NOTE 350

ON THE ANALYSIS OF GENERAL MULTICONDUCTOR TRANSMISSION-LINE NETWORKS

by

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ABSTRACT

Starting from the graph describing the topology of a transmission-line network in terms of junctions and tubes, this note addresses the formation of the overall network equation referred to as the BLT equation. The matrix equation of propagation on a tube consisting of a multiconductor transmission line with sources is formulated in terms of the combined voltage vector (a special combination of the voltage and current vectors) which reduces the differential equation to first order; this readily incorporates the combined voltage sources and boundary conditions into the solution of the tube propagation. Utilizing interconnection matrices (defined by the topology) which related waves on the tubes to junctions (tube ends) which the waves leave and enter, the scattering supermatrices for the junctions are converted into a scattering supermatrix for the waves on the transmission-line network. Appropriate supermatrices describing the delay on the tubes, the source terms on the tubes, and the identity are also defined. Together with combined voltage supervectors for waves and sources, the BLT equation is constructed in one of its possible forms. Various properties of the BLT equation and the associated supermatrices, or dimatrices (matrices of matrices) in this case, as well as the corresponding supervectors are developed.

FOREWORD

This note has been planned for quite some time now. Some of the results were included in two papers [1.6,7] presented at the USNC/URSI Meeting in Amherst, Mass., October 1976. The basic concepts include the topology of the transmission-line network, the propagation on multiwire transmission lines, and supermatrix/supervector forms for representing the variables so as to produce the BLT equation. This note is then the first in perhaps a series concerning a general kind of approach to transmission-line network theory. It allows the consideration of the analytical properties of the network, besides the properties of the network components.

"About half way up is a cave,
A gloomy cavern facing the West and Erebus,
And beneath this cave, my gallant Odysseus, you
Must steer your ship. It will be so high above you
That not even the strongest man could reach it with an arrow
Shot from the deck of his hollow ship below.
In it lives Scylla, yelping terribly, with a voice
That sounds no stronger than that of a puppy just born.
But she herself is an evil monster that no one
Would be glad to see, not though a god should meet her.
She has twelve feet in all, horribly dangling,
And six necks, tremendously long, on each of which
Is a terrible head with teeth in triple tiers,
Close set and chocked with black death. From her waist down
The deep cave hides her, but her heads sway out from the awful
Abyss, and with them, around the rock, she avidly
Fishes for dolphin and dog-fish and what greater beast
She may catch of the countless creatures that Amphitrite,
Deeply moaning, feeds. Never yet have sailors
Been able to boast that they got by her unscathed
In their ship, for with each of her heads she snaps up a man
From the dark-prowed ship."

From The Odyssey, Book XII, by Homer, translated by Ennis Rees, Modern Library, Random House, 1960, p. 198.

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I. INTRODUCTION

Transmission-line theory has been with us for quite some time [1.1]. Its impact on communication technology should be obvious. However, its expression in one-dimensional scalar form for a single voltage-current pair has rather limited application to modern complex electronic systems. In the analysis of EMP interaction with electronic systems, transmission-line theory is commonplace [1.2]; however, its practical use is still in a rudimentary form, usually being applied in a one-dimensional scalar form as in the case of a coaxial cable or some simple approximation of a more complex system in one-dimensional scalar form [1.3].

Recent investigations have considered multiconductor transmission lines as an extension of transmission-line theory applicable to complex systems problems such as involved in EMP and EMC [1.4,5]. However, one should recognize that such models are still quite simplistic in the context of the total system response in typical cases. This note addresses the problem of networks of such multiconductor transmission lines.

Basic to the analysis of transmission-line networks is the network topology based on junctions and tubes, each tube being a representation of a multiconductor transmission line, and tube terminations (including connections to other tubes) occurring at junctions. So first we consider the network topology and the associated interconnection matrices which will be used to construct the network equation. This transmission-line network topology is compared to other kinds, such as those used for lumped-element networks and electromagnetic scatterers.

Next the equations describing a single tube or multiconductor transmission line are considered. The problem is reduced to a first-order matrix differential equation through the introduction of a combined voltage vector which is a special linear combination of the voltage and current vectors. This equation is readily solved for given boundary conditions at the tube ends and source vectors along the tube. The propagation matrix is assumed diagonalizable and the resulting eigenmodes and eigenvalues are used to give representations of the various parameters describing the tube and its response.

The remainder of the note then integrates the result of a single tube with the scattering matrices of the junctions using the interconnectivity of the transmission-line network topology and its associated wave indexing. This forms the overall equation of the multiconductor transmission-line network, referred to as the BLT equation [1.6,7]. For this purpose it is useful to introduce the concept of supermatrices, or matrices of matrices, and corresponding supervectors. This separates the indices in a manner which associates different indices with different physical aspects of the multiconductor transmission-line network and its associated topology. In addition, the supermatrices correspond to a symmetric partioning of matrices in a manner which makes them block sparse.

The supermatrices used in this note can also be referred to as dimatrices corresponding to a single level of partition resulting in two pairs of indices or subscripts to describe the dimatrix elements. This concept has already been generalized to higher order partitions and hence higher order supermatrices in applications concerning the topology of complex electromagnetic scatterers [1.8]. So when the reader has waded through the present tome he can cheerfully contemplate that more is to come (or he may need cheering up for the same reason).

References

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1.2 J.A. Cooper, D.E. Merewether, and R.L. Parker (eds.), "Electromagnetic Pulse Handbook for Missiles and Aircraft in Flight", EMP Interaction 1-1, Sept. 1972.

1.3 F.M. Tesche and T.K. Liu, "Selected Topics in Transmission-Line Theory for EMP Internal Interaction Problems", Interaction Note 318, March 1977.

1.4 S. Frankel, Multiconductor Transmission-Line Analysis, Artech House, 1977.

1.5 C. Paul, "On Uniform Multimode Transmission Lines", IEEE Trans. MTT, vol. MTT-21, Aug. 1973, pp. 556-558.

1.6 C.E. Baum, "Coupling into Coaxial Cables from Currents and Charges on the Exterior", USNC/URSI Meeting, Amherst, Massachusetts, Oct. 1976.

1.7 F.M. Tesche, "A General Multiconductor Transmission-Line Model", USNC/URSI Meeting, Amherst, Massachusetts, Oct. 1976.

1.8 C.E. Baum, "The Treatment of the Problem of Electromagnetic Interaction with Complex Systems", IEEE 1978 International Symposium on Electromagnetic Compatibility, Atlanta, Georgia, June 1978.

II. TOPOLOGY

Network topology is a generic name given to the topological properties of a network. It is studied widely in lumped circuit theory to gain reliable knowledge concerning the number of independent equations in a circuit of arbitrary structural complexity [2.1]. For electromagnetic problems, a more general type of topology is required to describe the three-dimensional volumes and surfaces that are generally associated with the scatterers. Indeed, the scatterer topology has been introduced by Baum [2.2, 2.3], Tesche [2.4] and has found useful applications in considering practical shielding and grounding problems [2.5, 2.6].

For transmission-line networks, the topological description falls between that of lumped circuits and that of scatterers. Similar to a lumped circuit, there are well-defined material paths along which energy propagates, and there are positions in the transmission-line network where energy is distributed according to Kirchhoff's laws. Unlike a lumped circuit, energy may be coupled between the material paths, and the path characteristics (length, geometry, etc.) alter the ways energy propagates. Energy sources may also be induced along the lengths of the paths. These latter properties which are attributed to the distributed nature of transmission lines are more similar to those of scatterers. In fact, it is more appropriate to describe the transmission-line behavior as wave phenomena.

The specialized topological description of a transmission-line network has been called the transmission-line topology [2.3]. In the following, we first review the concept of circuit graphs and circuit topology. The transmission-line topology is then described. A brief outline of other related topologies is also included.

A. Graphs

The basic elements of a graph are edges and vertices. They are termed differently in specific types of topology, and are summarized in Table 2.1 under Section 2F.

A graph is thus made up of a set of vertices which are interconnected by edges. It is structured to represent, in a simplified form, the electrical connections and/or signal flow paths of the network or system.

Often, for a complicated network, one cannot represent the network comprehensively by a single graph. Parts of the network are then represented by subgraphs. A subgraph fulfills all requirements of being a graph, but is limited to represent only part of the network. An example is that of a multiconductor transmission-line network where a network graph is used to show the transmission-line connections using tube and junction representations, but the detailed electrical connections within junctions are represented by separate subgraphs [2.6].

On the other hand, a network graph may be a subset to a graph which represents a larger system. The latter is called a supergraph. In the previous example, the transmission-line network graph is a supergraph of that of the junctions and tubes.

The concepts of supergraph, graph and subgraph define a hierarchical order of representing a system. However, the naming of super- and sub- are only relative when compared to a smaller or larger part of the network.

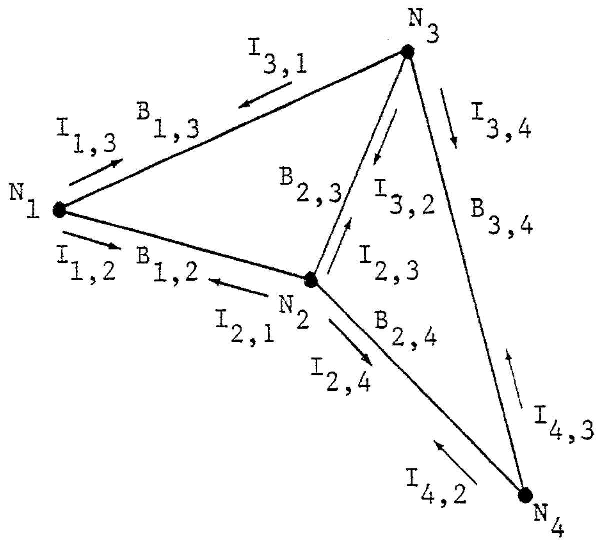
B. Electrical Circuits

The most common network topological concepts have been applied to lumped circuits. The construction of the network graph and its associated development of cut sets and tie sets are assumed to be familiar to the reader. In transmission-line networks, often the junctions contain lumped elements, or the transmission-line discontinuities may be accurately modeled by lumped circuits at low frequencies [2.7 - 2.10]. Hence, in tranmission-line network analysis, it is often necessary to include lumped circuit analysis. We summarize the essential points in the following.

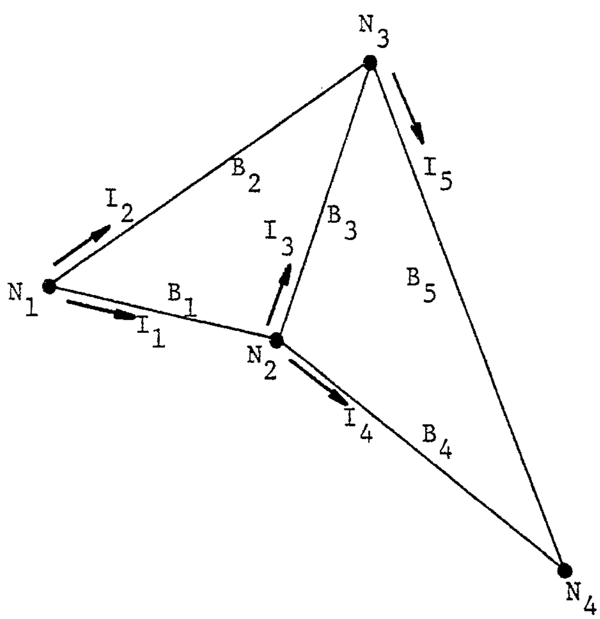
The basic elements of a lumped circuit network graph are branches and nodes. A branch is a component part of a circuit characterized by two terminals to which connections can be made. A node is formed where two or more branches are connected. The graphic symbols for branches and nodes are respectively lines and dots. The nodes in a circuit are numbered and the nth node is denoted by N_n. The total number of nodes is N_N. The branch connecting nodes N_n and N_m is labeled $B_{n,m}$. Hence, $B_{n,m}$ and $B_{m,n}$ denote the same branch. The total number of branches is N_B. If between a node pair there is more than one branch, then by parallel combinations one can reduce this to a single branch.

As an example, a circuit graph is shown in Fig. 2.1a, representing a four node, five branch circuit. The branches are numbered by double subscripts using the rule outlined above.

It is useful to introduce three topology matrices which define the topological structure of a graph. They describe node-node (or node interconnection), node-branch and branch-branch (or branch interconnection) connections. These matrices contain somewhat redundant information and usually only one of them is sufficient to specify the associated graph. However, the last type (branch-branch) does not give unique node numbering.



(a)



(b)

Fig. 2.1 A circuit graph with (a) double-subscripted branch numbers, (b) single-subscripted branch numbers.

For a circuit with N_N nodes, the node-node interconnection matrix $(C_{n,m})_{N-N}$ is an $N_N \times N_N$ matrix. The elements are defined by

$n \neq m$	$C_{n,m;N-N} = 1$ if nodes N_n and N_m are connected $= 0$ if no connection	(2.1)
$n = m$	$C_{n,n;N-N} = 1$ to denote self connection	

Note that the node-node interconnection matrix is symmetric, i.e.,

$$C_{n,m;N-N} = C_{m,n;N-N} \tag{2.2}$$

For example in Fig. 2.1, the node-node matrix is

$$(C_{n,m})_{N-N} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{2.3}$$

Similarly, the node-branch matrix $(C_{n,m})_{N-B}$ is $N_N \times N_B$, where N_B is the total number of branches in the circuit. The elements $C_{n,m;N-B}$ are defined by

$C_{n,m;N-B} = 1$ if node N_n is connected to branch B_m
$C_{n,m;N-B} = 0$ if N_n is not connected to B_m

(2.4)

This matrix is in general rectangular instead of square. A single-subscripted denotation of the branches is necessary for matrix manipulations. The numbering of the branches starts at node N_1 for the branch going to the node with the lowest node number. The rest of the branches connected to node N_1 are numbered consecutively with the increase of node numbers they are connected to at the other ends. The sequential numbering continues for branches connected to node N_2, again according to the increase of node numbers they are connected to at the other ends. Here, branches already assigned a branch number are not renumbered. This process repeats for all nodes.

The single-subscripted branch numbering for the graph in Fig. 2.1a is depicted in Fig. 2.1 b. The node-branch matrix is given by

$$(C_{n,m})_{N-B} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (2.5)$$

The branch-branch interconnection matrix $(C_{n,m})_{B-B}$ is an $N_B \times N_B$ matrix describing branch-to-branch connections. The elements are defined by

$n \neq m$	$C_{n,m;B-B}$	= number of connections between the ends of branches B_n and B_m = 0 implies no connections = 1 implies one end of each branch is connected
$n = m$	$C_{n,m;B-B}$	= 0 excluding branches with both ends connected to the same node

(2.6)

Note that this matrix is symmetric, i.e.

$$C_{n,m;B-B} = C_{m,n;B-B} \quad (2.7)$$

The branch-branch matrix corresponding to the graph in Fig. 2.1b is

$$(C_{n,m})_{B-B} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad (2.8)$$

Note that the node-node matrix specifies the connections for a given set of nodes. The node-branch matrix specifies the connection for a given set of nodes and branches. Either of these two matrices can regenerate the graph with the same node and branch numbering. This may be much more difficult for the branch-branch matrix. However, all the matrices are derived from the given network (graph) consisting of nodes and branches.

The Kirchhoff's laws can be written down comprehensively. One defines voltages $\tilde{V}_n(s)$ at each node N_n and currents $\tilde{I}_{n,m}(s)$ leaving N_n to N_m along the branch $B_{n,m}$ with the condition

$$\sum_m \tilde{I}_{n,m} = 0 \quad , \quad \tilde{I}_{n,n} = 0 \tag{2.9}$$

The voltages and currents are related by

$$\tilde{V}_n - \tilde{V}_m = \tilde{Z}_{n,m} \tilde{I}_{n,m} - \tilde{V}^{(s)}_{n,m} \tag{2.10}$$

where $\tilde{V}^{(s)}_{n,m}$ is a voltage source along branch $B_{n,m}$ (and increases from n to m) and $\tilde{Z}_{n,m}$ is some impedance (assumed linear) on the same branch. In a degenerate case the branch current might be specified by a current source. For a closed loop, $\tilde{V}_n - \tilde{V}_n \equiv 0$, and (2.10) becomes

$$\tilde{Z}_{n,m_1} \tilde{I}_{n,m_1} + \tilde{Z}_{m_1,m_2} \tilde{I}_{m_1,m_2} + \ldots + \tilde{Z}_{m_i,n} \tilde{I}_{m_i,n} - (\tilde{V}^{(s)}_{n,m_1} + \tilde{V}^{(s)}_{m_1,m_2} + \ldots + \tilde{V}^{(s)}_{m_i,n}) = 0 \tag{2.11}$$

Appropriate applications of (2.9) and (2.11) to a given circuit yield the network equations. There are many forms of network equations which are derived according to the cut sets or tie sets chosen [2.1]. It is not intended to go into this subject here, but instead one form of the network equations for the example illustrated in Fig. 2.1 is given. Currents are labeled by single subscripts in the same way as the single-subscripted branch numbers. Application of (2.9) to N_1, N_2 and N_3 yields

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ \tilde{I}_3 \\ \tilde{I}_4 \\ \tilde{I}_5 \end{pmatrix} = (0_n) \tag{2.12}$$

For the mesh containing branches B_1, B_2, B_3 and the mesh containing B_3, B_4, B_5, (2.11) becomes

$$\begin{pmatrix} \tilde{Z}_{1,2} & -\tilde{Z}_{1,3} & \tilde{Z}_{2,3} & 0 & 0 \\ 0 & 0 & \tilde{Z}_{2,3} & -\tilde{Z}_{2,4} & \tilde{Z}_{3,4} \end{pmatrix} \cdot \begin{pmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ \tilde{I}_3 \\ \tilde{I}_4 \\ \tilde{I}_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}^{(s)}_{1,2} \\ \tilde{V}^{(s)}_{1,3} \\ \tilde{V}^{(s)}_{2,3} \\ \tilde{V}^{(s)}_{2,4} \\ \tilde{V}^{(s)}_{3,4} \end{pmatrix} \tag{2.13}$$

For dimensional consistency, using an arbitrary impedance $\tilde{Z}_{ref}$, combination of (2.12) and (2.13) gives one form of the network equations, viz.,

$$\begin{pmatrix} \tilde{Z}_{ref} & \tilde{Z}_{ref} & 0 & 0 & 0 \\ -\tilde{Z}_{ref} & 0 & \tilde{Z}_{ref} & \tilde{Z}_{ref} & 0 \\ 0 & -\tilde{Z}_{ref} & -\tilde{Z}_{ref} & 0 & \tilde{Z}_{ref} \\ \tilde{Z}_{1,2} & -\tilde{Z}_{1,3} & \tilde{Z}_{2,3} & 0 & 0 \\ 0 & 0 & \tilde{Z}_{2,3} & -\tilde{Z}_{2,4} & \tilde{Z}_{3,4} \end{pmatrix} \cdot \begin{pmatrix} \tilde{I}_1 \\ \tilde{I}_2 \\ \tilde{I}_3 \\ \tilde{I}_4 \\ \tilde{I}_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \tilde{V}^{(s)}_{1,2} \\ \tilde{V}^{(s)}_{1,3} \\ \tilde{V}^{(s)}_{2,3} \\ \tilde{V}^{(s)}_{2,4} \\ \tilde{V}^{(s)}_{3,4} \end{pmatrix} \tag{2.14}$$

The currents are readily obtained by inverting the 5×5 matrix.

It is the purpose of this note to present similar network equations for transmission-line networks. These equations are complicated by the wave nature of the voltages and currents, and their dependence on positions and modal properties.

C. Transmission-Line Networks

Concepts similar to those of circuit topology are developed for transmission-line networks to help summarize the network configurations, to define topology matrices and to set up network equations.

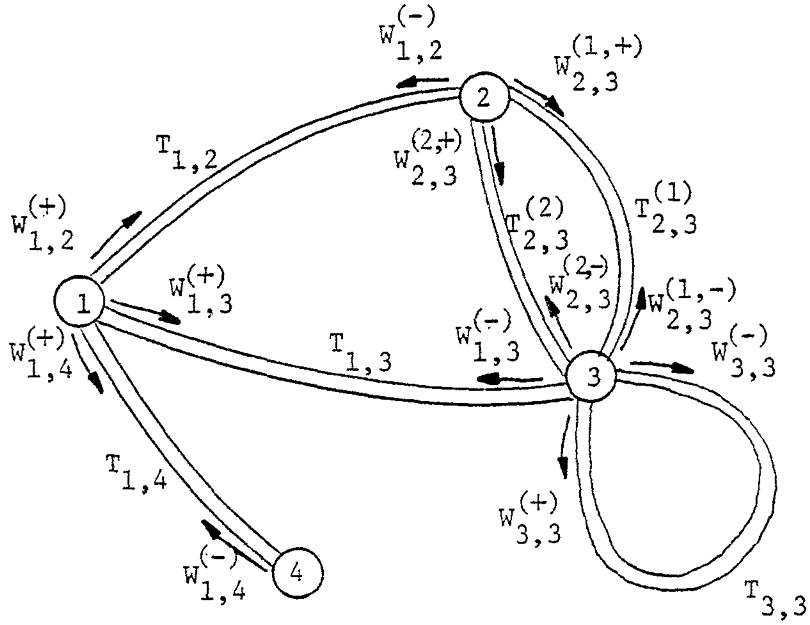
The basic elements of the transmission-line network graphs are tubes and junctions. A tube is a collection of wires characterized by two ends to which electrical connections can be made. A junction is where wires terminate. Usually a bundle of wires is considered as a tube which may be terminated by a circuit. Branching of a bundle of wires can be considered as a tube divided into a few tubes with the position of branching as a junction within which only direct electrical connections occur.

The graphic symbols for tubes and junctions are respectively "parallel" lines and circles. The νth junction is denoted as J_ν for $\nu = 1,\ldots,N_J$ where N_J is the number of junctions in the network. For transmission-line networks, it is possible to have more than one tube between two junctions. The pth tube between junctions J_ν and $J_{\nu'}$ is labeled $T^{(p)}_{\nu,\nu'}$. If there is only one tube between J_ν and $J_{\nu'}$, the simplified notation $T_{\nu,\nu'}$ $(=T^{(1)}_{\nu,\nu'})$ is often used.

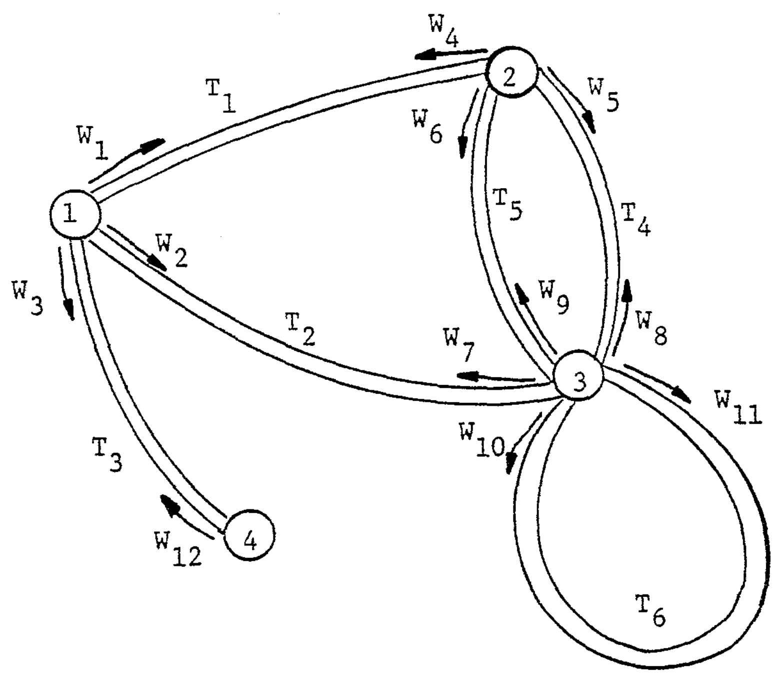
Each tube can be characterized by two sets of waves: the forward traveling wave and the backward traveling wave. The waves on the pth tube between junctions J_ν and $J_{\nu'}$ are labeled $W^{(p,+)}_{\nu,\nu'}$ and $W^{(p,-)}_{\nu,\nu'}$, where $W^{(p,+)}_{\nu,\nu'}$ travels from J_ν to $J_{\nu'}$ and $W^{(p,-)}_{\nu,\nu'}$ travels from $J_{\nu'}$ to J_ν. Thus, $W^{(p,+)}_{\nu,\nu'} = W^{(p,-)}_{\nu',\nu}$. There are thus two waves traveling in opposite directions on a given tube.

As an example, a transmission-line graph is shown in fig. 2.2a. There are four junctions and six tubes. The tubes and waves are numbered with double subscripts according to the rules outlined above. The parallel tubes and the self tube* are unique for transmission-line networks as there are no corresponding elements for the circuits. Topology matrices similar to those used for lumped circuits can be defined here involving

*A self tube is one that has both ends terminating in the same junction.



(a) Double-subscripted



(b) Single-subscripted

Fig. 2.2 A transmission line graph with (a) double-subscripted numbering and (b) single-subscripted numbering junctions, tubes and waves. Specifically, there are six useful interconnection matrices: junction-junction (or junction interconnection), junction-tube, tube-tube (or tube interconnection), junction-wave, wave-wave (or wave interconnection), and tube-end-wave.

For a transmission-line network with N_J junctions, the junction-junction interconnection matrix $(t_{\nu,\nu'})_{J-J}$ is an $N_J \times N_J$ matrix. The elements are defined as:

$\nu \neq \nu'$	$t_{\nu,\nu';J-J}$ = number of tubes connecting junctions J_ν and J_ν
	$t_{\nu,\nu';J-J} = 0$ implies no connection between the two junctions
$\nu = \nu'$	$t_{\nu,\nu;J-J} = 1$ denotes self connection (since a junction is always connected to itself)
	$t_{\nu,\nu;J-J} > 1$ denotes existence of self tubes
	$= 1 + 2 \times$(number of self tubes)

(2.15)

For the example in Fig. 2.2a, the junction-junction matrix is

$$(t_{\nu,\nu'})_{J-J} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{2.16}$$

Similarly, the junction-tube interconnection matrix $(t_{\nu,n})_{J-T}$ has elements defined by

$$t_{\nu,n;J-T} = 1 \quad \text{if junction } J_\nu \text{ is connected to tube } T_n$$
$$= 2 \quad \text{if junction } J_\nu \text{ is connected to self tube } T_n$$
$$= 0 \quad \text{if junction } J_\nu \text{ is not connected to tube } T_n \qquad (2.17)$$

The single-subscripted denotation of the tubes T_n is useful for matrix manipulations. The numbering system is similar to that for branches in the circuit topology. Consecutive numbering starts at the first tube linking junction J_1 to the junction that has the lowest node number. (If there is a self tube at J_1 this would be first.) The tube number increases for other parallel tubes going from J_1 to the same junction until all these tubes are labeled. This process continues for tubes going to the junction with the next higher number until all junctions connected to J_1 are exhausted. The procedure continues at J_2 except for tubes which are already labeled: they are not repeated (i.e., tubes $T^{(p)}_{1,2}$ are already labeled and are left out here). The process continues until all tubes are numbered.

One may note here that the tube labeling is not oriented, i.e., $T^{(p)}_{\nu,\nu'} = T^{(p)}_{\nu',\nu}$.

For the example in Fig. 2.2b, the junction-tube matrix is:

$$(t_{\nu,n})_{J-T} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad (2.18)$$

The tube-tube interconnection matrix $(t_{n,m})_{T-T}$ is defined in the following table:

$n \neq m$	$t_{n,m;T-T}$ = number of connections between the ends of tube T_n and tube T_m = 0 implies no connections = 1 implies one end of each tube is connected = 2 implies either (i) two parallel tubes or (ii) one is a self tube = 4 implies both are self tubes
$n = m$	$t_{n,n;T-T}$ = 0 for a simple tube (normal situation) = 2 for a self tube

(2.19)

The $(t_{n,m})_{T-T}$ matrix for the case of Fig. 2.2b is

$$(t_{n,m})_{T-T} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 2 & 2 \\ 1 & 1 & 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 & 2 & 2 \end{pmatrix} \tag{2.20}$$

Each tube is also characterized by two waves. The junction-wave matrix $(t_{\nu,u})_{J-W}$ describes the waves that are incident or reflected from a junction. One defines

$$t_{\nu,u;J-W} = \begin{cases} 1 & \text{if wave } W_u \text{ is leaving junction } J_\nu \text{ (transmitted and/or reflected)} \\ 1 & \text{if wave } W_u \text{ is entering junction } J_\nu \text{ (incident)} \\ 0 & \text{if junction } J_\nu \text{ is not associated with wave } W_u \\ 2 & \text{if wave } W_u \text{ is on a self tube} \end{cases} \tag{2.21}$$

The single-subscripted denotation of a wave, W_u, is numbered similar to that of a tube. However, there are two waves on a tube, oriented to propagate in opposite directions. Thus, numbering starts at junction J_1 for a wave leaving J_1 on tube T_1 until all tubes are exhausted. Numbering continues at junction J_2 for all tubes (in ascending tube numbers), again for waves leaving J_2. This is repeated for all junctions. This results in different numbering as compared to the tubes. In fact, for N_T tubes, there are N_W waves given by

$$N_W = 2\ N_T \tag{2.22}$$

The junction-wave matrix for the example in Fig. 2.2b is

$$(t_{\nu,u})_{J-W} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.23}$$

The wave-wave matrix $(W_{u,v})$ describes interconnection of waves (via junctions). Elements are defined by

$W_{u,v} = 1$ if wave W_v scatters into wave W_u, i.e., if W_v is connected to W_u via a junction into which W_v is incoming and W_u is outgoing.

$$W_{u,v} = \begin{cases} 1 & \text{for a self tube} \\ 0 & \text{otherwise (normal situation).} \end{cases} \tag{2.24}$$

For the example in Fig. 2.2b, $(W_{u,v})$ is

$$(W_{u,v}) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.25}$$

Another matrix of interest is the tube-end-wave interconnection matrix. We denote by the index r in $J_{\nu;r}$ the tube ends reaching the junction J_ν ; then, for a wave entering J_ν via $J_{\nu;r}$, it is labeled as $J_{\nu;r,-}$, and for a wave leaving J_ν via $J_{\nu;r}$, it is labeled $J_{\nu;r,+}$.

The tube-end-wave interconnection matrix $(t_{r,u})_{\nu;E-W}$ is defined as follows:

$$t_{r,u;\nu;E-W} = \begin{cases} 0 \text{ if wave } W_u \text{ does not connect to } J_\nu \text{ via end } J_{\nu;r} \\ -1 \text{ if wave } W_u \text{ enters } J_\nu \text{ via } J_{\nu;r} \text{ (i.e., } J_{\nu;r,-}) \\ 1 \text{ if wave } W_u \text{ leaves } J_\nu \text{ via } J_{\nu;r} \text{ (i.e., } J_{\nu;r,+}) \end{cases} \tag{2.26}$$

Thus, for junction J_3 of fig. 2.2b, a new illustration is depicted in fig. 2.3. Here the tube-end-wave interconnection matrix is

$$(t_{r,u})_{\nu;E-W} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{pmatrix} \tag{2.27}$$

Note that the junction-wave interconnection matrix is formed by

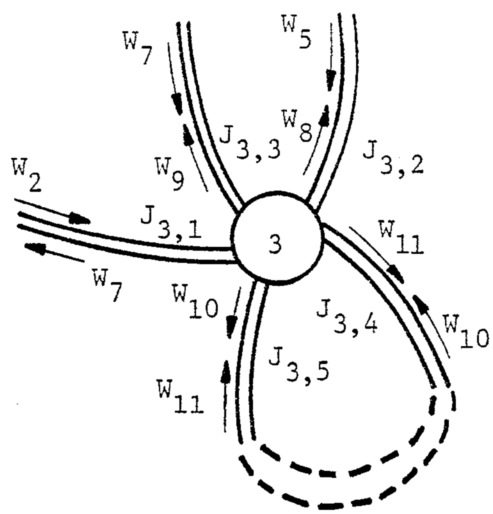


Fig. 2.3. Tube-end labeling for Junction J_3 .

$$t_{\nu,u;J-W} = \sum_{r=1}^{r_\nu} \left| t_{r,u;\nu;E-W} \right| \tag{2.28}$$

where

$$r_\nu \equiv \text{maximum value of } r$$
$$= \text{number of tube ends at } J_\nu \tag{2.29}$$
$$|\ | \equiv \text{absolute value}$$

D. Equivalent Circuits of Junctions and Tubes

Most junctions of interest either consist of physical lumped circuits or are transmission-line discontinuities modeled by lumped circuit elements [2.8 - 2.10]. Topological descriptions of a junction can thus be similar to those for a lumped circuit, as outlined in Section 2B.

Junctions can be classified according to their complexities. The simplest one involves only one tube terminated by an impedance network (including sources). This includes the special cases of open-circuited and short-circuited terminations. The voltage-current relation is given by

$$(\tilde{V}_n(s)) + (\tilde{V}_n^{(s)}(s)) = (\tilde{Z}_{n,m}(s)) \cdot [(\tilde{I}_n(s)) + (\tilde{I}_n^{(s)}(s))] \tag{2.30}$$

The dual relationship is

$$(\tilde{I}_n(s)) + (\tilde{I}_n^{(s)}(s)) = (\tilde{Y}_{n,m}(s)) \cdot [(\tilde{V}_n(s) + (\tilde{V}_n^{(s)}(s))] \tag{2.31}$$

The configuration is depicted in fig. 2.4; note that current is taken positive into the junction.

For the short-circuited case

$$(\tilde{V}_n(s)) = (0_n) \tag{2.32}$$

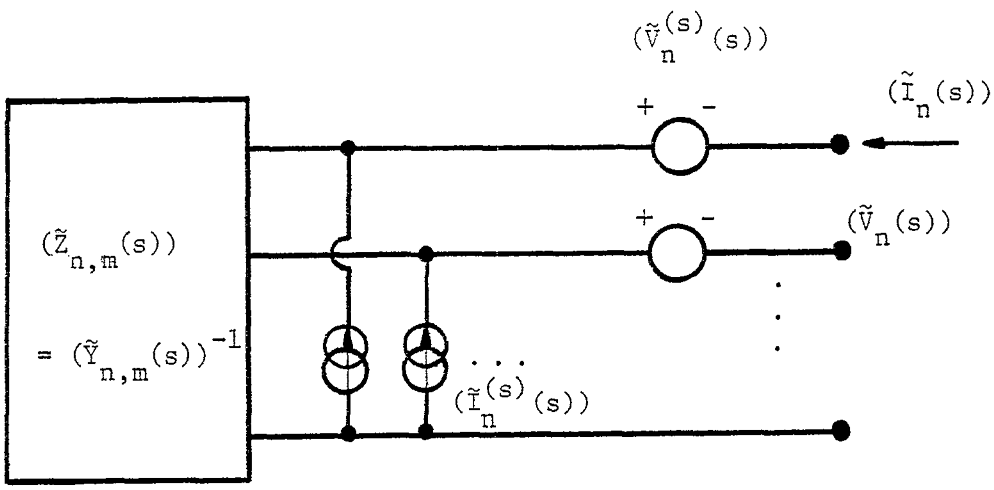


Fig. 2.4. Terminating Network

and for the open-circuited case

$$(\tilde{I}_n(s)) = (0_n) \tag{2.33}$$

A more general type of junction is one that connects to many tubes, and connections between wires of the tubes are by direct electrical contacts. This type is extremely relevant in modeling the case of branching. As is well known, branches connected at a node have equal voltages and the sum of currents leaving the node is zero. Thus, each connecting point within the junction is a node and the above voltage and current relations (2.9-11) apply [2.7].

More general forms of junctions are multitube junctions. While the equations take the same form as in (2.34,35), these equations need to be partitioned according to the different tubes (and associated waves) that connect to the junction. This partitioning is considered in some detail in Section VI; it utilizes the topology matrices discussed in the previous subsection.

A tube is characterized by its physical construction and geometry. These are in turn described by the per-unit-length quantities such as the per-unit-length impedance matrix $(\tilde{Z}'_{n,m}(s))$ and the per-unit-length admittance matrix $(\tilde{Y}'_{n,m}(s))$ for the general case, or the per-unit-length inductance matrix $(L'_{n,m})$ and the per-unit-length capacitance matrix $(C'_{n,m})$ for the lossless case.

Based on the per-unit-length series impedance $(\tilde{Z}'_{n,m}(s))$ and shunt admittance $(\tilde{Y}'_{n,m}(s))$, a per-unit-length electrical network can be developed. It illustrates the electrical model of the tube at a point on the tube. This is shown in Fig. 2.5 where for completeness two sets of distributed sources are also shown.

Note that the per-unit-length sources $(\tilde{V}_n^{(s)'})$ and $(\tilde{I}_n^{(s)'})$, as well as the voltage $(\tilde{V}'_n)$ and current $(\tilde{I}'_n)$ vectors, are functions of the coordinate z along the tube as well as the complex frequency s. The equations governing these variables on a tube are considered in detail in Section III.

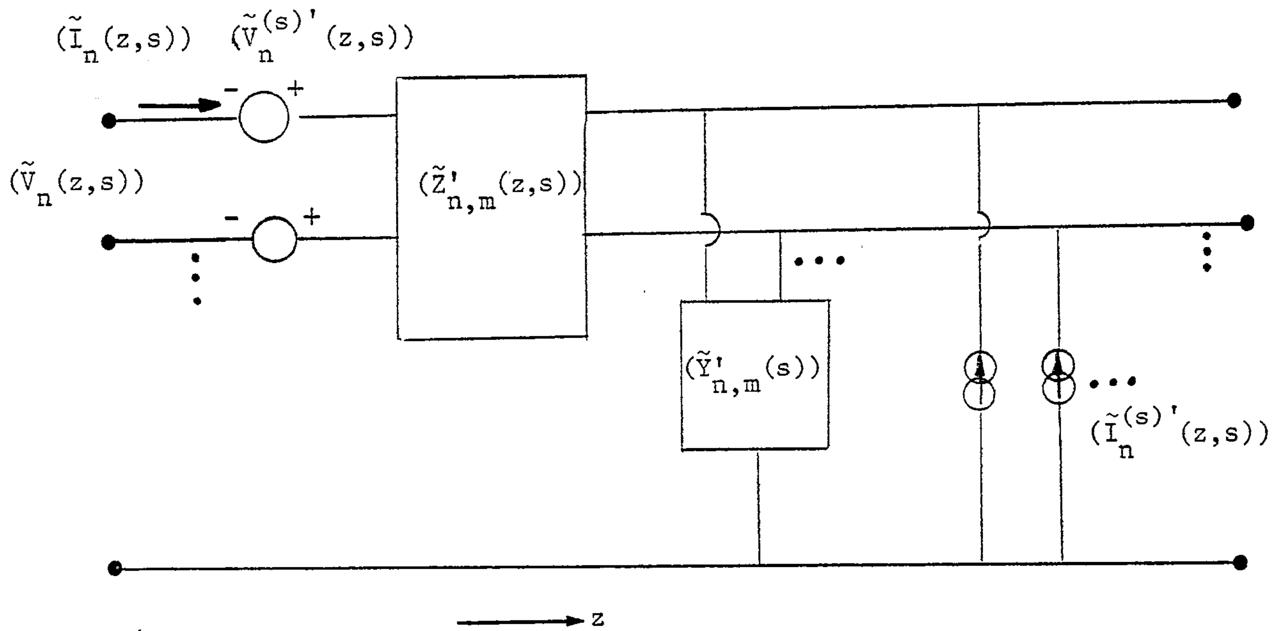


Fig. 2.5 The per-unit-length model of a multiconductor transmission line.

E. Other Related Topologies

The development of scatterer topology and the hierarchical scatterer topology is useful in dealing with scattering and penetration problems [2.2, 2.3, 2.4]. For an aircraft, missile or other systems, there are many cable bundles enclosed inside the walls of the system. The use of hierarchical scatterer topological concepts is well-suited to aid in the solution of these kinds of problems. Here one deals with the problem layer-by-layer, dividing it into many subproblems of coupling, propagation and penetration. The details are treated in the above-cited references and are not described here.

F. Topology Summary

The various types of topologies and their associated quantities are summarized in Table 2.1, which was first presented in ref. [2.3].

As mentioned earlier, the hierarchical scatterer problem can be divided into the following subproblems corresponding to the electromagnetic processes associated with each layer (principal volume) in the transport of signals into the system:

1. coupling

   This relates the response of each system layer to the electromagnetic fields coming from the layer external to the one under consideration. Quantitatively coupling can be identified with the source terms in the equations used to describe the response of the layer of interest.

2. propagation

   This deals with the distribution of signals within the layer of interest in the system. It is concerned with the operator (integral, differential, etc.) in the equations describing the response of the layer as well as the resulting response itself.

3. penetration

   This deals with the excitation of the signals in the next layer (going to the interior). Specifically penetration is concerned with the conversion of the response within a layer into an appropriate set of parameters which can be used for the coupling process in the next layer. It is then the transition from one layer to the next.

In the hierarchical decomposition of a system one or more of the layers of interest may be represented as transmission line networks, in which case the above breakdown is relevant to transmission-line problems. The above breakdown within a layer is summarized in Table 2.2.

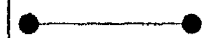
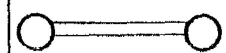
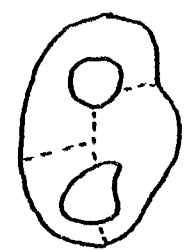
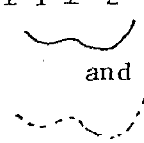
	Topology	Basic Topological Quantity and Symbol	Interconnecting Topological Quantity and Symbol	Diagramatic form of Topology
1	Graph (Generalized)	Vertex	Edge	
2	Circuit	Node N_n	Branch $B^{(T)}_{n,m}$	
3	Transmission Line	Junction J_n	Tube $T^{(T)}_{n,m}$	
4	Scatterer	Volume V_n 	Surface $S_{n,m}$ 	
5	Hierarchical Scatterer	Principal Volume $(V)^{(\lambda')}_{\lambda}$ 	Principal Surface $(S)^{(\lambda')}_{\lambda}$  (closed but sometimes in more than one part)	
		Elementary Volume $V^{(\lambda')}_{\lambda,\tau}$ 	Elementary Surface $S^{(\lambda'_1,\lambda'_2)}_{\lambda_1,\tau_1;\lambda_2,\tau_2}$ (usually open but sometimes closed) 	

Table 2.1 Various topologies

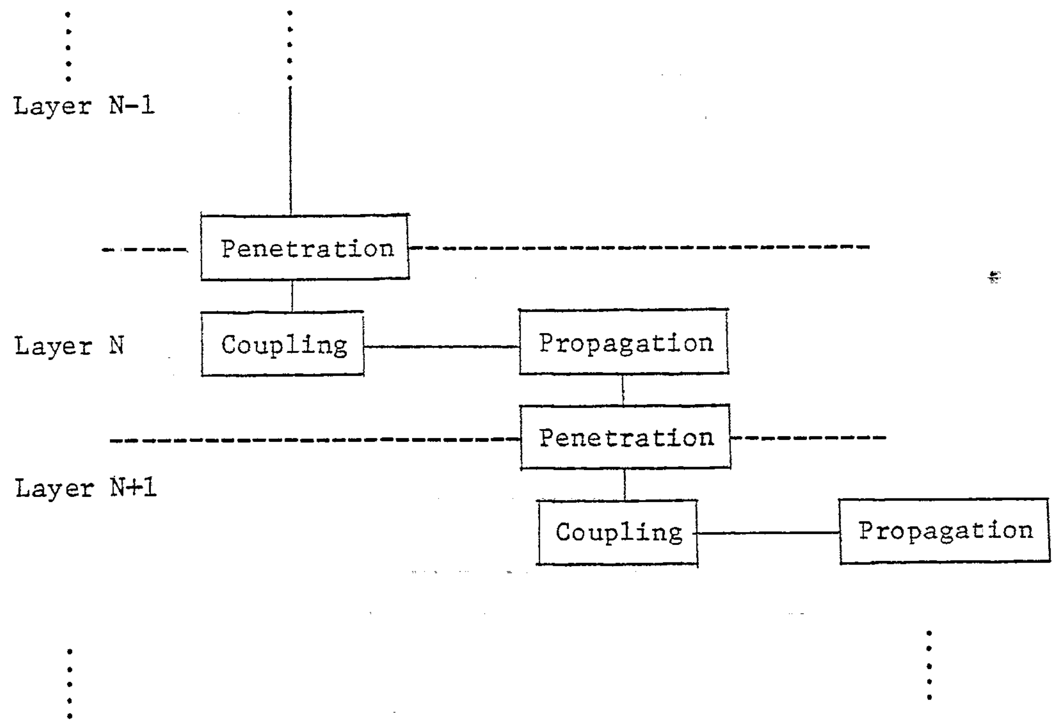


Table 2.2 Hierarchical decomposition of a system

References

2.1 See, for example, N. Balabanian, Fundamentals of Circuit Theory (Boston, Allyn and Bacon, Inc., 1962).

2.2 C.E. Baum, "How to Think About EMP Interaction," Proc. 1974 Spring FULMEN Meet., pp. 12-23, April 1974.

2.3 C.E. Baum, "The Role of Scattering Theory in Electromagnetic Interference Problems," in P.L.E. Uslenghi (ed.) Electromagnetic Scattering, Academic Press, 1978.

2.4 F.M. Tesche, M.A. Morgan, B. Fishbine and E.R. Parkinson, "Internal Interaction Analysis: Topological Concepts and Needed Model Improvements," Interaction Note 248, July 1975.

2.5 E.F. Vance, "Shielding and Grounding Topology for Interference Control," Interaction Note 306, April 1977.

2.6 F.M. Tesche, "Topological Concepts for EMP Internal Interaction," in Special Issue on the Nuclear Electromagnetic Pulse, IEEE Trans. on Antennas and Propagat., Vol. AP-26, January 1978, and IEEE Trans. on Electromagnetic Compatibility, Vol. EMC-20, February 1978.

2.7 C.E. Baum, T.K. Liu, F.M. Tesche and S.K. Chang, "Numerical Results for Multiconductor Transmission-Line Networks," Interaction Note 322, September 1977.

2.8 F.M. Tesche and T.K. Liu, "An Electric Model for a Cable Clamp," Interaction Note 307, December 1976.

2.9 S. Coen, T.K. Liu and F.M. Tesche, "Calculation of the Equivalent Capacitance of a Rib near a Single Wire Transmission Line," Interaction Note 310, February 1977.

2.10 K.S.H. Lee and F.C. Yang, "A Wire Passing by a Circular Aperture in an Infinite Ground Plane," Interaction Note 317, February 1977.

III. PROPAGATION ON AN N-WIRE TRANSMISSION LINE TUBE

In this section the phenomena of waves propagating on a single section, N-wire transmission-line tube are considered. An N-wire transmission line is one that consists of n conductors and a reference which may be infinity or ground. As will be derived later, such a system has N modes of propagation.

The equations governing the voltage and current propagation on an N-wire transmission line, i.e., the generalized transmission equations are the current change equation

$$\frac{d}{dz}(\tilde{I}_n(z,s)) = -(\tilde{Y}'_{n,m}(s))\cdot(\tilde{V}_n(z,s)) + (\tilde{I}_n^{(s)'}(z,s)) \tag{3.1}$$

and the voltage change equation

$$\frac{d}{dz}(\tilde{V}_n(z,s)) = -(\tilde{Z}'_{n,m}(s))\cdot(\tilde{I}_n(z,s)) + (\tilde{V}_n^{(s)'}(z,s)) \tag{3.2}$$

where

$$\begin{aligned}
z &= \text{position along the tube} \\
(\tilde{I}_n(z,s)) &= \text{current vector at } z \\
(\tilde{V}_n(z,s)) &= \text{voltage vector at } z \\
(\tilde{Y}'_{n,m}(s)) &= \text{per-unit-length shunt admittance matrix} \\
(\tilde{Z}'_{n,m}(s)) &= \text{per-unit-length series impedance matrix} \\
(\tilde{I}_n^{(s)'}(z,s)) &= \text{per-unit-length shunt current source vector} \\
(\tilde{V}_n^{(s)'}(z,s)) &= \text{per-unit-length series voltage source vector.}
\end{aligned} \tag{3.3}$$

It is noted that all vectors are of dimension N, and all matrices are $N \times N$. The per-unit-length equivalent circuit has been given in Figure 2.3.

There are a few ways of solving (3.1) and (3.2). One could reduce the equations to a second order differential equation in either the voltage vector or the currect vector, or one could express the voltage-current relations in terms of a transmission supermatrix [3.1]. Still another way is to solve for the unknown propagation vectors that are associated with the waves. Here, the derivation is in terms of a yet undefined combined voltage. This approach, as will be illustrated later, has many definite advantages over other methods.

A. Combined Voltage Equation

Pre-multiplying (3.1) by a matrix $(\tilde{A}_{n,m}(s))$ which is $N \times N$ and non-singular and adding (3.2), then

$$\frac{d}{dz}[\,(\tilde{A}_{n,m}(s))\cdot(\tilde{I}_n(s)) + (\tilde{V}_n(s))\,]$$

$$= [\,-\,(\tilde{A}_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))\cdot(\tilde{V}_n(s)) - (\tilde{Z}'_{n,m}(s))\cdot(\tilde{I}_n(s))]$$

$$+ [\,(\tilde{A}_{n,m}(s))\cdot(\tilde{I}_n^{(s)'}(s)) + (\tilde{V}_n^{(s)'}(s))] \qquad (3.4)$$

Defining the following quantities:

$$(\tilde{V}_n(z,s))_q \equiv (\tilde{V}_n(z,s)) + (\tilde{A}_{n,m}(s))\cdot(\tilde{I}_n(z,s)) \qquad (3.5)$$

and

$$(\tilde{V}_n^{(s)'}(z,s))_q \equiv (\tilde{V}_n^{(s)'}(z,s)) + (\tilde{A}_{n,m}(s))\cdot(\tilde{I}_n^{(s)'}(z,s)) \qquad (3.6)$$

(3.4) becomes

$$\frac{d}{dz}(\tilde{V}_n(s))_q = -\,(\tilde{C}_{n,m}(s))\cdot(\tilde{V}_n(s))_q + (\tilde{V}_n^{(s)'}(s))_q \qquad (3.7)$$

where

$$(\tilde{C}_{n,m}(s))\cdot(\tilde{V}_n(z,s))_q = (\tilde{A}_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))\cdot(\tilde{V}_n(z,s))$$

$$+ (\tilde{Z}'_{n,m}(s))\cdot(\tilde{I}_n(z,s)) \qquad (3.8)$$

Using (3.8) and definition (3.5), one obtains

$$(\tilde{C}_{n,m}(s)) = (\tilde{A}_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s)) \qquad (3.9)$$

and

$$(\tilde{C}_{n,m}(s))\cdot(\tilde{A}_{n,m}(s)) = (\tilde{Z}'_{n,m}(s)) \qquad (3.10)$$

Equating $(\tilde{A}_{n,m}(s))$ in (3.9) and (3.10)

$$(\tilde{C}_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))^{-1} = (\tilde{C}_{n,m}(s))^{-1}\cdot(\tilde{Z}'_{n,m}(s)) \tag{3.11}$$

i.e.

$$(\tilde{C}_{n,m}(s))^{2} = (\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s)) \tag{3.12}$$

One can also write

$$(\tilde{C}_{n,m}(s)) = [(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))]^{1/2} \tag{3.13}$$

which has many values. One may define a principal value (or matrix)* $(\tilde{\gamma}_{c_{n,m}}(s))$, i.e.

$$(\tilde{\gamma}_{c_{n,m}}(s)) = \text{principal value of } [(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))]^{1/2} \tag{3.14}$$

$(\tilde{\gamma}_{c_{n,m}}(s))$ is called the propagation matrix. Expressing (3.7) in this new form, one obtains

$$[(1_{n,m})\frac{d}{dz} + q(\tilde{\gamma}_{c_{n,m}}(s))]\cdot(\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)'}(z,s))_q$$

$$1_{n,m} = \begin{cases} 1 \text{ for } n=m \\ 0 \text{ for } n\neq m \end{cases} \tag{3.15}$$

$q = \pm 1$ for forward and backward traveling combined N-vector waves, respectively.

Essentially, $(\tilde{C}_{n,m}(s))$ has been restricted to $q(\tilde{\gamma}_{c_{n,m}}(s))$ in (3.5).

From (3.9), (3.10) and (3.14)

$$\begin{aligned}(\tilde{A}_{n,m}(s)) &= q(\tilde{\gamma}_{c_{n,m}}(s))\cdot(\tilde{Y}'_{n,m}(s))^{-1} \\ &= q(\tilde{\gamma}_{c_{n,m}}(s))^{-1}\cdot(\tilde{Z}'_{n,m}(s))\end{aligned} \tag{3.16}$$

* see definition under Section III-B.

Thus, $(\tilde{A}_{n,m}(s))$ is a characteristic of the transmission line and has the dimensions of an impedance. Define

$$(\tilde{Z}_{c_{n,m}}(s)) \equiv (\tilde{\gamma}_{c_{n,m}}(s)) \cdot (\tilde{Y}'_{n,m}(s))^{-1}$$
$$= (\tilde{\gamma}_{c_{n,m}}(s))^{-1} \cdot (\tilde{Z}'_{n,m}(s)) \tag{3.17}$$

which is called the characteristic impedance matrix. Now, definitions (3.5) and (3.6) are rewritten to be

$$(\tilde{V}_n(z,s))_q \equiv (\tilde{V}_n(z,s)) + q(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s))$$
$$(\tilde{V}_n^{(s)'}(z,s))_q \equiv (\tilde{V}_n^{(s)'}(z,s)) + q(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n^{(s)'}(z,s)) \tag{3.18}$$

One can also define the characteristic admittance matrix to be the inverse of the characteristic impedance matrix, viz.

$$(\tilde{Y}_{c_{n,m}}(s)) \equiv (\tilde{Z}_{c_{n,m}}(s))^{-1} \tag{3.19}$$

Putting $q = +1$ and $q = -1$ in (3.18), one can obtain the following relations

$$(\tilde{V}_n(z,s)) = \frac{1}{2}[(\tilde{V}_n(z,s))_+ + (\tilde{V}_n(z,s))_-] \tag{3.20}$$

$$(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s)) = \frac{1}{2}[(\tilde{V}_n(z,s))_+ - (\tilde{V}_n(z,s))_-] \tag{3.21}$$

Thus, for forward traveling waves

$$(\tilde{V}_n(z,s))_+ = (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s))_+ \tag{3.22}$$

and for backward traveling waves

$$(\tilde{V}_n(z,s))_- = -(\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(z,s))_- \tag{3.23}$$

This is an important result which allows one to easily separate the voltage and current vectors into forward and backward waves and easily reconstruct the voltage and current vectors from the waves.

B. Eigenmode Expansion

Expression (3.14) for $(\tilde{Y}_{c_{n,m}}(s))$ clearly indicates the necessity of eigenmode expansion of the matrix product $(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))$. The eigenvalues would yield the values of the propagation constants of the eigenmodes. Corresponding to each eigenmode, there is a complete set of eigenvectors for the combined voltage. These properties are examined in this section.

1. Positive real properties

a. Definitions

A rational function f(s) which is real for real values of s and whose real part is positive for all values of s with a positive real part is called a positive real function (p. r. function) [3.2]. A positive real matrix is one whose eigenvalues are all p.r. functions. Let $(\tilde{P}_{n,m}(s))$ be a p.r. matrix of size $N\times N$, then the eigenvalue problem becomes

$$(\tilde{P}_{n,m}(s))\cdot(\tilde{P}_n^{(r)}(s))_\delta = \tilde{P}_\delta(s)(\tilde{P}_n^{(r)}(s))_\delta$$

$$(\tilde{P}_n^{(\ell)}(s))_\delta\cdot(\tilde{P}_{n,m}(s)) = \tilde{P}_\delta(s)(\tilde{P}_n^{(\ell)}(s))_\delta \tag{3.24}$$

where $\delta = 1,2,\ldots,N$ is the eigenindex and $\tilde{p}_\delta(s)$ are the eigenvalues, and $(\tilde{P}_n^{(r)}(s))_\delta$ and $(\tilde{P}_n^{(\ell)}(s))_\delta$ are the eigenvectors. Since $(\tilde{P}_{n,m}(s))$ is a p.r. matrix, $\tilde{p}_\delta(s)$ is a p.r. function of s. If $\tilde{p}_\delta$ is independent of s, then $\tilde{p}_\delta$ is real and $\tilde{p}_\delta \geq 0$ for all δ. For the oft encountered case of symmetric p.r. matrices we can set

$$(\tilde{P}_n^{(r)}(s))_\delta \equiv (\tilde{P}_n^{(\ell)}(s))_\delta$$

$$(\tilde{P}_{n,m}(s)) = (\tilde{P}_{n,m}(s))^T \tag{3.25}$$

b. Eigenmode expansion

A p.r. matrix can be expanded as follows [3.3]:

$$(\tilde{P}_{n,m}(s)) = \sum_{\delta} \tilde{p}_{\delta}(s)(P_n^{(r)}(s))_{\delta}(\tilde{P}_n^{(\ell)}(s))_{\delta}[(\tilde{P}_n^{(r)}(s))_{\delta}\cdot(\tilde{P}_n^{(\ell)}(s))_{\delta}]^{-1} \qquad (3.26)$$

If the matrix is the argument of a scalar function F , then

$$F[(\tilde{P}_{n,m}(s))] = \sum_{\delta} F[\tilde{p}_{\delta}(s)](\tilde{P}_n^{(r)}(s))_{\delta}(\tilde{P}_n^{(\ell)}(s))_{\delta}[(\tilde{P}_n^{(r)}(s))_{\delta}\cdot(\tilde{P}_n^{(\ell)}(s))_{\delta}]^{-1} \qquad (3.27)$$

One assumes that the scalar function F is single-valued for all complex $\tilde{p}_{\delta}(s)$. Otherwise the principal value of $F[p_{\delta}(s)]$ is defined to define the principal value (or matrix) of $F[(\tilde{P}_{n,m}(s))]$.

For example, powers of the matrix can be expressed as

$$(\tilde{P}_{n,m}(s))^{\varepsilon} = \sum_{\delta} [\tilde{p}_{\delta}(s)]^{\varepsilon}(\tilde{P}_n^{(r)}(s))_{\delta}(\tilde{P}_n^{(\ell)}(s))_{\delta}[(\tilde{P}_n^{(r)}(s))_{\delta}\cdot(\tilde{P}_n^{(\ell)}(s))_{\delta}]^{-1} \qquad (3.28)$$

where for $Re(s) > 0$ and $Im(\varepsilon) = 0$, $[\tilde{p}(s)]^{\varepsilon} > 0$ defines the principal value of the ε-th power of a p.r. matrix.

c. Normalized eigenvectors

The normalized eigenvector is defined by

$$(\tilde{p}_n^{\binom{r}{\ell}}(s))_{\delta} = \frac{(\tilde{P}_n^{\binom{r}{\ell}}(s))_{\delta}}{\sqrt{(\tilde{P}_n^{(r)}(s))_{\delta}\cdot(\tilde{P}_n^{(\ell)}(s))_{\delta}}} \qquad (3.29)$$

In terms of the normalized quantities, (3.26) and (3.27) become

$$\begin{aligned}(\tilde{P}_{n,m}(s)) &= \sum_{\delta} \tilde{p}_{\delta}(s)(\tilde{p}_n^{(r)}(s))_{\delta}(\tilde{p}_n^{(\ell)}(s))_{\delta} \\ F[(\tilde{P}_{n,m}(s))] &= \sum_{\delta} F[\tilde{p}_{\delta}(s)](\tilde{p}_n^{(r)}(s))_{\delta}(\tilde{p}_n^{(\ell)}(s))_{\delta}\end{aligned} \qquad (3.30)$$

Note for symmetric p.r. matrices the normalized eigenvectors can be reduced to

$$(\tilde{p}_n^{(r)}(s))_\delta = (\tilde{p}_n^{(\ell)}(s))_\delta \equiv (\tilde{p}_n(s))_\delta \tag{3.31}$$

d. Transmission line p.r. matrices

Assume that the per-unit-length impedance and admittance matrices $(Z'_{n,m}(s))$, $(Y'_{n,m}(s))$ are passive. Then they are p.r. matrices.

In the special case of a lossless transmission line,

$$\begin{aligned}(Z'_{n,m}(s)) &= s(L'_{n,m})\\ (Y'_{n,m}(s)) &= s(C'_{n,m})\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}(L'_{n,m}) &\equiv \text{per-unit-length inductance matrix}\\ (C'_{n,m}) &\equiv \text{per-unit-length capacitance matrix.}\end{aligned} \tag{3.33}$$

which we also assume to be frequency independent (dispersionless) and, hence, constant matrices. The elements $L'_{n,m}$ and $C'_{n,m}$ are real, being derivable from quasi-static boundary-value problems (Laplace equation). The p.r. property of the per-unit-length impedance and admittance matrices then implies that the per-unit-length inductance and capacitance matrices are both p.r. and positive semidefinite. Thus, $(L'_{n,m})$ and $(C'_{n,m})$ have real non-negative eigenvalues.

If we further assume that $(L'_{n,m})$ is symmetric, then

$$\begin{aligned}(L'_{n,m})\cdot(L'_n)_\delta &= \ell'_\delta\ (L'_n)_\delta\\ (L'_n)_\delta(L'_{n,m}) &= \ell'_\delta\ (L'_n)_\delta\end{aligned} \tag{3.34}$$

In terms of the normalized eigenvectors $(\ell'_n)_\delta$, $(\tilde{Z}'_{n,m}(s))$ is expressible as

$$(\tilde{Z}'_{n,m}(s)) = s\sum_\delta \ell'_\delta\ (\ell'_n)_\delta(\ell'_n)_\delta \tag{3.35}$$

Similar expressions exist for symmetric $(\tilde{Y}'_{n,m}(s))$ and $(C'_{n,m})$.

2. Propagation matrix $(\tilde{\gamma}_{c_{n,m}}(s))$

The squared quantity of the propagation matrix is equal to the product of two matrices, each of which is typically symmetric (reciprocity), i.e.,

$$(\tilde{\gamma}_{c_{n,m}}(s))^2_V = (\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s)) \tag{3.36}$$

Let $(\tilde{V}_{c_n}(s))_\delta$ be the right eigenvector of $(\tilde{\gamma}_{c_{n,m}}(s))^2_V$

$$(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))\cdot(\tilde{V}_{c_n}(s))_\delta = \tilde{\gamma}^2_{V_\delta}(s)(\tilde{V}_{c_n}(s))_\delta \tag{3.37}$$

The corresponding quantity for the combined current mode is given by

$$(\tilde{\gamma}_{c_{n,m}}(s))^2_I = (\tilde{Y}'_{n,m}(s))\cdot(\tilde{Z}'_{n,m}(s)) \tag{3.38}$$

Let $(\tilde{I}_{c_n}(s))_\delta$ be the left eigenvector of $(\tilde{\gamma}_{c_{n,m}}(s))^2_V$

$$(\tilde{I}_{c_n}(s))_\delta\cdot(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s)) = \tilde{\gamma}^2_{I_\delta}(s)(\tilde{I}_{c_n}(s))_\delta \tag{3.39}$$

In this note, the following simplified notation is used:

$$(\tilde{\gamma}_{c_{n,m}}(s)) \equiv (\tilde{\gamma}_{c_{n,m}}(s))_V \tag{3.40}$$

Defining the normalized eigenvectors by

$$(\tilde{v}_{c_n}(s))_\delta = \frac{(\tilde{V}_{c_n}(s))_\delta}{\sqrt{(\tilde{V}_{c_n}(s))_\delta\cdot(\tilde{I}_{c_n}(s))_\delta}} \tag{3.41}$$

and

$$(\tilde{i}_{c_n}(s))_\delta = \frac{(I_{c_n}(s))_\delta}{\sqrt{(\tilde{V}_{c_n}(s))_\delta \cdot (I_{c_n}(s))_\delta}} \tag{3.42}$$

it is possible to expand the squared propagation matrix into

$$(\tilde{\gamma}_{c_{n,m}}(s))^2 = \sum_\delta \gamma_\delta^2(s)(\tilde{v}_{c_n}(s))(\tilde{i}_{c_n}(s)) \tag{3.43}$$

and using (3.31) the propagation matrix is

$$(\tilde{\gamma}_{c_{n,m}}(s)) = \sum_\delta \tilde{\gamma}_\delta(s)(\tilde{v}_{c_n}(s))(\tilde{i}_{c_n}(s)) \tag{3.44}$$

where $\tilde{\gamma}_\delta(s)$ is the principal value of $[\gamma_\delta^2(s)]^{1/2}$.

Here principal value means for

$$\tilde{f}(s) \geq 0 \qquad \text{for} \quad s \geq 0 \tag{3.45}$$

and then

$$\tilde{f}^{1/2}(s) \geq 0 \qquad \text{for} \quad s \geq 0 \tag{3.46}$$

with analytic continuation away from the $s \geq 0$ axis. We then assume

$$\tilde{\gamma}_\delta^2(s) \geq 0 \qquad \text{for} \quad s \geq 0 \tag{3.47}$$

so that we may choose

$$\tilde{\gamma}_\delta(s) \geq 0 \qquad \text{for} \quad s \geq 0\ ,\ \delta = 1,2,\ldots,N \tag{3.48}$$

We further assume that the $\tilde{\gamma}_\delta(s)$ are p.r. functions so that

$$\left.\begin{array}{ll} \mathrm{Re}[\tilde{\gamma}_\delta(s)] \geq 0 & \text{for} \quad s \geq 0 \\ \tilde{\gamma}_\delta(s) \ \text{analytic for} \ s > 0 & \end{array}\right\} \cdot \quad \delta = 1,2,\ldots,N \tag{3.49}$$

Note that $\tilde{\gamma}_\delta(s)$ having this property corresponds to a + or right-going wave, since a positive real part corresponds to an attenuation in the + direction. A p.r. propagation constant is then a causal propagation constant. However, other more general forms are perhaps possible. For present purposes, p.r. propagation constants are assumed.

The necessity of choosing which square root to use for the propagation matrix is potentially troublesome. The matrix transmission-line equations may have buried in them certain mathematical problems, such as related to existence and uniqueness of solutions, representation of solutions, etc. The diagonalization of the square of the propagation matrix may depend on certain properties of the per-unit-length impedance and admittance matrices; this in turn influences the nature of the square root of the square of the propagation matrix. The problems in choice of the propagation constants may lead to some restrictions to situations that such choices are applicable or even possible. There are then some open questions requiring further research.

With the above definitions we obtain two sets of waves propagating in opposite directions along z. For all modes we have

$$\begin{aligned} &\exp[-(\tilde{\gamma}_{c_{n,m}}(s))z] \quad \text{is + propagating} \\ &\exp[(\tilde{\gamma}_{c_{n,m}}(s))z] \quad \text{is - propagating} \end{aligned} \tag{3.50}$$

For a function F of $(\tilde{\gamma}_{c_{n,m}}(s))$, assuming nondegenerate modes, one can express

$$F[(\tilde{\gamma}_{c_{n,m}}(s))] = \sum_\delta F[\tilde{\gamma}_\delta(s)]\ (\tilde{v}_{c_n}(s))(\tilde{i}_{c_n}(s)) \tag{3.51}$$

Specifically, we have

$$
\begin{aligned}
(\tilde{\gamma}_{c_{n,m}}(s)) &= \sum_{\delta} \tilde{\gamma}_{\delta}(s)(\tilde{v}_{c_n}(s))_{\delta}(\tilde{i}_{c_n}(s))_{\delta} \\
(\tilde{\gamma}_{c_{n,m}}(s))^{-1} &= \sum_{\delta} \tilde{\gamma}_{\delta}^{-1}(s)(\tilde{v}_{c_n}(s))_{\delta}(\tilde{i}_{c_n}(s))_{\delta} \\
e^{-q(\gamma_{c_{n,m}}(s))z} &= \sum_{\delta} e^{-q\tilde{\gamma}_{\delta}(s)z}(\tilde{v}_{c_n}(s))_{\delta}(\tilde{i}_{c_n}(s))_{\delta} \\
(\tilde{\gamma}_{c_{n,m}}(s))^{0} &= (1_{n,m}) = \sum_{\delta} (\tilde{v}_{c_n}(s))_{\delta}(\tilde{i}_{c_n}(s))_{\delta} \quad \text{(identity)}
\end{aligned} \tag{3.52}
$$

3. Properties of $(\tilde{v}_{c_n}(s))_{\delta}$, $(\tilde{i}_{c_n}(s))_{\delta}$

The two normalized eigenvectors as defined in (3.36), (3.37), (3.41), and (3.42) possess unique properties, which are exploited in this section.

Rewriting (3.36) and (3.37)

$$
\begin{aligned}
(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))\cdot(\tilde{v}_{c_n}(s))_{\delta} &= \tilde{\gamma}_{\delta}^{2}(s)(\tilde{v}_{c_n}(s))_{\delta} \\
(\tilde{i}_{c_n}(s))_{\delta'}\cdot(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s)) &= \tilde{\gamma}_{\delta'}^{2}(s)(i_{c_n}(s))_{\delta'}
\end{aligned} \tag{3.53}
$$

First, premultiply the first by $(\tilde{i}_{c_n}(s))_{\delta'}$, then postmultiply the second by $(\tilde{v}_{c_n}(s))_{\delta}$ (both in dot product sense). The difference of these two new equations becomes

$$
\begin{aligned}
&(\tilde{i}_{c_n}(s))_{\delta'}\cdot[(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s)) - (\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))]\cdot(\tilde{v}_{c_n}(s))_{\delta} \\
&= [\tilde{\gamma}_{\delta}^{2}(s) - \tilde{\gamma}_{\delta'}^{2}(s)]\ (\tilde{i}_{c_n}(s))_{\delta'}\cdot(\tilde{v}_{c_n}(s))_{\delta} \\
&= 0
\end{aligned} \tag{3.54}
$$

There are two possible cases. First, if $\gamma_{\delta}^{2} \neq \gamma_{\delta'}^{2}$ then

$$
(\tilde{i}_{c_n}(s))_{\delta'}\cdot(\tilde{v}_{c_n}(s))_{\delta} = 1_{\delta,\delta'} \tag{3.55}
$$

where

$$1_{\delta,\delta'} = \begin{cases} 0 & \text{for} \quad \delta \neq \delta' \\ 1 & \text{for} \quad \delta = \delta' \end{cases} \tag{3.56}$$

are elements of the $N \times N$ identity matrix $(1_{\delta,\delta'})$ (or Kronecker delta). Equation (3.55) is called the biorthonormal relation; $(\tilde{i}_{c_n}(s))_\delta$ and $(\tilde{v}_{c_n}(s))_\delta$ are the biorthonormal eigenvectors. Second, if $\gamma_\delta^2 = \gamma_\delta'^2$, i.e., the degenerate case, the orthonormal vectors are constructed by other means such as the Gram-Schmidt procedure.

C. Solution of Combined Voltage Equations

1. Integration of combined voltage equation

The combined differential equation (3.15), i.e.,

$$\frac{d}{dz} (\tilde{V}_n(s))_q + q(\tilde{\gamma}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)'}(z,s))_q \tag{3.57}$$

can be readily solved [3.4] to give

$$(\tilde{V}_n(z,s))_q = \exp\{-q(\tilde{\gamma}_{c_{n,m}}(s))[z-z_0]\} \cdot (\tilde{V}_n(z_0,s))_q + \int_{z_0}^{z} \exp\{-q(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_q \, dz' \tag{3.58}$$

For a + wave (i.e., a wave propagating in the + z direction), let us assume that $(\tilde{V}_n(0,s))_+$ is specified, giving

$$(\tilde{V}_n(z,s))_+ = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z\} \cdot (\tilde{V}_n(0,s))_+ + \int_0^z \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_+ \, dz' \tag{3.59}$$

Similarly for a − wave with $(\tilde{V}_n(L,s))_-$ assumed specified, we have

$$(\tilde{V}_n(z,s))_- = \exp\{(\tilde{\gamma}_{c_{n,m}}(s))[z-L]\}\cdot(\tilde{V}_n(L,s))_- + \int_L^z \exp\{(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\}\cdot(\tilde{V}_n^{(s)'}(z',s))_-\,dz' \tag{3.60}$$

These results illustrate one aspect of the simplification introduced by the combined voltage in that the + wave depends only on the left boundary condition and the - wave depends only on the right boundary condition in a very compact way. Note that for the minus wave if we replace z by $L-z$ as the coordinate variable, then the - wave has precisely the same form as the + wave, which one would expect by symmetry.

Using (3.52), equation (3.58) can be written in terms of eigenmodes, i.e.,

$$\begin{aligned}(\tilde{V}_n(z,s))_q &= \sum_\delta \Big\{ e^{-q\tilde{\gamma}_\delta(s)[z-z_0]}[(\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{V}_n(z_0,s))_q](\tilde{v}_{c_n}(s))_\delta \\ &\quad + \int_{z_0}^z e^{-q\tilde{\gamma}_\delta(s)(z-z_0)}[(\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{V}_n^{(s)'}(z',s))]dz'\Big\}(\tilde{v}_{c_n}(s))_\delta \\ &= \sum_\delta [(\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{V}_n(z,s))_q]\ (\tilde{v}_{c_n}(s))_\delta \end{aligned} \tag{3.61}$$

Define coefficients of expansion as

$$\tilde{C}_{V_{\delta,q}}(z,s) = (\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{V}_n(z,s))_q \tag{3.62}$$

For a + wave, i.e., one that travels from $z=0$ to $z=L$ along the transmission line, let us assume that $(\tilde{V}_n(0,s))_+$ is specified. Then

$$(\tilde{V}_n(z,s))_+ = \sum_\delta C_{V_{\delta,+}}(z,s)(\tilde{v}_{c_n}(s))_\delta \tag{3.63}$$

and

$$C_{V_{\delta,+}}(z,s) = (\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{V}_n(z,s))_+$$

$$= e^{-\tilde{\gamma}_\delta(s)z} [(\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{V}_n(0,s))_+]$$

$$+ \int_0^z e^{-\tilde{\gamma}_\delta(s)(z-z')} [(\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{V}_n^{(s)'}(z',s))_+] \, dz' \qquad (3.64)$$

Similarly for a - wave with $(\tilde{V}_n(L,s))_-$ specified

$$(\tilde{V}_n(z,s))_- = \sum_\delta C_{V_{\delta-}}(z,s)(\tilde{v}_{c_n}(s))_\delta \qquad (3.65)$$

and

$$C_{V_{\delta,-}}(z,s) = (\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{V}_n(z,s))_-$$

$$= e^{\tilde{\gamma}_\delta(s)(z-L)} [(\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{V}_n(L,s))_-]$$

$$+ \int_L^z e^{\tilde{\gamma}_\delta(s)(z-z')} [(\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{V}_n^{(s)'}(z',s))_-] \, dz' \qquad (3.66)$$

Equations (3.63) and (3.65) show that there are 2N eigenwaves for a N-wire transmission line (plus a reference). These waves are characterized by $\tilde{C}_{V_{\delta,+}}(\tilde{v}_{c_n}(s))_\delta$ and $\tilde{C}_{V_{\delta,-}}(\tilde{v}_{c_n}(s))_\delta$, $\delta = 1,2,\ldots,N$. One could define an eigenmatrix as follows

$$(\tilde{E}_{n,m}(s))_V = ((\tilde{v}_{c_n}(s))_1 \, , \, (\tilde{v}_{c_n}(s))_2 \, , \ldots, \, (\tilde{v}_{c_n}(s))_N) \qquad (3.67)$$

where the columns are the voltage eigenvectors. The eigenmode coefficient vector is defined by

$$(\tilde{C}_{V_n}(z,s))_q = (\tilde{C}_{V_{1,q}}(z,s) \, , \, \tilde{C}_{V_{2,q}}(z,s) \, , \ldots, \, \tilde{C}_{V_{N,q}}(z,s)) \qquad (3.68)$$

Equation (3.61) can be rewritten as

$$(V_n(z,s))_q = (E_{n,m}(s))_V \; (C_{V_n}(z,s))_q \tag{3.69}$$

2. Semi-infinite transmission line

Many transmission-line properties can be learned by studying the semi-infinite line where the complications due to reflections do not exist. As discussed earlier, the per-unit-length electrical model of a transmission line is given in Figure 2.5.

Assuming that $(\tilde{V}_n(0,s))_+$ is given and there are no other sources along the line so that only + waves propagate, (3.59) gives

$$(\tilde{V}_n(z,s))_+ = \exp[-(\tilde{\gamma}_{c_{n,m}}(s))z] \cdot (\tilde{V}_n(0,s))_+ \tag{3.70}$$

At $z = 0$,

$$\begin{aligned} &(\tilde{V}_n(0,s))_+ \text{ is specified} \\ &(\tilde{V}_n(0,s))_- = (0_n) \end{aligned} \tag{3.71}$$

from (3.19)

$$(\tilde{V}_n(0,s))_- = (\tilde{V}_n(0,s)) - (\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(0,s)) = (0_n) \tag{3.72}$$

Thus

$$(\tilde{V}_n(0,s)) = (\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(0,s)) \tag{3.73}$$

Or, from (3.20)

$$(\tilde{I}_n(0,s)) = (\tilde{Y}_{c_{n,m}}(s))\cdot(\tilde{V}_n(0,s)) \tag{3.74}$$

As is well-known, waves propagate in only one direction on a semi-infinite line driven at the one end, with voltage and current related by the characteristic impedance (Equation 3.73). Thus, the effect of a semi-infinite multiconductor transmission line can be represented by an equivalent impedance network that is equivalent to the characteristic impedance matrix $(\tilde{Z}_{c_{n,m}}(s))$.

3. Normalization relation of $(\tilde{v}_{c_n}(s))_\delta$ and $(\tilde{i}_{c_n}(s))_\delta$ in terms of $(\tilde{Z}_{c_{n,m}}(s))$ and $(\tilde{Y}_{c_{n,m}}(s))$ and associated modal expansions

For forward traveling waves only, (3.22) gives

$$(\tilde{V}_n(z,s)) = (\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(z,s)) \tag{3.75}$$

For $(\tilde{V}_n(z,s))$ chosen as a single mode, (3.63) gives

$$(\tilde{V}_n(z,s)) = C_{v_{\delta,+}}(\tilde{v}_{c_n}(s))_\delta \tag{3.76}$$

Hence

$$(\tilde{I}_n(z,s)) = C_{v_{\delta,+}}(\tilde{Y}_{c_{n,m}}(s))\cdot(\tilde{v}_{c_n}(s))_\delta \tag{3.77}$$

Therefore, the δ-th mode for the current can be normalized as

$$\begin{aligned}(\tilde{i}_{c_n}(s))_\delta &\equiv (\tilde{Y}_{c_{n,m}}(s))\cdot(\tilde{v}_{c_n}(s))_\delta \\ (\tilde{v}_{c_n}(s))_\delta &\equiv (\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{i}_{c_n}(s))_\delta\end{aligned} \tag{3.78}$$

Together with (3.55) this specifies $(\tilde{v}_{c_n}(s))$ and $(\tilde{i}_{c_n}(s))$ including their units.

For nondegenerate modes (3.53), (3.55), and (3.78), give

$$
\begin{aligned}
(\tilde{v}_{c_n}(s))_\delta \cdot (\tilde{i}_{c_n}(s))_{\delta'} &= (\tilde{v}_{c_n}(s))_\delta \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{v}_{c_n}(s))_{\delta'} = 1_{\delta,\delta'} \\
(\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{v}_{c_n}(s))_{\delta'} &= (\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{i}_{c_n}(s))_{\delta'} = 1_{\delta,\delta'}
\end{aligned} \tag{3.79}
$$

For the first of these equations left dyadic by $(\tilde{v}_{c_n}(s))_\delta$ and right dyadic multiply by $(\tilde{i}_{c_n}(s))_{\delta'}$ and sum over δ,δ'.

$$
\begin{aligned}
\sum_{\delta,\delta'} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_{\delta'} 1_{\delta,\delta'} &= \sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta \\
&= (1_{n,m}) \\
&= \sum_{\delta,\delta'} (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{v}_{c_n}(s))_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'} \\
&= \left[\sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right] \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot \left[\sum_{\delta'} (\tilde{v}_{c_n}(s))_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'} \right] \\
&= \left[\sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right] \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot (1_{n,m}) \\
&= \left[\sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right] \cdot (\tilde{Y}_{c_{n,m}}(s))
\end{aligned} \tag{3.80}
$$

From which we conclude

$$
(\tilde{Z}_{c_{n,m}}(s)) = (\tilde{Y}_{c_{n,m}}(s))^{-1} = \sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \tag{3.81}
$$

In a similar manner from the second of (3.78) dyadic multiplication by $(\tilde{v}_{c_n}(s))_\delta$ on the left and $(\tilde{i}_{c_n}(s))_{\delta'}$ on the right and summing over δ,δ' gives

$$\sum_{\delta,\delta'} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_{\delta'} 1_{\delta,\delta'} = \sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_{\delta'}$$

$$= (1_{n,m})$$

$$= \sum_{\delta,\delta'} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{i}_{c_n}(s))_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'}$$

$$= \left[\sum_{\delta} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta\right] \cdot (\tilde{Z}_{c_{n,m}}(s)) \cdot \left[\sum_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'}\right]$$

$$= (1_{n,m}) \cdot (\tilde{Z}_{c_{n,m}}(s)) \cdot \left[\sum_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'} (\tilde{i}_{c_n}(s))_{\delta'}\right]$$

$$= (\tilde{Z}_{c_{n,m}}(s)) \cdot \left[\sum_{\delta} (\tilde{i}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta\right] \tag{3.82}$$

from which we conclude

$$(\tilde{Y}_{c_{n,m}}(s)) = (\tilde{Z}_{c_{n,m}}(s))^{-1} = \sum_{\delta} (\tilde{i}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta \tag{3.83}$$

These results are quite illuminating. Specifically, they show that the characteristic impedance matrices are symmetric, i.e.,

$$\begin{aligned} (\tilde{Z}_{c_{n,m}}(s))^T &= (\tilde{Z}_{c_{n,m}}(s)) \\ (\tilde{Y}_{c_{n,m}}(s))^T &= (\tilde{Y}_{c_{n,m}}(s)) \end{aligned} \tag{3.84}$$

This is evident from (3.81) and (3.83) which expand these as sums of symmetric dyads. The form in (3.84) is usually referred to as reciprocity, but this property was not assumed in the beginning, but is required by our results. Again, this is possibly associated with the assumed characteristics of the propagation matrix.

4. Expansion of $(\tilde{Z}'_{n,m}(s))$ and $(\tilde{Y}'_{n,m}(s))$ in terms of $(\tilde{v}_{c_n}(s))_\delta$ and $(\tilde{i}_{c_n}(s))_\delta$

From (3.58) for the case of no sources along the semi-infinite line, $z \geq 0$, with only + waves, we have

$$
\begin{aligned}
(\tilde{V}_n(z,s))_+ &= \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z\}\cdot(\tilde{V}_n(0,s))_+ \\
(\tilde{V}_n(z,s))_- &= (0_n) \qquad (3.85)\\
(\tilde{V}_n(z,s)) &= (\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(z,s))
\end{aligned}
$$

Expanded in modal form we have

$$
\begin{aligned}
(\tilde{V}_n(z,s))_+ &= \left\{\sum_\delta e^{-\tilde{\gamma}_\delta(s)z}(\tilde{v}_{c_n}(s))_\delta(\tilde{i}_{c_n}(s))_\delta\right\}\cdot(\tilde{V}_n(0,s))_+ \qquad (3.86)\\
\frac{d}{dz}(\tilde{V}_n(z,s))_+ &= -\left\{\sum_\delta \tilde{\gamma}_\delta(s)\, e^{-\tilde{\gamma}_\delta(s)z}(\tilde{v}_{c_n}(s))_\delta(\tilde{i}_{c_n}(s))_\delta\right\}\cdot(\tilde{V}_n(0,s))_+
\end{aligned}
$$

Noting the special relation between the voltage and current vectors we can then write equations for the voltage vector the same as for the combined voltage vector, i.e.,

$$
\begin{aligned}
(\tilde{V}_n(z,s)) &= \left\{\sum_\delta e^{-\tilde{\gamma}_\delta(s)z}(\tilde{v}_{c_n}(s))_\delta(\tilde{i}_{c_n}(s))_\delta\right\}\cdot(\tilde{V}_n(0,s)) \qquad (3.87)\\
\frac{d}{dz}(\tilde{V}_n(z,s)) &= -\left\{\sum_\delta \tilde{\gamma}_\delta(s)\, e^{-\tilde{\gamma}_\delta(s)z}(\tilde{v}_{c_n}(s))_\delta(\tilde{i}_{c_n}(s))_\delta\right\}\cdot(\tilde{V}_n(0,s))
\end{aligned}
$$

Similarly, for the current vector we have by multiplication (dot product) by the characteristic admittance matrix

$$
(\tilde{I}_n(z,s)) = (\tilde{Y}_{c_{n,m}}(s))\cdot\left\{\sum_\delta e^{-\tilde{\gamma}_\delta(s)z}(\tilde{v}_{c_n}(s))_\delta(\tilde{i}_{c_n}(s))_\delta\right\}\cdot(\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(0,s)) \qquad (3.88)
$$

Then using the modal expansions for the characteristic admittance and impedance matrices in (3.83) and (3.81) respectively, together with the biorthonormal modal relation in (3.55) we have

$$(\tilde{I}_n(z,s)) = \left\{ \sum_{\delta} e^{-\tilde{\gamma}_\delta(s)z} (\tilde{i}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right\} \cdot (\tilde{I}_n(0,s))$$
$$\frac{d}{dz}(\tilde{I}_n(z,s)) = - \left\{ \sum_{\delta} \tilde{\gamma}_\delta(s)\, e^{-\tilde{\gamma}_\delta(s)z} (\tilde{i}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right\} \cdot (\tilde{I}_n(0,s)) \tag{3.89}$$

Recall (3.1) and (3.2) without sources

$$\frac{d}{dz}(\tilde{I}_n(z,s)) = -(\tilde{Y}'_{n,m}(s)) \cdot (\tilde{V}_n(z,s))$$
$$\frac{d}{dz}(\tilde{V}_n(z,s)) = -(\tilde{Z}'_{n,m}(s)) \cdot (\tilde{I}_n(z,s)) \tag{3.90}$$

Comparing these to the above modal expansions of the derivatives we have first, considering the current derivative

$$- \left\{ \sum_{\delta} \tilde{\gamma}_\delta(s)\, e^{-\tilde{\gamma}_\delta(s)z} (\tilde{i}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right\} \cdot (\tilde{Y}_{c_{n,m}}(s)) \cdot (\tilde{V}_n(0,s))$$
$$= - \left\{ \sum_{\delta} \tilde{\gamma}_\delta(s)\, e^{-\tilde{\gamma}_\delta(s)z} (\tilde{i}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta \right\} \cdot (\tilde{V}_n(0,s))$$
$$= -(\tilde{Y}'_{n,m}(s)) \cdot (\tilde{V}_n(z,s)) \tag{3.91}$$

Evaluating this at $z=0$ and noting that $(\tilde{V}_n(0,s))$ is an arbitrary N vector, we have

$$(\tilde{Y}'_{n,m}(s)) = \sum_{\delta} \tilde{\gamma}_\delta(s)\, (\tilde{i}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta \tag{3.92}$$

Similarly, using the voltage derivative equations

$$- \left\{ \sum_\delta \tilde{\gamma}_\delta(s) \, e^{-\tilde{\gamma}_\delta(s) z} (\tilde{v}_{c_n}(s))_\delta (\tilde{i}_{c_n}(s))_\delta \right\} \cdot (\tilde{Z}_{c_{n,m}}(s)) \cdot (\tilde{I}_n(0,s))$$

$$= - \left\{ \sum_\delta \tilde{\gamma}_\delta(s) \, e^{-\tilde{\gamma}_\delta(s) z} (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \right\} \cdot (\tilde{I}_n(0,s))$$

$$= (\tilde{Z}'_{n,m}(s)) \cdot (\tilde{I}_n(z,s)) \qquad (3.93)$$

Evaluating this at $z=0$ and noting that $(\tilde{I}_n(0,s))$ is an arbitrary N vector, we have

$$(\tilde{Z}'_{n,m}(s)) = \sum_\delta \tilde{\gamma}_\delta(s) \, (\tilde{v}_{c_n}(s))_\delta (\tilde{v}_{c_n}(s))_\delta \qquad (3.94)$$

Note now that (3.92) explicitly illustrates that $(\tilde{Y}'_{n,m}(s))$ is symmetric and (3.94) does the same for $(\tilde{Z}'_{n,m}(s))$, i.e.,

$$\begin{aligned} (\tilde{Z}'_{n,m}(s))^T &= (\tilde{Z}'_{n,m}(s)) \\ (\tilde{Y}'_{n,m}(s))^T &= (\tilde{Y}'_{n,m}(s)) \end{aligned} \qquad (3.95)$$

As in (3.84) for the characteristic impedance and admittance matrices, this symmetry is a statement of reciprocity for the impedance and admittance per-unit-length matrices. While this was not explicitly assumed at the start, it is a consequence of the development. This may be associated with the assumed diagonalization characteristics of the propagation matrix. Then let us consider the reciprocity (symmetry) of the impedance and admittance per-unit-length matrices as one of our assumptions for the present development.

Taking (3.94) and left or right dot multiplying by a current eigenmode, we have

$$\begin{aligned} \tilde{\gamma}_\delta(s)(\tilde{v}_{c_n}(s))_\delta &= (\tilde{Z}'_{n,m}(s)) \cdot (\tilde{i}_{c_n}(s))_\delta \\ &= (\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{Z}'_{n,m}(s)) \end{aligned} \qquad (3.96)$$

A dot product with the δ'-th current mode gives

$$\tilde{\gamma}_\delta(s)\, 1_{\delta,\delta'} = (\tilde{i}_{c_n}(s))_\delta \cdot (\tilde{Z}'_{n,m}(s)) \cdot (\tilde{i}_{c_n}(s))_{\delta'} \tag{3.97}$$

Given $\tilde{\gamma}_\delta(s)$ this normalizes the $(\tilde{i}_{c_n}(s))_\delta$ in terms of $(\tilde{Z}'_{n,m}(s))$. Note also the relationship of the voltage and current modes via $(\tilde{Z}'_{n,m}(s))$ and $\tilde{\gamma}_\delta(s)$.

Similarly, dot multiplying (3.92) on left or right by a voltage eigenmode gives

$$\begin{aligned}\tilde{\gamma}_\delta(s)\, (i_{c_n}(s))_\delta &= (\tilde{Y}'_{n,m}(s)) \cdot (\tilde{v}_{c_n}(s)_\delta \\ &= (\tilde{v}_{c_n}(s))_\delta \cdot (\tilde{Y}'_{n,m}(s))\end{aligned} \tag{3.98}$$

A dot product with the δ'-th voltage mode gives

$$\tilde{\gamma}_\delta(s)\, 1_{\delta,\delta'} = (\tilde{v}_{c_n}(s))_\delta \cdot (\tilde{Y}'_{n,m}(s)) \cdot (\tilde{v}_{c_n}(s))_{\delta'} \tag{3.99}$$

This normalizes the $(\tilde{v}_{c_n}(s))_\delta$ in terms of $(\tilde{Y}'_{n,m}(s))$ and the $\tilde{\gamma}_\delta(s)$. Note also the relationship of the voltage and current modes via $(\tilde{Y}'_{n,m}(s))$ and $\tilde{\gamma}_\delta(s)$.

5. Termination condition of a tube

A transmission line is usually terminated at the two ends $z=0$ and $z=L$. The termination could be a lumped impedance, a distributed network, open-circuit or short-circuit. If sources are included, these conditions can be represented by a generalized Thévenin equivalent network or a generalized Norton equivalent network.

Passive terminations can be specified as an impedance matrix $(\tilde{Z}_{T_{n,m}}(z,s))$ or an admittance matrix $(\tilde{Y}_{T_{n,m}}(z,s))$ where $z=0$ or L. The condition $(\tilde{Z}_{T_{n,m}}(L,s)) = (\tilde{Z}_{c_{n,m}}(s))$, or equivalently $(\tilde{Y}_{T_{n,m}}(L,s)) = (\tilde{Y}_{c_{n,m}}(s))$ specifies a perfectly matched line and the transmission line behaves like a semi-infinite line for $0 \le z \le L$ (with an equivalent single end at $z=0$).

Alternatively, the terminating conditions can be specified by scattering matrices $(\tilde{S}_{n,m}(z,s))$ where $z=0$ or L. Consider at $z=L$ (see Figure 3.1); let the incoming waves be designated by a superscript $-$ and the outgoing waves $+$. The scattering matrix is defined by

$$(\tilde{W}_n^{(+)}(s)) = (\tilde{S}_{n,m}(z,s))\cdot(\tilde{W}_n^{(-)}(s)) \tag{3.100}$$

For the case illustrated in Figure 3.1, one observes that if this termination is taken as $z=L$, then

$$\begin{aligned} (\tilde{W}_n^{(+)}(s)) &= (\tilde{V}_n(L,s))_- \\ (\tilde{W}_n^{(-)}(s)) &= (\tilde{V}_n(L,s))_+ \end{aligned} \tag{3.101}$$

One can then rewrite (3.94) as

$$(\tilde{V}_n(L,s))_- = (\tilde{S}_{n,m}(L,s))\cdot(\tilde{V}_n(L,s))_+ \tag{3.102}$$

which in this terminating case is the same as the definition of a reflection matrix given by

$$(\tilde{S}_{n,m}(L,s)) = [(\tilde{Z}_{T_{n,m}}(L,s)) + (\tilde{Z}_{c_{n,m}}(s))]^{-1}\cdot[(\tilde{Z}_{T_{n,m}}(L,s)) - (\tilde{Z}_{c_{n,m}}(s))] \tag{3.103}$$

Similarly, at $z=0$ the termination conditions are

$$(\tilde{V}_n(0,s))_+ = (\tilde{S}_{n,m}(0,s))\cdot(\tilde{V}_n(0,s))_- \tag{3.104}$$

and

$$(\tilde{S}_{n,m}(0,s)) = [(\tilde{Z}_{T_{n,m}}(0,s)) + (\tilde{Z}_{c_{n,m}}(s))]^{-1}\cdot[(\tilde{Z}_{T_{n,m}}(0,s)) - (\tilde{Z}_{c_{n,m}}(s))] \tag{3.105}$$

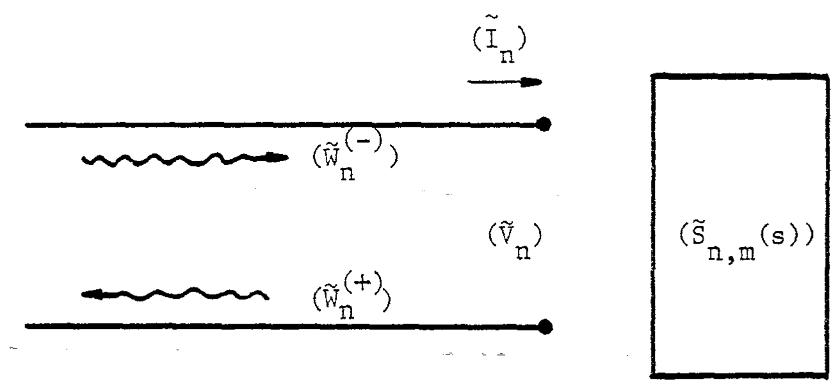


Figure 3.1. Incoming and outgoing wave at a junction

6. Solution of combined voltages

Combination of (3.58), (3.102) and (3.104) gives the solution of the combined voltage equation. Rewriting these equations

$$(\tilde{V}_n(z,s))_+ = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s)z\} \cdot (\tilde{V}_n(0,s))_+ + \int_0^z \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\} \cdot (\tilde{V}_n^{(s)'}(z',s))_+ \, dz'$$

$$(\tilde{V}_n(z,s))_- = \exp\{(\tilde{\gamma}_{c_{n,m}}(s))[z-L]\} \cdot (\tilde{V}_n(L,s))_- + \int_L^z \exp\{(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\} \cdot (\tilde{V}_L^{(s)'}(z',s))_- \, dz'$$

$$(\tilde{V}_n(0,s))_+ = (\tilde{S}_{n,m}(0,s)) \cdot (\tilde{V}_n(0,s))_-$$

$$(\tilde{V}_n(L,s))_- = (\tilde{S}_{n,m}(L,s)) \cdot (\tilde{V}_n(L,s))_+ \tag{3.106}$$

These equations can be solved by substitutions, or can be arranged in a matrix form, as described later in the BLT equation. As written here, these correspond to the special case of a transmission-line network consisting of two junctions (or terminations) connected by a single tube.

7. Reconstruction of total voltages and total currents

Once the combined voltages are evaluated, the total voltages and total currents are readily obtained.

From (3.18), one obtains

$$\begin{aligned} (\tilde{V}_n(z,s)) &= \frac{1}{2}[(\tilde{V}_n(z,s))_+ + (\tilde{V}_n(z,s))_-] \\ (\tilde{I}_n(z,s)) &= \frac{1}{2}(\tilde{Y}_{c_{n,m}}(s)) \cdot [(\tilde{V}_n(z,s))_+ - (\tilde{V}_n(z,s))_-] \end{aligned} \tag{3.107}$$

Hence, if one knows $(\tilde{V}_n(z,s))_+$ and $(\tilde{V}_n(z,s))_-$ for a given tube, as well as $(\tilde{Y}_{c_{n,m}}(s))$ or $(\tilde{Z}_{c_{n,m}}(s))$ (being measurable or conceivably even calculable), then the measurable voltage and current vectors are directly reconstructable.

D. Sign Convention of q

It is noted that in the definitions of the combined voltage, the convention $q = +1$ is chosen to represent the wave propagating from $z=0$ to $z=L$. Correspondingly, $q = -1$ represents the wave propagating from $z=L$ to $z=0$.

Let us further denote the above quantities with a subscript u, i.e., $q_u = +1$ corresponds to wave propagating from left to right, i.e., from $z_u = 0$ to $z_u = L$. This is shown in Figure 3.2.

It is also permissible to choose, on the same tube, a different convention. Let $z_v = L - z_u$ be a new coordinate, as shown in Figure 3.2. The new q convention, q_v, is now oppostie to q_u. Here, $q_v = +1$ corresponds to a wave traveling from right to left (i.e., $z_v = 0$ to $z_v = L$).

This convention will prove useful and is recalled in deriving the BLT equation.

The subsequent sections deal with multitube multiconductor transmission-line networks. More general notations, primarily in the form of additional subscripts denoting either the tube or the junction, are used.

E. Summary

Since there are so many expressions, relations, etc., introduced in this section, it is useful to summarize them in tabular form, as presented in Tables 3.1-4.

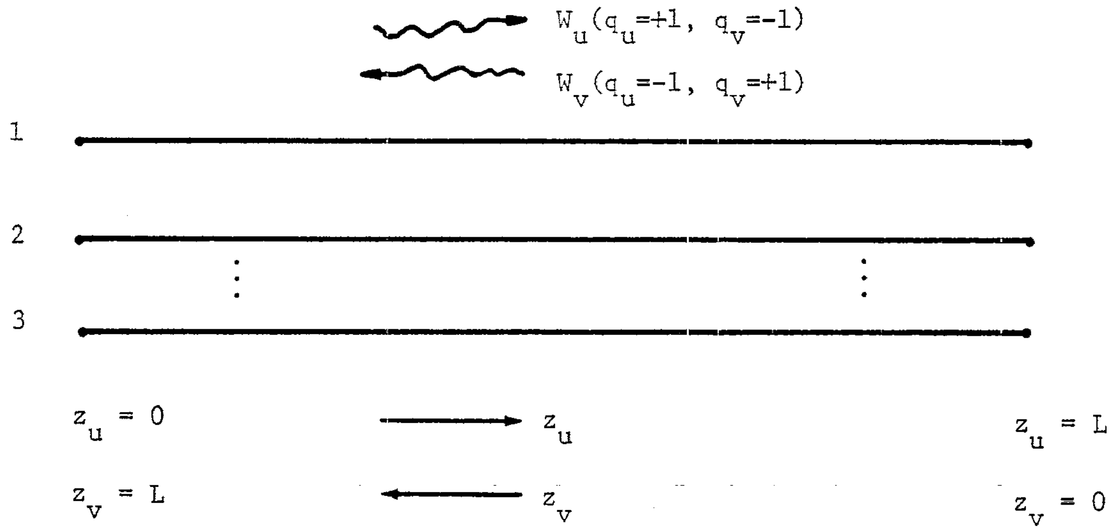


Figure 3.2 Left and right traveling waves

Table 3.1 Transmission-Line Equations

Name	Symbol	Relation
Trans. line eqns. (telegrapher eqns.)		$\frac{d}{dz}(\tilde{V}_n(z,s)) = -(\tilde{Z}'_{n,m}(s))\cdot(\tilde{I}_n(z,s)) + (\tilde{V}_n^{(s)'}(z,s))$ $\frac{d}{dz}(\tilde{I}_n(z,s)) = -(\tilde{Y}'_{n,m}(s))\cdot(\tilde{V}_n(z,s)) + (\tilde{I}_n^{(s)'}(z,s))$
Combined Voltage Vector	$(\tilde{V}_n(z,s))_q$	$= (\tilde{V}_n(z,s)) + q(\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(z,s))$
Combined Per-Unit-Length Source Vector	$(\tilde{V}_n^{(s)'}(z,s))_q$	$= (\tilde{V}_n^{(s)'}(z,s)) + q(\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{I}_n(z,s))$
Voltage Vector Reconstruction	$(\tilde{V}_n(z,s))$	$= \frac{1}{2}\left[(\tilde{V}_n(z,s))_+ + (\tilde{V}_n(z,s))_-\right]$
Current Vector Reconstruction	$(\tilde{I}_n(z,s))$	$= \frac{1}{2}(\tilde{Y}_{c_{n,m}}(s))\cdot\left[(\tilde{V}_n(z,s))_+ - (\tilde{V}_n(z,s))_-\right]$
Separation Index	q	$= \pm 1$
Combined Voltage Equation		$\left[(1_{n,m})\frac{d}{dz} + q(\tilde{\gamma}_{c_{n,m}}(s))\right]\cdot(\tilde{V}_n(z,s))_q = (\tilde{V}_n^{(s)'}(z,s))_q$
Propagation Matrix	$(\tilde{\gamma}_{c_{n,m}}(s))$	$= \left[(\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))\right]^{1/2}$ (principal or p.r. value)
Characteristic Impedance Matrix	$(\tilde{Z}_{c_{n,m}}(s))$	$= (\tilde{\gamma}_{c_{n,m}}(s))\cdot(\tilde{Y}'_{n,m}(s))^{-1} = (\tilde{\gamma}_{c_{n,m}}(s))^{-1}\cdot(\tilde{Z}'_{n,m}(s))$
Characteristic Admittance Matrix	$(\tilde{Y}_{c_{n,m}}(s))$	$= (\tilde{Z}_{c_{n,m}}(s))^{-1} = (\tilde{Y}'_{n,m}(s))\cdot(\tilde{\gamma}_{c_{n,m}}(s))^{-1}$ $= (\tilde{Z}'_{n,m}(s))^{-1}\cdot(\tilde{\gamma}_{c_{n,m}}(s))$
General Solution (referenced to arbitrary position z_o)	$(\tilde{V}_n(z,s))_q$	$= \exp\left\{-q(\tilde{\gamma}_{c_{n,m}}(s))\left[z-z_o\right]\right\}\cdot(\tilde{V}_n(z_o,s))_q$ $+ \int_{z_o}^{z} \exp\left\{-q((\tilde{\gamma}_{c_{n,m}}(s))\left[z-z'\right]\right\}\cdot(\tilde{V}_n^{(s)'}(z',s))_q\, dz'$
Solution for tube $0 \le z \le L$ in terms of boundary values: + or right wave	$(\tilde{V}_n(z,s))_+$	$= \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))z\right\}\cdot(\tilde{V}_n(0,s))_+ +$ $+ \int_{0}^{z} \exp\left\{-(\tilde{\gamma}_{c_{n,m}}(s))\left[z-z'\right]\right\}\cdot(\tilde{V}_n^{(s)'}(z',s))_+\, dz'$
Solution for tube $0 \le z \le L$ in terms of boundary values: − or left wave	$(\tilde{V}_n(z,s))_-$	$= \exp\left\{(\tilde{\gamma}_{c_{n,m}}(s))\left[z-L\right]\right\}\cdot(\tilde{V}_n(L,s))_-$ $+ \int_{L}^{z} \exp\left\{(\tilde{\gamma}_{c_{n,m}}(s))\left[z-z'\right]\right\}\cdot(\tilde{V}_n^{(s)'}(z',s))_-\, dz'$

Table 3.2 Diagonalization of Propagation Matrix

Name	Symbol	Relation
Square of Propagation Matrix	$(\tilde{\gamma}_{c_{n,m}}(s))^2$	$= (\tilde{Z}'_{n,m}(s))\cdot(\tilde{Y}'_{n,m}(s))$
Normalized Voltage Eigenvector	$(\tilde{v}_{c_n}(s))_\delta$	$(\tilde{\gamma}_{c_{n,m}}(s))^2\cdot(\tilde{v}_{c_n}(s))_\delta = \tilde{\gamma}_\delta^2(s)\ (\tilde{v}_{c_n}(s))_\delta$
Normalized Current Eigenvector	$(\tilde{i}_{c_n}(s))_\delta$	$(\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{\gamma}_{c_{n,m}}(s))^2 = \tilde{\gamma}_\delta^2(s)\ (\tilde{i}_{c_n}(s))_\delta$
Eigenvalue of Propagation Matrix	$\tilde{\gamma}_\delta(s)$	$= \left[\tilde{\gamma}_\delta^2(s)\right]^{1/2}$ (principal or p.r. value assumed)
Eigenindex	δ	$= 1,2,\ldots,N$ (N×N matrices)
Biorthonormal Property (used for normalization)		$(\tilde{v}_{c_n}(s))_\delta\cdot(\tilde{i}_{c_n}(s))_{\delta'} = 1_{\delta,\delta'}$ (N independent eigenvectors assumed of both voltage and current types)
Function of Propagation Matrix	$F((\tilde{\gamma}_{c_{n,m}}(s)))$	$= \sum_\delta F(\tilde{\gamma}_\delta(s))\ (\tilde{v}_{c_n}(s))_\delta\ (\tilde{i}_{c_n}(s))_\delta$
Special Cases: Propagation Matrix	$(\tilde{\gamma}_{c_{n,m}}(s))$	$= \sum_\delta \tilde{\gamma}_\delta(s)\ (\tilde{v}_{c_n}(s))_\delta\ (\tilde{i}_{c_n}(s))_\delta$
Special Cases: Transpose	$(\tilde{\gamma}_{c_{n,m}}(s))^T$	$= \sum_\delta \tilde{\gamma}_\delta(s)\ (\tilde{i}_{c_n}(s))_\delta\ (\tilde{v}_{c_n}(s))_\delta$
Special Cases: Inverse	$(\tilde{\gamma}_{c_{n,m}}(s))^{-1}$	$= \sum_\delta \tilde{\gamma}_\delta^{-1}(s)\ (\tilde{v}_{c_n}(s))_\delta\ (\tilde{i}_{c_n}(s))_\delta$
Special Cases: Identity	$(1_{n,m})$	$= (\tilde{\gamma}_{n,m}(s))^0 = \sum_\delta (\tilde{v}_{c_n}(s))_\delta\ (\tilde{i}_{c_n}(s))_\delta = \sum_\delta (\tilde{i}_{c_n}(s))_\delta\ (\tilde{v}_{c_n}(s))_\delta$

Table 3.3 Normalization of Voltage and Current Eigenmodes

Name	Symbol	Relation
Normalization via Characteristic Impedance and Admittance Matrices	Interrelation of voltage and current eigenmodes	$(\tilde{v}_{c_n}(s))_\delta = (\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{i}_{c_n}(s))_\delta = (\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{Z}_{c_{n,m}}(s))$ $(\tilde{i}_{c_n}(s))_\delta = (\tilde{Y}_{c_{n,m}}(s))\cdot(\tilde{v}_{c_n}(s))_\delta = (\tilde{v}_{c_n}(s))_\delta\cdot(\tilde{Y}_{c_{n,m}}(s))$
	Separate voltage and current eigenmode normalization	$1_{\delta,\delta'} = (\tilde{i}_{c_n}(s))_\delta\cdot(\tilde{Z}_{c_{n,m}}(s))\cdot(\tilde{i}_{c_n}(s))_{\delta'}$ $1_{\delta,\delta'} = (\tilde{v}_{c_n}(s))_\delta\cdot(\tilde{Y}_{c_{n,m}}(s))\cdot(\tilde{v}_{c_n}(s))_{\delta'}$
Normalization via Per-Unit-Length Impedance and Admittance Matrices	Interrelation of voltage and current eigenmodes	$\tilde{\gamma}_\delta(s)(\tilde{v}_{c_n}(s))_\delta = (\tilde{Z}'_{n,m}(s))\cdot(\tilde{I}_{c_n}(s))_\delta = (\tilde{I}_{c_n}(s))_\delta\cdot(\tilde{Z}'_{n,m}(s))$ $\tilde{\gamma}_\delta(s)(\tilde{I}_{c_n}(s))_\delta = (\tilde{Y}'_{n,m}(s))\cdot(\tilde{v}_{c_n}(s))_\delta = (\tilde{v}_{c_n}(s))_\delta\cdot(\tilde{Y}'_{n,m}(s))$
	Separate voltage and current eigenmode normalization	$1_{\delta,\delta'} = \tilde{\gamma}_\delta^{-1}(s)(\tilde{I}_{c_n}(s))_\delta\cdot(\tilde{Z}'_{n,m}(s))\cdot(\tilde{I}_{c_n}(s))_{\delta'}$ $1_{\delta,\delta'} = \tilde{\gamma}_\delta^{-1}(s)(\tilde{v}_{c_n}(s))_\delta\cdot(\tilde{Y}'_{n,m}(s))\cdot(\tilde{v}_{c_n}(s))_{\delta'}$

Table 3.4 Representation of Other Matrices in Terms of Voltage and Current Normalized Eigenmodes (indicating assumed reciprocity)

Name	Symbol	Relation
Characteristic Impedance Matrix	$(\tilde{Z}_{c_{n,m}}(s))$	$= \sum_\delta (\tilde{v}_{c_n}(s))_\delta\ (\tilde{v}_{c_n}(s))_\delta$
Characteristic Admittance Matrix	$(\tilde{Y}_{c_{n,m}}(s))$	$= \sum_\delta (\tilde{i}_{c_n}(s))_\delta\ (\tilde{i}_{c_n}(s))_\delta$
Per-Unit-Length Impedance Matrix	$(\tilde{Z}'_{n,m}(s))$	$= \sum_\delta \tilde{\gamma}_\delta(s)\ (\tilde{v}_{c_n}(s))_\delta\ (\tilde{v}_{c_n}(s))_\delta$
Per-Unit-Length Admittance Matrix	$(\tilde{Y}'_{n,m}(s))$	$= \sum_\delta \tilde{\gamma}_\delta(s)\ (\tilde{I}_{c_n}(s))_\delta\ (\tilde{I}_{c_n}(s))_\delta$

References

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[3.3] C. E. Baum, "On the Eigenmode Expansion Method for Electromagnetic Scattering and Antenna Problems, Part I: Some Basic Relations for Eigenmode Expansions and Their Relation to the Singularity Expansion," Interaction Note 229, January 1975.

[3.4] E.A. Coddington and H. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1965.

IV. SUPERMATRICES AND SUPERVECTORS

Define a supermatrix, or more specifically, a dimatrix or tensor of rank four, as a partitioned matrix or matrix of matrices in the form

$$((D_{n,m})_{u,v}) \tag{4.1}$$

with elementary matrices or blocks

$$(D_{n,m})_{u,v} \tag{4.2}$$

and elements

$$D_{n,m;u,v} \tag{4.3}$$

such that the blocks or elementary matrices are $N_u \times M_v$, i.e.,

$$\begin{aligned} n &= 1,2,\ldots,N_u \\ m &= 1,2,\ldots,M_v \end{aligned} \tag{4.4}$$

and the dimatrix is $N \times N$, i.e.,

$$\begin{aligned} u &= 1,2,\ldots,N \\ v &= 1,2,\ldots,M \end{aligned} \tag{4.5}$$

Note that this corresponds to a matrix with

$$\begin{aligned} &\sum_{n=1}^{N} N_u \quad \text{rows} \\ &\sum_{v=1}^{M} M_v \quad \text{columns} \end{aligned} \tag{4.6}$$

which has been partitioned into blocks or elementary matrices by a partitioning of the row and column indices. Pictorially this corresponds to drawing horizontal and vertical lines completely through the matrix between selected adjacent rows and selected adjacent columns.

For our purposes, the dimatrices will be square, i.e.,

$$N = M \tag{4.7}$$

Furthermore, the partitioning will be symmetric, i.e.,

$$N_u = M_v \quad \text{for} \quad u = v \tag{4.8}$$

Hence the diagonal blocks

$$(D_{n,m})_{u,u} \text{ , size } N_u \times N_u \tag{4.9}$$

are square and off-diagonal blocks are symmetrically rectangular, i.e.,

$$\begin{aligned} &(D_{n,m})_{u,v} \text{ , size } N_u \times N_v \\ &(D_{n,m})_{v,u} \text{ , size } N_v \times N_u \end{aligned} \tag{4.10}$$

Supervectors or divectors are similarly defined in the form

$$((V_n)_u) \tag{4.11}$$

with elementary vectors as

$$\begin{aligned} &(V_n)_u \\ &n = 1,2,\ldots,N_u \\ &u = 1,2,\ldots,N \end{aligned} \tag{4.12}$$

remembering that n,m and u,v are merely dummy indices. Note that the elements are designated as

$$V_{n;u} \quad (\text{not } V_{n,u}) \tag{4.13}$$

Define supermatrix multiplication in the dot product or contraction sense as

$$((A_{n,m})_{u,v}) : ((B_{n,m})_{u,v})$$

$$= \left(\sum_{u'=1}^{N} (A_{n,m})_{u,u'} \; (B_{n,m})_{u',v}\right)$$

$$= \left(\left(\sum_{u'=1}^{N} \sum_{n'=1}^{N_{u'}} A_{n,n';u,u'} \; B_{n',m;u',v}\right)\right)$$

$$= ((C_{n,m})_{u,v}) \tag{4.14}$$

Here we note contraction is done twice involving the second indices of the two pairs of indices for the first matrix, and the first indices of the two pairs of indices for the second matrix; this is denoted by two levels of dot product : , noting the two dots one above the other.

In (4.14) the two dimatrices are not necessarily square. It is merely required that the second indices m,v of the first dimatrix have the same range (hence same partitioning) as the first indices of the second dimatrix. Two dimatrices with this property are said to be of compatible order, for multiplication in the double dot product sense in this case, with order of multiplication specified.

In the present note all dimatrices are taken as being of symmetric compatible order, i.e., (4.7,8) apply and N and the N_u have the same values for all dimatrices in the particular discussion (i.e., describing a given physical situation). Furthermore, the divectors are also taken as having the same compatible order. Thus we can form any such operations as

$$
\begin{array}{ll}
((A_{n,m})_{u,v}) + ((B_{n,m})_{u,v}) & \text{dimatrix} \\
((A_{n,m})_{u,v}) : ((B_{n,m})_{u,v}) & \text{dimatrix} \\
((B_{n,m})_{u,v}) : ((A_{n,m})_{u,v}) & \text{dimatrix} \\
((A_{n,m})_{u,v}) : ((V_n)_u) & \text{divector} \\
((V_n)_u) : ((A_{n,m})_{u,v}) & \text{divector} \\
((V_n)_u) : ((W_n)_u) & \text{scalar}
\end{array}
\tag{4.15}
$$

where dimatrix-divector and divector-divector multiplication in the double dot product sense are obvious specializations of (4.14).

V. IDENTITY SUPERMATRIX

Before continuing the supermatrices of the previous sections to yield an equation, it is necessary to define an identity supermatrix $((1_{n,m})_{u,v})$.

The identity supermatrix is such that its diagonal element matrices are all identity matrices, and all off-diagonal element matrices are zero matrices, i.e.,

$$(1_{n,m})_{u,v} = 1_{u,v}(1_{n,m})$$

$$1_{n,m} = \begin{cases} 1 \text{ for } n=m \\ 0 \text{ for } n\neq m \end{cases} \tag{5.1}$$

involving Kronecker deltas. The individual elements can be written as

$$1_{n,m;u,v} = \begin{cases} 1 \quad \text{for both } n=m \text{ and } u=v \\ 0 \quad \text{for either } n\neq m \text{ or } u\neq v \end{cases} \tag{5.2}$$

as a sort of super Kronecker delta or superidentity element. Note the identity supermatrix is then a symmetric dimatrix as in (4.7,8).

For a supermatrix $((M_{n,m})_{u,v})$ of symmetric compatible order, then

$$((1_{n,m})_{u,v}) : ((M_{n,m})_{u,v}) = ((M_{n,m})_{u,v})) : ((1_{n,m})_{u,v}) = ((M_{n,n})_{u,v}) \tag{5.3}$$

Also, an inverse $((M_{n,m})_{u,v})^{-1}$ of $((M_{n,m})_{u,v})$ exists such that

$$((M_{n,m})_{u,v})^{-1} : ((M_{n,m})_{u,v}) = ((M_{n,m})_{u,v}) : ((M_{n,m})_{u,v})^{-1}$$

$$= ((1_{n,m})_{u,v}) \tag{5.4}$$

provided

$$\det[((M_{n,n})_{u,v})] \neq 0 \tag{5.5}$$

Note that $((M_{n,m})_{u,v})^{-1}$ is of symmetric compatible order with $((M_{n,m})_{u,v})$.

VI. SCATTERING SUPERMATRIX

The concept of scattering matrices introduced in Section III for a terminated tube is extended here for junctions where more than one tube is connected. Collections and suitable ordering of scattering matrices at all junctions of the transmission-line network form a scattering supermatrix.

A. Junction Scattering Supermatrix

Consider the νth junction J_ν with tube ends denoted by $J_{\nu;r}$ with index r as discussed in subsection IID. Let this junction be characterized by an impedance matrix

$$(\tilde{Z}_{n,m}(s))_\nu = (\tilde{Y}_{n,m}(s))_\nu^{-1} \tag{6.1}$$

The junction scattering matrix is defined so that

$$(\tilde{V}_n(s))_{\nu,+} = (\tilde{S}_{n,m}(s))_\nu \cdot (\tilde{V}_n(s))_{\nu,-} \tag{6.2}$$

where the subscripts + and − refer to the aggregate of respectively outgoing and incoming waves (N-waves) on the various tubes in the form of combined voltage vectors; remember that the current convention for outgoing waves is positive direction outward, and for incoming waves is positive direction inward.

In the supermatrix form partition according to waves on the r_ν tube ends connected to J_ν as

$$\begin{gathered} ((\tilde{V}_n^{(0)}(s))_r)_\nu = ((\tilde{Z}_{n,m}(s))_{r,r'})_\nu \odot ((\tilde{I}_n^{(0)}(s))_r)_\nu \\ ((\tilde{Y}_{n,m}(s))_{r,r'})_\nu \equiv ((\tilde{Z}_{n,m}(s))_{r,r'})_\nu^{-1} \end{gathered} \tag{6.3}$$

where

$$(\tilde{V}_n^{(0)}(s))_{r;\nu}\ ,\ (\tilde{I}_n^{(0)}(s))_{r;\nu}$$
$$r = 1,2,\ldots,r \tag{6.4}$$

are the voltage and current vectors on the rth tube ends at J_ν with current convention into J_ν .

The tube associated with the rth tube end at J_ν has characteristic impedance and admittance matrices which can be put in supermatrix form for J_ν as

$((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu \equiv$ tube-end characteristic-impedance supermatrix for J_ν

$((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu \equiv$ tube-end characteristic-admittance supermatrix for J_ν

$$= ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu^{-1} \tag{6.5}$$

where

$$(\tilde{Z}_{c_{n,m}}(s))_{r,r';\nu} \equiv \begin{cases} \text{characteristic-impedance matrix for } r\text{th tube end at } J_\nu \text{ for } r=r' \text{ (square)} \\ (0_{n,m}) \text{ for } r\neq r' \text{ (rectangular)} \end{cases}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{r,r';\nu} \equiv \begin{cases} \text{characteristic-admittance matrix for } r\text{th tube end at } J_\nu \text{ for } r=r' \text{ (square)} \\ (0_{n,m}) \text{ for } r\neq r' \text{ (rectangular)} \end{cases}$$

$$(\tilde{Y}_{c_{n,m}}(s))_{r,r;\nu} = (\tilde{Z}_{c_{n,m}}(s))^{-1}_{r,r;\nu} \tag{6.6}$$

Thus, these impedance and admittance supermatrices for the tube ends at a given junction are block diagonal and may be represented in terms of the direct sum $\oplus$ as

$$\left(\left(\tilde{Z}_{c_{n,m}}(s) \right)_{r,r'} \right)_{\nu} \equiv \left(\tilde{Z}_{c_{n,m}}(s) \right)_{1,1;\nu} \oplus \left(\tilde{Z}_{c_{n,m}}(s) \right)_{2,2;\nu} \oplus \cdots \oplus \left(\tilde{Z}_{c_{n,m}}(s) \right)_{r_\nu,r_\nu;\nu}$$

$$\equiv \bigoplus_{r=1}^{r_\nu} \left(\tilde{Z}_{c_{n,m}}(s) \right)_{r,r;\nu}$$

$$\left(\left(\tilde{Y}_{c_{n,m}}(s) \right)_{r,r'} \right)_{\nu} \equiv \left(\tilde{Y}_{c_{n,m}}(s) \right)_{1,1;\nu} \oplus \left(\tilde{Y}_{c_{n,m}}(s) \right)_{2,2;\nu} \oplus \cdots \oplus \left(\tilde{Y}_{c_{n,m}}(s) \right)_{r_\nu,r_\nu;\nu}$$

$$\equiv \bigoplus_{r=1}^{r_\nu} \left(\tilde{Y}_{c_{n,m}}(s) \right)_{r,r;\nu} \tag{6.7}$$

where the convention used here is to maintain the partitioning according to the two pairs of indices $(n,m$ and $r,r')$ instead of combining them in one pair as in a regular matrix (or monomatrix). Note the subscript ν on the supermatrices; the elementary matrices are also identified with ν and the r,r' indices range over the tube ends at J_ν, not over the wave indices u,v.

The scattering supermatrix for J_ν is defined by

$$\left(\left(\tilde{V}_n(s) \right)_r \right)_{\nu,+} \equiv \left(\left(\tilde{S}_{n,m}(s) \right)_{r,r'} \right)_{\nu} : \left(\left(\tilde{V}_n(s) \right)_r \right)_{\nu,-}$$

$$\left(\left(\tilde{V}_n(s) \right)_r \right)_{\nu,+} \equiv \left(\left(\tilde{V}_n^{(0)}(s) \right)_r \right)_{\nu} - \left(\left(\tilde{Z}_{c_{n,m}}(s) \right)_{r,r'} \right)_{\nu} : \left(\left(\tilde{I}_n^{(0)}(s) \right)_r \right)_{\nu}$$

$$\equiv \text{outgoing wave supervector at } J_\nu$$

$$\left(\left(\tilde{V}_n(s) \right)_r \right)_{\nu,-} \equiv \left(\left(\tilde{V}_n^{(0)}(s) \right)_r \right)_{\nu} + \left(\left(\tilde{Z}_{c_{n,m}}(s) \right)_{r,r'} \right)_{\nu} : \left(\left(\tilde{I}_n^{(0)}(s) \right)_r \right)_{\nu}$$

$$\equiv \text{incoming wave supervector at } J_\nu \tag{6.8}$$

Again note that the J_ν current convention is positive current into J_ν so that the usual Ohm's law convention in (6.3) holds for J_ν. Solving (6.8) for the voltage and current supervectors at J_ν

$$((\tilde{V}_n^{(0)}(s))_r)_\nu = \frac{1}{2}\,[((\tilde{V}_n(s))_r)_{\nu,+} + ((\tilde{V}_n(s))_r)_{\nu,-}]$$

$$((\tilde{I}_n^{(0)}(s))_r) = \frac{1}{2}\,((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu : [((\tilde{V}_n^{(0)}(s))_r)_{\nu,-} - ((\tilde{V}_n^{(0)}(s))_r)_{\nu,+}] \tag{6.9}$$

Now we can compute the junction scattering supermatrix for J_ν by combining (6.8) and (6.9) with (6.3) to give

$$\begin{aligned}((\tilde{S}_{n,m}(s))_{r,r'})_\nu &= [((\tilde{Z}_{n,m}(s))_{r,r'})_\nu : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu + ((1_{n,m})_{r,r'})_\nu]^{-1} \\ &\quad : [((\tilde{Z}_{n,m}(s))_{r,r'})_\nu : ((\tilde{Y}_{c_{n,m}}(s))_{r,r'})_\nu - ((1_{n,m})_{r,r'})_\nu] \\ &= [((1_{n,m})_{r,r'})_\nu + ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{Y}_{n,m}(s))_{r,r'})_\nu]^{-1} \\ &\quad : [((1_{n,m})_{r,r'})_\nu - ((\tilde{Z}_{c_{n,m}}(s))_{r,r'})_\nu : ((\tilde{Y}_{n,m}(s))_{r,r'})_\nu]\end{aligned} \tag{6.10}$$

Note the identity supermatrix corresponding to J_ν ; it is of course partitioned in the same symmetric compatible order as are the various impedance and admittance supermatrices and the scattering supermatrix for J_ν.

For the junction J_ν let the rth tube end have $N_{\nu;r}$ conductors (plus the reference) so that the wave on this tube end is a vector of dimension $N_{\nu;r}$. The supervectors then have dimension for J_ν as

$$N_\nu = \sum_{r=1}^{r_\nu} N_{\nu;r} \tag{6.11}$$

The associated supermatrices are $r_\nu \times r_\nu$ in terms of the blocks or elementary matrices; the corresponding matrices (unpartitioned) are $N_\nu \times N_\nu$. The reader may consult fig. 2.3 for an example of tube ends connecting to a junction.

B. Reindexing of Elementary Matrices in the Collection of Junction Scattering Supermatrices

Having considered the junction scattering supermatrix for J_ν and noting that $\nu = 1,2,\ldots,N_J$ gives all the junctions, we then have the elementary scattering matrices from one tube to another wherever there is such a connection at any junction. The problem is one of rearranging the equations so as to combine the results for junctions and tubes to obtain a description of the overall transmission-line network.

To convert the junction scattering supermatrix to a network scattering supermatrix, consider the tube-end-wave matrix $(t_{r,u})_{\nu;E-W}$ which relates the tube ends (r) at junction J_ν to the waves W_u on those tubes. Recall the definition from (2.26) of the elements of the tube-end-wave matrix as

$$t_{r,u;\nu;E-W} \equiv \begin{cases} 0 \text{ if } W_u \text{ does not connect to } J_\nu \text{ via the } r\text{th tube end (i.e., } J_{\nu;r}) \\ -1 \text{ if } W_u \text{ enters } J_\nu \text{ via the } r\text{th tube end (i.e., } J_{\nu;r,-}) \\ +1 \text{ if } W_u \text{ leaves } J_\nu \text{ via the } r\text{th tube end (i.e., } J_{\nu;r,+}) \end{cases} \tag{6.12}$$

Once can then construct this matrix for each J_ν for $\nu = 1,2,\ldots,N_J$ from the topological diagram (graph) for the transmission-line network giving the junction J_ν numbering and wave W_u numbering (as in the example in fig. 2.2b), and from the corresponding diagram for each junction J_ν including tube-end labeling (r or $J_{\nu;r}$) (as in the example for J_3 in fig. 2.3).

Now to associate an elementary scattering matrix $(\tilde{S}_{n,m}(s))_{r,r';\nu}$ for two tube ends, $J_{\nu;r}$ and $J_{\nu;r'}$, with $(\tilde{S}_{n,m}(s))_{u,v}$ corresponding to two waves, W_u and W_v, in the overall network is straightforward; one must associate

$$\begin{aligned} &r' \text{ (or } J_{\nu;r'}) \rightarrow v \text{ (or } W_v) \text{ for the incoming wave} \\ &r \text{ (or } J_{\nu;r}) \rightarrow u \text{ (or } W_u) \text{ for the outgoing wave} \end{aligned} \tag{6.13}$$

However, this is what the tube-end-wave matrix does.

Consider incoming waves corresponding to the second index, v, in $(\tilde{S}_{n,m}(s))_{u,v}$. For one and only one J_ν there is a negative entry in $(t_{r',v})_{\nu;E-W}$ under the vth column; the corresponding row is the value of r'. Hence, for each v

$$r' \text{ (in } J_{\nu;r'}) \text{ is that } r' \ni t_{r',v;\nu;E-W} = -1 \tag{6.14}$$

which is readily found and even automated on a computer. Said another way, v is a function (an integer function) of ν and r'. To aid in the search for $J_{\nu;r'}$ the value of ν (or junction J_ν) is found from the junction-wave matrix $(t_{\nu,v})_{J-W}$ (as in (2.21)) by finding those values of ν for which $t_{\nu,v;J-W}$ is nonzero; there are at most two such values of ν corresponding to the W_v leaving one junction and entering another junction, except in the case of a self tube where W_v both leaves and enters the same junction. Considering the one or two possible J_ν the value of ν and r' are readily found as in (6.14) or via a diagram. After going through $v = 1,2,\ldots,N_W$ one can construct a table in the form

v	ν	r'
1		
2		
⋮		
N_W		

(6.15)

Table of Correspondence of incoming waves W_v to junctions J_ν and tube ends $J_{\nu;r'}$

with the values of ν and r' filled in for every v.

Similarly for outgoing waves corresponding to the first index u in $(\tilde{S}_{n,m}(s))_{u,v}$, we have a value of r given by

$$r \text{ (in } J_{\nu;r}) \text{ is that } r \ni t_{r,u;\nu;E-W} = +1 \tag{6.16}$$

Hence, u is a function of ν and r . Again utilizing the junction-wave matrix $(t_{\nu,j})_{J-W}$ and finding the values of $t_{\nu,u;J-W}$ which are nonzero, one reduces the consideration to at most two values of the junction index ν . The values of ν and r are then readily found from the tube-end-wave matrices as in (6.16). After going through $u = 1,2,\ldots,N_W$ one can construct a table in the form

u	ν	r
1		
2		
$\vdots$		
N_W		

Table of correspondence of outgoing waves W_u to junctions J_ν and tube ends $J_{\nu;r}$

with the values of ν and r filled in for every u.

Hence, with each pair (u,v) we associate the pair $(J_{\nu_1;r}, J_{\nu_2;r'})$. Now we have

$$\begin{aligned}
&\nu_1 = \nu_2 \equiv \nu \quad \text{for } W_v \text{ scattering into } W_u \text{ at junction } J_\nu \\
&\nu_1 \neq \nu_2 \quad \text{for } W_v \text{ not scattering into } W_u \text{ (no interconnection) at any } J_\nu
\end{aligned} \tag{6.18}$$

Then we form the network elementary scattering matrices as

$$(\tilde{S}_{n,m}(s))_{u,v} \equiv \begin{cases} (\tilde{S}_{n,m}(s))_{r,r';\nu} & \text{for } \nu_1 = \nu_2 = \nu \text{ or } W_v \text{ scattering into } W_u \text{ at } J_\nu \\ (0_{n,m}) = (0_{n,m})_{u,v} & \text{for } \nu_1 \neq \nu_2 \text{ or } W_v \text{ not scattering into } W_u \end{cases} \tag{6.19}$$

This gives an explicit algorithm for constructing the $(\tilde{S}_{n,m}(s))_{u,v}$ from the collection of junction scattering supermatrices $((\tilde{S}_{n,m}(s))_{r,r'})_\nu$.

The reader will also note the correspondence of these results with the wave-wave matrix $(W_{u,v})$ as defined in (2.24) as

$$W_{u,v} = \begin{cases} 1 \text{ for } \nu_1 = \nu_2 \equiv \nu \text{ and } W_v \text{ scattering into} \\ \quad W_u \text{ at } J_\nu \\ 0 \text{ for } \nu_1 \neq \nu_2 \text{ or } W_v \text{ not scattering} \\ \quad \text{into } W_u \end{cases} \tag{6.20}$$

The wave-wave matrix then indicates which (u,v) pairs must be considered for finding nonidentically zero scattering matrices, thereby simplifying the search among the elementary matrices comprising the junction scattering supermatrices.

C. Scattering Supermatrix

The proper ordering of all the junction scattering matrices into one large matrix forms the system (or network) scattering supermatrix $((\tilde{S}_{n,m}(s))_{u,v})$. This supermatrix is a collection of the junction scattering matrices, which themselves are collections of individual tube scattering matrices. The latter are matrices containing reflection and transmission coefficients of individual wires within the tubes. Thus, $((\tilde{S}_{n,m}(s))_{u,v})$ is a dimatrix (or tensor of rank four).

The wave-wave matrix $(W_{u,v})$ gives the structure of the scattering supermatrix since the scattering supermatrix is in general block sparse as

$$(\tilde{S}_{n,m}(s))_{u,v} = (0_{n,m})_{u,v} \quad \text{for} \quad W_{u,v} = 0 \tag{6.21}$$

Hence, also, the scattering supermatrix is $N_W \times N_W$ in terms of the u,v indices, i.e.,

$$u,v = 1,2,\ldots,N_W \tag{6.22}$$

The elementary scattering matrices $(\tilde{S}_{n,m}(s))_{u,v}$ are $N_u \times N_v$, i.e.,

$$n = 1,2,\ldots,N_u$$
$$m = 1,2,\ldots,N_v \tag{6.23}$$

where

$$N_u = \text{number of conductors (not including reference) on the tube with } u\text{th wave} \tag{6.24}$$

and likewise for N_v.

As a special case, it is interesting to note that if there are no self tubes (with both ends connected to the same junction), then

$$W_{u,u} = 0 \quad \text{for } u = 1,2,\ldots N_W \text{ for no self tubes}$$
$$(\tilde{S}_{n,m}(s))_{u,u} = (0_{n,m})_{u,u} \quad \text{for } n,m = 1,2,\ldots,N_u \text{ (square)} \tag{6.25}$$

In this case the scattering supermatrix has zero matrices for its diagonal blocks; this will complement the identity supermatrix which has, as its only nonzero elementary matrices, the diagonal blocks which are identity matrices (as discussed in Section V). This case is anticipated to be quite common in practice.

VII. DEFINITIONS OF SOME IMPORTANT SUPERMATRIX AND SUPERVECTOR QUANTITIES BASED ON RESULTS FOR WAVES ON A TUBE

This section takes the results for the combined voltages on a tube and separates them into the wave variables for the network. The resulting equation for a general combined voltage wave W_u is used to relate the combined voltages at both ends of the tube with the sources along the tube. Each term is generalized to a form appropriate to the transmission-line network, i.e., supermatrices and supervectors, by aggregating the results for all W_u for $u = 1,2,\ldots,N_W$.

A. Common Equation for the Two Waves on a Tube

Let us take the results for the propagation on a single tube developed in subsection IIIC from (3.59) and (3.60) as

$$
\begin{aligned}
(\tilde{V}_n(z,s))_+ &= \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))z\}\cdot(\tilde{V}_n(0,s))_+ \\
&\quad + \int_0^z \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\}\cdot(\tilde{V}_n^{(s)'}(z',s))_+ \, dz' \\
(\tilde{V}_n(z,s))_- &= \exp\{(\tilde{\gamma}_{c_{n,m}}(s))[z-L]\}\cdot(\tilde{V}_n(L,s))_- \\
&\quad + \int_L^z \exp(\tilde{\gamma}_{c_{n,m}}(s))[z-z']\}\cdot(\tilde{V}_n^{(s)'}(z',s))_- \, dz'
\end{aligned}
\tag{7.1}
$$

Then, as discussed in subsection IIID, let us identify the two waves on the tube with two waves of the transmission-line network, say W_u and W_v .

Consider the + wave; call this W_u and set the coordinate and dimension variables as

$L_u \equiv L \equiv$ length of path for W_u

$z_u \equiv z \equiv$ wave coordinate for W_u

$0 \le z_u \le L_u$

$N_u \equiv N \equiv$ number of conductors (less reference) on tube and dimension of vectors for W_u (7.2)

The wave and source conventions are then

$$(\tilde{V}_n(z_u,s))_u \equiv (\tilde{V}_n(z,s))_+ = (\tilde{V}_n(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n(z_u,s))$$

$\equiv$ combined voltage for W_u

$$(\tilde{V}_n^{(s)'}(z_u,s))_u \equiv (\tilde{V}_n^{(s)'}(z,s))_+ = (\tilde{V}_n^{(s)'}(z_u,s)) + (\tilde{Z}_{c_{n,m}}(s))_u \cdot (\tilde{I}_n^{(s)'}(z_u,s))$$

$\equiv$ combined voltage source per unit length for W_u

$(\tilde{Z}_{c_{n,m}}(s))_u \equiv (\tilde{Y}_{c_{n,m}}(s))_u^{-1} \equiv$ characteristic impedance matrix for W_u

$(\tilde{\gamma}_{c_{n,m}}(s))_u \equiv (\tilde{\gamma}_{c_{n,m}}(s)) \equiv$ propagation matrix for W_u (7.3)

Note that for W_u the current $(\tilde{I}_n(z_u,s))$ convention is taken as positive in the direction of increasing z_u (as in fig. 2.5). Likewise, the voltage source per unit length $(\tilde{V}_n^{(s)'}(z_u,s))$ (including any discrete voltage sources) is taken as positive increasing in the direction of increasing z_u. The first of (7.1) then takes the form for W_u as

$$(\tilde{V}_n(z_u,s))_u = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u \, z_u\} \cdot (\tilde{V}_n(0,s))_u$$
$$+ \int_0^{z_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [z_u - z_u']\} \cdot (\tilde{V}_n^{(s)'}(z_u',s))_u \, dz_u' \quad (7.4)$$

Next consider the - wave; call this W_v and set the coordinate dimension variables as

$$L_v \equiv L \equiv \text{length of path for } W_v$$

$$z_v \equiv L - z \equiv \text{wave coordinate for } W_v$$

$$0 \leq z_v \leq L_v$$

$$N_v \equiv N \equiv \text{number of conductors (less reference) on tube and and dimension of vectors for } W_v \tag{7.5}$$

The wave and source conventions are then

$$(\tilde{V}_n(z_v,s))_v \equiv (\tilde{V}_n(z,s))_- = (\tilde{V}_n(z_v,s)) + (\tilde{Z}_{c_{n,m}}(s))_v \cdot (\tilde{I}_n(z_v,s))$$

$$\equiv \text{combined voltage for } W_v$$

$$(\tilde{V}_n^{(s)'}(z_v,s))_v \equiv -(\tilde{V}_n^{(s)'}(z,s))_- = (\tilde{V}_n^{(s)'}(z_v,s)) + (\tilde{Z}_{c_{n,m}}(s))_v \cdot (\tilde{I}_n^{(s)'}(z_v,s))$$

$$\equiv \text{combined voltage source per unit length for } W_v$$

$$(\tilde{Z}_{c_{n,m}}(s))_v \equiv (\tilde{Y}_{c_{n,m}}(s))_v^{-1} \equiv \text{characteristic impedance matrix for } W_v$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_v \equiv (\tilde{\gamma}_{c_{n,m}}(s)) \equiv \text{propagation matrix for } W_v \tag{7.6}$$

Now for W_v the current $(\tilde{I}_n^{(s)}(z_v,s))$ convention is taken as positive in the direction of increasing z_v and, hence, of decreasing z (opposite to that for W_u). Similarly, the voltage source per unit length is taken as positive in the direction of increasing z_v which is the direction of decreasing z. This is so that for W_v the conventions are defined with respect to z_v in the same manner as for W_u they have been defined with respect to z_u. The second of (7.1) then takes the form for W_v as

$$(\tilde{V}_n(z_v,s))_v = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_v\; z_v\}\cdot(\tilde{V}_n(0,s))_v$$

$$+ \int_0^{z_v} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_v[z_v - z_v']\}\cdot(\tilde{V}_n^{(s)'}(z_v',s))_v\; dz_v' \qquad (7.7)$$

which is exactly the same as for W_u in (7.4). Hence, only one such equation need be considered; it is applicable for all $u = 1,2,\ldots,N_W$ thereby applying to all waves in the transmission-line network.

The chosen conventions for W_u and W_v to have the same form with respect to z_u and z_v are then important to simplifying the formulation of the network equations. With these choices we have relations between the two waves W_u and W_v on the same tube for coordinates and dimensions as

$$L_u \equiv L_v \equiv L$$

$$z_u + z_v = L \qquad (7.8)$$

$$N_u = N_v = N$$

The wave and source relations are (for uncombined quantities)

$$(\tilde{V}_n(z_u,s)) = (\tilde{V}_n(z_v,s))$$

$$(\tilde{I}_n(z_u,s)) = -(\tilde{I}_n(z_v,s))$$

$$(\tilde{V}_n^{(s)'}(z_u,s)) = -(\tilde{V}_n^{(s)'}(z_v,s))$$

$$(\tilde{I}_n^{(s)'}(z_u,s)) = (\tilde{I}_n^{(s)'}(z_v,s))$$

$$(\tilde{Z}_{c_{n,m}}(s))_u = (\tilde{Z}_{c_{n,m}}(s))_v = (\tilde{Z}_{c_{n,m}}(s))$$

$$(\tilde{\gamma}_{c_{n,m}}(s))_u = (\tilde{\gamma}_{c_{n,m}}(s))_v = (\tilde{\gamma}_{c_{n,m}}(s)) \qquad (7.9)$$

B. Relation of Combined-Voltage Waves on Both Ends of a Tube

Now in (7.4) (or equivalently, (7.7)) we have the combined voltage at any z_u in terms of the value (boundary condition) at $z_u = 0$. Setting $z_u = L_u$ we introduce the boundary value there as giving

$$(\tilde{V}_n(L_u,s))_u = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u\, L_u\}\cdot(\tilde{V}_n(0,s))_u + \int_0^{L_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u[L_u - z_u']\}\cdot(\tilde{V}_n^{(s)'}(z_u',s))_u\, dz_u' \qquad (7.10)$$

This evidently relates $(\tilde{V}_n(0,s))_u$ which is an outgoing wave from the junction at $z_u = 0$, to $(\tilde{V}_n(L_u,s))$ which is an incoming wave to the junction at $z_u = L_u$. This is used later with the scattering supermatrix to form the BLT equation for the transmission-line network.

As a matter of convention, let all sources be considered as being present in the tubes instead of the junctions. If one has a junction with an equivalent circuit containing sources, as for example in fig. 2.4, then the sources can be moved just across the terminals into the tube, a movement of zero distance. Note then that the boundary values $(\tilde{V}_n(0,s))_u$ and $(\tilde{V}_n(L_u,s))_u$ are combined voltages on the junction "side" of the connections to the junction. Given this convention again, note the different conventions for sources for the two different waves on a tube, as discussed above.

C. Propagation Characteristic Supermatrix

Considering the various terms in (7.10), let us first aggregate all the propagation terms not associated with the sources into a block diagonal propagation supermatrix as

$$((\tilde{\Gamma}_{n,m}(s))_{u,v}) \equiv \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_1 L_1\} \oplus \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_2 L_2\} \oplus \cdots \oplus \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_{N_W} L_{N_W}\}$$

$$\equiv \bigoplus_{u=1}^{N_W} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\} \qquad (7.11)$$

$\equiv$ propagation supermatrix

where the elementary matrices (blocks) are given by

$$(\tilde{\Gamma}_{n,m}(s))_{u,v} = \begin{cases} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\} & \text{for } u = v \\ (0_{n,m}) & \text{for } u \neq v \end{cases}$$

$$= 1_{u,v} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u L_u\} \tag{7.12}$$

D. <u>Source Supervector and Supermatrix Integral Operator</u>

Again from (7.10) let us define a source vector for W_u in traveling from $z_u = 0$ to $z_u = L_u$ as

$$(\tilde{V}_n^{(s)}(s))_u \equiv \int_0^{L_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [L_u - z'_u]\} \cdot (\tilde{V}_n^{(s)'}(z'_u,s))_u \, dz'_u \tag{7.13}$$

The source supervector is then merely

$$((\tilde{V}_n^{(s)}(s))_u) \equiv \left(\int_0^{L_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [L_u - z'_u]\} \cdot (\tilde{V}_n^{(s)'}(z'_u,s))_u \, dz'_u \right) \tag{7.14}$$

Once can factor the above result by the use of a supermatrix integral operator. Define the elementary matrix blocks of this operator as

$$(\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot)))_{u,v}$$

$$\equiv \begin{cases} \int_0^{L_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [L_u - z'_u]\}(\cdot) \, dz'_u & \text{for } u = v \\ (0_{n,m}) & \text{for } u \neq v \end{cases}$$

$$= 1_{u,v} \int_0^{L_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [L_u - z'_u](\cdot) \, dz'_u \tag{7.15}$$

where the argument $(\cdot)$ indicates the place to put the expression following the operator in order to perform the operation. This is defined so that

$$\left(\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot))\right)_{u,v}\cdot\left(\tilde{V}_n^{(s)'}(z'_u,s)\right)_v$$

$$= 1_{u,v}\,\left(\tilde{V}_n^{(s)}(s)\right)_v$$

$$= 1_{u,v}\int_0^{L_v}\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_v[L_v-z'_v]\}\cdot\left(\tilde{V}_n^{(s)'}(z'_v,s)\right)\,dz'_v \tag{7.16}$$

with the multiplication which is part of the operation taken in the dot product sense. We can then readily form

$$\left(\left(\tilde{V}_n^{(s)}(s)\right)_u\right) = \left(\left(\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot))\right)_{u,v}\right):\left(\left(\tilde{V}_n^{(s)'}(z'_u,s)\right)_u\right)$$

$$\left(\left(\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot))\right)_{u,v}\right)$$

$$\equiv\bigoplus_{u=1}^{N_W}\left(\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot))\right)_{u,v}$$

$$\equiv\bigoplus_{u=1}^{N_W}\int_0^{L_u}\exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u[L_u-z'_u]\}(\cdot)\,dz'_u$$

$\equiv$ propagation supermatrix integral operator

$$\left(\left(\tilde{V}_n^{(s)'}(z'_u,s)\right)_u\right)\equiv\text{distributed source supervector} \tag{7.17}$$

Note that the propagation supermatrix integral operator in (7.17) is a generalization of the propagation supermatrix in (7.11) to allow for continuous combined voltage sources along the wave coordinates instead of just the boundary conditions (equivalent sources) at the set of $z_u = 0$.

E. Combined Voltage Supervector

For completeness we have the aggregate of combined voltage vectors in (7.10) as

$((\tilde{V}_n(0,s))_u) \equiv$ combined voltage supervector of outgoing waves at the junctions

$((\tilde{V}_n(L_u,s))_u) \equiv$ combined voltage supervector of incoming waves at the junctions (7.18)

In this note we will formulate the BLT equation in terms of the outgoing waves at the junctions, but other forms are also possible.

VIII. BLT EQUATION

Combining the results of the previous derivations we can write the BLT equation for the description of the transmission-line network. We begin with the scattering supermatrix in Section VI which relates the incoming waves to the outgoing waves as

$$((\tilde{V}_n(0,s))_u) = ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{V}_n(L_u,s)_u) \tag{8.1}$$

using the combined voltage supervectors from subsection VII E. Note the distinction between incoming waves $(z_u = L_u)$ and outgoing waves $(z_u = 0)$ at the set of junctions or tube ends.

Next, relate the incoming waves at the output ends of the tubes $(z_u = L_u)$ to the same waves at the input end of the same tubes $(z_u = L_u)$, albeit at different junctions in general. Taking (7.10) in supermatrix form, we have

$$((\tilde{V}_n(L_u,s))_u)$$

$$= ((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{V}_n(0,s))_u) + ((\tilde{V}_n^{(s)}(s))_u)$$

$$= ((\tilde{\Gamma}_{n,m}(s))_u):((\tilde{V}_n(0,s))_u) + ((\tilde{\Lambda}_{n,m}(z_u',s;(\bullet)))_{u,v}):((\tilde{V}_n^{(s)'}(z_u',s))_u) \tag{8.2}$$

Combining (8.1) and (8.2) we have

$$((\tilde{V}_n(0,s))_u)$$

$$= ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v}):((\tilde{V}_n(0,s))_u) + ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{V}_n^{(s)}(s))_u) \tag{8.3}$$

That is rearranged by use of the supermatrix identity as

$$
\begin{aligned}
&[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v})]:((\tilde{V}_n(0,s))_u) \\
&\qquad = ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{V}_n^{(s)}(s))_u) \\
&\qquad = ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot)))_{u,v}):((\tilde{V}_n^{(s)'}(z'_u,s))_u)
\end{aligned}
\tag{8.4}
$$

This is one form of the BLT equation with the unknowns taken as the combined voltage waves leaving the junctions. Note again that all sources are given a convention as being on the tubes in the wave coordinates $0 \le z_u \le L_u$ so that they are picked up in the integration along the wave coordinates and are not included in the combined voltages at the junctions $((\tilde{V}_n(0,s))_u)$ which are being computed.

For computational purposes the BLT equation is one large matrix equation with square matrices of size $N \times N$ and vectors of dimension N where

$$
N = \sum_{u=1}^{N_W} N_u \tag{8.5}
$$

Noting, however, the sparse nature of these matrices with blocks of zeros, one may be able to take advantage of the partitioning used to construct the supermatrices to simplify computations.

IX. RECONSTRUCTION OF VOLTAGES AND CURRENTS

Having solved the BLT equation as in (8.4) in some form or other, we have a set of combined voltages such as the outgoing combined voltages $((\tilde{V}_n(0,s))_u)$ at the junctions. From these one can find voltages and currents essentially everywhere, including at the junction terminals (tube ends) and at arbitrary positions on the tubes.

Consider the important case of voltages and currents at the tube ends (junctions). Let the two waves on a particular tube be W_u and W_v as in subsection VIIA. Using the conventions established there we have

$$N_u = N_v = N \equiv \text{dimension of vectors}$$

$$L_u = L_v = L \equiv \text{length}$$

$$z_u + z_v = L \equiv \text{relation between two wave coordinates}$$

$$z = z_u = L - z_v \equiv \text{tube coordinate} \tag{9.1}$$

Then we have at $z=0$

$$(\tilde{V}_n(0,s)) = \frac{1}{2}\,[(\tilde{V}_n(0,s))_u + (\tilde{V}_n(L_v,s))_v]$$

$$(\tilde{I}_n(0,s)) = \frac{1}{2}\,(\tilde{Y}_{c_{n,m}}(s))\cdot[(\tilde{V}_n(0,s))_u - (\tilde{V}_n(L_v,s))_v] \tag{9.2}$$

with the current positive in the $+z$ direction or out of the junction at $z=0$. At the other end with $z=L$ we have

$$(\tilde{V}_n(L,s)) = \frac{1}{2}\,[(\tilde{V}_n(L_u,s))_u + (\tilde{V}_n(0,s))_v]$$

$$(\tilde{I}_n(L,s)) = \frac{1}{2}\,(\tilde{Y}_{c_{n,m}}(s))\cdot[(\tilde{V}_n(L_u,s))_u - (\tilde{V}_n(0,s))_v] \tag{9.3}$$

with current positive in the $+z$ direction or into the junction at $z=L$. For substitution into the above equations, one uses

$$(\tilde{V}_n(L_u,s))_u = (\tilde{\Gamma}_{n,m}(s))_{u,u} \cdot (\tilde{V}_n(0,s))_u + (\tilde{\Lambda}_{n,m}(z'_u,s;(\cdot)))_{u,u} \cdot (\tilde{V}_n^{(s)'}(z'_u,s))_u$$

$$(\tilde{V}_n(L_v,s))_v = (\tilde{\Gamma}_{n,m}(s))_{v,v} \cdot (\tilde{V}_n(0,s))_v + (\tilde{\Lambda}_{n,m}(z'_v,s;(\cdot)))_{v,v} \cdot (\tilde{V}_n^{(s)'}(z'_v,s))_v \tag{9.4}$$

so that W_u has $z_u = L_u$ related to $z_u = 0$ and W_v has $z_v = L_v$ related to $z_v = 0$. In this form the current and voltage vectors at the tube ends can be computed from the combined voltage vectors leaving the junctions (i.e., $((\tilde{V}_n(0,s))_u)$ from the BLT equation (8.4) and the combined sources along the tube via (9.4) for the two waves on the tube.

For more general positions along the tube of interest we have

$$(\tilde{V}_n(z,s)) = \frac{1}{2} [(\tilde{V}_n(z_u,s))_u + (\tilde{V}_n(z_v,s))_v]$$

$$(\tilde{I}_n(z,s)) = \frac{1}{2} (\tilde{Y}_{c_{n,m}}(s)) \cdot [(\tilde{V}_n(z_u,s)_u - (\tilde{V}_n(z_v,s))_v] \tag{9.5}$$

with current positive in the $+z$ direction which is equivalent to the $+z_u$ direction and to the $-z_v$ direction. For substituting into (9.5), one uses (7.4) and (7.7) repeated here as

$$(\tilde{V}_n(z_u,s))_u = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u \, z_u\} \cdot (\tilde{V}_n(0,s))_u + \int_0^{z_u} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_u [z_u - z'_u]\} \cdot (\tilde{V}_n^{(s)'}(z'_u,s))_u \, dz'_u$$

$$(\tilde{V}_n(z_v,s))_v = \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_v \, z_v\} \cdot (\tilde{V}_n(0,s))_v + \int_0^{z_v} \exp\{-(\tilde{\gamma}_{c_{n,m}}(s))_v [z_v - z'_v]\} \cdot (\tilde{V}_n^{(s)'}(z'_v,s))_v \, dz'_v \tag{9.6}$$

In this form the combined voltage supervectors $((\tilde{V}_n(0,s))_u)$ leaving the junctions as computed from the BLT equation (8.4) and the combined sources $((\tilde{V}_n^{(s)'}(z'_u,s))_u)$ along the tubes (or waves) can be used to compute the combined voltages and thereby the voltages and currents at any position $z = z_u = L_v - z_v$ along any tube of interest.

X. SOME FORMS OF SOLUTIONS OF BLT EQUATIONS

Having formulated the BLT equation (8.4), one can represent its solution in various ways. The reader should note that the particular form in (8.4) is only one of many forms the BLT equation can take; in this case the unknowns are the combined voltages scattered from (outward propagation from) the junctions.

Since the BLT equation has been cast in the form of a supermatrix equation, the solution can be written directly as

$$((\tilde{V}_n(0,s))_u)$$
$$= [((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v}]^{-1}$$
$$:((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Lambda}_{n,m}(z_u',s;(\cdot)))_{u,v}):((\tilde{V}_n^{(s)'}(z_u',s))_u) \qquad (10.1)$$

For each complex frequency s this solution can be directly computed via integration (for the distributed sources), supermatrix multiplication, and supermatrix inversion, typically by computer. However, this approach may have limited utility for some kinds of problems due to a desire for the transient behavior and/or the characterization of the solution (such as bounding it) for a large class of excitations $((\tilde{V}_n^{(s)'}(z_u',s))_u)$.

Considerable work has been done in representing the solution of electromagnetic scattering problems, as formulated in integral equations, in terms of the eigenmode expansion method (EEM) and the singularity expansion method (SEM). The literature on SEM and EEM is quite extensive and the reader can consult two review book chapters [10.1,2] concerning this subject and obtain a bibliography. While the SEM and EEM concepts have been cast in terms of electromagentic integral equations, there is a direct connection to matrix equations because of the moment method (MoM) which is used to matricize the integral equations, i.e., put the integral equations in a form for numerical evaluation as on a computer [10.3]. In fact, some of the original developments in SEM and EEM theory and application used matrix concepts to arrive at the needed ideas and techniques [10.4-6]. Hence, SEM and EEM are directly applicable to the

BLT equation, as in (8.4) or in other related forms of it. A few of the results are presented here to indicate the forms of some of the basic results as applicable to the supermatrix BLT equation.

In EEM form one defines eigenmode supervectors and eigenvalues via

$$[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v})]:((\tilde{V}_n(s))_u)_\beta = \tilde{\lambda}_\beta(s)((\tilde{V}_n(s))_u)_\beta$$

$$((\tilde{L}_n(s))_u)_\beta:[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v})] = \tilde{\lambda}_\beta(s)((L_n(s))_u)_\beta$$

$$\beta = 1,2,\ldots,N \tag{10.2}$$

where for distinct eigenvalues we have the biorthogonal property

$$((\tilde{L}_n(s))_u)_\beta:((\tilde{V}_n(s))_u)_{\beta'} = 0 \quad \text{for} \quad \beta \neq \beta' \tag{10.3}$$

This result also applies in the weaker case of independent eigensupervectors. From (8.5) we have N eigenvalues and assume the existence of N independent eigensupervectors (of both left and right kinds separately). The right eigenmodes are used to expand $(\tilde{V}_n(0,s))$ which gives the outgoing waves at the junctions. The left eigenmodes appear to be related to the incoming waves at the junctions, and this aspect will hopefully be considered in a future note.

Defining normalized eigensupervectors as

$$\begin{aligned} ((\tilde{v}_n(s))_u)_\beta &\equiv [((\tilde{L}_n(s))_u)_\beta:((\tilde{V}_n(s))_u)_\beta]^{-\frac{1}{2}} ((\tilde{V}_n(s))_u)_\beta \\ ((\tilde{\ell}_n(s))_u)_\beta &\equiv [((\tilde{L}_n(s))_u)_\beta:((\tilde{V}_n(s))_u)_\beta]^{-\frac{1}{2}} ((\tilde{L}_n(s))_u)_\beta \end{aligned} \tag{10.4}$$

we have the biorthonormal property

$$((\tilde{\ell}_n(s))_u)_\beta:((\tilde{v}_n(s))_u)_{\beta'} = 1_{\beta,\beta'} \tag{10.5}$$

This allows us to write the expansion

$$((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v})$$

$$= \sum_{\beta} \tilde{\lambda}_{\beta}(s) \ [((\tilde{L}_n(s))_u)_{\beta}:((\tilde{V}_n(s))_u)_{\beta}]^{-1}((\tilde{V}_n(s))_u)_{\beta}((\tilde{L}_n(s))_u)_{\beta}$$

$$= \sum_{\beta} \tilde{\lambda}_{\beta}(s) \ ((\tilde{v}_n(s))_u)_{\beta} \ ((\tilde{\ell}_n(s))_u)_{\beta} \tag{10.6}$$

which is an example of a dyadic expansion using a dyadic or outer product of supervectors. The inverse is

$$[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v})]^{-1}$$

$$= \sum_{\beta} \tilde{\lambda}_{\beta}^{-1}(s) \ ((\tilde{v}_n(s))_u)_{\beta} \ ((\tilde{\ell}_n(s))_u)_{\beta} \tag{10.7}$$

and the identity is

$$((1_{n,m})_{u,v}) = \sum_{\beta} \ ((\tilde{v}_n(s))_u)_{\beta} \ ((\tilde{\ell}_n(s))_u)_{\beta}$$

$$= \sum_{\beta} \ ((\tilde{\ell}_n(s))_u)_{\beta} \ ((\tilde{v}_n(s))_u)_{\beta} \tag{10.8}$$

Combining (10.6) with (10.8) also gives

$$((\tilde{S}_{n,m}(s))_{u,v}):((\tilde{\Gamma}_{n,m}(s))_{u,v}) = \sum_{\beta} [1 - \tilde{\lambda}_{\beta}(s)] \ ((\tilde{v}_n(s))_u)_{\beta} \ ((\tilde{\ell}_n(s))_u)_{\beta} \tag{10.9}$$

The solution of the BLT equation (8.4) can then be written as a sum of eigensupervector contributions as

$$((\tilde{V}_n(s))_u)$$

$$= \sum_\beta \tilde{\lambda}_\beta^{-1}(s)[((\tilde{\ell}_n(s))_u)_\beta : ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Lambda}_{n,m}(z_u',s;(\cdot)))_{u,v}) : ((\tilde{V}_n^{(s)'}(z_u',s))_u)]$$

$$((\tilde{v}_n(s))_u)_\beta \qquad (10.10)$$

Note that this solution expresses the outgoing combined voltages at the junctions in terms of eigenmodes at the junctions. These eigenmodes can be extended throughout the tubes of the transmission-line network by the techniques discussed in Section IX; these extended eigenmodes can then be used to construct the combined voltages and voltages and currents throughout the tubes. However, these eigenmodes are not anticipated to be simply related to the tube eigenmodes (Section III) which may be more appropriate for extending the combined voltages at the junctions to the combined voltages, voltages, and currents throughout the network tubes.

Concerning the SEM representation of the solution, there is much that can be adapted from the work on electromagnetic scattering and antenna problems. The general form of the solution of the BLT equation in the form expressed in (8.4) is

$$((\tilde{V}_n(0,s))_u) = \sum_\alpha \tilde{f}(s_\alpha)\, \tilde{\eta}_\alpha(s)\, ((v_n)_u)_\alpha\, (s-s_\alpha)^{-n_\alpha}$$

$$+ \text{ other singularity terms} \qquad (10.11)$$

where $\tilde{f}(s)$ (or $f(t)$) is some excitation waveform which appears in the combined sources $((\tilde{V}_n^{(s)'}(z_u,s))_u)$ and which is taken out so as to give some equivalent delta-function response in defining the coupling coefficients $\tilde{\eta}_\alpha(s)$. For present purposes, we can set $\tilde{f}(s) = 1$ assuming that the excitation has been appropriately normalized. The order of the pole is $n_\alpha = 1,2,\ldots,$ but here only the first order is considered; second order can be adapted from [10.2].

The natural mode supervectors are found from

$$[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s_\alpha))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s_\alpha))_{u,v}] : ((v_n)_u)_\alpha = ((0_n)_u)$$

$$((\ell_n)_u)_\alpha : [((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s_\alpha))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s_\alpha))_{u,v})] = ((0_n)_u) \qquad (10.12)$$

where the left mode supervectors are also referred to as the coupling supervectors. The natural frequencies are found as the solutions of

$$\det[((1_{n,m})_{u,v}) = ((\tilde{S}_{n,m}(s_\alpha))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s_\alpha))_{u,v})] = ((0_n)_u) \qquad (10.13)$$

These can be related to the eigenvalues via

$$D(s,\lambda) = \det[(1-\lambda)((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v})] \qquad (10.14)$$

for which we have

$$D(s,\tilde{\lambda}_\beta(s)) = 0$$

$$D(s_\alpha,0) = 0 \qquad (10.15)$$

from which we set

$$\alpha = (\beta,\beta') \qquad (10.16)$$

such that

$$\tilde{\lambda}_\beta(s_{\beta,\beta'}) = 0 \qquad (10.17)$$

associates the natural frequencies with the zeros of eigenvalues. The natural modes are similarly related to the eigenmodes as

$$((v_n)_u)_{\beta,\beta'} = N_{\beta,\beta'}((\tilde{v}_n(s_{\beta,\beta'}))_u)_\beta$$

$$((\ell_n)_u)_{\beta,\beta'} = M_{\beta,\beta'}((\tilde{\ell}_n(s_{\beta,\beta'}))_u)_\beta \qquad (10.18)$$

where $N_{\beta,\beta'}$ and $M_{\beta,\beta'}$ are complex constants related to the normalization chosen for the natural modes.

The class 1 coupling coefficients are given by

$$\tilde{\eta}_\alpha(s)$$

$$= \frac{((\ell_n)_u)_\alpha : ((\tilde{\Lambda}_{n,m}(z'_u, s_\alpha; (\cdot))))_{u,v}) : ((\tilde{V}_n^{(s)'}(z'_u, s_\alpha))_u)\; e^{-(s-s_\alpha)t_o}}{((\ell_n)_u)_\alpha : \frac{\partial}{\partial s}\left[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v})\right]\Big|_{s-s_\alpha} : ((v_n)_u)_\alpha}$$

$$= N^{-1}_{\beta,\beta'}\left[\frac{\partial}{\partial_s}\lambda_\beta(s)\Big|_{s=s_{\beta,\beta'}}\right]^{-1} ((\tilde{\ell}_n(s_{\beta,\beta'}))_u) : ((\tilde{\Lambda}_{n,m}(z'_u, s_{\beta,\beta'}; (\cdot))))_{u,v})$$

$$: ((\tilde{V}_n^{(s)'}(z'_u, s_{\beta,\beta'}))_u)\; e^{-(s-s_{\beta,\beta'})t_o} \qquad (10.19)$$

where the turn-on time t_o can be taken as a function of position (n and u indices) in the network. With this class 1 coupling coefficient, the time-domain form of (10.11) is

$$((V_n(0,t))_u) = \sum_\alpha \tilde{f}(s_\alpha)\,\tilde{\eta}_\alpha(s_\alpha)((v_n)_u)_\alpha\, e^{s_\alpha t}\, u(t-t_o)$$

$$+ \text{ other singularity terms} \qquad (10.20)$$

The class 2 coupling coefficients (corresponding to the SEM representation of the inverse matrix in (10.1)) are given by

$$\tilde{\eta}_\alpha(s)$$

$$= \frac{((e^{-(s-s_\alpha)t_o}\ell_n)_u)_\alpha : ((\tilde{\Lambda}_{n,m}(z'_u, s; (\cdot))))_{u,v}) : ((\tilde{V}_n^{(s)'}(z'_u, s))_u)}{((\ell_n)_u)_\alpha : \frac{\partial}{\partial_s}\left[((1_{n,m})_{u,v}) - ((\tilde{S}_{n,m}(s))_{u,v}) : ((\tilde{\Gamma}_{n,m}(s))_{u,v})\right]\Big|_{s=s_\alpha} : ((V_n)_u)_\alpha}$$

$$= N^{-1}_{\beta,\beta'}\left[\frac{\partial}{\partial_s}\tilde{\lambda}_\beta(s)\Big|_{s=s_{\beta,\beta'}}\right]^{-1} ((e^{-(s-s_{\beta,\beta'})t_o}\tilde{\ell}_n(s_{\beta,\beta'}))_u) : ((\tilde{\Lambda}(z'_u, s; (\cdot))))_{u,v})$$

$$: ((\tilde{V}_n^{(s)'}(z'_u, s))) \qquad (10.21)$$

where the turn-on time t_o can be taken as a function of two sets of position variables (n and u) in the network corresponding to both the summation with the left mode supervectors and the position of observation. In time domain the class 2 coupling coefficients give more complicated results than (10.20) for class 1 due to the appearance of a time convolution.

Like the eigenmodes, the natural modes can be extended throughout the transmission-line network and made a function of the z_u coordinates. These can then be used for representing voltage and current supervectors throughout the tubes in the nework.

This section has merely indicated some of the properties of BLT equations, particularly due to their formal similarity to electromagnetic integral equations. This analogy should provide much insight and future results.

References

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XI. CONCLUSION

This has been a long quest. While we have found a few things of apparent significance, the quest is not finished. As with many results the answers raise as many, if not more, questions. There are several general areas for future development that come to mind.

The BLT equation (including its alternate forms) expresses the characteristics of a multiconductor transmission-line network in a single supermatrix equation. In this form various properties of the network can be explored. Various properties related to energy and reciprocity can be formulated. In this regard, the symmetry properties of the various impedance and admittance matrices in the network need to be explored. This appears to have some relation to the diagonalizability properties of the propagation matrices.

A development parallel to transmission-line network topology is scatterer topology. In scatterer topology a hierarchical topology related to shielding concepts has been introduced. Perhaps this hierarchical topology can be introduced into some kinds of transmission-line networks to simplify their analysis and/or synthesis. Turning the question around, perhaps the transmission-line network topology and the BLT equation can aid in developing new insights into scatterer topology and the associated equations describing the electromagnetic scattering.

EPILOGUE

"Then he subdued the Pisidians who made head against him, and conquered the Phrygians, at whose chief city, Gordium, which is said to be the seat of the ancient Midas, he saw the famous chariot fastened with cords made of the rind of the cornel-tree, which whosoever should untie, the inhabitants had a tradition, that for him was reserved the empire of the world. Most authors tell the story that Alexander finding himself unable to untie the knot, the ends of which were secretly twisted round and folded up within it, cut it asunder with his sword. But Aristobulus tells us it was easy for him to undo it, by only pulling the pin out of the pole, to which the yoke was tied, and afterwards drawing off the yoke itself from below."

From The Lives of the Noble Grecians and Romans, by Plutarch, translated by John Dryden, revised by Arthur Hugh Clough, Modern Library, Random House, reprint of Clough edition (1864), from the section on Alexander the Great.

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