Interaction Notes

Note 356

Statistical Modeling of EMP Interaction

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ABSTRACT

This report summarizes the results of our effort under the program "Statistical Modeling of EMP Interaction," an effort carried out in a time span from October 1977 to December 1978. In what follows, Section I describes the technical motivations and approaches, and Section II presents technical status summaries for the tasks. Appendix A provides the derivation of a few key formulas for the task on random mutual coupling for which the main results are elaborated in the text.

PREFACE

The RDA Staff members whose technical efforts contributed to this program and the final report are L. L. DeRaad, W. R. Graham, W. J. Karzas, C. T. C. Mo, G. K. Schlegel, and A. D. Varvatsis. Other staff members whose discussions were helpful are J. Marcum and J. A. Wright.

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SECTION I

TECHNICAL BACKGROUND, MOTIVATION, AND APPROACHES

1. BACKGROUND

The background against which a statistical method in EMP interaction analysis is pursued is the simple inadequacy of the traditional deterministic method. The deterministic method calls for the modeling and solution of the interaction problem as an electromagnetic (EM) boundary value problem (or an equivalent circuit version). The inadequacy lies in that the solution so obtained does not always agree well with the EMP coupling occurring in a real system and that an uncertainty measure for applying the predicted deterministic results is lacking. In fact, even for some EMP tests with purposefully-simplified and tightly-controlled configuration the agreement between the measurement and the prediction is poor enough to strongly exhibit that inadequacy [Ref 1], let alone for a complicated real system.

The causes for such an inadequacy are: (A) the necessary simplification in the modeling to maintain mathematical tractability and conform with resource constraints; (B) the uncertainties and variations in the values of the parameters used in the model; and most importantly (C) the complexity of the real system in its multiple coupling paths and mutual interactions among different parts to EMP--a phenomenon which practically defies deterministic modeling.

2. MOTIVATION

In searching for new approaches and technologies to compensate the above-mentioned inadequacies, there at least three reasons that motivate and favor the selection of and investigation into a statistical one. First, from the view of analysis, the applicational errors and uncertainties due to modeling simplifications and parameter estimates which were not treatable in a purely deterministic approach are naturally amenable to a statistical one. Example cases are when the compounded uncertainties in successive submodelings render the overall deterministic result meaningless and when the uncontrolled probabilistic nature of the EMP stress itself and the detailed operational state of the system become the prevailing factors in determining the system's performance.

Second, from the view of synthesis, the EMP prediction-data comparison and the subsequent modifications and calibrations of the prediction capability are always imperfect. Such processes are invariably based upon a finite number of computations of finite modeling complexities and a finite number of data under simulated less-than-full testing conditions. Both the reconciliation between prediction and data and the further use of prediction for untested cases necessarily encompass inferences of a statistical nature. That nature becomes more pronounced and more accurately subject to a statistical quantification when the interaction becomes more complex and disorderly. But this is the situation we usually face in the EMP interaction problem.

Third, for the many cases of systems with functional redundancies it is the probabilistic distribution and its various averages of the component responses to the EMP on which the system performance depends. For such cases, the much less information-loaded distribution of responses is needed rather than the individual responses themselves. Thus, a statistical description can by-pass the insurmountable difficulty in obtaining details and still provide very useful information.

In short, a statistical approach squarely recognizes and seeks to deal with the realistic uncertainties and complexities of the EMP system interaction problem from the very start. This approach is diagonally opposite to the conventional deterministic analysis and, judging from the maturity and difficulties of the latter, is clearly needed. The present exploratory investigation intends to and should be but a beginning effort in this new and promising area of EMP-system interaction technology.

3. APPROACHES

Consistent with the nature of this effort, significant attention was directed toward asking the "right questions." As such, the question regarding what types of systems and what kinds of EMP interaction quantities are amenable to a statistical approach must be considered at the outset before any particular approaches can be selected and pursued. In this regard, at least three guidelines were reached and used:

A. The system has to "degrade gradually" in its "relevant performance."

B. There must be a "large number" of components or elements involved in the physical responses which contribute and control those performances.

C. There must be a prevailing "randomness" in the EMP coupling mechanism for those responses.

With these guidelines in mind, we set out to pursue three types of approaches. They are (a) the aggregate of random interacting dipoles as a crude model for the many complicated EMP-interacting wires/struts in an aircraft or communication center; (b) the effect of uncertain perturbations in the generic shapes of conductor geometry on the SEM (singularity expansion method) and EEM (eigenfunction expansion method) characterization of system EMP responses: and (c) the statistical POE (point of entry) formalism to quantify uncertainties in and distributions of the internal interaction paths. These approaches were investigated separately in this effort and were successful to different extents. Their technical status is summarized in what follows.

SECTION II

TECHNICAL STATUS SUMMARY OF TASKS

1. STATISTICAL INTERACTION OF DIPOLE ELEMENTS (SOW TASKS 4.2 AND 4.3)

a. Concept and Objective

To the uncontrolled EMP signal, the many wires, loops, and conducting struts in a real system such as an aircraft or communication center, indeed exhibit a high degree of disorder and complexity. The induced currents and voltages on these individual internal structures may be so sensitive and vary so much, depending on the details of the incident EM fields and the system's in-operation states, that only statistical descriptions of these responses and their various averages are the practically meaningful quantities we can gain a firm grasp and make use of in coping with the EMP threat to the system.

On the other hand, dipoles are the simplest elements one can use to facilitate and build up EM interaction. Although the individual structures are not exactly dipoles, they as a whole do resemble somewhat an aggregate of such, especially for the relatively low frequency content ($\lesssim$ several mega-Hertz) of a typical EMP. Our concept is to model the complicated structures as an aggregate of random dipoles and to obtain the resulting statistical distribution of the coupling responses on these dipoles. The input randomizations to the model are provided via those of the dipole sizes and shapes, orientations, impedance loadings, and spatial distribution (see Figure 1). The so-obtained statistical distribution of the responses and its various averages will be compared with "corresponding" data from real systems to gain understanding of and insight to the complex-system coupling problem. The

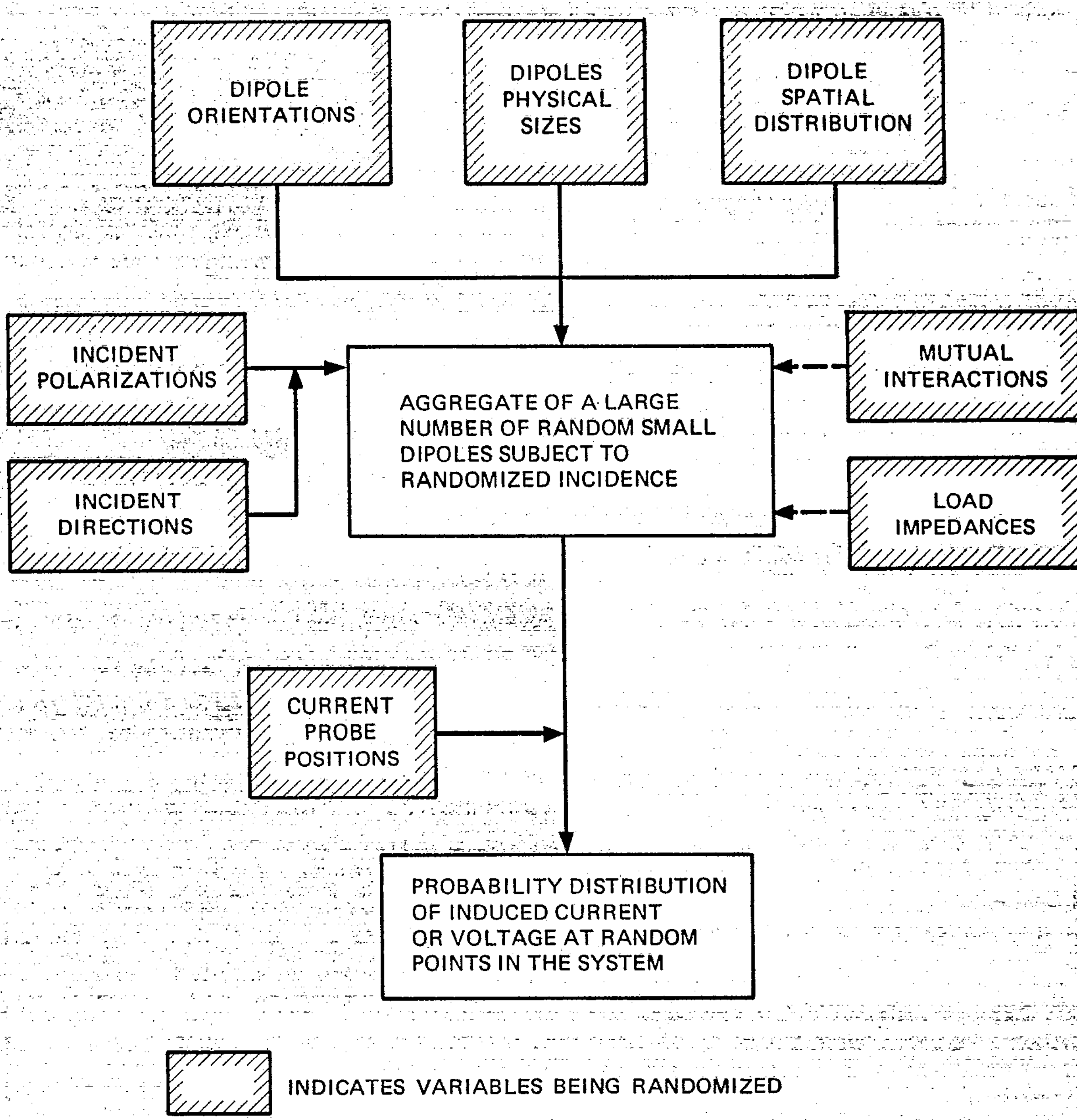

Figure 1. Structure of Statistical Dipole Model

result will then be used to make further predictions and direct more definitive approaches.

The present task is devoted to the effects of mutual coupling among the random dipoles. In a previous effort of ours [Ref 2], which did reveal interesting and encouraging results, such effects were only accounted for at an "assumed-parameter" level. Neither the validity nor the physical constituency of the parameters was investigated. The objective of this task is to obtain the conditions and generalization of the dipole mutual interaction effect.

b. Technical Results

(i) The General Model

The inquiry into the induced currents' distribution and their averages on the collection of dipoles is a special case of representing microscopic average phenomena in terms of macroscopic ones. The formalism of representing such for the interacting dipoles is, of course, known [Ref 3]. Although that formalism contains no information other than defining concepts, it is briefly presently here to exhibit the nature of the specific model that followed.

Consider electric dipoles, for example, we have first

$$\underset{\sim}{p}_i = \varepsilon_o \underset{\approx}{P}_i \cdot \left(\underset{\sim}{E}_i^{\,o} + \underset{\sim}{E}_i^{\,M}\right) \tag{1}$$

where $\underset{\sim}{p}_i$ = microscopic electric dipole moment for i^{th} dipole element (Coulomb-meter)

$\underset{\sim}{E}_i^{\,o}$, $\underset{\sim}{E}_i^{\,M}$ = respectively the incident and mutual coupled electric field (volt/meter)

$\underset{\approx}{p}$ = microscopic dipole polarizability tensor

Second, if all interactions are linear, we have

$$\underset{\sim}{E}_i^M = \underset{\approx}{\lambda}_i \cdot \underset{\sim}{E}_i \qquad (2)$$

where $\underset{\approx}{\lambda}$ = electric mutual interaction coefficient tensor (dimensionless)

Third, within the linearity and considering only the induced dipole moments (i.e., no permanent dipole moments), $\underset{\sim}{p}_i$ is independent of field strengths. Fourth, the linkage between microscopic to macroscopic description is facilitated via

$$\underset{\sim}{P} \equiv \varepsilon_o \underset{\approx}{\chi}_e \cdot \underset{\sim}{E}^o \equiv \left\langle \frac{d\Sigma_i \underset{\sim}{p}_i}{d^3x} \right\rangle = \langle n \rangle \langle \underset{\sim}{p}_i \rangle$$

where $\underset{\sim}{P}$ = average macroscopic polarization (Coulomb/meter^2)

$\underset{\approx}{\chi}_e$ = average macroscopic electric susceptibility tensor (dimensionless)

$\underset{\sim}{E}_o$ = macroscopic electric field without mutual interaction (volt/meter)

$\langle\ \rangle$ = denoting macroscopic average

$\langle n \rangle$ = average number of dipoles per unit volume (meter^{-3})

Fifth, as a result of the above and denoting $U \equiv$ unit dyadic, we have

$$\underset{\approx}{X}_e = \langle n \rangle \cdot \left\langle \underset{\approx}{p} \cdot (\underset{\approx}{U} + \underset{\approx}{\lambda}) \right\rangle$$

expressing the macroscopic susceptibility in terms of the averages of its microscopic constituents.

Four remarks are in line here. One, the magnetic dipoles can be formulated just similarly. Two, the induced current I_i on the i^{th} electric dipole is related to its moment $\underset{\sim}{p}_i$ simply by

$$\underset{\sim}{I}_i = \frac{-i\omega \underset{\sim}{p}_i}{\ell_i}$$

and the averages of the former can be easily related to those of the latter. Three, the full quantity of interest is the distribution of the $\underset{\sim}{I}_i$ or $\underset{\sim}{p}_i$ for the collection of dipoles, not merely their first moments (the average of $\underset{\sim}{I}_i$ or $\underset{\sim}{p}_i$). Four, the randomized dipole model presented in the following is actually a special case of the above and is investigating the properties of $\underset{\approx}{p}_i$ and $\underset{\approx}{\lambda}$.

(ii) Statistical Dipole Mutual Coupling Model

The geometries of the small electric dipole and the small magnetic dipoles are shown in Figures 2 and 3, respectively. The induced currents probability densities [Ref 4] for a random

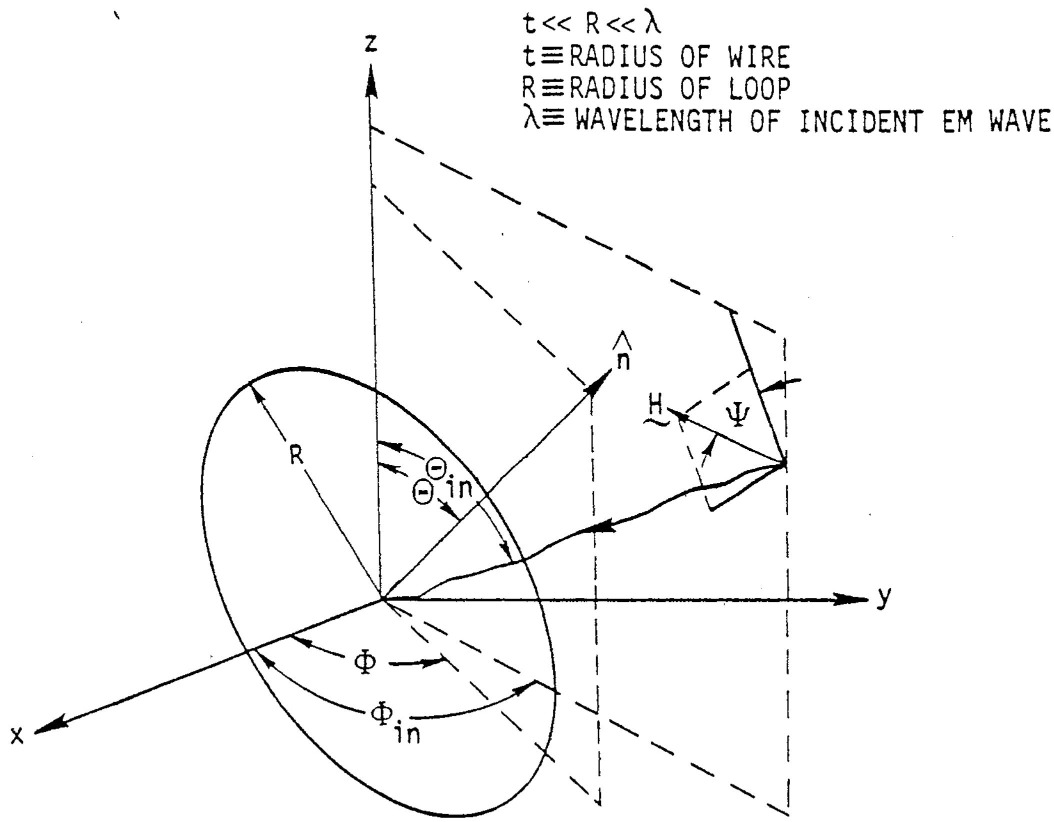

Figure 2. Orientations of the Incident Field and the Loop with Normal $\hat{n}$.

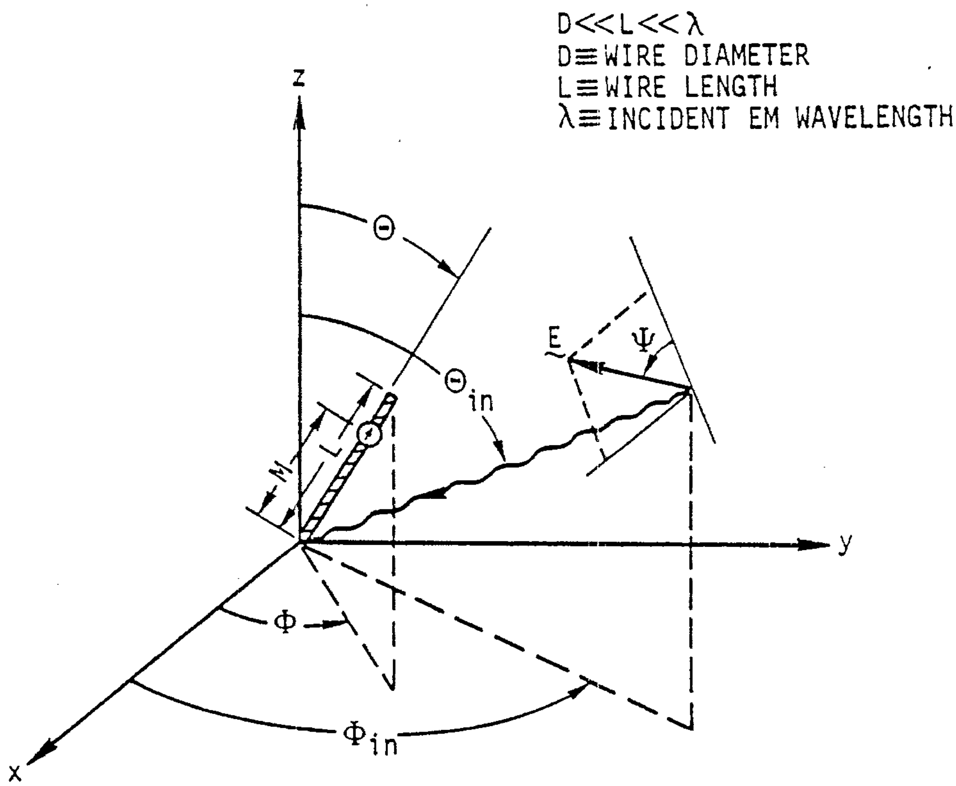

Figure 3. Orientations of the Incident Field and the Wire.

aggregate of such small dipoles, without mutual coupling effects, are given by the respective expressions:

$$p_I^{(wire)}(i) = \frac{d}{di} F^{(wire)}\{I \leq i\} \tag{6}$$

where

$$F^{(wire)}\{I \leq i\} = \int_{\frac{sx^2\eta}{\ln x} \leq i \frac{b^2}{\ln b}} p_{\Theta,\Phi}(\theta,\phi)\, d\theta d\phi\, p_{\Theta_{in},\Phi_{in}}(\theta_{in},\phi_{in})$$

$$\cdot\, d\theta_{in} d\phi_{in}\, p_\psi(\psi)\, d\psi\, p_S(s)\, ds\, p_\chi(x)\, dx \tag{7}$$

and

$$p_I^{(loop)}(i) = \frac{d}{di} F^{(loop)}\{I \leq i\}. \tag{8}$$

where

$$F^{(loop)}\{I \leq i\} = \int_{r \leq \frac{ir_m}{\eta}} p_{\Theta,\Phi}(\theta,\phi)\, d\theta d\phi\, p_{\Theta_{in},\Phi_{in}}(\theta_{in},\phi_{in})$$

$$\cdot\, d\theta_{in} d\phi_{in}\, p_\Psi(\psi)\, d\psi\, p_R(r) dr \tag{9}$$

In Equations (6) to (9), the induced current has been normalized and the probability densities in the integrands are where the model randomization enters as inputs to the analysis.

with respect to the shape of the induced currents' probability distributions, the main results previously obtained were [Ref 5] (A) the load impedance effects are negligible if

$$Z_1^{(\text{wire})} << 2.3\text{x}10^{10} \text{ x } \frac{\ell n\left(\frac{\ell}{d}\right)}{f_{\text{MHz}}\ \ell_{\text{meter}}} \tag{10-a}$$

$$Z_1^{(\text{loop})} << 47\text{x}f_{\text{MHz}} \text{ x } R_{\text{meter}} \text{ x } \frac{\ell n\left(\frac{8R}{t}\right)}{\ell n(400)}, \tag{10-b}$$

but tend to smooth and prolong the high value distribution tail of currents and enhance the low value tail if condition (10) does not hold; (B) the mutual coupling effects under the simple parameter assumption can be handled via

$$I = I^{(o)} \cdot (1+M) \cdot |\ G\ | \tag{11}$$

where $I^{(o)}$ is the current r.v. (random variable) without mutual coupling and G is normally distributed,

$$G \ \varepsilon\ N(1,\ M_G), \tag{12}$$

which broadens the resulting distribution via

$$1 \leq \frac{1 + \left(\frac{\sigma}{\mu}\right)_I^2}{1 + \left(\frac{\sigma}{\mu}\right)_{I_o}^2} \leq \pi/_2\ ; \tag{13}$$

and (C) the resulting currents statistical distributions resemble somewhat but are not lognormal with the central part of approximate slope ∿5 to ∿8 dB.

This present effort concentrates on the detail of the mutual coupling effect and obtains the values, with their full scale factors, of the induced currents' first two moments.

(iii) Specific Results*

Consider the aggregate of small electric dipoles. The induced currents on the dipoles satisfy the eqs. (see Appendix A for details of Section 4.1) (see Fig. 4 for notations):

$$I_i = \frac{\pi\omega\varepsilon}{4} \cdot \frac{\ell_i^{\,2}}{\ell n\left(\frac{\ell_i}{d_i}\right)} \cdot \hat{i} \bullet \left(\underset{\sim}{E}_i^{o} + \sum_{j\neq i} \underset{\sim}{E}_{ij}\right) \tag{14}$$

$$\underset{\sim}{E}_{ij} \equiv \frac{I_j \ell_j}{4\pi\varepsilon_o \omega r_{ij}^3} \cdot \left(\hat{j} - \hat{e}_{ij}\,\hat{e}_{ij}\cdot\hat{j}\right) \tag{15}$$

The "weak" mutual coupling case, under the condition of small dipoles located not close to each other,

$$\frac{\ell_j^3}{16 r_{ij}^3 \,\ell n\left(\frac{\ell_j}{d_i}\right)} << 1 \quad , \tag{16}$$

*See Appendix A for some derivations of formulas in Section II.1.

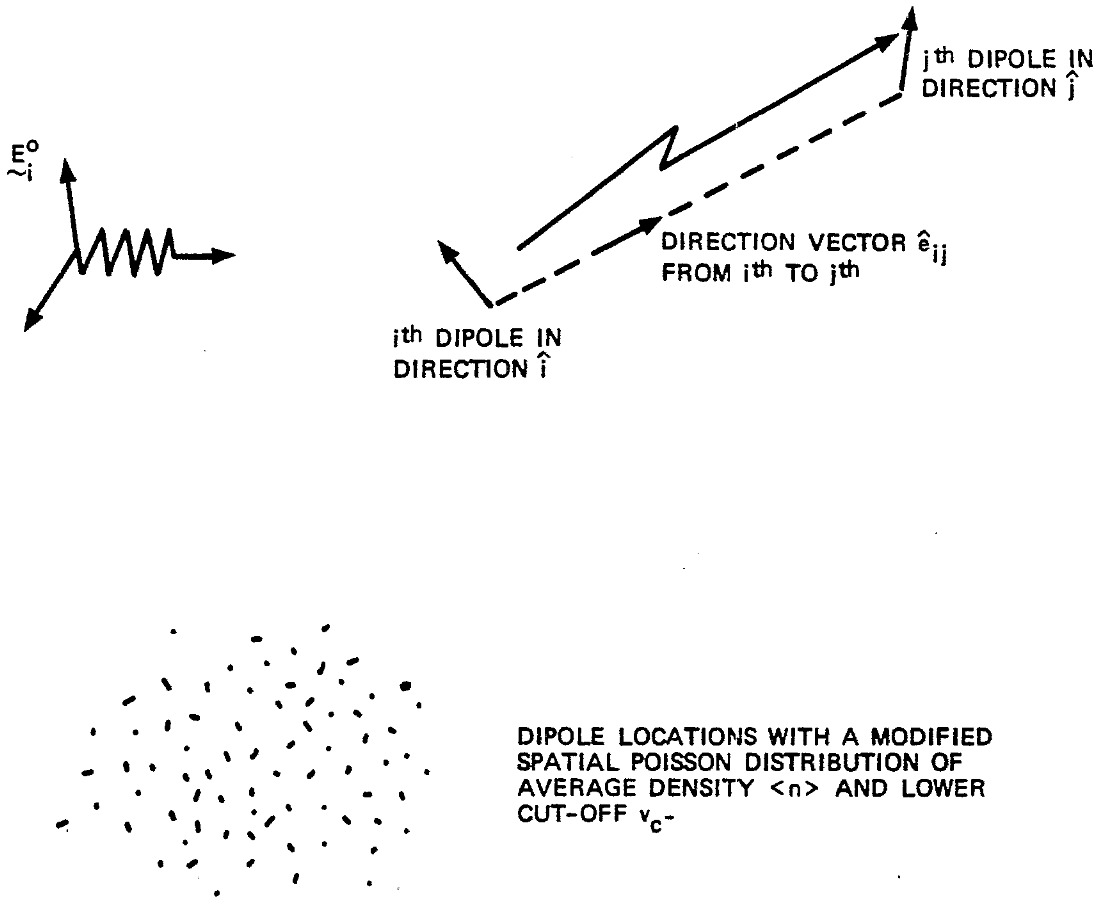

Figure 4. Aggregate of Mutually Interacting Dipoles

leads to a first-order perturbation result for the mutual coupling effect:

$$I_i = \frac{\pi\varepsilon\omega}{4} \cdot \frac{\ell_i^2}{\ell n\left(\frac{\ell_i}{d_i}\right)} \cdot \left\{ \hat{i} \cdot \underset{\sim}{E}_i + \sum_{j \neq i} \hat{j} \cdot \underset{\sim}{E}_j^o \right.$$

$$\left. \cdot \frac{\ell_j^3 \; \hat{i} \cdot \left(\underset{\approx}{U} - \hat{e}_{ij}\hat{e}_{ij}\right) \cdot \hat{j}}{16 r_{ij}^3 \;\; \ell n \left(\frac{\ell_j}{d_j}\right)} \right\}$$

$$\equiv I_i^o + I_i^M \equiv I_i^o \; (1 + X_i) \tag{17}$$

Now, for this "weak" coupling case, at a given i^{th} dipole the mutual interaction terms in the summation of (17) are indeed independent and the central limit theory dictates their approach to a normal distribution. After careful and lengthy calculations, the major results obtained are (see Appendix A)

$$X_i \longrightarrow N\left(\alpha,\ \beta \sqrt{1 + \frac{1}{1 + \left(\hat{i} \cdot \hat{E}_i^o\right)^2}}\right) \tag{18}$$

where the parameters α and β expressed in terms of their physical model constituencies are

$$\alpha \equiv \frac{\pi}{2 \cdot 3^3} \cdot \langle \ell^3 \rangle \cdot \left\langle \frac{1}{\ell n g} \right\rangle \cdot \langle n \rangle \cdot \ell n\left(\frac{v_{c+}}{v_{c-}}\right) \equiv \langle n \rangle \, c_1 \tag{19}$$

$$\beta^2 \equiv \frac{\pi^2}{2^3 \cdot 3^4 \cdot 5} \cdot \langle \ell^6 \rangle \cdot \left\langle \frac{1}{(\ell n q)^2} \right\rangle \cdot \frac{\langle n \rangle}{v_{c+}} \cdot \left(\frac{v_{c+}}{v_{c-}} - 1 \right) \equiv \langle n \rangle c_2 \tag{20}$$

where $q \equiv \ell_i / d_i$ (21a)

$$v_{c\pm} \equiv \frac{4\pi}{3} r_{\pm}^3 \text{ (the upper \& lower cutoff volumes)} \tag{21b}$$

$$r_{c+} \sim \min (\lambda, L_{system}) \tag{21c}$$

$$r_{c-} \sim \langle \ell \rangle \tag{21d}$$

In obtaining the above results, we used the following approximations or assumptions:

(a) dipoles sizes << average interdipole distances and wavelength;

(b) interdipole distances << wavelength; and

(c) the dipoles are spatially randomly located with a Poisson distribution of lower cutoff volume v_{c-}; i.e., dipoles are located independent of each other and are nonoverlapping with a minimum occupied volume v_{c-} and an average number density $\langle n \rangle$.

It should be noted that more general results without approximation (b) were obtained in Appendix A; approximation (b) neglects the phase delays among dipoles at different locations and thus gives an upper bound to the mutual coupling effect.

Consequently, the mutual coupling among dipoles has indeed, at least for the "weak" coupling case, the effect of modifying the statistical distribution of the currents on the collection of dipoles as we had assumed previously. The modification enhances, on the average, the induced currents as

$$\langle I_i \rangle = \langle I_i^o \rangle \ (1+\alpha) \qquad (22)$$

which follows Equations (17) to (19). Notice that the mutual coupling contribued part α is proportional to the average number of dipoles per unit volume. This is intuitively clear, because with the phase delays between dipoles neglected and with the dipole's lack of polarity (a dipole, of no permanent dipole moment, flipping 180° is identical to itself) the dipoles' coupling to the incident field is strengthened by the scattering. In addition, the factor $\ell n(v_{c+}/v_{c-})$ in α is also intuitively clear, because with uniform Poisson spatial density the number of dipoles in $r \rightarrow r+dr$ goes as $\sim r^2 dr$, but the near zone interaction strength goes as $\sim r^{-3}$.

The modification also has the effect of broadening the statistical distribution of the induced currents; i.e., the range over which those currents vary becomes wider:

$$\text{var} \left\{ I_i \right\} = \text{var} \left\{ I_i^\circ \right\} \cdot \left[(1+\alpha)^2 + 4\beta^2 \right] + 4_\beta{}^2 \langle I_i^\circ \rangle^2 \qquad (23)$$

Viewed slightly differently, this broadening reads

$$\frac{1 + \left(\frac{\sigma}{\mu}\right)_I^2}{1 + \left(\frac{\sigma}{\mu}\right)_{I^\circ}^2} = 1 + \frac{4\beta^2}{(1+\alpha)^2} \qquad (24)$$

where

$$o \leq \frac{4\beta^2}{(1+\alpha)^2} \leq \frac{c_2}{c_1} = \frac{\pi < \frac{1}{(\ell nq)^2} >}{60 < \frac{1}{\ell nq} >} \cdot \frac{<\ell^6>}{<\ell^3> v_{c-}} \cdot \frac{1}{\ell n\left(\frac{v_{c+}}{v_{c-}}\right)} \tag{25}$$

Applying Equation (25) to plausible geometrical dimensions of a typical systems, say, with $q \sim 50$, ℓ uniform in (o, ℓ_{max} $\sim$1 meter), L_{system} $\sim$10 meter, the factor in (25) is of $\sim 3 \times 10^{-3}$. This small number means that there is very little relative broadening as far as the dependence on $<n>$ is concerned. This indicates that when plotted on a lognormal graphic paper the slope of the distribution, which is essentially $\ell n[1 + (\sigma/\mu)^2]$, should be insensitive to the detailed parameters of the model.

(iv) Some Applications of the Technical Results

The weak coupling technical results bear out a previous conjecture as to the trend and the insensitivity of the distribution. Further, since we now have the physical model parameters for the μ_I and σ_I, we can fit data to the model and obtain the various average physical parameters contained in α and β. Then we can use them to predict trend, averages of further coupling responses, and probabilistic bounds to their range of variation. One useful average, for example, is

$$\begin{pmatrix}\text{average absorbed} \\ \text{energy per unit} \\ \text{volume of system}\end{pmatrix} = \frac{<\sum_{i\epsilon V} I_i^2 R_i>_V}{V} = <n> \cdot <R_i> \cdot \left[\operatorname{var}\{I_i\} + <I_i>^2\right] \tag{26}$$

(v) Unanswered Questions and Needed Pursuits

One of the most important questions which we have not been able to address within this effort is the other limit, the "strong" mutual coupling case. Whether the general trend and the (σ/μ) ratio of the distribution will be altered or shifted significantly is not now understood. This has a strong bearing to a real system since wire and strut components in such are often rather closely coupled to each other.

The second crucial question is: since a real system such as an aircraft is not literally made of a collection of random dipoles even in the sense of EMP-interaction, how do we generalize and hedge the dipole results to make them more applicable to real complicated systems, more directly and further quantitatively and conditionally? A third question is to experimentally verify that the theoretical investigation is in the right direction and truly useful in practice. These and other technical questions urgent to the development of this random coupling aspect of the statistical EMP interaction technology are beyond the scope of this effort but definitely need to be pursued.

2. VARIATION OF EEM AND SEM PARAMTERS DUE TO GEOMETRICAL PERTURBATION (SOW TASK 4.4)

a. Concept and Objective

While the Singularity Expansion Method (SEM) is a very useful technique in characterizing, recognizing, and storing a system's response to EMP, its physical foundation both in terms of relation to the well-established Eigenfunction Expansion Method (EEM) and sensitivity to the variations in geometrical structures has not been clarified. The concept of the statistical parameter investigation for SEM and EEM is first to translate geometrical deviations (from canonical shapes) to changes in those parameters; second, to obtain the statistical variations of the latter in terms of the uncertainties of the former; and third, to generalize the results judiciously for more complex and noncanonical geometrical shapes to estimate at least the probabilistic error bounds of their relevant SEM and EEM parameters.

The <u>objective</u> of the present task is to realize the first part of the concept mentioned above. The approach is to utilize the only known analytical solution for the EM scattering from a conducting body, a sphere, in both the SEM and EEM formulations to obtain the perturbational effects due to slight deviations from the spherical shape. The second and third parts, their results necessarily built upon those of the first part, of the concept in randomizing the perturbations and obtaining the corresponding statistical distribution and estimating their bounds could not be and were not pursued here. Those should be pursued in a succeeding effort.

b. Technical Results

The details of the technical results are included in [Ref. 9]. A brief summary is given in the following.

(i) The Model

Consider the magnetic field integral equation on the surface of a perfect conducting sphere

$$L_o J_{(o)} \equiv \frac{1}{2} \underset{\sim}{J}_{(o)}(\underset{\sim}{r}) - \int_{S_o} \hat{n}_o(\underline{r}) \times \left[\nabla G \times \underset{\sim}{J}_{(o)}(\underset{\sim}{r}')\right] dS' = J^{inc}(\underset{\sim}{r}) \qquad (27)$$

where S_o is the surface of the sphere with radius vector $\underset{\sim}{r} = \underset{\sim}{r}_o \equiv r_o\hat{r}$, $\underset{\sim}{J}_{(o)}$ the surface current density, $\underset{\sim}{J}^{inc} \equiv \hat{n}_o \times \underset{\sim}{H}^{inc}$, in which $\hat{n}_o \equiv \hat{r}$ and $\underset{\sim}{H}^{inc} \equiv$ incident magnetic field, and $G \equiv \exp(-kR)/(4\pi R)$ being the free space Green's function (see page 6 of Ref. 9) for other notations) in the Laplace domain, s cγ. Now, consider that the spherical surface S_o was perturbed into S via

$$\underset{\sim}{r} = \underset{\sim}{r}_o(\theta,\phi)\ [1 + \varepsilon\ f\ (\theta,\phi)] \tag{28}$$

where ε is the small perturbation indicator with respect to which we shall apply a first-order perturbation theory and (θ,ϕ) is an arbitrary function. Since we knew for the sphere S_o its EEM parameters: the eigenvalues, eigenfunctions, and expansion coefficients, and its EEM parameters: the natural frequencies, natural modes, and coupling coefficients, the goal is to obtain their perturbed values for S as a result of the geometrical perturbation $S_o \longrightarrow S$ specified by Eq. (28). These perturbed values are obtained via a first order perturbation method and are valid only up to and including order of ε.

(ii) Specific Results

After a very lengthy and somewhat subtle analysis, the main first-order perturbational results for the EEM and SEM parameters of the perturbed-sphere S are summarized here. First, for S the perturbed magnetic eigenvalue problem becomes

$$(L_o + \varepsilon p)\ \underset{\sim}{J}_{pt} = \lambda_p\ \underset{\sim}{J}_{pt} \tag{29}$$

where P is the shifted surface integral operator evaluated on S_o (see page 12 of Ref. 9) $\underset{\sim}{J}_{pt}$ is the part of the perturbed eigenmode on S tangential to S_o, and λ_p the perturbed eigenvalue. Notice that being a first order perturbation result, all quantities in Eq. (29) are evaluated on the unperturbed surface S_o.

Second, the perturbed eigenvalues are

$$\lambda^{R}_{nm} = \lambda^{(o)R}_{n} + \varepsilon\lambda^{(1)R}_{nm} \tag{30a}$$

$$\lambda^{R}_{nm} = \lambda^{(o)Q}_{n} + \varepsilon\lambda^{(1)Q}_{nm} \tag{30b}$$

Here, the subscript p is replaced by the double-index set nm, with $m = -n, -n+1, \ldots o, \ldots n-1, n$ and $n = 1.2\ldots\ldots$ because the EEM for S_o has two independent angular dependences, respectively, for θ and ϕ. The unperturbed eigenvalues (see page 14 of Ref. 10) $\lambda^{(o)}_{n}$ have only one subscript because the EEM for S_o has a degeneracy that makes its eigenvalues independent of m. The superscripts R and Q stand respectively for the magnetic (TE) and the electric (TM) parts of the surface eigencurrents $\underset{\sim}{R}_{nm}$ and $\underset{\sim}{Q}_{nm}$ on S_o (see page 14 of Ref. 9). [Ref 6]. Further, the degeneracy is assumed to be removed by the perturbation; but this point was not qualified in general and will be remarked upon later.

Third, the shifts in the eigenvalue, $\varepsilon\lambda^{(1)R}_{nm}$, and the coefficients, b^{R}_{nms}, are to be obtained from the new eigenvalue problem:

$$\sum_{k} P^{R}_{nsk}\, b^{R}_{nmk} = n(n+1)\, \lambda^{(1)R}_{nm}\, b^{R}_{nms} \tag{31a}$$

where

$$P^{R}_{nsk} \equiv \int_{S_o} \underset{\sim}{Q}_{ns} \cdot \left[\hat{n} \times P\, \underset{\sim}{R}_{nk} \right] ds \equiv \{\underset{\sim}{Q}_{ns},\, P\underset{\sim}{R}_{nk}\} \tag{31b}$$

Similarly, the $\varepsilon\lambda_{nm}^{(1)Q}$ and coefficients b_{nms}^{Q} are obtained from the set of equations

$$\sum_{k} P_{nsk}^{Q} \, b_{nmk}^{Q} = -n(n+1) \, \lambda_{nm}^{(1)Q} \, b_{nms}^{Q} \tag{32a}$$

where

$$P_{nsk}^{Q} \equiv \{ \underset{\sim}{R}_{ns}, \, P \, \underset{\sim}{Q}_{nk} \} \tag{32b}$$

Notice that as a result of Eqs. (31) and (32), $\lambda_{nm}^{(1)R} = -\lambda_{nm}^{(1)Q}$.

Fourth, the tangential (to S_o) part of the perturbed eigenmodes of the surface currents are (p. 15, 16, Appendix B)

$$\underset{\sim}{J}_{nmt}^{R} = \underset{\sim}{\check{R}}_{nm} + \varepsilon \sum_{p} \sum_{\ell} \left[a_{nmp\ell}^{RR} \, \underset{\sim}{R}_{p\ell} + a_{nmp\ell}^{RQ} \, \underset{\sim}{Q}_{p\ell} \right], \tag{33}$$

etc.

Here the $\underset{\sim}{\check{R}}_{nm}$ is a "natural" collection of degenerate modes for a given n grouped by the perturbed operator (28),

$$R_{nm} \equiv \Sigma \, b_{nmk}^{R} \, \underset{\sim}{R}_{nk}, \tag{34}$$

and the perturbed eigenmodes obey the orthogonal condition

$$\{ \underset{\sim}{Q}_{nm}, R_{p\ell} \} = n(n+1) \, \delta_{np} \delta_{m\ell} \tag{35}$$

etc.

Further, the first order eigenmode shifts in (33) are governed by the coefficients

$$a^{RR}_{nmrs} = \frac{\{\underset{\sim}{Q}_{rs}, \; P\overset{\vee}{\underset{\sim}{R}}_{nm}\}}{\left(\lambda_n^{(o)R} - \lambda_r^{(o)R}\right) n(n+1)} \quad , \; (n \neq r) \tag{36a}$$

$$a^{RQ}_{nmrs} = \frac{-\{\underset{\sim}{R}_{rs}, \; P\overset{\vee}{\underset{\sim}{R}}_{nm}\}}{\left(\lambda_n^{(o)R} - \lambda_r^{(o)Q}\right) n(n+1)} \quad , \; (n \neq r) \tag{36b}$$

Fifth, the normal (to S_o) part of the perturbed eigenmodes $\underset{\sim}{J}^R_{nmt}$ start with a first-order term in ε and are given by

$$-\varepsilon\underset{\sim}{n}^{(1)} \cdot \underset{\sim}{J}^R_{nmt} \text{ or } -\varepsilon\underset{\sim}{n}^{(1)} \cdot \underset{\sim}{R}_{nm} \tag{37a}$$

where $\varepsilon\underset{\sim}{n}^{(1)}$ is the difference between the new (to S) and old (to S_o) normals:

$$\varepsilon\underset{\sim}{n}^{(1)} \equiv \hat{n} - \hat{n}_o \tag{37b}$$

Sixth, for SEM the natural pole location to the perturbed S becomes

$$\gamma_{nmn'} = \gamma^{(o)}_{nn'} + \varepsilon\gamma^{(1)}_{nmn'} \tag{38}$$

Here the $\gamma^{(o)}_{nm'}$ is the unperturbed SEM natural pole: the root of $\lambda_n^{(o)}\left(\gamma_{nn'}^{(o)}\right) \equiv 0$. Also, the shifted part is

$$\varepsilon\gamma^{(1)}_{nmn'} = -\,\varepsilon\left[\frac{\lambda^{(1)}_{nm}(\gamma)}{\frac{d}{d\gamma}\left(\lambda_n^{(o)}(\gamma)\right)}\right]_{\gamma\,=\,\gamma^{(o)}_{nn'}} \tag{39}$$

which applies to both the R and Q types of surface currents, with that superscripts respectively inserted for all quantities of Eq. (39). This is an important and simple result. The shifts in the SEM natural poles, in the first order, are values of the bracketed quantity in Eq. (39) but are evaluated at the unperturbed pole location.

Seventh, the perturbed natural modes are

$$\underset{\sim}{\nu}_{nmn'} = \overset{\vee}{\underset{\sim}{J}}{}^{(o)}_{nm} + \varepsilon\, \underset{\sim}{J}{}^{(1)}_{nm} \left(\gamma^{(o)}_{nm'}\right) \tag{40}$$

where the second term is from the Eqs. (33) and (37a) evaluated at unperturbed natural poles $\gamma^{(o)}_{nm}$.

Eighth, the SEM coupling coefficients are rather laborious to calculate in general. But the δ-function-plane-incidence SEM response is found to be (p. 29, Appendix B)

$$\underset{\sim}{J}\,(\underset{\sim}{r},\ \gamma) = \sum_{\alpha} \left\{ \eta^{(o)}_{\alpha} \cdot \frac{\underset{\sim}{\nu}{}^{(o)}_{\alpha}}{\gamma-\gamma^{(o)}_{\alpha} - \varepsilon\gamma^{(1)}_{\alpha}} + \varepsilon\eta^{(1)}_{\alpha} \cdot \frac{\underset{\sim}{\nu}{}^{(o)}_{\alpha}}{\gamma-\gamma^{(1)}_{\alpha}} + \varepsilon\eta^{(o)}_{\alpha} \cdot \frac{\underset{\sim}{\nu}{}^{(1)}_{\alpha}}{\gamma-\gamma^{(o)}_{\alpha}} \right\} \tag{41}$$

Here, α is the shorthand notation of all the sub- and superscripts.

(iii) Unanswered Questions, Remarks, and Areas to Pursue

Of course, the usefulness of the perturbational results in a statistical EMP coupling approach should be judged by further carrying the above results in EEM and SEM to the second and third parts of the concept outlined in Section II.2.a. However, even without that having been done here, the perturbational analysis itself shows some interesting results. First, from Eqs. (30), (31), and (39), clearly some natural poles and modes are more perturbed by, and therefore more sensitive to, a given type of perturbation than others. This could be seen explicitly if one can express the general perturubation response in terms of a δ-function-perturbation response. Therefore, obtaining such an expression should be an important step for the usefulness of the perturbational procedure. Second, the dependence of degeneracy removal on the nature of perturbation is not understood. This has a significant bearing on the modes excitation and should be obtained. Overall, we have laid the necessary ground for quantifying the statistical errors and uncertainties in EEM and SEM, but that statistical quantification itself has yet to be obtained.

3. A STATISTICAL FORMALISM OF POE INTERACTION PATH UNCERTAINTIES (SOW TASK 4.5)

a. Concept and Objective

The prediction of EMP responses at internal points in a complicated system is often, and obviously, subject to large uncertainties. The uncertainties come not merely from those in the external exciting environment, but also from those in many interwoven interaction paths along which the externally coupled excitations at the POEs penetrate and propagate to the response point and then couple there locally to generate the response signal. Of course, the uncertainties in these paths dominate the response errors for cases of well specified environments. For general cases, they either are at least as important as the environment uncertainties or remain still the dominant type; as is strongly evidenced by the discrepancies among predictions and experimental data for some EMP tests [Ref 7] whose environments and geometrical configurations are designed and implemented under tight control.

The cause for those POE interaction path uncertainties is clearly the difficulty, even within the linearity region, in attributing which paths to which POEs are the true principal contributors to the response. This difficulty cannot be resolved by experimentally and/or theoretically refining the prediction capability for selected and isolated interaction paths. The response may come from some unidentified POEs,* or result from the phase cancelling effect of some identified ones. The practical and repeatedly confronted problem is that a good deterministic prediction capability (analytical formulas or numerical codes) for one configuration and some particular response does not guarantee the same quality when applied next. The <u>concept</u> of the present approach is to statistically quantify the response variations or uncertainties due to the uncertainties in the POE interaction paths such that usefully improved and bounded predictions can be made for untested cases.

The objective of the task is to obtain the general trend and "average" effects of such response uncertainties due to identified and unidentified POEs via simple but realistic analysis, compare and contrast the effects and extract and use the information by building an indicative formalism of parameters.

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*In this report, the identified or unidentified POE represents really the whole respective interaction paths from the POE to the response point.

b. Technical Results

(i) The Model

Within the linearity region, an EMP response $R_i(\omega)$ at the i^{th} internal point can be written as the sum of an identified and an unidentified part:

$$R_i(\omega) = \sum_{j=1}^{n} T^I_{ij}(\omega)E^I_j(\omega) + \sum_{j=1}^{K} T^u_{ij}(\omega)E^u_j(\omega) \equiv R^I_i + R^u_i \qquad (42)$$

where $T_{ij}(\omega)$ is the transfer function that links the externally coupled driving (to internal) field strength $E_j(\omega)$ at the j^{th} POE to the i^{th} internal point (see page 6 of Ref. 10). Here and in what follows we shall use the analysis-simplifying but nonfundamental assumptions: all quantities being dealt with are real scalars; POE drivers $E_j(\omega)$ are independent of each other; interaction paths are simple (not composite, see page 10 of Ref. 10) algebraic, and independent of each other; there are n identified known POEs and K unidentified "hidden" POEs. With this simple model Eq. (42), we will treat the T_i's, E_j's, and K as stochastic variables with parametrically assigned properites and obtain the resulting characterization of $R(\omega)$. The types of properties to be parametrically assigned will be selected based on our physical intuition and technical experience and judgment; there is no mature technology to warrant a first-principle approach to this, although the efforts under the preceding two task areas (Sec. II.1 and Sec. II.2) result in initial steps toward this direction.

(ii) Identified POEs

In the case that all POEs are identified ($K = o$ and $R_i^u \equiv o$ in (42)), the resulting response prediction is*

$$R \sim \langle R \rangle \pm \sigma \equiv \sum_j t_j e_j \pm \sqrt{\sum_{j=1}^{M} \left(t_j^2 \sigma_j^2 + e_j^2 \delta_j^2 + \delta_j^2 \sigma_j^2 \right)} \qquad (43)$$

Here, $t_j \equiv \langle T_j \rangle$, $e_j \equiv \langle E_j \rangle$, and δ_j and σ_j are respectively the s.d. (standard deviation or one-sigma uncertainty range) of T_j and E_j. First, the relative error or noise-to-signal-ratio for the absolute value of the resulting response prediction is $\sigma/\langle |R| \rangle$ and bounded by

$$\frac{\tilde{\sigma}}{\langle |R| \rangle} \leqslant \frac{\sigma}{\langle |R| \rangle} \qquad (44)$$

where the upper bound is itself bounded by

$$\frac{\sigma}{\sqrt{\langle R^2 \rangle}} \leqslant \frac{\sigma}{\langle |R| \rangle} \leqslant \frac{\sigma}{\langle R \rangle} \qquad (45)$$

For the simple but plausible case that all individual products $T_j^I E_j^I$ have the same mean μ_o and s.d. error σ_o, the bound (45) reduces to

$$\frac{\sigma_o}{|\mu_o|\sqrt{n}} \cdot \frac{1}{\sqrt{1 + \left(\frac{\sigma_o}{\mu_o \sqrt{n}} \right)^2}} \leqslant \frac{\sigma}{\langle |R| \rangle} \leqslant \frac{\sigma_o}{|\mu_o|\sqrt{n}} \qquad (46)$$

*The index i is suppressed; see Reference 10 for details of this and subsequent results.

This important result suggests that the relative error of the resulting amplitude goes down as $n^{-1/2}$ to tolerable limit even though individually each of the many similar and contributing interaction paths may encompass sizeable relative errors. A central limit case explicitly solved also offer clear support to this suggestion (see page 20 of Ref. 10).

Second, if the responses at a fixed point for M similar (in strength) but independent (in random orientations) EMP exposures are made, then the response for another, M+1th, EMP exposure can be inferred approximately as

$$|R(M+1)| \sim \langle |X| \rangle \pm \sqrt{\sigma_{|x|}{}^2 + \langle Y \rangle} \tag{47}$$

Here, $\langle |X| \rangle$ is the (sample) mean of the absolute value of the predicted M responses, $\sigma_{|x|}$ the (sample) s.d., and $\langle Y \rangle$ the mean of the square errors σ_α (uncertainty variance) in those M predictions. The relative error in the inferred result (47) either is constant or decreases as $n^{-1/2}$ when the number of similar contributing interaction paths of the response increases. Further, we reached the general inferrence that the probability is very small that the relative error ever grows with n (see page 16-18 of Reference 10). On the other hand, if the M+1th response is predicted just as were the other M responses, but not its error range, then we conclude

$$|R(M+1)| \sim |\text{predicted M+1}^{th}\text{ response}| \pm \left(\frac{\sum_1^M \sigma_\alpha}{M} \right) \tag{48}$$

Here, only the error part is inferred. It naturally should be and always is smaller than the error in the fully inferred result (47) into which both exposure variation and path prediction uncertainty enter.

Third, we expect that the above results for responses at a fixed point under many "similar" EMP exposures hold equally for responses at many "similar" points under one EMP response. Such an ergadic nature is probably a good approximation, but was not proved.

Fourth, when amplitudes of responses are of primary interest and since it is difficult to mathematically manipulate absolute values, we may square the data and examine everything based on them. These data of pure positive numbers may be more amenable to useful statistical analysis and interpretation. But such was not carried out here.

(iii) Unidentified POEs

The error contributed to a prediction by the lack of inclusion of some unidentified POEs can be characterized via the expression

$$R^U = R^M - R^I \longleftrightarrow \gamma^U = \gamma^M \pm \sigma^M - (\gamma \pm \sigma) \qquad (49)$$

The idea is to "queeze out" information about R^u using known information in the measured response R^M and predicted response R^I. The latter information is obtained via methods such as controlled data prediction comparison delineated in a previous effort [Ref 8] and in the above subsection (ii).

First, assuming that external excitations at those unidentified POIs are similar to those of the identified ones (i.e., as being sampled from the same population of excitations) and assuming the same for the interaction paths, we obtain either results (see page 24-27 of Ref. 10).

$$\langle |R^u| \rangle = c^{II}\ \varepsilon \begin{pmatrix} \langle K \rangle / n \\ \langle \sqrt{K} \rangle / \sqrt{n} \end{pmatrix} \langle | R^I | \rangle \tag{50}$$

where the C is a constant of order unit and the ε is the typical value of the maximum of t_i^u divided by the similar typical maximum of t_i^I and is less than one.

Second, if we assume that the fraction (out of total number) of POEs unidentified on the average is p and their actual number is Poisson-distributed with average (1-p)n/p, then the unidentified "error Eq. (50) becomes either of

$$\langle |R^u| \rangle = c^{II}\ \varepsilon \begin{pmatrix} (1-p)/p \\ \sqrt{1-p)/p} \end{pmatrix} \tag{51}$$

Third, by pushing the results harder than rigorous mathematics presently warrants, we arrive at a plausible and simple unidentified POE-improvement formula

$$\gamma \pm \sigma \longrightarrow (\gamma \pm \sigma)\ (1 + \eta\rho),\ \eta \equiv \text{sign}\ \gamma \tag{52}$$

which enables us to modify the EMP prediction, $\gamma \pm \sigma$, to accommodate the expected effect due to unidentified POEs. Here

$$\frac{\hat{\mu} \pm \hat{\alpha}}{\langle |R| \rangle \pm \left(\frac{\sum_1^m \sigma_\alpha}{m} \right)} \tag{53}$$

measures the relative importance of the unidentified and the identified responses and should be independent of the scale of excitation in the linear response region. The $\hat{u}$ and $\hat{\sigma}$ are respectively the sample mean and same s.d. of $R_\alpha^M - R_\alpha^I$ for a succession of EMP exposure experiments of $\alpha = 1...m$. As explained for (48), the response for each of these experiments is both measured and predicted using known POE paths.

(iv) Unanswered Questions and Needed Pursuit

The first and most obvious question that needs to be pursued is the effects of removal of those simple restrictive assumptions made in Section II.3.a.(i): the real scalar modeling and the non-composite and independent interacting paths. Some of their effects may only be a straightforward increase of algebraic complexity, and others may significantly alter the general conclusion we reached without their removal. The second but practically interesting question is the inclusion and impact of nonlinearity effects from the point of view of statistical uncertainties and quantification. At present, this seems not so much as appropriate for analytical efforts as for experimental ones: more data are needed.

The third area of concern is a more careful and rigorous mathematical analysis to justify the many loose steps made in our investigation. Such an effort will not only validate our results upon a firm ground but also provide the conditions under which those results can be applied to practical situations.

Fourth and finally, a most useful effort to be pursued next is using the results already obtained to analyze and interpret some existing EMP-system interaction data, e.g., the E-3A test data. This will provide a necessary check on the correctness and usefulness of the direction of the present theoretical effort. In addition, and more importantly, this will further reveal the practically relevant problems that need be addressed intensively by a statistical approach to EMP interaction.

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3. See any typical text book on Electromagnetic Theory, e.g., J. D. Jackson, Classical Electrodynamics, Secs. 4.5 to 4.7, (Wiley Sons, 1962).

4. The probability theory used here is very elementary. For reference, see e.g., W. Feller, An Introduction to Probability Theory, Vol. I (3rd edition, John Wiley & Sons, 1968).

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APPENDIX A
THE DERIVATION OF SOME FORMULAS IN MUTUAL RANDOM COUPLING

At the location of the i-th electric dipole $\underset{\sim}{p}_i$ and under the assumption $k\ell \ll kr \ll 1$, the near-zone electric field generated by the jth electric dipole $\underset{\sim}{p}_j$ is

$$\underset{\sim}{E}_{ij} = \frac{\underset{\approx}{U} - \hat{e}_{ij}\,\hat{e}_{ij}}{4\pi\varepsilon_o r^3_{ij}} \cdot \underset{\sim}{p}_j \tag{A-1}$$

where $\underset{\approx}{U} \equiv$ unit dyadic, $\hat{e}_{ij} \equiv$ unit vectors pointing from position i to position j, and γ_{ij} the distance between them. Now, the induced current on the i-th short wire dipole is

$$I_i \simeq \frac{V_i}{Z_A + Z_{iL}} \simeq \frac{\underset{\sim}{\ell}_i \cdot \underset{\sim}{E}_i}{Z_{iL} + \dfrac{1}{i\omega\left[\dfrac{\pi E_o \ell_i}{4\ln\left(\dfrac{\ell_i}{d_i}\right)}\right]}} \tag{A-2}$$

$$\simeq \frac{i\pi\varepsilon}{4} \cdot \frac{\omega\ell_i^2 \ \hat{\ell}_i \cdot \underset{\sim}{E}_i}{\ln\left(\dfrac{\ell_i}{d_i}\right)} \equiv \alpha_i \ \hat{i}\cdot\underset{\sim}{E}_i \quad \text{if } Z_{iL} \ll \frac{4\ln\left(\dfrac{\ell_i}{d_i}\right)}{\omega\pi E_o \ell} \tag{A-3}$$

But $\underset{\sim}{E}_i = \underset{\sim}{E}^o{}_i + \underset{\sim}{E}_{ij}$ and $\underset{\sim}{p} = iI\underset{\sim}{\ell}/\omega$, thus we have the system of equations for the dipole currents

$$I_i - \sum_{j\neq i} \lambda_{ij}\, I_j = \alpha_i \ \hat{i}\cdot\underset{\sim}{E}^o{}_i \tag{A-4}$$

where

$$\lambda_{ij} \equiv \frac{\alpha_i \; \hat{i} \cdot \left(\underset{\approx}{U} - e_{ij} \; e_{ij} \right) \cdot \hat{j} \; \ell_j}{4\pi\varepsilon_o \omega r_{ij}^3} \tag{A-5}$$

Now, (A-4) can be solved by interation. Under the condition of small dipoles sparcely distributed,

$$\frac{\ell^3}{16r^3 \; \ln\left(\frac{\ell}{d}\right)} << 1 \; , \tag{A-6}$$

the first order solution is

$$I_i = \frac{\pi\varepsilon_o\omega}{4} \; \frac{\ell_i^2}{\ln\left(\frac{\ell_i}{d_i}\right)} \cdot \left\{ \hat{i} \cdot \underset{\sim}{E}_i^o + \sum_{j \neq i} \; \frac{\hat{i} \cdot \left(\underset{\approx}{U} - \hat{e}_{ij} \hat{e}_{ij} \right) \cdot \hat{j} \ell^3_i}{16 r_{ij}^3 \; \ln\left(\frac{\ell_i}{d_i}\right)} \; \hat{j} \cdot \underset{\sim}{E}^o_j \right. \tag{A-7}$$

This is (17) in the text.

To find the average of I_i, we have to average over the random distribution of dipole orientations $\hat{j}$, the directions $\hat{e}_{ij}$ for relative locations, the dipole lengths ℓ_j, the dipole aspect ratio ℓ_j/d_j, and the distances r_{ij} in the random sum for $j \neq r$. Although tedious, the averages are straightforward to perform if one assumes the independencies of these random distributions, except for the r_{ij}. The average over r_{ij} involves the spatial distribution of the locations of the dipoles which enters in both the r_{ij} and the random sum $\sum_{j \neq r}$. A simple way to account for such is, after averaging over all other distributions, to re-label the dipoles: the j-th being the j-th encountered when one expands a spherical volume centered at the i-th dipole. For definiteness, we assume a simple spatial Poissen distribution with a lower cut-off volume v_{c-}. That is, the dipoles are distributed independently and uniformly in space starting with volume v_{c-} and with an average density $\langle n \rangle$.

As a result, we have first

$$\left\langle I_i \right\rangle_{\hat{j},\hat{e}_{ij},\ell_j,\frac{\ell_j}{d_j}} = \frac{\pi\omega\varepsilon_o}{4} \cdot \frac{\ell^2_i}{\ln\left(\frac{\ell_i}{d_i}\right)} \left(\hat{i}\cdot \underset{\sim}{E}^o_i\right) \left\{ 1 + \sum_{j\neq i} \frac{1}{72 r^3_{ij}} \cdot \left\langle \ell^3_j \right\rangle \cdot \left\langle \frac{1}{\ln\left(\frac{\ell_i}{d_j}\right)} \right\rangle A_{ij} \right\} \tag{A-8}$$

where

$$A_{ij} \equiv \frac{3\left[\left(k^2 r^2_{ij} - 1\right) \text{Sin } kr_{ij} + kr_{ij} \text{ Cos } kr_{ij}\right]}{2\left(kr_{ij}\right)^3} \xrightarrow[kr_{ij} << 1]{} 1 \tag{A-9}$$

Then, averaging over the positions as described in the preceeding paragraph for the case $kr_{ij} << 1$ gives

$$\left\langle I_i \right\rangle_{\text{not } \underline{i}} = \frac{\pi\omega\varepsilon_o}{4} \cdot \frac{\ell^2_i}{\ln \frac{\ell_i}{d_i}} \cdot \left(\hat{i}\cdot \underset{\sim}{E}^o_i\right) \left\{ 1 + \frac{\pi}{54} \lambda \left\langle \ell^3 \right\rangle \left\langle \frac{1}{\ln q} \right\rangle \left[\ln\left(\frac{v_{c+}}{v_{c-}}\right) - 1\right] \right\} \tag{A-10}$$

This gives (19) and (22) in the text.

To find the variance of I_i, we take averages on the random distributions successively, but first on a fixed i and then on i. The computation is similarly to that for the I_i, except that the product terms encompassing a double summation make the calculation much more tedious. The result is (23) of the text. Algebraic details of above as well as further investigation into the effects of strong random mutual couplings shall be documented and pursued separately under another effort.