Interaction Notes

Note 355

October 1978

Pseudosymmetric Eigenmode Expansion for the Magnetic Field Integral Equation and SEM Consequences

Maurice I. Sancer*
A. D. Varvatsis
Scott Siegel

TDR, INC.
Los Angeles, CA 90049

ABSTRACT

An Eigenmode expansion for the magnetic field integral equation (MFIE) is derived which eliminates the requirement that an adjoint solution be explicitly sought. Instead, an orthogonality relation is derived which only involves the eigenmodes of the original MFIE operator. To promote confidence in the validity of the resulting expansion, two analyses based on this expansion are presented which lead to known results. First, the expansion is applied to the problem of determining the surface current density induced on a perfectly conducting sphere by a plane wave and the known solution for this problem is duplicated by the expansion. The second analysis shows that for a general perfectly conducting body, the eigenmode expansion coefficient numerator evaluated at the purely imaginary frequency corresponding to an interior resonance is zero. This result is necessary in order to relate the eigenmode expansion to SEM.

Viewing the SEM as a change of representation of the eigenmode expansion intended to facilitate the inverse Laplace transform of that expansion and taking advantage of our detailed sphere calculations we obtained an important SEM result. We found that MFIE class 2 coupling coefficients give the wrong answer for the sphere. This result caused us to examine class 2 coupling coefficients corresponding to the electric field integro-differential equation (EFIDE). We examined the symmetric eigenmode expansion corresponding to that equation and found that EFIDE class 2 coupling coefficients also give the wrong answer for the sphere. This detailed examination of the sphere solution led us to postulate a set of SEM assumptions that have potential application to a general class of closed surfaces and at the same time are consistent with the sphere solution. Finally, we present numerical results that utilize our pseudosymmetric expansion and illustrate the capability of determining SEM quantities by patch zoning the MFIE.

*All authors now with R & D Associates, Los Angeles, CA 90291

PREFACE

This work sponsored by the Defense Nuclear Agency under: Subtask Code EB200, Work Unit 71.

CONTENTS

		Page
I.	Introduction	4
II.	Pseudosymmetric Eigenmode solution to the Magnetic Field Integral Equation	8
III.	Consistency of Zero Residue at Interior Resonances and the Pseudosymmetric Eigenmode Solution	16
IV.	MFIE Pseudosymmetric Eigenmode Solution to the Sphere	19
V.	SEM Coupling Coefficients for the Sphere Via the Pseudosymmetric Eigenmode Solution to the MFIE	26
VI.	Eigenmode Solution to the Electric Field Interodifferential Equation	33
VII.	A Set of Assumptions for the Generalization of the SEM Sphere Solution to an Arbitrarily Shaped Conducting Body	38
VIII.	Numerical Determination of Pole Locations and Eigenfunctions for the Sphere	45
	Appendix A. Proof of Equation (14)	50
	Appendix B. Proof that	52

$$N_i \, d\lambda_i/d\gamma = \int dS \left[\tilde{\underline{J}}_i \cdot \left(\hat{n} \times \frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i \right) \right]$$

	References	54

SECTION I
INTRODUCTION AND SUMMARY

The original objective of this study was to determine whether quantities required by suggested singularity expansion method (SEM) expansions could be obtained through numerical procedures that include patch zoning the magnetic field integral equation (MFIE). We found that this could be accomplished at the expense of very lengthy computer runs. Furthermore, we were motivated to study the underlying theory of SEM by the fact that there was more than one suggested SEM expansion.

The predominant effort to provide a theoretical basis for SEM has been directed toward the problem of determining the current density induced on a perfectly conducting closed surface by an incident electromagnetic wave (i.e., the EMP external interaction problem). This problem is treated by studying equations that result from taking the Laplace transform of Maxwell's equations. This is the problem area that we also address in our study of SEM theory. The approach we took was to study the relationship between eigenmode expansion method (EEM) solutions and SEM expansions.

We begin our effort by studying the EEM as applied to the MFIE. The reason for this is that EEM and SEM can be related and EEM does not suffer from the same uncertainty as does SEM. This is the case since EEM has been studied as a special case of the spectral theory of operators. This does not imply that EEM should replace SEM, because the suggested SEM expansions have certain advantages over EEM expansion.

The ability to relate EEM to SEM was facilitated by recognizing that it was possible to use special properties of the MFIE in order to operationally simplify and interpret the standard EEM expansion. Specifically, the standard EEM expansion requires that eigenmodes of both the MFIE operator as well as its adjoint be determined since the MFIE operator is not self adjoint. The special properties of the MFIE operator allow exactly the same EEM expansion to be obtained as the described standard solution by utilizing only the eigenmodes of the original MFIE operator. Because the MFIE operator is not self adjoint and we are still able to obtain an EEM expansion that requires only the eigenmodes of the original MFIE operator, we term the expansion obtained, a pseudosymmetric eigenmode expansion for the MFIE. The procedure for grouping and then utilizing these eigenmodes are, however, considerably different from the procedure that would be employed for a symmetric operator. The standard EEM solution and the pseudosymmetric EEM solution are always termwise identical if explicitly evaluated. In fact, the eigenmodes used in both EEM expansions are theoretically identical, independent of evaluation. It is the expansion coefficients that are operationally different (i.e., different representations of the same quantities). It is this difference that allows the pseudosymmetric representation to more readily yield a result that is necessary to relate EEM to SEM.

In order to discuss this result, we will describe the three important classes of quantities that are required by any of the suggested SEM expansions. They are the natural frequencies, the natural modes, and the coupling coefficients. The natural frequencies are the values of the complex Laplace frequency, γ, for which the eigenvalues of the MFIE operator, λ_i, is zero. The natural modes are the EEM eigenmodes evaluated at the natural frequencies. The question of just what is a coupling coefficient is an open question that will shortly be addressed in more detail. The necessary SEM result is related to the fact that λ_i has zeros corresponding to both the interior, γ^{I}_{ij}, and the exterior, γ_{ij}, closed surface problem and λ_i always appears in the denominator of either form of the EEM expansion coefficients. It is essential for SEM purposes that the potential singularity in the EEM expansion coefficient at γ^{I}_{ij} be eliminated from the final expansion. The pseudosymmetric representation of the expansion coefficient is such that the numerator in that representation is readily shown to vanish at the γ^{I}_{ij} thus permitting a cancellation of the unwanted singularity and this is the described necessary result. Marin and Latham (ref. 1) in their SEM investigation also addressed the interior resonance issue and obtained the same result.

We now address the coupling coefficient issue according to the principle that EEM expansions and SEM expansions should be identical in the Laplace complex frequency domain. The fact that we are eventually interested in the time domain response corresponding to the inverse Laplace transforms, provides no theoretical justification for two different complex frequency domain solutions of the same induced surface current density. There is essentially a one to one correspondence between transform pairs (i.e., the real function of time and its Laplace transform). The inverse transform of a given Laplace transform function can result in time functions that differ only on a set of measure zero in the time domain (e.g., to correspond to the uncertainty of the function value at a jump discontinuity). Two different Laplace transform functions must necessarily result in two different inverse Laplace transform time signals.

---

1. Marin, L. and R. W. Latham, Analytical Properties of the Field Scattered by a Perfectly Conducting Finite Body, Interaction Note 92, Air Force Weapons Laboratory, 1972.

This leads to two questions that must be addressed concerning the relationship between the EEM expansion and suggested SEM expansions. Are they identical? If not, is the difference significant?

This effort is primarily directed toward the first question. We address this question by obtaining explicit EEM results and SEM results for the case where the closed surface is a sphere. These explicit results for EEM are obtained by using the pseudosymmetric expansion coefficient. The SEM representations explicitly evaluated are those employing class 1 and class 2 coupling coefficients. Obtaining the EEM results was facilitated by employing the explicit eigenmodes and eigenvalues for the sphere problem presented by Marin (ref. 2). The benefits of the pseudosymmetric expansion coefficient representation became apparent in this sphere calculation as it allowed us to use only a limited portion of the detailed sphere results obtained by Marin (ref. 2). The resulting pseudosymmetric EEM expansion was in exact agreement with known sphere results. Specifically, it duplicated the standard Mie solution and Baum's original SEM results (ref. 3). This duplication of results can only be seen after each representation is rearranged, but it is only rearrangement that should be permitted in the theoretical comparison of solutions. The agreement of the pseudosymmetric EEM solution accomplished two purposes. It promoted confidence in the pseudosymmetric EEM theory. Secondly, it enabled a

---

2. Marin, L., Natural-Mode, Representation of Transient Scattering from Rotationally Symmetric, Perfectly Conducting Bodies and Numerical Results for a Prolate Spheroid, Interaction Note 119, Air Force Weapons Laboratory, 1972.
3. Baum, C. E., On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems, Interaction Note 88, Air Force Weapons Laboratory, 1971.

direct evaluation of SEM class 1 and class 2 coupling coefficients (according to their pseudosymmetric description). We found class 1 to yield an exact rearrangement of the known solution while class 2 led to an erroneous expansion. To conclude that the class expansion is erroneous, we argue that each sum, corresponding to the SEM expansion employing either class 1 or class 2 coupling coefficients, must necessarily yield different results in the limit as more terms are added. This can be concluded without even examining the limiting process since each class of coupling coefficients leads to a termwise different expansion coefficient multiplying the exact same linearly independent function. (See end of section V.)

Having found that class 2 coupling coefficients based on the MFIE gave theoretically incorrect results, we decided to examine coupling coefficients for the sphere problem based on the electric field integro-differential equation (EFIDE). Again we found class 1 to yield an exact rearrangement of the known solution while class 2 led to an erroneous expansion. Our investigation of the sphere problem allowed us to arrive at a set of SEM assumptions, to be further investigated, that are applicable to a general class of closed surfaces and are not in conflict with the known sphere solution.

At this point we can conclude that SEM expansions based on class 2 coupling coefficients are not identical with EEM expansions while SEM expansions employing class 1 coupling coefficients are not in conflict with the general assumptions that we have identified. The question of whether the theoretical error caused by employing class 2 coupling coefficients is significant was not specifically addressed in this report. In relation to this issue, it is known that the basic SEM representation is of questionable value for high frequencies/ early times. This early time concern is further enhanced because the time dependent behavior of each term, resulting from the inverse Laplace transform of an SEM expansion employing class 1 coupling coefficients, would exhibit a propagating turn on time faster than the speed of light. Whether this deficiency is important is related to whether for these early times the total SEM expansion is of utility.

A possible rationale for using the theoretically wrong class 2 coupling coefficients is that they do not result in this termwise propagation speed problem. As a consequence the corresponding SEM expansions have the potential to yield better approximations for early time applications than theoretically correct expansions. The works of Tesche (ref. 4) and Marin (ref. 2) were not specifically directed at this issue; however, they provide some evidence that SEM expansions utilizing class 2 coupling coefficients potentially can yield meaningful time domain solutions. Both references considered time domain solutions for the total current induced on an object, a thin wire in (ref. 4) and a prolate spheroid in (ref. 2). In either case they compared the time domain solutions obtained by an alternate procedure to the solution obtained by an SEM expansion employing class 2 coupling coefficients and they obtained good agreement. At this point, the use of class 2 coupling coefficients should be viewed as an empirical approximation.

We view the status of SEM as providing suggested representations having potential benefits to a variety of wave interaction applications as well as to electromagnetic pulse (EMP) interaction problems; however, there is at present, insufficient knowledge to decide a priori how well any suggested SEM expansion represents even external interaction quantities.

---

4. Tesche, F. M., On the Singularity Expansion Method as Applied to Electromagnetic Scattering from Thin Wires, Interaction Note 102, Air Force Weapons Laboratory, 1972.

SECTION II

PSEUDOSYMMETRIC EIGENMODE SOLUTION TO THE MAGNETIC FIELD INTEGRAL EQUATION

In this section we derive an eigenfunction expansion solution to the Magnetic Field Integral Equation (MFIE) by employing a procedure that allows the determination of the expansion coefficients without explicitly calculating the eigenfunctions of the adjoint operator. We call this solution the pseudo-symmetric eigenmode expansion for the MFIE.

We start with the MFIE for the exterior problem

$$\frac{1}{2}\,\underline{J}(\underline{r}) - \int_S \hat{n}(\underline{r}) \times \left[\nabla G(\underline{r},\underline{r}') \times \underline{J}(\underline{r}')\right] dS' = \underline{J}^{inc}(\underline{r}) \tag{1}$$

where $\underline{J}(\underline{r})$ is the induced surface current density on the perfectly conducting surface S, $\hat{n}(\underline{r})$ is the outward unit normal of S, $\underline{J}^{inc} = \hat{n} \times \underline{H}^{inc}$, $\underline{H}^{inc}$ is the incident magnetic field and G is the free space Green's function,

$$G(\underline{r},\underline{r}') = \frac{1}{4\pi|\underline{r} - \underline{r}'|}\, e^{-\gamma|\underline{r}-\underline{r}'|}$$

We rewrite equation (1) as

$$\mathscr{L}\underline{J} = \underline{J}^{inc} \tag{2}$$

with

$$\mathscr{L} \equiv \frac{I}{2} - L \tag{3}$$

where I is the identity operator. The solution to equation (2) is given by

$$\underline{J} = \sum_i a_i \underline{J}_i \tag{4}$$

where

$$\mathscr{L}\underline{J}_i = \lambda_i \underline{J}_i \tag{5}$$

The expansion coefficients a_i involve an inner product for $\underline{J}^{inc}$ and the eigenfunctions of the adjoint operator $\mathscr{L}^\dagger$. As we mentioned in the beginning of the section we need not determine these eigenfunctions explicitly because we can employ the pseudo-symmetric method which we will now outline. We begin by defining operators M and Q through

$$\begin{aligned} M\underline{f} &\equiv \hat{n} \times \underline{f} \\ Q &\equiv ML \end{aligned} \tag{6}$$

where $\underline{f}$ is a surface vector. Noting that $M^2\underline{f} = \hat{n} \times \hat{n} \times \underline{f} = -\underline{f}$ leads to the following property

$$M^2 = -I \tag{7}$$

Next we operate on equation (5) with M and obtain the eigenvalue equation

$$\left.\begin{aligned} M\mathscr{L}\underline{J}_i &= \lambda_i M\underline{J}_i \\ M\mathscr{L} &= \frac{1}{2} M - Q \end{aligned}\right\} \tag{8}$$

In conjunction with equation (8) we define the adjoint equation

$$(M\mathscr{L})^\dagger \underline{F}_i = \lambda_i^\dagger M^\dagger \underline{F}_i \tag{9}$$

where $(M\mathscr{L})^\dagger$, $M^\dagger$ obey the usual adjointness relationship

$$(\underline{f},A\underline{g}) = (A^\dagger \underline{f},\underline{g}) \tag{10}$$

with an inner product defined as

$$(\underline{a},\underline{b}) = \int_S \underline{a}^* \cdot \underline{b}\, dS$$

With the aid of equations (8), (9), and (10) we can derive the relationship

$$(\lambda_i - \lambda_j^{\dagger *})(\underline{F}_j,M\underline{J}_i) = 0 \tag{11}$$

For generality we assume that the eigenvalues are degenerate and rewrite equation (11) as

$$(\lambda_i - \lambda_j^{\dagger *})(\underline{F}_{j\ell},M\underline{J}_{im}) = 0 \tag{12}$$

where ℓ, m signify degeneracy. Using this one can show that

$$\begin{aligned} \lambda_i &= \lambda_i^{\dagger *} \\ (\underline{F}_{j\ell},M\underline{J}_{im}) &= N_{j\ell}\, \delta_{ji}\, \delta_{\ell m} \end{aligned} \tag{13}$$

where the biorthogonality relationship for $i = j$, $\ell \neq m$ is obtained via a Gram-Schmidt biorthogonalization procedure. We are now in a position to derive an important result by noting that

$$(M\mathscr{L})^\dagger = \left(\frac{1}{2} M - Q\right)^\dagger = \frac{1}{2} M^\dagger - Q^\dagger = -\frac{1}{2} M - Q^* \tag{14}$$

(see appendix A) and invoking the first of equations (13) to rewrite equation (9) as

$$\left(-\frac{1}{2}M - Q^*\right)\underline{F}_{i\ell} = -\lambda_i^* M\underline{F}_{i\ell} \qquad (15)$$

By first operating on equation (15) with M and then complex conjugating the resulting equation, we obtain

$$\left(\frac{I}{2} + L\right)\underline{F}_{i\ell}^* = \lambda_i \underline{F}_{i\ell}^* \qquad (16)$$

where we have used equations (6) and (7). Equation (16) can be rewritten as

$$\mathscr{L}\underline{\tilde{J}}_{i\ell} = (1 - \lambda_i)\underline{\tilde{J}}_{i\ell} \qquad (17)$$

$$\underline{\tilde{J}}_{i\ell} \equiv \underline{F}_{i\ell}^* \qquad (18)$$

where $\mathscr{L} = I/2 - L$ is the MFIE operator for the exterior problem. Thus $\underline{\tilde{J}}_{i\ell} \equiv \underline{F}_{i\ell}^*$ is an eigenfunction of the MFIE operator with an eigenvalue $1 - \lambda_i$ where λ_i corresponds to $\underline{J}_{im}$. According to the second of equations (13)

$$(\underline{F}_{j\ell}, M\underline{J}_{im}) = \int_S \underline{F}_{j\ell}^* \cdot \hat{n} \times \underline{J}_{im}\, dS$$

$$= \int_S \underline{\tilde{J}}_{j\ell} \cdot \hat{n} \times \underline{J}_{im}\, dS = N_{i\ell}\, \delta_{ij}\, \delta_{\ell m} \qquad (19)$$

and we define this integral form as a pseudo inner product

$$\{\underline{\tilde{J}}_{j\ell}, \underline{J}_{im}\} \equiv \int \underline{\tilde{J}}_{j\ell} \cdot \hat{n} \times \underline{J}_{im}\, dS = N_{i\ell}\, \delta_{ij}\, \delta_{\ell m} \qquad (20)$$

We now return to equation (2) to obtain the eigenfunction expansion solution.

First, we expand $\underline{J}^{inc}$ in terms of the $\underline{J}_{i\ell}$

$$\underline{J}^{inc} = \sum_i \sum_\ell \beta_{i\ell} \underline{J}_{i\ell} \qquad (21)$$

Taking the pseudo inner product of both sides of (21) and using (20), we can determine the $\beta_{i\ell}$'s as

$$\beta_{i\ell} = \{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\}/N_{i\ell} \qquad (22)$$

$$\{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\} = \int_S \underline{\tilde{J}}_{i\ell} \cdot \hat{n} \times \underline{J}^{inc}\, dS \qquad (23)$$

If we now expand $\underline{J}$ as

$$\underline{J} = \sum_i \sum_\ell a_{i\ell} \underline{J}_{i\ell}$$

where

$$\mathscr{L}\underline{J}_{i\ell} = \lambda_i \underline{J}_{i\ell} \qquad (5)$$

we can rewrite equation (2) as

$$\mathscr{L}\underline{J} = \sum_i \sum_\ell a_{i\ell} \lambda_i \underline{J}_{i\ell} = \underline{J}^{inc} = \sum_i \sum_\ell \beta_{i\ell} \underline{J}_{i\ell}$$

and conclude that

$$a_{i\ell} = \beta_{i\ell}/\lambda_i$$

Using equation (22) we obtain

$$\underline{J} = \sum_i \sum_\ell \frac{\{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\}}{N_{i\ell}\ \lambda_i} \underline{J}_{i\ell} \tag{24}$$

as the pseudosymmetric eigenmode expansion solution to the MFIE or equivalently to equation (2). The eigenmodes required by the expansion are eigenmodes of the original operator $\mathscr{L}$ taken in pairs, one corresponding to the eigenvalue λ_i and the other corresponding to the eigenvalue $1-\lambda_i$.

SECTION III

CONSISTENCY OF ZERO RESIDUE AT INTERIOR RESONANCES AND THE PSEUDOSYMMETRIC EIGENMODE SOLUTION

In this section we show that the set of eigenmodes for the exterior problem is identical to the set of eigenmodes for the interior problem, the eigenvalues λ_i of the exterior eigenvalue problem have zeros at the *interior* resonances (in addition to the exterior resonances) and that the coefficients $\{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\}$ in the pseudosymmetric eigenmode solution given by equation (24) are zero at these interior resonances. The last result is necessary in order to relate the EEM to the SEM. Thus the eigenmode series may be rearranged to be written as a singularity expansion (which is a special case of the Mittag-Leffler theorem stated in section VII, equation (103)) which will not involve interior resonances since the excitation coefficients $\{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\}$ will be shown to be zero.

We start by recalling equations (5), (16), (18) and the eigenvalue equation for the interior problem.

$$\mathscr{L}\underline{J}_i \equiv \left(\frac{I}{2} - L\right)\underline{J}_i = \lambda_i \underline{J}_i \tag{5}$$

$$\left(\frac{I}{2} + L\right)\underline{\tilde{J}}_i = \lambda_i \underline{\tilde{J}}_i \tag{25}$$

$$\mathscr{L}^I \underline{J}_i^I \equiv \left(\frac{I}{2} + L\right)\underline{J}_i^I = \lambda_i^I \underline{J}_i^I \tag{26}$$

where the superscript "I" denotes "interior." To simplify the notation we will not supply a superscript "E" to exterior quantities and also ignore degeneracy because our results do not depend on degeneracy. The eigenvalue λ_i in equation (5) is any eigenvalue of the exterior operator and either the $\underline{J}_i$'s or $\underline{\tilde{J}}_i$'s comprise the entire set of eigenmodes of the exterior operator $\mathscr{L}$. Bearing this in mind, a comparison of equations (25) and (26) shows that the interior and exterior eigenvalues comprise the same set and that the interior and exterior eigenfunctions comprise the same set.

We now proceed to show that the excitation coefficient evaluated at the interior resonances is zero. The interior resonances λ_{ij}^I are solutions to the equation

$$\lambda_i(\gamma_{ij}^I) = 0$$

and the corresponding interior natural modes satisfy equation (26) with $\lambda_i^I = 0$ or equivalently equation (25) with $\lambda_i = 0$.

$$\mathscr{L}^I(\gamma_{ij}^I) \; \underline{\tilde{J}}_i(\gamma_{ij}^I) = 0 \tag{27}$$

The solution to equation (27) can, using an appropriate normalization, be given by

$$\begin{aligned} \underline{\tilde{J}}_i(\gamma_{ij}^I,\underline{r}) &= \hat{n}_I \times \underline{H}_i(\gamma_{ij}^I,\underline{r}) \\ &= -\hat{n} \times \underline{H}_i(\gamma_{ij}^I,\underline{r}) \;, \quad \underline{r} \in S \end{aligned} \tag{28}$$

where we have simply made the distinction

$$\hat{n}_I = -\hat{n}$$

and $\hat{n}$ is the outward normal to S. The equation satisfied by $\underline{H}_i(\gamma_{ij}^I,\underline{r})$, where $\underline{r} \in V$ and V is the interior region bounded by S, is

$$\nabla\times\nabla\times \underline{H}_i(\gamma^I_{ij},\underline{r}) + (\gamma^I_{ij})^2 \underline{H}_i(\gamma^I_{ij},\underline{r}) = 0 \ , \quad \underline{r} \in V \tag{29a}$$

$$\hat{n}_I \times \left[\nabla \times \underline{H}_i(\gamma^I_{ij},\underline{r})\right] = 0 \ , \quad \underline{r} \in S \ . \tag{29b}$$

The excitation coefficient in equation (23) evaluated at γ^I_{ij} is

$$\{\tilde{\underline{J}}_i(\gamma^I_{ij}),\ \underline{J}^{inc}(\gamma^I_{ij})\} = \int_S \tilde{\underline{J}}_i(\gamma^I_{ij}) \cdot \hat{n} \times \underline{J}^{inc}(\gamma^I_{ij})\ dS$$

$$= \int_S \hat{n}\cdot\left[\underline{H}_i(\gamma^I_{ij}) \times \underline{H}^{inc}(\gamma^I_{ij})\right] dS \tag{30a}$$

$$\equiv \{\gamma^I_{ij}\} \tag{30b}$$

In order to show that $\{\gamma^I_{ij}\} = 0$ we employ the following identity

$$\nabla \cdot\left[\underline{H}_i \times (\nabla \times \underline{E}^{inc}) - \underline{E}^{inc} \times (\nabla \times \underline{H}_i)\right]$$

$$= \underline{H}_i \left[\nabla \times \nabla \times \underline{E}^{inc} + (\gamma^I_{ij})^2\ \underline{E}^{inc}\right]$$

$$- \underline{E}^{inc} \cdot \left[\nabla \times \nabla \times \underline{H}_i + (\gamma^I_{ij})^2\ \underline{H}_i\right] \quad \underline{r} \in V \tag{31}$$

Since the source for the incident electric field $\underline{E}^{inc}$ lies outside V and $\underline{H}_i$ satisfies equation (29a), it follows that an integral of the righthand side of equation (31) over V yields zero. The volume integral of the lefthand side of equation (31) can be converted to a surface integral using the divergence theorem to obtain

$$\int_S \hat{n} \cdot \left[\underline{H}_i \times \left(\nabla \times \underline{E}^{inc}\right) - \underline{E}^{inc} \times \left(\nabla \times \underline{H}_i\right)\right] dS = 0$$

Substituting the Maxwell equation $\nabla \times \underline{E}^{inc} = -\gamma Z_0 \underline{H}^{inc}$ into this relationship, we obtain

$$\{\gamma^I_{ij}\} = 0$$

SECTION IV

MFIE PSEUDOSYMMETRIC EIGENMODE SOLUTION TO THE SPHERE

In this section we apply the pseudosymmetric eigenmode solution developed in Section II to obtain the induced surface current on the surface of a perfectly conducting sphere illuminated by a plane wave that corresponds to the Laplace transform of a delta function plane wave pulse. The pseudosymmetric approach produces the Mie solution in the form given by equation (B-68) in reference 3.

We repeat here some of the equations in Section II needed in this section.

The solution to the MFIE

$$\mathscr{L}\underline{J} = \underline{J}^{inc} \tag{2}$$

is

$$\underline{J} = \sum_i \sum_\ell \frac{\{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\}}{N_{i\ell}\lambda_i(\gamma)} \underline{J}_{i\ell} \tag{23}$$

where

$$\mathscr{L}\underline{J}_{i\ell} = \lambda_i \underline{J}_{i\ell} \tag{5}$$

$$\mathscr{L}\underline{\tilde{J}}_{i\ell} = (1 - \lambda_i)\underline{\tilde{J}}_{i\ell} \tag{17}$$

$$\{\underline{\tilde{J}}_{i\ell}, \underline{J}_{jm}\} \equiv \int_S \underline{\tilde{J}}_{i\ell} \cdot \hat{n} \times \underline{J}_{jm} \, dS$$

$$= N_{i\ell}\,\delta_{ij}\,\delta_{\ell m} \tag{18}$$

$$\{\underline{\tilde{J}}_{i\ell}, \underline{J}^{inc}\} = \int_S \underline{\tilde{J}}_{i\ell} \cdot \hat{n} \times \underline{J}^{inc} \, dS \tag{33}$$

Notice that the second index signifies degeneracy. For a sphere (ref. 3)

$$\underline{J}_{i\ell} = \begin{cases} \underline{R}_{n,m,\sigma}(\theta,\phi) & n = 1,2, \ldots, \infty \\ & m \leq n \\ \underline{Q}_{n,m,\sigma}(\theta,\phi) & \sigma = o,e \text{ (odd,even)} \end{cases}$$

where

$$\left.\begin{aligned} \underline{R}_{n,m,\sigma}(\theta,\phi) &= -\frac{\partial Y_{n,m,\sigma}}{\partial\theta}\,\hat{e}_\phi + \frac{1}{\sin\theta}\,\frac{\partial Y_{n,m,\sigma}}{\partial\phi}\,\hat{e}_\theta \\ \underline{Q}_{n,m,\sigma}(\theta,\phi) &= \frac{1}{\sin\theta}\,\frac{\partial Y_{n,m,\sigma}}{\partial\phi}\,\hat{e}_\phi + \frac{\partial Y_{n,m,\sigma}}{\partial\phi}\,\hat{e}_\theta \end{aligned}\right\} \quad (34)$$

The corresponding eigenvalues are given by (ref. 2)

$$\left.\begin{aligned} \lambda_n^R &= \left[\gamma a i_n(\gamma a)\right]' \left[\gamma a k_n(\gamma a)\right] \\ \lambda_n^Q &= -\left[\gamma a i_n(\gamma a)\right] \left[\gamma a k_n(\gamma a)\right]' \end{aligned}\right\} \quad (35)$$

i.e.,

$$\left.\begin{aligned} \mathscr{L}\underline{R}_{n,m,\sigma} &= \lambda_n^R\,\underline{R}_{n,m,\sigma} \\ \mathscr{L}\underline{Q}_{n,m,\sigma} &= \lambda_n^Q\,\underline{Q}_{n,m,\sigma} \end{aligned}\right\} \quad (36)$$

An interesting property for the sphere that will be used shortly is

$$\begin{aligned} \underline{Q}_{n,m,\sigma} &= \hat{e}_r \times \underline{R}_{n,m,\sigma} \\ \underline{R}_{n,m,\sigma} &= -\hat{e}_r \times \underline{Q}_{n,m,\sigma} \end{aligned} \quad (37)$$

where $\hat{e}_r \equiv \hat{n}$. Next we observe that because of the Wronskian relationship

$$\zeta i_n(\zeta)\left[\zeta k_n(\zeta)\right]' - \zeta k_n(\zeta)\left[\zeta i_n(\zeta)\right]' = -1$$

the following is true

$$\lambda_n^R + \lambda_n^Q = 1 \tag{38}$$

Thus according to equations (17a) and (38)

$$\mathscr{L}\underline{\tilde{R}}_{n,m,\sigma} = \left(1 - \lambda_n^R\right) \underline{\tilde{R}}_{n,m,\sigma} = \lambda_n^Q \underline{\tilde{R}}_{n,m,\sigma} \tag{39}$$

$$\mathscr{L}\underline{\tilde{Q}}_{n,n,\sigma} = \left(1 - \lambda_n^Q\right) \underline{\tilde{Q}}_{n,n,\sigma} = \lambda_n^R \underline{\tilde{Q}}_{n,n,\sigma} \tag{40}$$

Comparing equations (39) and (40) to equations (36) we conclude that

$$\begin{aligned}\underline{\tilde{R}}_{n,m,\sigma} &= \sum_{m'=0}^{n} \sum_{\sigma'=o,e} c_{m',\sigma'} \underline{Q}_{n,m',\sigma'} \\ \underline{\tilde{Q}}_{n,m,\sigma} &= \sum_{m'=0}^{n} \sum_{\sigma'=o,e} d_{m',\sigma'} \underline{R}_{n,m',\sigma'}\end{aligned} \tag{41}$$

We next show that the only nonzero coefficients in equation (41) are $c_{m,\sigma}$ and $d_{m,\sigma}$. This can be accomplished by first recalling equations (18)

$$\begin{aligned}\int_S \underline{\tilde{R}}_{n,m,\sigma} \cdot (\hat{e}_r \times \underline{R}_{n,m'',\sigma''})\, dS &= N^R_{n,m,\sigma}\, \delta_{m,m''}\, \delta_{\sigma,\sigma''} \\ \int_S \underline{\tilde{Q}}_{n,m,\sigma} \cdot (\hat{e}_r \times \underline{Q}_{n,m'',\sigma''})\, dS &= N^Q_{n,m,\sigma}\, \delta_{m,m''}\, \delta_{\sigma,\sigma''}\end{aligned} \tag{42}$$

which with the aid of equations (37) become

$$\int_S \tilde{\underline{R}}_{n,m,\sigma} \cdot \underline{Q}_{n,m'',\sigma''}\, dS = N^R_{n,m,\sigma}\, \delta_{m,m''}\, \delta_{\sigma,\sigma''}$$

$$\int_S \tilde{\underline{Q}}_{n,m,\sigma} \cdot \underline{R}_{n,m'',\sigma''}\, dS = -N^Q_{n,m,\sigma}\, \delta_{m,m''}\, \delta_{\sigma,\sigma''} \tag{43}$$

Substituting equations (41) into equations (43) and employing the orthogonality properties

$$\int_S \underline{R}_{n,m',\sigma'} \cdot \underline{R}_{n,m'',\sigma''}\, dS = \int_S \underline{Q}_{n,m',\sigma'} \cdot \underline{Q}_{n,m'',\sigma''}\, dS$$

$$= M_{n,m''\sigma''}\, \delta_{m'',m'}\, \delta_{\sigma'',\sigma'} \tag{44}$$

given by equations (B-19) in reference 3, we obtain

$$c_{m'',\sigma''}\, M_{n,m'',\sigma''} = N^R_{n,m,\sigma}\, \delta_{m,m''}\, \delta_{\sigma,\sigma''}$$

$$d_{m'',\sigma''}\, M_{n,m'',\sigma''} = -N^Q_{n,m,\sigma}\, \delta_{m,m''}\, \delta_{\sigma,\sigma''} \tag{45}$$

which prove that only $c_{m,\sigma}$ and $d_{m,\sigma}$ are nonzero. Thus we can replace (41) by

$$\tilde{\underline{R}}_{n,m,\sigma} = d_R\, \underline{Q}_{n,m,\sigma}$$

$$\tilde{\underline{Q}}_{n,m,\sigma} = d_Q\, \underline{R}_{n,m,\sigma} \tag{46}$$

where d_R, d_Q are arbitrary constants.

Next we calculate $N^R_{n,m,\sigma}$, $N^Q_{n,m,\sigma}$ (which appear in expansion equation (23)) by invoking equations (45) and (46)

$$N^{R}_{n,m,\sigma} = c_{m,\sigma}\, M_{n,m,\sigma} = d_R\, M_{n,m,\sigma}$$

$$N^{Q}_{n,m,\sigma} = -d_{m,\sigma}\, M_{n,m,\sigma} = -d_Q\, M_{n,m,\sigma} \qquad (47)$$

In order to calculate the expansion coefficients in equation (23) we first recall equation (33) and employ equation (46)

$$\{\underline{\tilde{R}}_{n,m,\sigma}, \underline{J}^{inc}\} = \int \underline{\tilde{R}}_{n,m,\sigma} \cdot (\hat{e}_r \times \underline{J}^{inc})\, dS$$

$$= -d_R \int \underline{Q}_{n,m,\sigma} \cdot \underline{H}^{inc}\, dS \qquad (48)$$

since

$$\hat{e}_r \times \underline{J}^{inc} = \hat{e}_r \times \hat{e}_r \times \underline{H}^{inc} = -\underline{H}^{inc} + \hat{e}_r(\hat{e}_r \cdot \underline{H}^{inc})$$

and $\underline{Q}_{n,m,\sigma}$ is a surface vector. Similarly,

$$\{\underline{Q}_{n,m,\sigma}, \underline{J}^{inc}\} = -d_Q \int \underline{R}_{n,m,\sigma} \cdot \underline{H}^{inc}\, dS \qquad (49)$$

The form of the incident magnetic field can be found by using equations (B-55) and (B-58) in reference 3

$$\underline{H}^{inc}_p = \sum_{n=1}^{\infty} \sum_{m'=0}^{n} \sum_{\sigma'=e,0} \Big[A_{2,n',m',\sigma',p}\, \underline{M}^{(1)}_{n',m',\sigma'} - A_{1,n',m',\sigma',p}\, \underline{N}^{(1)}_{n',m',\sigma'} \Big] \qquad (50)$$

where p = 2,3 represents the two possible polarization directions defined in reference 3 and

$$\left.\begin{aligned} \underline{M}^{(1)}_{n,m,\sigma} &= i_n(\gamma r)\ \underline{R}_{n,m,\sigma}(\theta,\phi) \\ \underline{N}^{(1)}_{n,m,\sigma} &= n(n+1)\ \frac{i_n(\gamma r)}{\gamma r}\ \underline{P}_{n,m,\sigma}(\theta,\phi) \\ &\quad + \frac{\left[\gamma r\ i_n(\gamma r)\right]'}{\gamma r}\ \underline{Q}_{n,m,\sigma}(\theta,\phi) \end{aligned}\right\} \qquad (51)$$

$\underline{P}_{n,m,\sigma}$ is the third vector spherical harmonic (the other two being $\underline{Q}$ and $\underline{R}$; see equations (B-11), (B-12), and (B-13) in reference 3).

Employing the orthogonality properties among $\underline{P}$, $\underline{Q}$, and $\underline{R}$ given by equations (B-18 and (B-19) in reference 3 and equations (50) and (51), we can rewrite equations (48) and (49) as

$$\left.\begin{aligned} \{\underline{\tilde{R}}_{n,m,\sigma},\ \underline{J}_p^{inc}\} &= d_R\ A_{1,n,m,\sigma,p}\ M_{n,m,\sigma}\ \frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a} \\ \{\underline{\tilde{Q}}_{n,m,\sigma},\ \underline{J}_p^{inc}\} &= -d_Q\ A_{2,n,m,\sigma,p}\ M_{n,m,\sigma}\ i_n(\gamma a) \end{aligned}\right. \qquad (52)$$

where a is the radius of the sphere. We are now in a position to rewrite equation (23) in its final form by first recalling equations (47) and (52)

$$\underline{J}_p(\theta,\phi) = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\sigma=e,0}\left[\frac{A_{1,n,m,\sigma,p}\ d_R\ M_{n,m,\sigma}\left[\gamma a i_n(\gamma a)\right]'}{\gamma a\ \lambda_n^R\ d_R\ M_{n,m,\sigma}}\ \underline{R}_{n,m,\sigma}\right.$$

$$\left. - \frac{A_{2,n,m,\sigma,p}\ d_Q\ M_{n,m,\sigma}\ i_n(\gamma a)}{\lambda_n^Q(-d_Q\ M_{n,m,\sigma})}\ \underline{Q}_{n,m,\sigma}\right]$$

and using equation (35) to obtain

$$\underline{J}_p(\theta,\phi) = \sum_{n=1}^{\infty} \sum_{m=0} \sum_{\sigma=e,0} \left\{ \frac{A_{1,n,m,\sigma,p}}{(\gamma a)^2 \, k_n(\gamma a)} \underline{R}_{n,m,\sigma}(\theta,\phi) \right.$$

$$\left. - \frac{A_{2,n,m,\sigma,p}}{\gamma a \left[\gamma a \, k_n(\gamma a)\right]'} \underline{Q}_{n,m,\sigma}(\theta,\phi) \right\} \tag{53}$$

Equation (53) is identical to equation (B-68) in reference 3.

SECTION V

SEM COUPLING COEFFICIENTS FOR THE SPHERE VIA THE PSEUDOSYMMETRIC EIGENMODE SOLUTION TO THE MFIE

We first examine Class 1 coupling coefficients with $q = 1$ corresponding to the TM modes $\underline{R}_{n,m,\sigma}$. In appendix B we show that this coupling coefficient has the form

$$\eta^{R(1)}_{n,n',m,\sigma,p}(\gamma) \equiv e^{(\gamma^R_{n,n'}-\gamma)ct_o}\left\{\frac{\{\tilde{\underline{R}}_{n,m,\sigma},\ \underline{J}^{inc}_p\}}{N^R_{n,m,\sigma}\left[d\lambda^R_n/d\gamma\right]}\right\}_{\gamma=\gamma^R_{n,n'}} \tag{54}$$

where $t_o = -a/c$. From the previous section we recall equations (52), (47) and (35)

$$\{\tilde{\underline{R}}_{n,m,\sigma},\ \underline{J}^{inc}_p\} = A_{1,n,m,\sigma,p}\ d_R\ M_{n,m,\sigma}\ \frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a} \tag{55}$$

$$N^R_{n,m,\sigma} = d_R\ M_{n,m,\sigma}\quad \text{(independent of } \gamma) \tag{56}$$

$$\lambda^R_n = \left[\gamma a i_n(\gamma a)\right]'\left[\gamma a\ k_n(\gamma a)\right] \tag{57}$$

Recalling that for $q = 1$ modes $\left[\gamma a\ k_n(\gamma a)\right]_{\gamma=\gamma^R_{n,n'}} = 0$ we have

$$\begin{aligned}\left[d\lambda^R_n/d\gamma\right]_{\gamma=\gamma^R_{n,n'}} &= \Big\{a\left[\gamma a i_n(\gamma a)\right]''\left[\gamma a\ k_n(\gamma a)\right] \\ &\quad + a\left[\gamma a i_n(\gamma a)\right]'\left[\gamma a\ k_n(\gamma a)\right]'\Big\}_{\gamma=\gamma^R_{n,n'}} \\ &= a\left\{\left[\gamma a i_n(\gamma a)\right]'\ \left[\gamma a\ k_n(\gamma a)\right]'\right\}_{\gamma=\gamma^R_{n,n'}}\end{aligned} \tag{58}$$

In view of equations (55), (56), and (58), equation (54) gives

$$\eta^{R(1)}_{n,n',m,\sigma,p}(\gamma) = e^{(\gamma^R_{n,n'}-\gamma)ct_o} \frac{A_{1,n,m,\sigma,p}/a}{\left\{\gamma a\left[\gamma a\ k_n(\gamma a)\right]'\right\}_{\gamma=\gamma^R_{n,n'}}} \tag{59}$$

Following equation (B-76) in reference 3 in conjunction with equation (B-74) we find

$$(\gamma a)^2\ k_n(\gamma a) = e^{-\gamma a}\ C_{1,n}(\gamma a) \tag{60}$$

Noting that

$$\left\{\left[(\gamma a)^2\ k_n(\gamma a)\right]'\right\}_{\gamma=\gamma^R_{n,n'}} = \left\{\gamma a\ k_n(\gamma a) + \gamma a\left[\gamma a\ k_n(\gamma a)\right]'\right\}_{\gamma=\gamma^R_{n,n'}}$$

$$= \left\{\gamma a\left[\gamma a\ k_n(\gamma a)\right]'\right\}_{\gamma=\gamma^R_{n,n'}}$$

we can rewrite equation (60) as

$$\left\{\frac{1}{\gamma a\left[\gamma a\ k_n(\gamma a)\right]'}\right\}_{\gamma=\gamma^R_{n,n'}} = \frac{1}{\left[(\gamma a)^2\ k_n(\gamma a)\right]'_{\gamma=\gamma^R_{n,n'}}} = \frac{1}{\left[e^{-\gamma a}\ C_{1,n}(\gamma a)\right]'_{\gamma=\gamma^R_{n,n'}}}$$

$$= \frac{1}{\left\{e^{-\gamma a}\left[C_{1,n}(\gamma a)\right]' - e^{-\gamma a}\ C_{1,n}(\gamma a)\right\}_{\gamma=\gamma^R_{n,n'}}}$$

$$= \left\{\frac{1}{e^{-\gamma a}\left[C_{1,n}(\gamma a)\right]'}\right\}_{\gamma=\gamma^R_{n,n'}} \tag{61}$$

Combining equations (59) and (61) we obtain

$$\eta^{R(1)}_{n,n',m,\sigma,p}(\gamma) = e^{(\gamma^R_{n,n'}-\gamma)ct_o} \frac{A_{1,n,m,\sigma,p}/a}{e^{-\gamma^R_{n,n'}a}\left[C_{1,n}(\gamma a)\right]'_{\gamma=\gamma^R_{n,n'}}}$$

$$= \frac{A_{1,n,m,\sigma,p}/a}{\left[C_{1,n}(\gamma a)\right]'_{\gamma=\gamma^R_{n,n'}}} e^{-\gamma ct_o} \qquad (62)$$

since $t_o = -a/c$.

Employing equation (B-82) in reference 3 in conjunction with equation (B-98) we finally obtain

$$\eta^{R(1)}_{n,n',m,\sigma,p}(\gamma) = e^{-\gamma ct_o} C_{1,n,n',m,\sigma,p} \frac{1}{c} \qquad (63)$$

Recalling that $\gamma = s/c$ we understand that the coupling coefficient given by equation (63) is identical to the one that can be identified in equation (B-97) in reference 3 for $q = 1$.

Following the same procedure as above we obtain

$$\eta^{Q}_{n,n',m,\sigma,p}(\gamma) = e^{-\gamma ct_o} C_{2,n,n',m,\sigma,p} \frac{1}{c} \qquad (64)$$

Thus Class 1 coefficients give the correct result for the sphere, i.e., equation (B-97) in reference 3. For <u>Class 2</u> the defining relationship for $q = 1$ is

$$\eta^{R(2)}_{n,n',m,\sigma,p}(\gamma) = \frac{\{\underline{\tilde{R}}_{n,m,\sigma}, \underline{J}^{inc}_p\}}{\left\{N^R_{n,m,\sigma}\left[d\lambda^R_n/d\gamma\right]\right\}_{\gamma=\gamma^R_{n,n'}}} . \qquad (65)$$

Taking into account the manipulations that led to equation (63) we can rewrite equation (65) as

$$\eta^{R(2)}_{n,n',m,\sigma,p}(\gamma) = \frac{1}{c}\, e^{\gamma^{R}_{n,n'}a}\, C_{1,n,n',m,\sigma,p} \frac{\left[\gamma a i_n(\gamma a)\right]'/\gamma a}{\left\{\left[\gamma a i_n(\gamma a)\right]'/\gamma a\right\}_{\gamma=\gamma^{R}_{n,n'}}} \tag{66}$$

and similarly for q = 2

$$\eta^{Q(2)}_{n,n',m,\sigma,p}(\gamma) = \frac{1}{c}\, e^{\gamma^{Q}_{n,n'}a} C_{2,n,n',m,\sigma,p} \frac{i_n(\gamma a)}{i_n(\gamma^{Q}_{n,n'}a)} \tag{67}$$

Comparing these expressions with (63) and (64) we conclude that Class 2 does not give the correct answer for the sphere. (See end of this section.) Specifically let us consider q = 2, n = 1, n' = 1 and set $\gamma^{Q}_{n,n'} \equiv \gamma_1$. We have

$$i_1(\gamma a) = \frac{\cosh(\gamma a)}{\gamma a} - \frac{\sinh(\gamma a)}{(\gamma a)^2} = \frac{e^{\gamma a}}{2\gamma a}\left(1 - \frac{1}{\gamma a}\right) + \frac{e^{-\gamma a}}{2\gamma a}\left(1 + \frac{1}{\gamma a}\right) \tag{68}$$

To Laplace invert the form containing the Class 1 coupling coefficient for n = 1, n' = 1, we write

$$T^{(1)} \equiv \frac{1}{c}\, C_2\, \mathscr{L}^{-1}\left\{\frac{e^{\gamma a}}{\gamma - \gamma_1}\right\} = C_2\, e^{\gamma_1 c(t+a/c)}\, u(t + a/c) \tag{69}$$

where $C_2 \equiv C_{2,n,n',m,\sigma,p}$ with n = 1, n' = 1. We also invert the term n = 1, n' = 1 with the Class 2 coupling coefficient by considering the expression

$$T^{(2)} = \frac{1}{c}\frac{C_2 e^{\gamma_1 a}}{i_n(\gamma_1 a)}\mathscr{L}^{-1}\left\{\frac{e^{\gamma a}}{2\gamma a}\left(1 - \frac{1}{\gamma a}\right)\frac{1}{\gamma - \gamma_1} + \frac{e^{-\gamma a}}{2\gamma a}\left(1 + \frac{1}{\gamma a}\right)\frac{1}{\gamma - \gamma_1}\right\} \tag{70}$$

Now we can write

$$\left(\frac{a}{\gamma} - \frac{1}{\gamma^2}\right)\frac{1}{\gamma - \gamma_1} = \frac{1}{\gamma_1^2}\left(\frac{a\gamma_1 - 1}{\gamma - \gamma_1} - \frac{a\gamma_1 - 1}{\gamma} + \frac{\gamma_1}{\gamma^2}\right)$$

$$\left(\frac{a}{\gamma} + \frac{1}{\gamma^2}\right)\frac{1}{\gamma - \gamma_1} = \frac{1}{\gamma_1^2}\left(\frac{a\gamma_1 + 1}{\gamma - \gamma_1} - \frac{a\gamma_1 + 1}{\gamma} - \frac{\gamma_1}{\gamma^2}\right)$$

and equation (70) can be rewritten as

$$T^{(2)} = \frac{C_2\ e^{\gamma_1 a}}{2\gamma_1^2 a^2\ i_n(\gamma_1 a)}\left\{\left[(a\gamma_1 - 1)e^{\gamma_1 c(t+a/c)} + 1 + \gamma_1 ct\right]u(t + a/c)\right.$$

$$\left. + \left[(a\gamma_1 + 1)e^{\gamma_1 c(t-a/c)} - 1 - \gamma_1 ct\right]u(t - a/c)\right\}$$

$$= \frac{C_2 e^{\gamma_1 c(t+a/c)}}{2\gamma_1^2 a^2\ i_n(\gamma_1 a)}\left[a\gamma_1 - 1)e^{\gamma_1 a}\ u(t + a/c) + (a\gamma_1 + 1)\right.$$

$$\left. e^{-\gamma_1 a}\ u(t - a/c) + (1 + \gamma_1 ct)\left(u(t + a/c) - u(t - a/c)\right)\right] \tag{71}$$

For $-a/c < t < a/c$ equation (71) is different from equation (69). However, for $t > a/c$ equation (71) can be reduced to

$$T^{(2)} = C_2\ e^{\gamma_1 c(t+a/c)}$$

and Class 2 gives the correct answer only for $t > a/c$, i.e., after the wavefront has passed the sphere. Notice that the previous result is valid for a delta function incident plane wave. For an incident wave with a different functional dependence say $f_p(t - \hat{e}_1 \cdot r/c)\ u(t - \hat{e}_1 \cdot r/c)$ the response is split into the object response and waveform response, i.e.,

$$\underline{J}_{op} = \sum_{\alpha} f_p(s_\alpha)\ \eta_\alpha(s)\ \underline{\nu}_\alpha(\underline{r})\,(s - s_\alpha)^{-1} \tag{72}$$

$$\underline{J}_{wp} = \sum_{\alpha} \eta_\alpha(s)\ \underline{\nu}_\alpha(\underline{r})\ \frac{f_p(s) - f_p(s_\alpha)}{s - s_\alpha} \tag{73}$$

From equation (72) we see that Class 2 will again give the correct <u>object</u> response only for $t > a/c$. However, the <u>waveform</u> response will be wrong for all times since from equation (73) we have to convolve $\mathscr{L}^{-1}\left(\eta_a(s)/(s - s_a)\right)$ with $\mathscr{L}^{-1}\,(f_p(s) - f_p(s_\alpha))$ and $\mathscr{L}^{-1}\left(\eta_\alpha(s)/(s - s_\alpha)\right)$ is correct for $t > a/c$ only.

We conclude this section by answering the following question. What if, despite the fact that Class 1 and Class 2 coupling coefficients are different, the corresponding infinite sums produce identical responses? The answer is that this is impossible. To show this, we recall the orthogonality properties of the $\underline{\nu}_\alpha$'s $(\underline{R}_{nm\sigma}, \underline{Q}_{nm\sigma})$, i.e., equations (B-19) in reference 3 (which, incidentally, show the linear independence of the $\underline{\nu}_\alpha$'s as we mentioned in the introduction) and notice that if the responses were identical we would have

$$\sum_{n'} \frac{\Delta\eta_{n,n',m,\sigma,p}}{\gamma - \gamma_{nn'}} = 0$$

where $\Delta\eta_{n,n'm,\sigma,p}$ is the difference between the Class 1 and Class 2 coupling coefficients. In particular, for q = 1 ($\underline{R}_{nm\sigma}$ functions) and n = 1, we have n' = 0; i.e., there is only one pole and it lies on the real axis. Thus $\Delta\eta^{R}_{1,0,m,\sigma,p} = 0$. This is impossible as we have shown, and consequently the two sums must be different.

SECTION VI

EIGENMODE SOLUTION TO THE ELECTRIC FIELD INTERODIFFERENTIAL EQUATION

The Electric Field Integrodifferential Equation (EFIDE) for scattering from perfectly conducting bodies can be cast into the following operator form (ref. 2)

$$\underline{\underline{Z}} \cdot \underline{J} = Z_o^{-1} \underline{E}_t^{inc} \tag{74}$$

where

$$\underline{\underline{Z}} \cdot \underline{J} = \underline{\underline{I}}_t \cdot (\nabla\phi + \gamma\underline{A})$$

$$\underline{\underline{I}}_t = \underline{\underline{I}} - \hat{n}\hat{n}$$

$$\underline{A} = \int_S G(\underline{r},\underline{r}')\, \underline{J}(\underline{r}')\, dS'$$

$$\phi = -\frac{1}{\gamma}\int_S G(\underline{r},\underline{r}')\, \nabla' \cdot \underline{J}(\underline{r}')\, dS'$$

and $\underline{E}_t^{inc}$ is the tangential $(= \underline{\underline{I}}_t \cdot \underline{E}^{inc})$ component of the incident electric field. It can be shown that the following relationship is true

$$\int_S \underline{a} \cdot \underline{\underline{Z}} \cdot b\, dS = \int_S (\underline{\underline{Z}} \cdot \underline{a}) \cdot \underline{b}\, dS \tag{75}$$

Thus if we define a complex inner product

$$(\underline{f},\underline{g}) = \int_S \underline{f}^* \cdot \underline{g}\, dS$$

we have

$$(\underline{a},\underline{\underline{Z}} \cdot b) = (\underline{\underline{Z}}^* \cdot a,\underline{b}) \tag{76}$$

In order to utilize equation (76) in the derivation of the eigenmode solution to equation (74) we define the eigenvalue equations

$$\underline{\underline{Z}} \cdot \underline{J}_{i\ell} = \zeta_i \, \underline{J}_{i\ell} \tag{77}$$

$$\underline{\underline{Z}}^{\dagger} \cdot \underline{J}_{i\ell}^{\dagger} = \zeta_i^{\dagger} \, \underline{J}_{i\ell} \tag{78}$$

and one can show that

$$\left.\begin{aligned} \zeta_i &= \zeta_i^{\dagger *} \\ \int_S \underline{J}_i^{\dagger *} \cdot \underline{J}_{jm} \, dS &= N_{i\ell} \, \delta_{ij} \, \delta_{\ell m} \end{aligned}\right\} \tag{79}$$

Recalling equation (76) and the defining relationship

$$(\underline{a}, \underline{\underline{Z}} \cdot \underline{b}) = (\underline{\underline{Z}}^{\dagger} \cdot \underline{a}, \underline{b})$$

we understand that

$$\underline{\underline{Z}}^{\dagger} = \underline{\underline{Z}}^{*}$$

We can now rewrite equation (78) as

$$\underline{\underline{Z}}^{*} \cdot \underline{J}_{i\ell}^{\dagger} = \zeta_i^{*} \, \underline{J}_{i\ell}^{\dagger}$$

or

$$\underline{\underline{Z}} \cdot \underline{J}_{i\ell}^{\dagger *} = \zeta_i \, \underline{J}_{i\ell}^{\dagger *}$$

i.e.,

$$\underline{J}_{i\ell}^{\dagger} = \underline{J}_{i\ell}^{*} \tag{80}$$

Thus in view of equations (79) and (80) the eigenmode solution to equation (74) is

$$\underline{J} = \frac{1}{Z_o} \sum_{\ell} \sum_{i} \frac{(\underline{J}^*_{i\ell}, \underline{E}^{inc}_t)}{N_{i\ell}\ \zeta_i} \underline{J}_{i\ell} \tag{81}$$

where

$$\left.\begin{aligned} (\underline{J}^*_{i\ell}, \underline{E}^{inc}_t) &= \int_S \underline{J}_{i\ell} \cdot \underline{E}^{inc}_t \, dS \\ N_{i\ell} &= \int_S \underline{J}_{i\ell} \cdot \underline{J}_{i\ell} \, dS \end{aligned}\right\} \tag{82}$$

As an example for equation (81) we consider the case of a sphere. From reference 2 we have

$$\underline{J}_{i\ell} = \left\{\begin{matrix} \underline{R}_{n,m,\sigma} \\ \\ \underline{Q}_{n,m,\sigma} \end{matrix}\right. \qquad \left.\begin{matrix} n = 1,2, \ldots, \infty \\ m \leq n \\ \sigma = o,e \text{ (odd, even)} \end{matrix}\right\} \tag{83}$$

i.e., $\underline{\underline{Z}}$ has the same eigenfunctions as the MFIE operator $\mathscr{L}$. The corresponding eigenvalues are (ref. 2)

$$\begin{aligned} \zeta^R_n &= \left[\gamma a i_n(\gamma a)\right] \left[\gamma a \ k_n(\gamma a)\right] \\ \zeta^Q_n &= -\left[\gamma a i_n(\gamma a)\right]' \left[\gamma a \ k_n(\gamma a)\right]' \end{aligned} \tag{84}$$

and from equation (44)

$$\begin{aligned} N_{i\ell} = M_{n,m,\sigma} &= \int_S \underline{R}_{n,m,\sigma} \cdot \underline{R}_{n,m,\sigma} \, dS \\ &= \int_S \underline{Q}_{n,m,\sigma} \cdot \underline{Q}_{n,m,\sigma} \, dS \end{aligned} \tag{85}$$

The incident electric field $\underline{E}^{inc}$ has the form (equation (B59) in ref. 2)

$$\underline{E}^{inc} = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{\sigma=e,o} \left[A_{1,n,m,\sigma,p} \, \underline{M}^{(1)}_{n,m,\sigma}(\gamma \underline{r}) + A_{2,n,m,\sigma,p} \, \underline{N}^{(1)}_{n,m,\sigma}(\gamma \underline{r}) \right] \tag{86}$$

where $\underline{M}^{(1)}_{n,m,\sigma}$ and $\underline{N}^{(1)}_{n,m,\sigma}$ are given by equation (51) and the A's by equation (B-58) in reference 3. By using the orthogonality relationships given by equations (B-11), (B-12), and (B-13) in reference 3 in conjunction with equation (86) we find

$$Z_o^{-1}(\underline{R}^*_{n,m,\sigma}, \underline{E}^{inc}) = Z_o^{-1}(\underline{R}^*_{n,m,\sigma}, \underline{E}^{inc}_t) = i_n(\gamma a) \, M_{n,m,\sigma} \, A_{1,n,m,\sigma,p} \tag{87a}$$

$$Z_o^{-1}(\underline{Q}^*_{n,m,\sigma}, \underline{E}^{inc}) = Z_o^{-1}(\underline{Q}^*_{n,m,\sigma}, \underline{E}^{inc}_t) = \frac{\left[\gamma a i_n(\gamma a)\right]'}{\gamma a} \, M_{n,m,\sigma} \, A_{2,n,m,\sigma,p} \tag{87b}$$

where $M_{n,m,\sigma}$ are given by equation (85). Employing equations (87) and (84) we can rewrite equation (81) as

$$\underline{J} = \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{\sigma=o,e} \left[\frac{A_{1,n,m,\sigma,p}\, i_n(\gamma a)\, M_{n,m,\sigma}}{M_{n,m,\sigma}\left[\gamma a i_n(\gamma a)\right]\left[\gamma a\, k_n(\gamma a)\right]} \underline{R}_{n,m,\sigma} \right.$$

$$\left. - \frac{A_{2,n,m,\sigma,p}\left\{\left[\gamma a i_n(\gamma a)\right]'/\gamma a\right\} M_{n,m,\sigma}}{M_{n,m,\sigma}\left[\gamma a i_n(\gamma a)\right]'\left[\gamma a\, k_n(\gamma a)\right]'} \underline{Q}_{n,m,\sigma} \right]$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{n} \sum_{\sigma=o,e} \left[\frac{A_{1,n,m,\sigma,p}}{(\gamma a)^2\, k_n(\gamma a)} \underline{R}_{n,m,\sigma} - \frac{A_{2,n,m,\sigma,p}}{\gamma a\left[\gamma a\, k_n(\gamma a)\right]'} \underline{Q}_{n,m,\sigma} \right] \tag{88}$$

Equation (88) as expected, is identical to equation (53) obtained via the pseudosymmetric eigenmode expansion for the Magnetic Field Integral Equation.

We conclude this section by noting two important factors; (a) if we examine the coupling coefficients for the sphere we can follow a procedure similar to the one employed in section IV and arrive at formulas that are identical to the ones in section IV, i.e., draw the same conclusions and (b) the excitation coefficient $(\underline{J}_i^*, Z_o^{-1}\, \underline{E}_t^{inc})$ can be shown to vanish at the interior resonances by employing a procedure similar to the one in section III of the MFIE.

SECTION VII

A SET OF ASSUMPTIONS FOR THE GENERALIZATION OF THE SEM SPHERE SOLUTION TO AN ARBITRARILY SHAPED CONDUCTING BODY

In this section we cast the eigenmode solution (to either the MFIE or EFIDE) for electromagnetic scattering from an arbitrarily shaped perfectly conducting body into a form that represents a generalization to the SEM sphere solution. Certain assumptions are made along the way that are motivated by the known sphere solution and the procedure leads to an SEM representation that involves Class 1 coupling coefficients with no additional entire function. If the same procedure is applied to the sphere, no assumptions are necessary, and one is inexorably led to the SEM solution with Class 1 coupling coefficients with no entire function to be added.

We begin with the eigenmode solution

$$\underline{J}(\underline{r},\gamma) = \sum \frac{b_i}{\lambda_i} \underline{J}_i \tag{89a}$$

where $b_i = \{\underline{\tilde{J}}_i, \underline{J}^{inc}\}/N_i$ for the MFIE (eq. 23) and $b_i = (\underline{J}_i^*, Z_o^{-1} \underline{E}_t^{inc})/N_i$ for the EFIDE (eq. 81). The eigenvalues and eigenfunctions are determined by solving the appropriate eigenvalue problems. It has been shown in reference 1 that for an incident delta function plane wave pulse (in the time domain) $\underline{J}(\underline{r},\gamma)$ is a meromorphic function of γ, i.e., in any finite region of the complex γ-plane $\underline{J}(\underline{r},\gamma)$ has a finite number of pole singularities. The pole locations correspond to the exterior and interior resonances of the body and can be determined by setting $\lambda_i(\gamma) = 0$. Recall that our incident wave in the time domain is a delta function plane wave pulse, i.e., the incident fields have no singularities in the finite γ-plane. In section III we showed that the excitation coefficient $\{\underline{J}_i, \underline{J}_i^{inc}\}$ is zero at the interior resonances γ_{ij}^I and in section VI we mentioned that we can similarly prove that $(\underline{J}_i^*, Z_o^{-1} \underline{E}^{inc})$ is also zero at γ_{ij}^I. This leads us to one of the assumptions necessary for our generalization; it is that $\underline{J}(\underline{r},\gamma)$ can be written as

$$\hat{s} \cdot \underline{J}(\underline{r},\gamma) = \sum_i L_i(\underline{r},\gamma) \tag{89b}$$

where $\hat{s}$ is a unit surface vector and the $L_i(\underline{r},\gamma)$ are meromorphic functions of γ. At this point, it is appropriate to relate equation (89b) to the Mittag-Leffler theorem (see refs. 5 and 6). This theorem is used to derive a general representation for meromorphic functions in terms of an infinite sum in which poles are explicitly represented. After we obtain more explicit representations for the $L_i(\underline{r},\gamma)$, we will show that equation (89b) does not violate this theorem. Because the excitation coefficients are zero at γ_{ij}^I, at this point we assume that the $L_i(\underline{r},\gamma)$ are such that their only singularities in the finite γ-plane are poles located only at exterior resonances. Notice that to each subscript i corresponds a set of exterior resonances γ_{ij}.

Equation (89b) is exact for the sphere as we can see by invoking equations (34), (35), and (52) for the MFIE and equations (83), (84), and (87) for the EFIDE. Next we invoke a corollary to the Weierstrass theorem concerning the representation of an entire function which states that "every function which is meromorphic in the whole

---

5. Carrier, G., M. Krook and C. Pearson, Functions of a Complex Variable, McGraw-Hill, New York, 1966.

6. Ahlfors, L., Complex Analysis, McGraw-Hill, New York, Second Edition, 1966.

finite plane is the quotient of two entire functions" (see for example, ref. 6). Thus

$$L_i(\underline{r},\gamma) = \frac{f_i(\underline{r},\gamma)}{g_i(\gamma)} \tag{90}$$

and the zeroes of $g_i(\gamma)$ correspond to the poles of L_i. Equation (90) is also true for the sphere. We now assume that both f_i and g_i have a finite number of zeroes for each i and consequently being entire functions they must have the form (ref. 6)

$$\begin{aligned} f_i(\underline{r},\gamma) &= e^{F_i(\underline{r},\gamma)}\, Q_i(\underline{r},\gamma) \\ g_i(\gamma) &= e^{G_i(\gamma)}\, P_i(\gamma) \end{aligned} \tag{91}$$

where F_i, G_i are entire functions and Q_i, P_i polynomials. Guided by the sphere solution we assume that F_i is a function of γ only and that the degree of Q_i is lower than the degree of P in order to be able to obtain a partial fraction expansion of the desired form. With the aid of equations (91) we can rewrite equation (90) as

$$L_i(\underline{r},\gamma) = e^{\phi_i(\gamma)}\, \frac{Q_i(\underline{r},\gamma)}{P_i(\gamma)} \tag{92}$$

where we have defined $\phi_i \equiv F_i - G_i$. Again equation (92) is exactly true for the sphere. The explicit form for a sphere can be obtained by invoking equations (B-74), (B-75), and (B-76) in reference 3. Thus $\phi_i(\gamma) = \gamma a$ and by noting that $Q_i(\underline{r},\gamma) = M_i(\gamma)\ (\hat{s} \cdot \underline{J}_i)$ where $\underline{J}_i$ is either $\underline{R}_{n,m,\sigma}$ or $\underline{Q}_{n,m,\sigma}$ we obtain

$$M_i(\gamma) \rightarrow (\gamma a)^{n-1}$$

$$P_i(\gamma) \rightarrow \text{Polynomial of } n^{th} \text{ degree}$$

We will now assume that $P_i(\gamma)$ has simple zeroes (true for the sphere) and set

$$\frac{b_i}{\lambda_i} \hat{s} \cdot \underline{J}_i = L_i = e^{\phi_i(\gamma)} \frac{Q_i(\underline{r},\gamma)}{P_i(\gamma)} = e^{\phi_i(\gamma)} \sum_{j=1}^{N(i)} \frac{A_{ij}}{\gamma - \gamma_{ij}} \quad (93)$$

From equation (93) one obtains

$$A_{ij} = \frac{b_i(\gamma_{ij}) \; \hat{s} \cdot \underline{J}_i(\underline{r},\gamma_{ij})}{\left[d\lambda_i/d\gamma\right]_{\gamma=\gamma_{ij}}} e^{-\phi_i(\gamma_{ij})} \quad (94)$$

and equation (89a) can be rewritten as

$$\underline{J}(\underline{r},\gamma) = \sum_i \sum_{j=1}^{N(i)} e^{\phi_i(\gamma)-\phi_i(\gamma_{ij})} \frac{b_i(\gamma_{ij})}{\left[d\lambda_i/d\gamma\right]_{\gamma=\gamma_{ij}}} \frac{\underline{J}_i(\underline{r},\gamma_{ij})}{\gamma - \gamma_{ij}} \quad (95)$$

For the sphere $\phi(\gamma) = a\gamma$ and consequently equation (95) shows that an appropriate rearrangement of the eigenmode solution (89a) inexorably leads to an SEM expansion with Class 1 coupling coefficients. In order to determine the form of the entire function $\phi_i(\gamma)$ for a general body we consider the inverse Laplace transform and we close in the left half-plane for times $t > t_o$ and in the right half-plane for $t < t_o$ where t_o is the instant at which the incident wavefront first hits the scattering body. All the poles are located in the left half-plane and for $t < t_o$ we should obtain $\underline{J}(\underline{r},t) = 0$, i.e., the integral on the large semicircle, $\text{Re}\gamma>0$, should approach zero as its radius approaches infinity. The integrand of interest has the form

$$i(\gamma) = \frac{e^{\phi_i(\gamma)+\gamma ct}}{\gamma - \gamma_{ij}} \quad (96)$$

with Re $\gamma_{ij} < 0$. Consequently we require that

$$\text{Re}\left[\phi_i(\gamma) + \gamma ct\right] < 0 \quad (97)$$

for all γ along the semicircle. Let us now define $\Psi_i(\gamma)$ by the equation

$$\psi_i(\gamma) \equiv \phi_i(\gamma) - \gamma c\varepsilon + \gamma ct$$

$$\varepsilon \equiv t - t_o .$$

For $t < t_o$ condition (97) can be rewritten

$$\text{Re}\ \psi_i(\gamma) < |\varepsilon|\ c\text{Re}\ \gamma . \quad (98)$$

Recalling that Re $\gamma < |\gamma|$ we can cast inequality (98) into the form

$$\text{Re}\ \frac{\psi_i(\gamma)}{c|\gamma|} < |\varepsilon| \quad (99)$$

with ε being an arbitrarily small finite value. Recalling that $\psi_i(\gamma)$ is an entire function one would be tempted to expand it in a McLaurin series based on the argument that equation (99) is only required to be true on a finite semicircle at this stage of the analysis and argue that equation (99) could only be satisfied if $\psi_i(\gamma)$ = constant with all remaining coefficients in the series necessarily equal to zero. We have not satisfied ourselves with this argument; however, we are willing to conjecture that $\psi_i(\gamma)$ = constant in view of the fact that any polynomial representation for $\psi_i(\gamma)$ can be shown to reduce to a constant. Making this conjecture, it follows that

$$\phi_i(\gamma) = -\gamma c t_o + \text{constant} \tag{100}$$

Such a choice of course allows us to close in the left-hand plane for $t > t_o$ since the requirement

$$\mathrm{Re}\ \psi_i(\gamma) < -\varepsilon c \mathrm{Re}\ \gamma = \varepsilon c|\mathrm{Re}\ \gamma|, \quad (\mathrm{Re}\ \gamma < 0,\ \varepsilon > 0)$$

is then satisfied for large $|\gamma|$ with $\mathrm{Re}\ \gamma > 0$.

Assuming that $\phi_i(\gamma)$ is given by equation (100) then $\phi_i(\gamma) - \phi_i(\gamma_{ij}) = (\gamma_{ij} - \gamma)t_o$ and equation (95) is an SEM solution with Class 1 coupling coefficients. Rewriting equation (95) after making this substitution and using equation (89), one obtains

$$\hat{s} \cdot \underline{J}(\underline{r},\gamma) = E(\gamma) \sum_i \sum_{j=1}^{N(i)} \frac{C_{ij}(\underline{r})}{\gamma - \gamma_{ij}} \tag{101a}$$

where

$$E(\gamma) = e^{-\gamma c t_o} \tag{101b}$$

$$C_{ij}(\underline{r}) = e^{\gamma_{ij} c t_o} \frac{b_i(\gamma_{ij})\ \hat{s} \cdot \underline{J}_i(\underline{r},\gamma_{ij})}{\left[d\lambda_i/d\gamma\right]_{\gamma=\gamma_{ij}}} \tag{101c}$$

Combining the two indices (i and j) into the index "n," we can write equation (101) as

$$\hat{s} \cdot \underline{J}(\underline{r},\gamma) = E(\gamma) \sum_{n=1}^{\infty} \frac{C_n}{\gamma - \gamma_n} \tag{102}$$

and this form is convenient for a discussion of the relationship between meromorphic functions and pole series expansions. A consequence of the Mittag-Leffler theorem is that the most general infinite pole series representation of a meromorphic function $m(\gamma)$ is

$$m(\gamma) = \sum_{n=1}^{\infty} \left[P_n\left(\frac{1}{\gamma - \gamma_n}\right) - \tilde{P}_n(\gamma) \right] + H(\gamma) \qquad (103)$$

where P_n and $\tilde{P}_n$ are finite degree polynomials not necessarily of degree n and $H(\gamma)$ is an entire function. There exist meromorphic functions for which $P_n\left(1/(\gamma - \gamma_n)\right) = C_n/(\gamma - \gamma_n)$ and $\tilde{P}_n(\gamma)$ and $H(\gamma)$ are identically zero. We also note that multiplying the meromorphic function $m(\gamma)$ by an entire function $E(\gamma)$, would still result in a meromorphic function. For the case where γ_n and C_n have the appropriate n dependence to eliminate the need for P_n and H, we would obtain a representation for the meromorphic function $M(\gamma) = E(\gamma)\, m(\gamma)$ as follows

$$M(\gamma) = E(\gamma) \sum_{n=1}^{\infty} \frac{C_n}{\gamma - \gamma_n} \qquad (104)$$

which is consistent with both equation (102) and the Mittag-Leffler theorem. This is also consistent with our assumed initial representation shown in equation (89).

SECTION VIII

NUMERICAL DETERMINATION OF POLE LOCATIONS AND EIGENFUNCTIONS FOR THE SPHERE

In this section we describe the procedure that was used to approximately find the pole locations for the sphere, i.e., those values of γ for which $L(\gamma) - (1/2)I$ has a zero eigenvalue. To do this we approximate the integral operator $L(\gamma) - (1/2)I$ by a matrix, $M(\gamma)$, according to the algorithm described in reference 7, and try to find those values of γ for which $M(\gamma)$ has a zero eigenvalue. Since the matrix is of finite order its determinant, $D(\gamma)$, equals zero if and only if at least one of the eigenvalues of $M(\gamma)$ is zero. This observation permits us to search for the zeroes of $D(\gamma)$ rather than the more difficult task of searching for the zeroes of the eigenvalues.

Our method of calculation of the elements of the matrix $M(\gamma)$ guarantees that $D(\gamma)$ is an analytic function of γ. Assuming that the errors due to the finite word size of the computer do not interfere with certain properties of analytic functions we may apply the method of Singaraju, Giri, and Baum (ref. 8) for determining the zeroes of an analytic function to find those values of γ for which $D(\gamma)$ is zero. A brief description of the method follows.

If both f(z) and g(z) are analytic functions of z in a simply connected open domain, Ω, then Caughy's theorem states that if C is the border of a rectangular region contained in Ω

---

7. Sancer, M. I., S. Siegel and A. D. Varvatsis, Foundation of the Magnetic Field Integral Equation Code for the Calculation of Electromagnetic Pulse External Interaction with Aircraft, Interaction Note 320, Air Force Weapons Laboratory, 1976.

$$\frac{1}{2\pi i} \oint_C g(z) \frac{f'(z)}{f(z)} dz = \sum_{i=1}^{N} g(z_i) \quad , \qquad (105)$$

where the z_i are the N zeroes of f(z), counted according to multiplicity, in the interior of C, provided that f has no zeroes on C. In particular we have that

$$N = \frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz \qquad (106)$$

and

$$\sum_{i=1}^{N} (z_i)^k = \frac{1}{2\pi i} \oint_C z^k \frac{f'(z)}{f(z)} \; ; \qquad k = 1,2, \ldots, N \qquad (107)$$

and from these equations we can determine the z_i.

To avoid the numerical difficulties associated with computing f'(z), Singaraju et al. (ref. 8) integrate equation (105) by parts to obtain

$$\sum_{i=1}^{N} g(z_i) = N\, g(z_{init}) + \frac{1}{2\pi i} \oint_C g'(z) \text{ adj log } (f(z)) \, dz \qquad (108)$$

where adj log (f(z)) is that branch of the locally defined log function whose only discontinuity along the contour occurs at z_{init} where it undergoes a jump of $2\pi i$ N. The problem of

---

8. Singaraju, B. K., D. V. Giri and C. E. Baum, Further Developments in the Application of Contour Integration to the Evaluation of the Zeros of Analytic Functions and Relevant Computer Programs, Mathematics Note 42, Air Force Weapons Laboratory, 1976.

calculating f'(z) has been replaced by the simpler problem of finding the phase of f(z) and guaranteeing that the phase change from point to point is less than π radians. Using equation (108), equations (107) become

$$A_k \equiv \sum_{i=1}^{N} z_i^k = N\, z_{init}^k + \frac{k}{2\pi i} \oint_C z^{k-1} \text{ adj } \log(f(z))\, dz \quad k=1,2,\ldots,N \tag{109}$$

For further details we refer the reader to reference 7.

In their paper, Singaraju et al. provide a computer program to determine the z_i if a contour contains up to three zeroes, however, the sphere problem has a high degeneracy and therefore the numerical approximation has the zeroes of $D(\gamma)$ clustering about their true locations. We therefore found it necessary to extend their procedure to arbitrary N. To accomplish this we transformed the system of equations (109) to the equivalent N^{th} degree polynomial equation

$$\sum_{k=0}^{N} B_k\, z^{N-k} = 0 \tag{110a}$$

where B_k are defined by the recursive formula

$$B_o = 1 \tag{110b}$$

$$kB_k + \sum_{j=1}^{k} A_j\, B_{k-j} = 0 \quad k = 1, \ldots, N$$

Equation (110b) was found in reference 9 as suggested by Dr. Giri.

---

9. Krylov, V. I., Approximate Calculation of Integrals, (Priblizhennoe Vychislenie Integralov), Moskva, Gos. Izd-Vo Fizika Matematicheskoi Lit., 1959.

We have chosen to use the subroutine package ZRPCC of the AFWL Scientific Program Library (ASPLIB) to find the zeroes of this polynomial. This routine forms the companion matrix whose characteristic equation is identical to the required polynomial and then returns the eigenvalues of that matrix.

For each γ that estimates a zero of $D(\gamma)$ and therefore a pole location of the sphere's response function we compute a new $M(\gamma)$. We "feed" this matrix to another ASPLIB subroutine package, CIVAA, which is documented as CG in the Eispack Guide (ref. 10). This package computes the eigenvalues and eigenvectors of $M(\gamma)$. We expect to find, and in practice have found that all of the eigenvalues occur in pairs which sum to one (see equation (24)). We search the list of eigenvalues for those pairs near (0,1) and identify the corresponding eigenvectors as the natural mode vector $\underline{J}$ and its dual vector $\underline{\tilde{J}}$. In theory, because we are solving the problem for a sphere, $\underline{J}$ and $\underline{\tilde{J}}$ are related by the formula $\underline{J}(\underline{r}) = \hat{n}(\underline{r}) \times \underline{\tilde{J}}(\underline{r})\, e^{i\theta}$ where the $e^{i\theta}$ is introduced to account for the arbitrariness in the phase of an eigenvalue; our numerical experiments showed that this relationship held to within the errors of our approximation.

Since the sphere problem has a high degeneracy we cannot expect our eigenfunctions to agree with those published in other sources but we can expect ours to be linear combinations of theirs. As an example, for the $n = 1$, $q = 1$ mode we found three eigenmodes $\underline{J}_1$, $\underline{J}_2$, $\underline{J}_3$ which can be related to those found in reference 2.

---

10. Smith, B. T., J. M. Boyle, J. J. Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema and C. B. Moler, Matrix Eigensystem Routines - EISPACK Guide, Springer-Verlag, New York, Second Edition, 1976.

$$\underline{J}_1 = \sin\theta\ \hat{e}\phi$$

$$\underline{J}_2 = -\cos\theta\ \cos(\phi - \pi/8)\ \hat{e}_\phi - \sin(\phi - \pi/8)\hat{e}_\theta$$

$$\underline{J}_3 = \cos\theta\ \sin(\phi - 3\pi/16)\hat{e}_\phi - \cos(\phi - 3\pi/16)\hat{e}_\theta$$

or

$$\underline{J}_1 = \underline{j}''_{0,1}$$

$$\underline{J}_2 = a\underline{j}''_{1,1} + b\underline{j}''_{-1,1}$$

$$\underline{J}_3 = c\underline{j}''_{1,1} + d\underline{j}_{-1,1}$$

$$a = b^* = -\frac{1}{2}\ e^{-i\pi/8}$$

$$c = d^* = -\frac{1}{2}\ e^{-i3\pi/16}$$

and $\underline{j}''_{mn}$ are given by equation (A-10) in reference 2.

APPENDIX A

PROOF OF EQUATION (14)

In this appendix we show that $Q^{\dagger} = Q^*$ and $M^{\dagger} = -M$, i.e., equation (14) is true. Recall that

$$Q\underline{J}(\underline{r}) \equiv \hat{n}(\underline{r}) \times (\hat{n}(\underline{r}) \times \int_S f(R)(\underline{r} - \underline{r}') \times \underline{J}(\underline{r}')\, dS')$$

$$\equiv -\underline{\underline{P}}(\underline{r}) \cdot \int_S f(R)(\underline{r} - \underline{r}') \times \underline{J}(\underline{r}')\, dS'$$

where

$$\underline{\underline{P}}(\underline{r}) \equiv \underline{\underline{I}} - \hat{n}(\underline{r})\, \hat{n}(\underline{r}) = -\hat{n}(\underline{r}) \times \left((\hat{n}(\underline{r}) \times \underline{\underline{I}} \right)$$

$$f(R) = -(1 + \gamma R)\left(\frac{e^{-\gamma R}}{4\pi R^3}\right), \qquad R = |\underline{r} - \underline{r}'|$$

Notice that for a tangential vector $\underline{a}$, $\underline{\underline{P}} \cdot \underline{a} = \underline{a}$. Consider now the inner product involving the tangential vector functions $\underline{\Phi}(\underline{r})$ and $\underline{\Psi}(\underline{r})$:

$$(\underline{\Phi}, Q\underline{\Psi}) \equiv \int_S \underline{\Phi}^*(\underline{r}) \cdot (Q\underline{\Psi}(\underline{r}))\, dS$$

$$= -\int_S dS \left\{ \underline{\Phi}^*(\underline{r}) \cdot \underline{\underline{P}}(\underline{r}) \cdot \int_{S'} \left[f(R)(\underline{r} - \underline{r}') \times \underline{\Psi}(\underline{r}') \right] dS' \right\}$$

$$= -\int_S \int_{S'} dS\, dS'\, f(R) \left\{ \left[\underline{\Phi}^*(\underline{r}) \times (\underline{r} - \underline{r}') \right] \cdot \underline{\Psi}(\underline{r}') \right\}$$

If we interchange $\underline{r}$ and $\underline{r}'$ the "double" integral does not change value and

$$(\underline{\Phi}, Q\underline{\Psi}) = -\int_{S'}\int_{S} dS'\, dS\, f(R)\left\{\left[\underline{\Phi}^*(\underline{r}') \times (\underline{r}' - \underline{r})\right] \cdot \underline{\Psi}(\underline{r})\right\}$$

$$= -\int_{S'}\int_{S} dS'\, dS\left[f(R)(\underline{r} - \underline{r}') \times \underline{\Phi}^*(\underline{r}')\right] \cdot \underline{\Psi}(\underline{r})$$

$$= -\int_{S} dS\, \underline{\Psi}(\underline{r}) \cdot \underline{\underline{P}}(\underline{r}) \cdot \int_{S'}\left[f^*(R)(\underline{r} - \underline{r}') \times \underline{\Phi}(\underline{r}')\right]^* dS'$$

$$= \int dS\left\{-\underline{\underline{P}}(\underline{r}) \cdot \int_{S'} f^*(R)(\underline{r} - \underline{r}') \times \underline{\Phi}(\underline{r}')\right\}^* dS' \cdot \Psi(\underline{r})$$

$$= \int (Q^*\, \underline{\Phi}(\underline{r}))^* \cdot \underline{\Psi}(\underline{r})\, dS$$

$$\equiv (Q^*\, \underline{\Phi}, \underline{\Psi})$$

Recalling the defining relationship for $Q^\dagger$ we conclude that $Q^\dagger = Q^*$.

Next we show that $M^\dagger = -M$.

$$(\underline{\Phi}, M\underline{\Psi}) \equiv \int_S dS\, \underline{\Phi}^* \cdot (\hat{n} \times \underline{\Psi}) = -\int_S dS\, (\hat{n} \times \underline{\Phi}^*) \cdot \underline{\Psi}$$

$$= \int_S dS\, (-\hat{n} \times \underline{\Phi})^* \cdot \underline{\Psi} \equiv (-M\underline{\Phi}, \underline{\Psi})$$

and this proves that $M^\dagger = -M$.

APPENDIX B

PROOF THAT $N_i \; d\lambda_i/d\gamma = \int_S ds\left[\underline{\tilde{J}}_i \cdot \left(\hat{n} \times \frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i\right)\right]$

In this appendix we show the validity of the formulas employed in section V for the coupling coefficients by demonstrating that

$$N_i \frac{d\lambda_i}{d\gamma} = \int_S ds\left[\underline{\tilde{J}}_i \cdot \left(\hat{n} \times \frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i\right)\right] \tag{B-1}$$

where we have ignored degeneracy since the proof is identical for the degenerate case. We start with equation

$$\mathscr{L}\underline{J}_i = \lambda_i \underline{J}_i$$

and differentiate both sides w.r.t. γ

$$\frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i + \mathscr{L}\frac{\partial \underline{J}_i}{\partial \gamma} = \frac{d\lambda_i}{d\gamma} \underline{J}_i + \lambda_i \frac{\partial \underline{J}_i}{\partial \gamma} .$$

We operate on both sides by M and consider the inner product of the resulting equation with $\underline{F}_i$ defined in equation (9).

$$\left(\underline{F}_i, M \frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i\right) + \left(\underline{F}_i, M\mathscr{L} \frac{\partial \underline{J}_i}{\partial \gamma}\right) = \frac{d\lambda_i}{d\gamma} (\underline{F}_i, M\underline{J}_i) + \lambda_i\left(F_i, M \frac{\partial \underline{J}_i}{\partial \gamma}\right) \tag{B-2}$$

By definition $(\underline{F}_i, M\mathscr{L}\, \partial \underline{J}_i/\partial\gamma) \equiv \left((M\mathscr{L})^{\dagger}\underline{F}_i, \partial \underline{J}_i/\partial \gamma\right)$ and from equation (18) $\underline{\tilde{J}}_i \equiv \underline{F}_i^*$. We can now rewrite equation (B-2) as

$$\int_S ds \left[\underline{\tilde{J}}_i \cdot \left(\hat{n} \times \frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i\right)\right] + \left((M\mathscr{L})^\dagger \underline{F}_i, \frac{\partial \underline{J}_i}{\partial \gamma}\right)$$

$$= \frac{d\lambda_i}{d\gamma} N_i + \lambda_i \left(\underline{F}_i, M \frac{\partial \underline{J}_i}{\partial \gamma}\right). \qquad \text{(B-3)}$$

Recalling equation (9) equation (B-3) is transformed into

$$\int_S ds\ \underline{\tilde{J}}_i \cdot \left[\left(\hat{n} \times \frac{\partial \mathscr{L}}{\partial \gamma} \underline{J}_i\right)\right] + \lambda_i \left(M^\dagger \underline{F}_i, \frac{\partial \underline{J}_i}{\partial \gamma}\right)$$

$$= \frac{d\lambda_i}{d\gamma} N_i + \lambda_i \left(\underline{F}_i, M \frac{\partial \underline{J}_i}{\partial \gamma}\right). \qquad \text{(B-4)}$$

If we now notice that $(M^\dagger \underline{F}_i, \partial \underline{J}_i/\partial\gamma) \equiv (\underline{F}_i, M\ \partial \underline{J}_i/\partial\gamma)$, equation (B-4) yields the desired result, i.e., equation (B-1).

REFERENCES

1. Marin, L. and R. W. Latham, Analytical Properties of the Field Scattered by a Perfectly Conducting Finite Body, Interaction Note 92, Air Force Weapons Laboratory, 1972.

2. Marin, L., Natural-Mode, Representation of Transient Scattering from Rotationally Symmetric, Perfectly Conducting Bodies and Numerical Results for a Prolate Spheroid, Interaction Note 119, Air Force Weapons Laboratory, 1972.

3. Baum, C. E., On the Singularity Expansion Method for the Solution of Electromagnetic Interaction Problems, Interaction Note 88, Air Force Weapons Laboratory, 1971.

4. Tesche, F. M., On the Singularity Expansion Method as Applied to Electromagnetic Scattering from Thin Wires, Interaction Note 102, Air Force Weapons Laboratory, 1972.

5. Carrier, G., M. Krook and C. Pearson, Functions of a Complex Variable, McGraw-Hill, New York, 1966.

6. Ahlfors, L., Complex Analysis, McGraw-Hill, New York, Second Edition, 1966.

7. Sancer, M. I., S. Siegel and A. D. Varvatsis, Foundation of the Magnetic Field Integral Equation Code for the Calculation of Electromagnetic Pulse External Interaction with Aircraft, Interaction Note 320, Air Force Weapons Laboratory, 1976.

8. Singaraju, B. K., D. V. Giri and C. E. Baum, Further Developments in the Application of Contour Integration to the Evaluation of the Zeros of Analytic Functions and Relevant Computer Programs, Mathematics Note 42, Air Force Weapons Laboratory, 1976.

9. Krylov, V. I., Approximate Calculation of Integrals, (Priblizhennoe Vychislenie Integralov), Moskva, Gos. Izd-Vo Fizika Matematicheskoi Lit., 1959.

10. Smith, B. T., J. M. Boyle, J. J. Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema and C. B. Moler, Matrix Eigensystem Routines - EISPACK Guide, Springer-Verlag, New York, Second Edition, 1976.