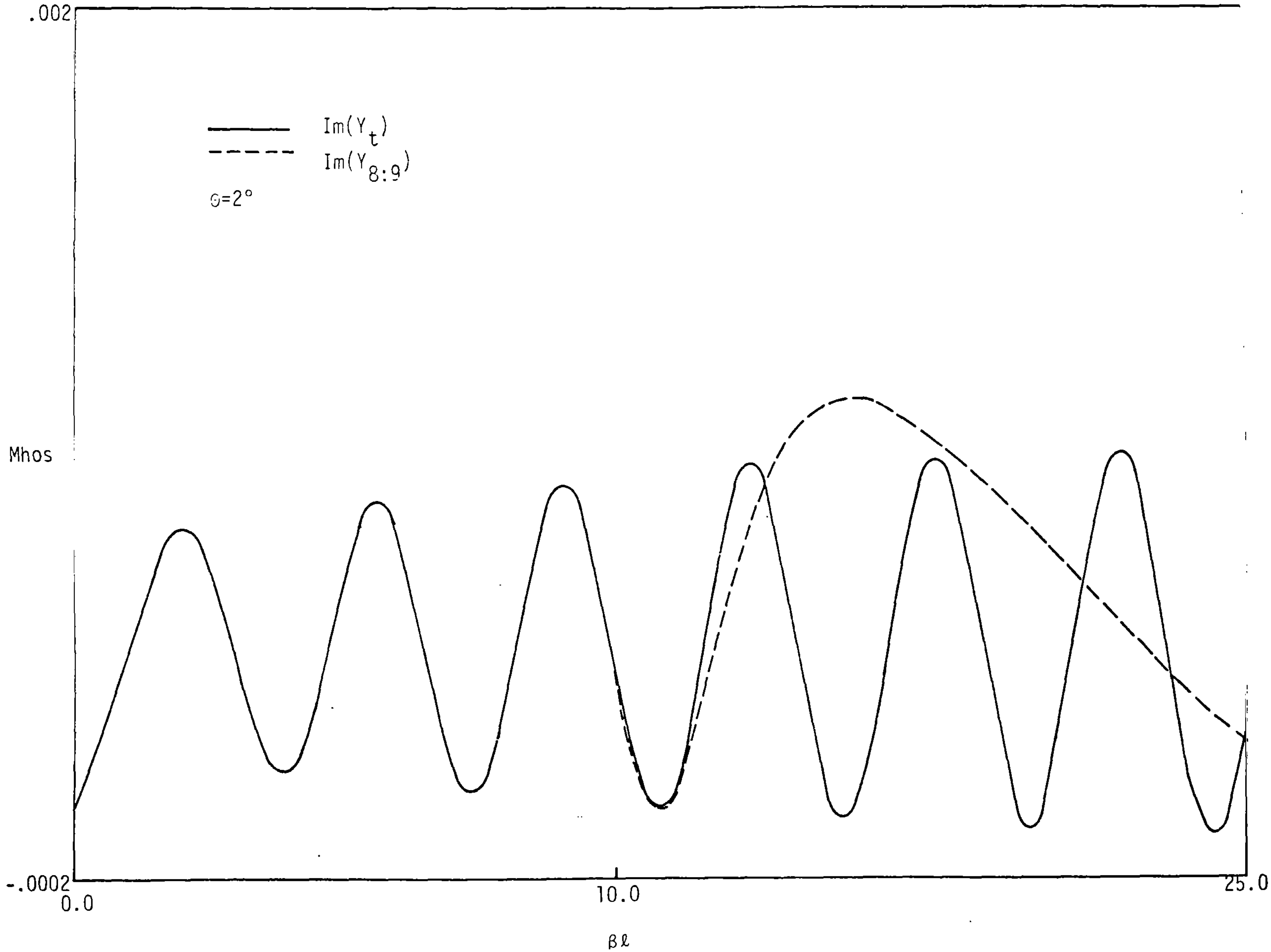


Figure 10

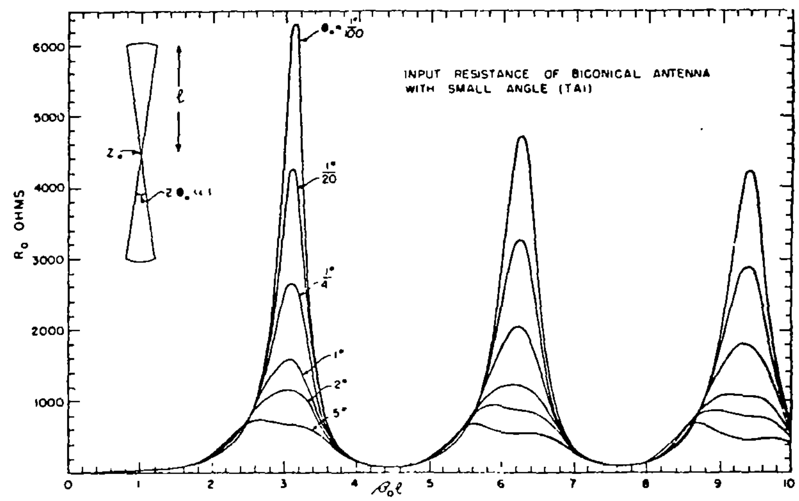


Fig. 10.3. Input resistance of biconical antenna with small angle (Tai).

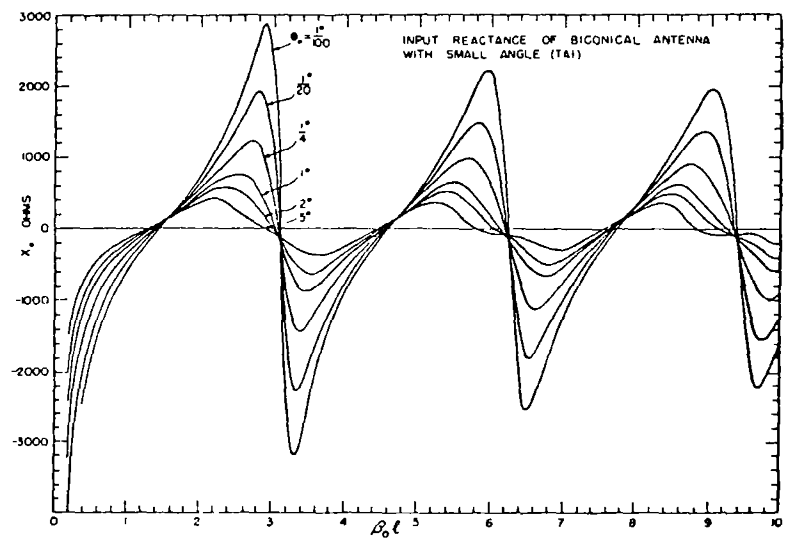


Fig. 10.4. Input reactance of biconical antenna with small angle (Tai).

Figure 11

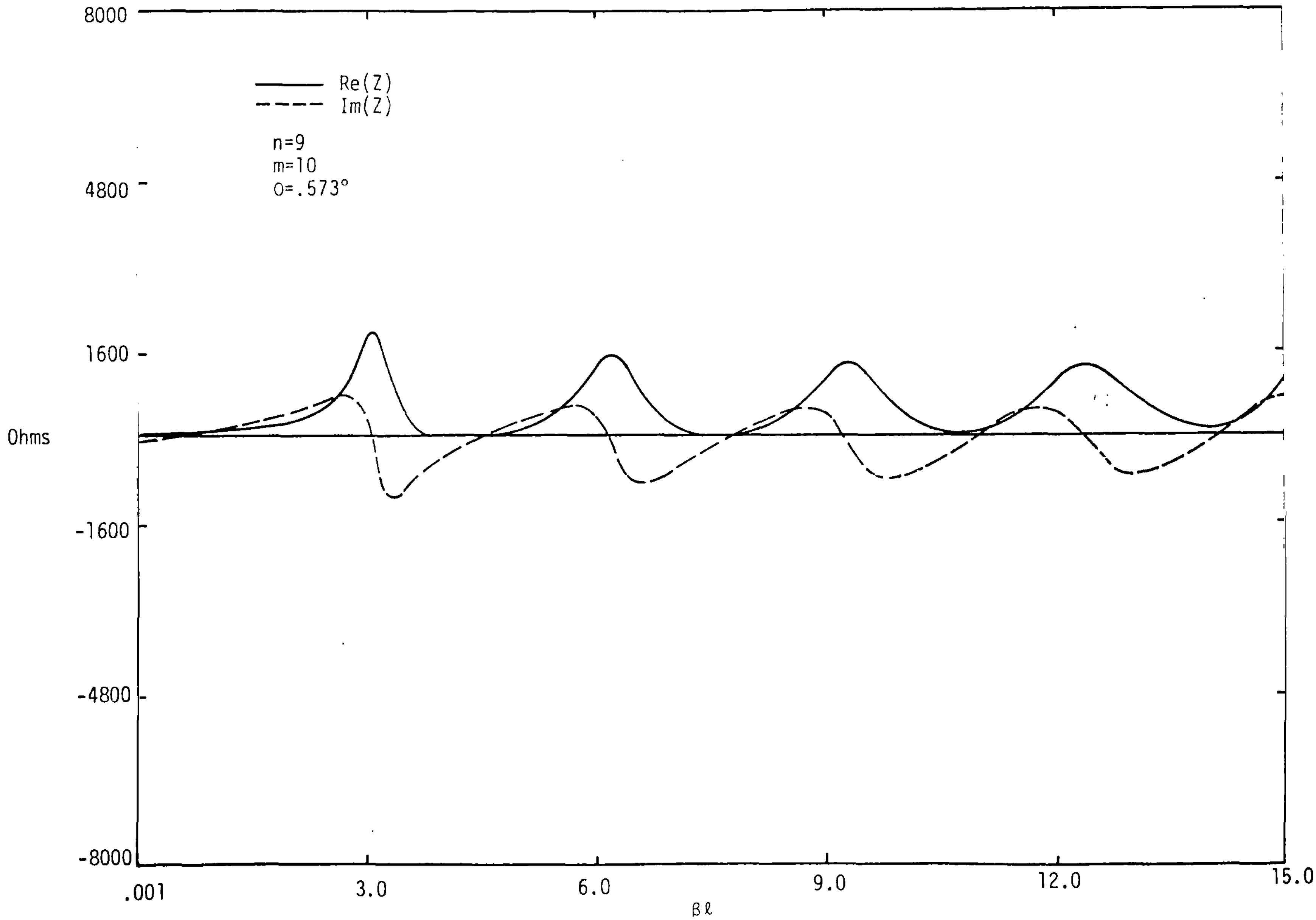

8000
4800
1600
Ohms
-1600
-4800
-8000
Re(Z)
Im(Z)
n=9
m=10
O=.573°
.001
3.0
6.0
9.0
12.0
15.0
βℓ

Figure 12

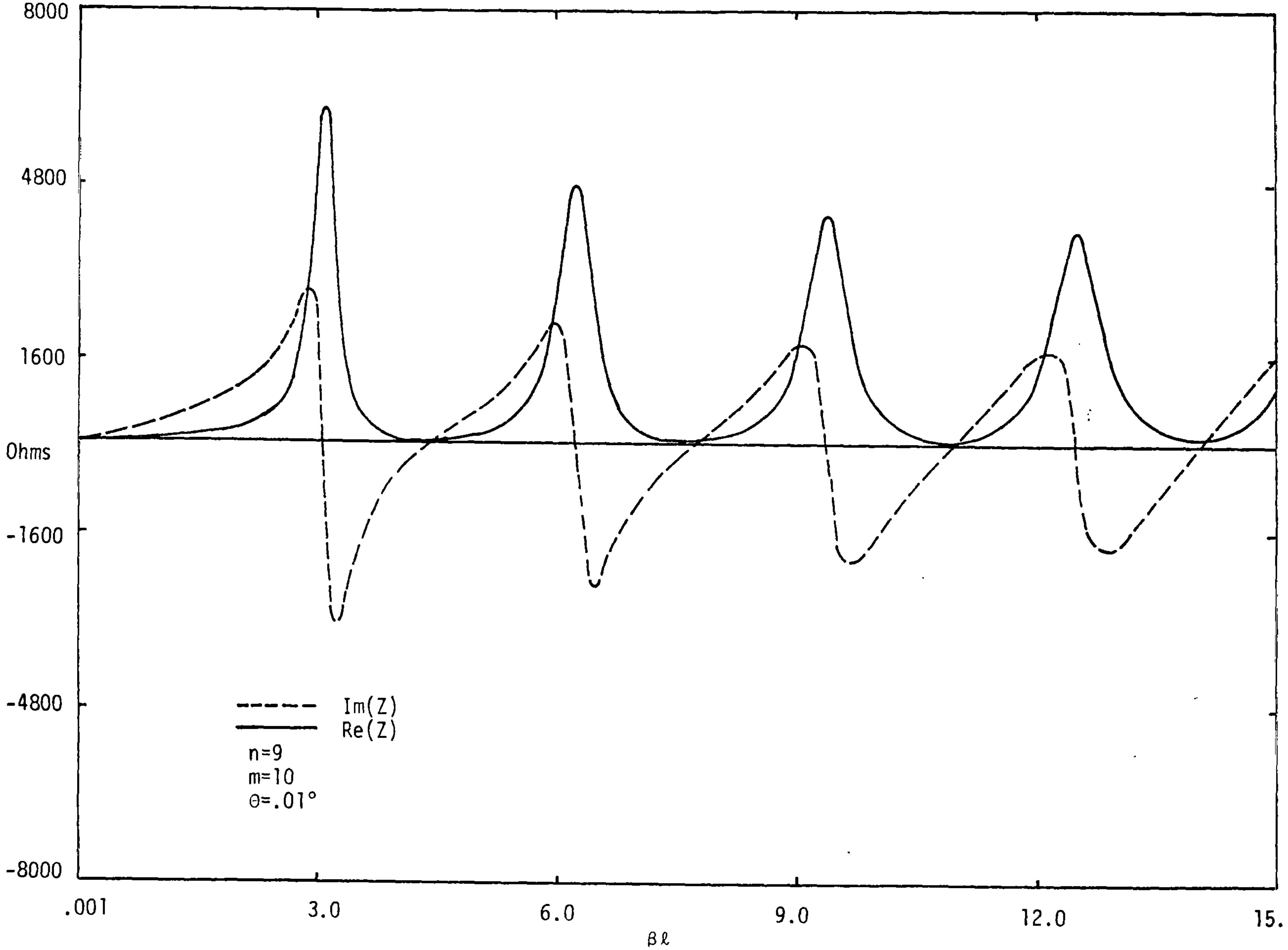

8000
4800
1600
Ohms
-1600
-4800
-8000
.001
3.0
6.0
9.0
12.0
15.0
$\beta\ell$
Im(Z)
Re(Z)
n=9
m=10
Θ=.01°

Figure 13

terminated in a resistance. The results of the Y_t approximation are shown in Figure 9 and Figure 10.

Another test was performed on data describing the input impedance of the biconical antenna as shown in Figure 11. This example was approximated by a ninth-order numerator and tenth-order denominator. The results are presented in Figure 12 and Figure 13. It is seen by comparing these results with Figure 11 that the approximated imaginary part of Z_{in} fails to fit the data near $\beta\ell=0$. This is due to a pole at zero which the data contained that the rational approximating function could not accomodate due to its chosen structure. This could have been corrected by changing the data and reinserting the pole later, a technique described by Levy [5].

The poles and zeros of the approximating function were extracted by standard techniques and compared with those found by Tai and Cho via a grid search. These results were later confirmed by Giri, Baum and Tai [8] using an application of Cauchy's residue theorem. The resulting pole and zero locations are shown in Figure 14. The poles and zeros reflect the closeness of the fit over the approximating range.

3.3 Conclusion

The rational function approximation method employed here has the advantage of being able to produce an analytic representation of data that is amenable to linear transform methods of solution. Furthermore, the implementation of the method is straightforward and computationally efficient. For the accuracy achieved here, the typical run took 2 CPU seconds (Amdahl 470) and cost $.50.

As a final remark, a more nearly mini-max approximation could be obtained by incorporating Lawson's algorithm [2] in the iterative procedure but our investigations to date do not indicate that this will be necessary.

Acknowledgment

It is a pleasure to thank C. T. Tai for his helpful advice and encouragement during the course of this study.

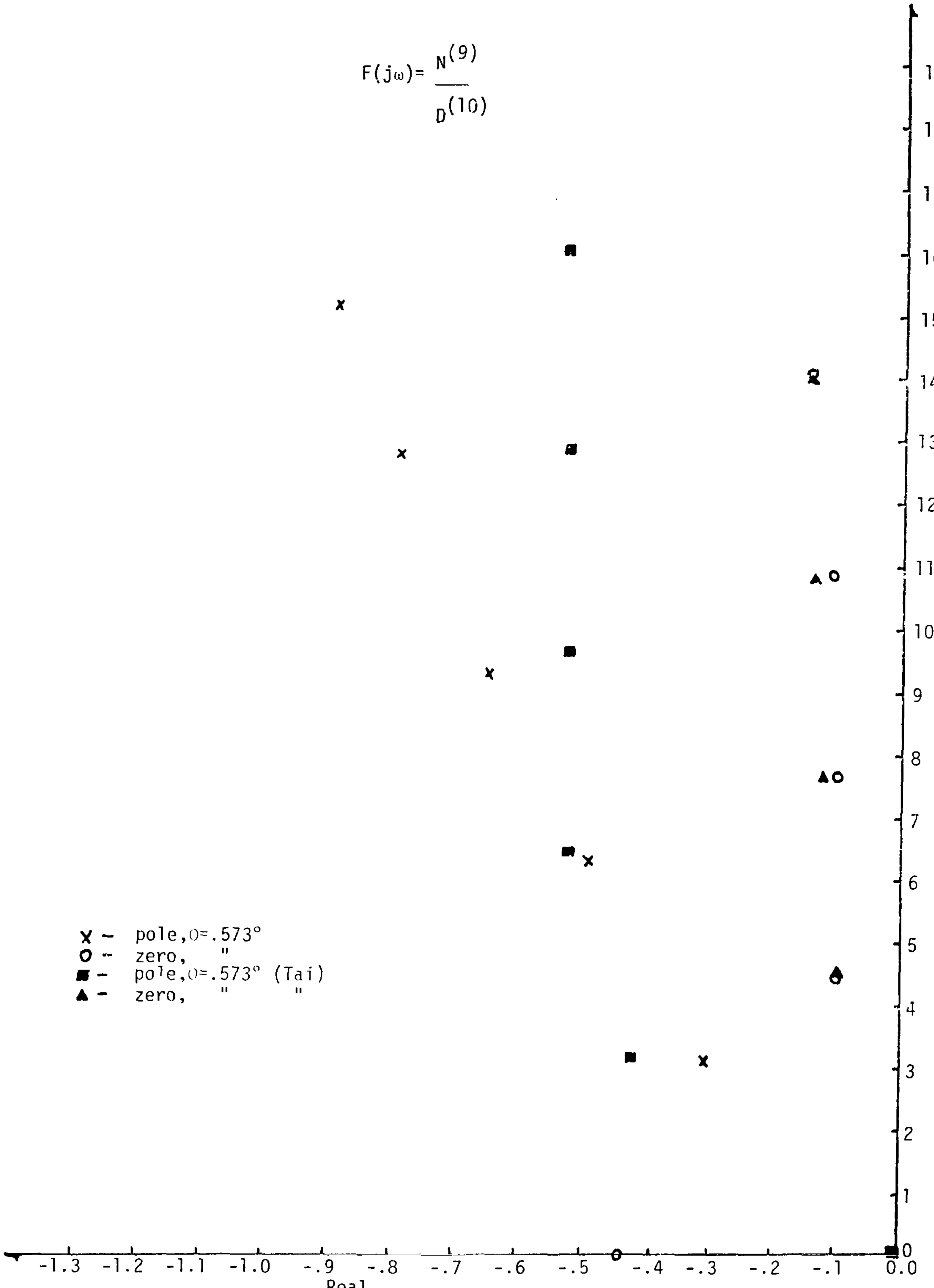

$F(j\omega) = \frac{N^{(9)}}{D^{(10)}}$
X – pole, θ=.573°
O – zero, "
■ – pole, θ=.573° (Tai)
▲ – zero, " "
Real
Imaginary
-1.3 -1.2 -1.1 -1.0 -.9 -.8 -.7 -.6 -.5 -.4 -.3 -.2 -.1 0.0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Figure 14

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