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Harvey J. Fletcher
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Abstract

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ABSTRACT

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1.0 INTRODUCTION

The problem of electromagnetic penetration of cavities has been discussed by many authors. L-W. Chen[1] has discussed the penetration of a rectangular cavity through a small elliptical aperture using a quasi-static approach. D.K. Cheng and C.A. Chen[11] have treated this problem for the case of an electromagnetic pulse incident on an aperture of arbitrary size and shape, but explicit expressions for the fields were not obtained. Other investigators have examined various kinds of nonrectangular or general cavities, for which the reader is referred to the bibliography of C.M. Butler et al.[12]

This paper includes an extension of L-W. Chen's work to give explicit expressions for the fields in a rectangular cavity as infinite series of the natural modes. The elliptical aperture is taken to be small so that the quasi-static approach may be used, but a dipole approximation is not used at the aperture. The position and orientation of the aperture are arbitrary. These results are extended to cavities with non-perfectly-conducting walls.

Fourier transform techniques are used to find explicitly the response of a perfectly conducting cavity to a plane EMP. The position and orientation of the aperture are again arbitrary.

2.0 ASSUMPTIONS

The fields in free space satisfy the relations

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{H} = 0$$

where $\vec{E}$ is the electric field and $\vec{H}$ is the magnetic field and ε_0 and μ_0 are constants.

By taking the curl of these equations, one obtains

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

and

$$\nabla^2 \vec{H} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

If one assumes that $\vec{E} = \vec{E}_0(x,y,z)e^{j\omega t}$, then $\nabla^2 \vec{E} = -k^2 \vec{E}$ where $k = \frac{\omega}{c}$. If the frequency is low, then the $k^2 \vec{E}$ term can be neglected and $\vec{E}$ satisfies the Laplace Equation. This is equivalent to saying $\nabla x \vec{E} = 0$ or $\vec{E} = -\nabla\phi$ and similarly $\vec{H} = -\nabla\phi^*$ where $\nabla^2\phi = \nabla^2\phi^* = 0$.

Consider a plane wave incident on an aperture of a cavity as in Figure 2.1.

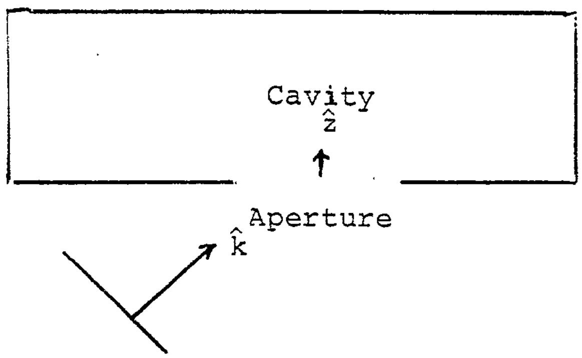

FIGURE 2.1 Incident Wave

Let the incident wave have the orientation with respect to the x,y, and z axes given by Euler angles[5]

$$\begin{pmatrix} \hat{E}^i \\ \hat{H}^i \\ \hat{k} \end{pmatrix} = \begin{pmatrix} C\psi C\phi - C\theta S\phi S\psi & C\psi S\phi + C\theta C\phi S\psi & S\psi S\theta \\ -S\psi C\phi - C\theta S\phi C\psi & -S\psi S\phi + C\theta C\phi C\psi & C\psi S\theta \\ S\theta S\psi & -S\theta C\psi & C\theta \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

where $\hat{z}$ is the unit vector perpendicular to a planar aperture of zero thickness. Let $\hat{x}$ and $\hat{y}$ be perpendicular to $\hat{z}$ and in the plane of the aperture. ϕ represents a rotation of the x,y,z axes about the z axis to form x', y', z axis. θ represents a subsequent rotation about the x' axis to form x', y", z' axes. Then,

ψ represents a rotation about z' to obtain x", y", z' axes which are in the direction of $\vec{E}^i$, $\vec{H}^i$, $\hat{k}$ as in Figure 2.2.

$\hat{k}$ is a unit vector in the direction of the Poynting vector and must not be confused with $\hat{z}$. $\vec{E}^i$ is the incident electric vector and $\vec{H}^i$ is the incident magnetic vector.

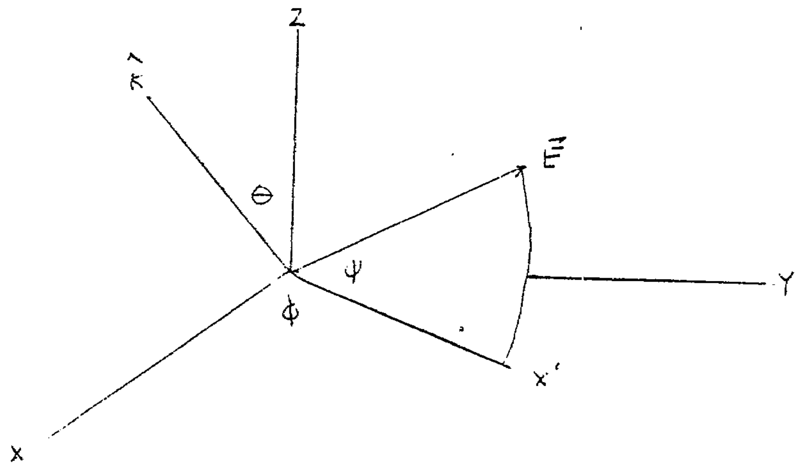

FIGURE 2.2 Euler Angles

From these definitions it is apparent that θ is the angle between the Poynting vector and the z axis. ψ is the angle between the intersection of the wave plane with the xy plane and $\vec{E}$. φ is the angle between the x axis and the intersection of the xy plane with the wave plane. Thus, θ and φ give the direction of the wave and ψ is the polarization angle. The incident fields can be expressed as

$$\vec{E}^i = E_0{}^i\ [\hat{x}(C\psi C\theta - C\theta S\phi S\psi) + \hat{y}\ (C\psi S\phi + C\theta C\phi S\psi) + \hat{z}\ S\psi S\theta]$$
$$f(t - \frac{\hat{k}\cdot\vec{r}}{c_0})$$

$$\vec{H}^i = H_0{}^i\ [\hat{x}(-S\psi C\phi - C\theta S\phi C\psi) + \hat{y}(-S\psi S\phi + C\theta C\phi C\psi) + \hat{z}\ C\psi S\theta]$$

$$\cdot f(t - \frac{\hat{k}\cdot\vec{r}}{c_0})$$

Where S, C stand for sine and cosine and c_0 is the velocity of light ($c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$). A pulse (EMP) would be a special case of the time factor $f(t)$.

If we first solve the problem with $f(t) = e^{j\omega t}$, we can find the general result of EMP by taking the Fourier Transform.

If $\hat{k}$, the unit vector in the direction of the incident wave, is equal to $k_1\hat{x} + k_2\hat{y} + k_3\hat{z}$, then the unit vector in the direction of the reflected wave must be $k_1\hat{x} + k_2\hat{y} - k_3\hat{z}$.

In order that the total wave have the proper boundary conditions at a conducting surface, the reflecting electric field must be of the form

$$\vec{E}^r = E_0{}^i\ [-\hat{x}(C\psi C\theta - C\theta S\phi S\psi)\ -\hat{y}(C\psi S\phi + C\theta C\phi S\psi) + \hat{z}\ S\psi S\theta]$$

$$\cdot\ f(t - k_1 x - k_2 y + k_3 z)$$

where $k_1 = S\theta S\phi$, $k_2 = -S\theta C\psi$, and $k_3 = C\theta$

$$\vec{E}^o = \vec{E}^i + \vec{E}^r$$

$$\vec{H}^o = \vec{H}^i + \vec{H}^r$$

These satisfy the boundary conditions

$$\hat{z}x\vec{E}^o = 0, \ \frac{\partial}{\partial z}\hat{z}\cdot\vec{E}^o = 0, \ \hat{z}\cdot\vec{H}^o = 0, \ \hat{z}\cdot\vec{E}^o = 2\hat{z}\cdot\vec{E}^i \text{ at } z=0$$

where

$$\vec{H}^i = \frac{1}{Z_0}\hat{k}^i x\vec{E}^i \qquad \text{and } \vec{H}^r = \frac{1}{Z_0}\hat{k}^r x\vec{E}^r$$

$$\vec{E}^i = -Z_0\hat{k}^i x\vec{H}^i \qquad \text{and } E = -Z_0\hat{k}^r x\vec{H}^r$$

where $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is the free wave impedance. Thus, the fields in the absence of an aperture are

$$\vec{E}^o = \hat{z}E_z^o$$

$$\vec{H}^o = \hat{x}H_x{}^o + \hat{y}H_y{}^o$$

E_z^o, $H_x{}^o$, and $H_y{}^o$ are constants given by

$$E_z{}^o = 2E_o{}^i \ S\psi S\theta$$

$$H_x{}^o = 2H_o{}^i \ (-S\psi C\phi - C\theta S\phi C\psi)$$

$$H_y{}^o = 2H_o{}^i \ (-S\psi S\phi + C\theta C\phi C\psi)$$

In these and all subsequent equations up to and including Section 10.0, the time dependence $e^{j\omega t}$ is implicitly understood. Let us seek a solution for a sinusoidal input which has a wavelength which is large compared to the diameter of the aperture. This can be approximated by letting f(t) = 1. In other words, the incident wave would produce a constant field across the aperture. We neglect also the departure from a plane wave due to the front face of the cavity being non-infinite. We also have assumed that the outside surface of the cavity is a perfect conductor. We cannot assume the inside is a perfect conductor or there would be infinite resonances when the incoming wave had a frequency equal to one of the natural frequencies of the cavity.

Let us assume the aperture thickness is infinitesimally small and that the field at the aperture is the same as that produced in an infinite sheet with the same shape aperture. The presence of a large cavity is assumed not to affect the field very much at the aperture.

3.0 THE ELECTRIC FIELD DUE TO AN ELLIPTIC APERTURE

To find the fields in the aperture we find the fields in a similar shape aperture in an infinite conducting screen and assume that this is a good approximation. We can thus find the electric potential which satisfies

$$\nabla^2 \phi = 0$$

$$\vec{E} = -\nabla \phi$$

$\vec{E} = \vec{E}_0 = 2E_0^{\,i}\ S\psi\ S\theta\ \hat{z}$ at large distances from the aperture on the incident side

$\vec{E} = 0$ at large distances from the aperture on the shadow side

$\hat{n} \times \vec{E} = 0$ on the inside and outside of the screen

$\vec{E}$ and $\frac{\partial \vec{E}}{\partial Z}$ are continuous everywhere particularly at the aperture

Ellipsoidal coordinates are the natural coordinates for this problem (see appendix).

The potential if no aperture were present would be

$$\phi_0 \equiv -E^0 z = -\frac{\sqrt{\xi \eta \zeta}}{ab}\ E_z^0 \ \text{sign}\ z$$

Assume the general potential to be

$$\phi = \phi_0 \ F(\xi)$$

Substituting this into Laplace Equation one gets that $F(\xi)$ must satisfy

$$\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} \ln (\xi R_\xi) = 0$$

or

$$F(\xi) = A \int_\xi^\infty \frac{d\xi}{\xi^{3/2}(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}} + B$$

Therefore

$$\phi_1(Z<0) = \phi_0^- \ [A \int_\xi^\infty + B]$$

$$\phi_2(Z>0) = \phi_0^+ \ [C \int_\xi^\infty + D]$$

with the boundary conditions

$$\phi_1(\xi=\infty) = \phi_0^-$$

$$\phi_2(\xi=\infty) = 0$$

$$\lim_{\xi\to 0} \phi_1 = \lim_{\xi\to 0} \phi_2$$

$$\lim_{\xi\to 0} \frac{\partial\phi_1}{\partial Z} = \lim_{\xi\to 0} \frac{\partial\phi_2}{\partial Z}$$

The condition that tangential $\vec{E}$ is zero on the screen is automatically satisfied.

The other boundary conditions are satisfied if D=0, B=1

$$A = -C = \frac{1}{2 \lim_{\xi\to 0} \left[\frac{2z^2(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}}{(\xi-\eta)(\xi-\zeta)\ \xi^{3/2}} - \frac{2(\xi+b^2)^{1/2}}{\xi^{1/2} b^2 (\xi+a^2)^{1/2}} \right.}$$

$$\left. + \frac{2E(\alpha,e)}{ab^2} \right]$$

$$\alpha = \cot^{-1} \frac{\xi^{1/2}}{a} \qquad e = \sqrt{1 - b^2/a^2}$$

E is the elliptic integral of the first kind. (See appendix for integrals.) Care must be taken since this is an indeterminate form. After taking the limit, we obtain

$$A = \frac{ab^2}{4E(\frac{\pi}{2}, e)}$$

$$\phi_2 = \frac{ab^2 E_z^0 z}{4E(\frac{\pi}{2}, e)} \int_\xi^\infty \frac{d\xi}{\xi^{3/2}(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}}$$

$$= \frac{ab^2 E_z^0 z}{2E(\frac{\pi}{2},e)} \left[- \frac{E(\alpha,e)}{ab^2} + \frac{(\xi+b^2)^{1/2}}{b^2 \xi^{1/2} (\xi+a^2)^{1/2}} \right]$$

The electric fields on the shadow side can be found from

$$\vec{E} = - \nabla\phi_2$$

If rectangular coordinates are used for the ∇ operator, then ξ can be thought of as an implicit function of x,y, and z. $\xi_x \equiv \frac{\partial\xi}{\partial z}$, $\xi_y \equiv \frac{\partial\xi}{\partial y}$, $\xi_z \equiv \frac{\partial\xi}{\partial z}$ can be found in the appendix.

These are obtained by implicit differentiation of the expressions $x = x(\xi,\eta,\zeta)$, $y = y(\xi,\eta,\zeta)$, and $z = z(\xi,\eta,\zeta)$. These give for $\vec{E}$

$$E_x = \frac{2E_z{}^0 Axz(\xi+b^2)^{1/2}}{\xi^{1/2}(\xi+a^2)^{1/2}(\xi-\zeta)(\xi-\eta)}$$

$$E_y = \frac{2E_z{}^0 Ayz(\xi+a^2)^{1/2}}{\xi^{1/2}(\xi+b^2)^{1/2}(\xi-\zeta)(\xi-\eta)}$$

$$E_z = \frac{2E_z{}^0 A}{a^2 b^2} \left[aE(\alpha, e) - \frac{\xi^{1/2}(\xi+b^2)^{1/2}(a^2\xi - a^2\eta - a^2\zeta - \eta\zeta)}{(\xi-\eta)(\xi-\zeta)(\xi+a^2)^{1/2}} \right]$$

where

$$A = \frac{ab^2}{4E(\frac{\pi}{2},e)}, \quad e = \sqrt{1 - \frac{b^2}{a^2}}, \quad \alpha = \cot^{-1} \frac{\xi^{1/2}}{a}$$

Special values of these fields are given in Section 5.0.

The fields on the incident side of the screen are given by symmetry conditions of Bethe:[10]

$$E_x(x,y,z) = E_x\ (x,y,z)$$

$$E_y(x,y,-z) = E_y(x,y,z)$$

$$E_z(x,y,-z) = E_z^{o}-E_z(x,y,z)$$

$$H_x(x,y,-z) = H_x^{o}-H_x(x,y,z)$$

$$H_y(x,y,-z) = H_y^{o}-H_y(x,y,z)$$

$$H_z(x,y,-z) = H_z(x,y,z)$$

4.0 THE MAGNETIC FIELD DUE TO AN ELLIPTIC APERTURE

In a similar fashion, the magnetic field is given by $\vec{H} = -\nabla\phi^*$ where $\nabla^2\phi^* = 0$. As $z \to -\infty$, $\phi^* \to -H_x^0 x - H_y^0 y$ where as before,

$$H_x^o = 2H_o^i\,(-S\psi C\phi - C\theta S\phi C\psi)$$

$$H_y^o = 2H_o^i\,(-S\psi S\phi + C\theta C\phi C\psi)$$

Let us treat each component independently.

$$\phi_o^* = -H_x^o x = -\left[\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{a^2(a^2-b^2)}\right]^{1/2} H_x^o \text{ Sign } x$$

Let $\phi^* = \phi_o^* F(\xi)$

After substituting this in Laplace Equation (see appendix) and solving for $F(\xi)$, we obtain

$$F(\xi) = A \int_\xi^\infty \frac{d\xi}{\xi^{1/2}(\xi+a^2)^{3/2}(\xi+b^2)^{1/2}} + B$$

$$\phi_1^*(z \leq 0) = (A \int_\xi^\infty + B)\,\phi_0^*$$

$$\phi_2^*(z \geq 0) = (C \int_\xi^\infty + D)\,\phi_0^*$$

The boundary conditions are

$$\phi_1^*(\xi=\infty) = \phi_0^*$$

$$\phi_2^*(\xi=\infty) = 0$$

$$\phi_1^* = \phi_2^* \quad \text{at } \xi=0$$

$$\lim_{z\to 0^-} \frac{\partial \phi_1^*}{\partial z} = \lim_{z\to 0^+} \frac{\partial \phi_2^*}{\partial z} \qquad \text{at } \xi=0$$

These equations determine the constants

$$B=1,\ D=0,\ -A=+C = \frac{1}{2\int_0^\infty \frac{d\xi}{\xi^{1/2}(\xi+a^2)^{3/2}(\xi+b^2)^{1/2}}}$$

These integrals are given in the appendix. A similar procedure is used to determine the components of $\vec{H}$ due to H_y^0. If we add the fields to H_x^0 and H_y^0, we get the total magnetic field as

$$H_x = \frac{H_x^0}{2}\left[F(\tfrac{\pi}{2},e) - E(\tfrac{\pi}{2},e)\right]^{-1}\left[F(\alpha,e) - E(\alpha,e)\right.$$

$$\left. - \frac{\xi^{1/2}(\xi+b^2)^{1/2}(\eta+a^2)(\zeta+a^2)}{a(\xi-\zeta)(\xi-\eta)(\xi+a^2)^{1/2}}\right]$$

$$- \frac{H_y^0\ ab^2(a^2-b^2)\ xy\xi^{1/2}}{2(\xi-\zeta)(\xi-\eta)(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}\left[a^2E(\tfrac{\pi}{2},e) - b^2F(\tfrac{\pi}{2},e)\right]}$$

$$H_y = \frac{-H_x^0 \, a^3 e^2 xy \xi^{1/2}}{2(\xi-\eta)(\xi-\zeta)\left[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)\right] (\xi+a^2)^{1/2}(\xi+b^2)^{1/2}}$$

$$- \frac{H_y^0 \, ab^2(a^2-b^2) \; y^2 \xi^{1/2}(\xi+a^2)^{1/2}}{2(\xi-\zeta)(\xi-\eta)(\xi+b^2)^{3/2}\left[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)\right]}$$

$$+ \frac{H_y^0}{2} \; \frac{\left[a^2 E(\alpha,e) - b^2 F(\alpha,e) - \frac{a\xi^{1/2}(a^2-b^2)}{(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}}\right]}{\left[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)\right]}$$

$$H_z = \frac{-H_x^0 \, a^3 e^2 \; xz \; (\xi+b^2)^{1/2}}{2\xi^{1/2}(\xi-\eta)(\xi-\zeta)(\xi+a^2)^{1/2}\left[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)\right]}$$

$$- \frac{H_y^0 \, ab^2(a^2-b^2) \; yz \; (\xi+a^2)^{1/2}}{2\xi^{1/2}(\xi-\eta)(\xi-\zeta)(\xi+b^2)^{1/2}\left[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)\right]}$$

where $\alpha = \cot^{-1} \frac{\xi^{1/2}}{a}$, $e = \sqrt{1 - b^2/a^2}$ and F and E are elliptical integrals of the first and second kind.

5.0 TABLE OF FIELDS FOR SPECIAL CASES

All expressions in this table are for the shadow ($z \geq 0$) side of the screen. Definitions are:

$$\alpha = \cot^{-1} \frac{\sqrt{\xi}}{a}$$

$$e = \sqrt{1 - b^2/a^2}$$

$F(\phi,k)$ and $E(\phi,k)$ are elliptic integrals of the first and second kind, respectively

$\lim_{b \to a} \xi = \xi_{ob}$ and $\lim_{b \to a} \eta = \eta_{ob}$, where the subscripts refer to the oblate spheroidal coordinates used with a circular aperture. Thus, we may consider these two oblate spheroidal coordinates as special cases of the corresponding ellipsoidal coordinates when b=a. For this reason, no distinction is made between ξ and ξ_{ob}, η and η_{ob} in the table.

In the circular aperture case, it has sometimes been useful to express formulas in terms of

$$\vec{\rho} = x\hat{x} + y\hat{y}$$

$$\rho = \sqrt{x^2+y^2} = \sqrt{\frac{(\xi+a^2)(\eta+a^2)}{a^2}}$$

and

$$\hat{\rho} = \frac{\vec{\rho}}{\rho}$$

See appendix for values of ξ, η and ζ on the aperture, screen and axis and at large distances.

CIRCLE

5.1 ELECTRIC AND MAGNETIC POTENTIALS

$$\Phi_E = -\frac{E_z^0\sqrt{-\eta}}{\pi}\left(\frac{\alpha\sqrt{\xi}}{a} - 1\right)$$

$$\Phi_M = \frac{\sqrt{(\xi+a^2)(\eta+a^2)}}{\pi a}\left(\alpha - \frac{a\sqrt{\xi}}{\xi+a^2}\right)\left(H_x^0 \cos\phi + H_y^0 \sin\phi\right)$$

5.2 ELECTRIC AND MAGNETIC FIELDS

$$\vec{E} = \frac{E_z^0 a^2\sqrt{-\eta}\ \vec{\rho}}{\pi(\xi-\eta)(\xi+a^2)} + \frac{\vec{E}^0}{\pi}\left(\alpha - \frac{a\sqrt{\xi}}{\xi-\eta}\right)$$

or

$$E_x = \frac{E_z^0 a^2\sqrt{-\eta}\ x}{\pi(\xi-\eta)(\xi+a^2)}$$

$$E_y = \frac{E_z^0 a^2\sqrt{-\eta}\ y}{\pi(\xi-\eta)(\xi+a^2)}$$

ELLIPSE

5.1 ELECTRIC AND MAGNETIC POTENTIALS

$$\Phi_E = \frac{E_z^0 z}{2E(\frac{\pi}{2},e)} \left[-E(\alpha,e) + \frac{a\sqrt{\xi+b^2}}{\sqrt{\xi(\xi+a^2}}\right]$$

$$\Phi_M = \frac{xH_x^0}{2}\,\frac{F(\alpha,e) - E(\alpha,e)}{F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)} - \frac{H_y^0 a^3 e^2 y}{2[a^2E(\frac{\pi}{2},e) - b^2F(\frac{\pi}{2},e)]}$$

$$\cdot \left[\frac{a^2E(\alpha,e) - b^2F(\alpha,e)}{a^3e^2} - \sqrt{\frac{\xi}{(\xi+a^2)(\xi+b^2)}}\right]$$

5.2 ELECTRIC AND MAGNETIC FIELDS

$$E_x = \frac{E_z^0 ab^2 xz\sqrt{\xi+b^2}}{2(\xi-\eta)(\xi-\zeta)\sqrt{\xi(\xi+a^2)}\,E(\frac{\pi}{2},e)}$$

$$E_y = \frac{E_z^0 ab^2 yz\sqrt{\xi+a^2}}{2(\xi-\eta)(\xi-\zeta)\sqrt{\xi(\xi+b^2)}\,E(\frac{\pi}{2},e)}$$

$$E_z = \frac{E_z^0}{\pi} \left(\alpha - \frac{a\sqrt{\xi}}{\xi-\eta}\right)$$

$$\vec{H} = \frac{\vec{H}^0}{\pi} \left(\alpha - \frac{a\sqrt{\xi}}{\xi+a^2}\right) - \frac{2a^2\vec{H}^0\cdot\vec{\rho}\,[a\sqrt{\xi}\;\vec{\rho} + (\xi+a^2)\sqrt{-\eta}\;\hat{z}]}{\pi(\xi+a^2)^2(\xi-\eta)}$$

or

$$H_x = \frac{H_x^0}{\pi} \left(\alpha - \frac{a\sqrt{\xi}}{\xi+a^2}\right) - \frac{2a(\eta+a^2)\sqrt{\xi}\cos\phi}{\pi(\xi+a^2)(\xi-\eta)} (H_x^0 \cos\phi + H_y^0 \sin\phi)$$

$$H_y = \frac{H_y^0}{\pi} \left(\alpha - \frac{a\sqrt{\xi}}{\xi+a^2}\right) - \frac{2a\sqrt{\xi}(\eta+a^2)\sin\phi}{\pi(\xi+a^2)(\xi-\eta)} (H_x^0 \cos\phi + H_y^0 \sin\phi)$$

$$E_z = \frac{E_z^0}{2E(\frac{\pi}{2},e)} \left[E(\alpha,e) - \frac{(a^2\xi - a^2\eta - a^2\zeta - \eta\zeta\sqrt{\xi(\xi+b^2)}}{a(\xi-\eta)(\xi-\zeta)\sqrt{\xi+a^2}}\right.$$

$$H_x = \frac{H_x^0}{2}\left[F(\tfrac{\pi}{2},e) - E(\tfrac{\pi}{2},e)\right]^{-1}\left[F(\alpha,e) - E(\alpha,e) - \frac{(\eta+a^2)(\xi+a^2)\sqrt{\xi(\xi+b^2)}}{a(\xi-\eta)(\xi-\zeta)\sqrt{\xi+a^2}}\right]$$

$$- \frac{H_y^0 a^3 b^2 e^2 xy\sqrt{\xi}}{2(\xi-\eta)(\xi-\zeta)\sqrt{(\xi+a^2)(\xi+b^2)}\left[a^2E(\frac{\pi}{2},e) - b^2F(\frac{\pi}{2},e)\right]}$$

$$H_y = - \frac{H_x^0 a^3 e^2 xy\sqrt{\xi}}{2(\xi-\eta)(\xi-\zeta)\sqrt{(\xi+a^2)(\xi+b^2)}\left[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)\right]}$$

$$- \frac{H_y^0 a^3 b^2 e^2 y^2\sqrt{\xi(\xi+a^2)}}{2(\xi-\eta)(\xi-\zeta)\sqrt{(\xi+b^2)^3}\left[a^2E(\frac{\pi}{2},e) - b^2F(\frac{\pi}{2},e)\right]}$$

$$+ \frac{H_y^0}{2}\left[a^2E(\tfrac{\pi}{2},e) - b^2F(\tfrac{\pi}{2},e)\right]^{-1}\left[a^2E(\alpha,e) - b^2F(\alpha,e)\right.$$

$$\left. - \frac{a^3e^2\sqrt{\xi}}{\sqrt{(\xi+a^2)(\xi+b^2)}}\right]$$

$$H_z = - \frac{2a^2\sqrt{-\eta}\,(xH_x^0 + yH_y^0)}{\pi(\xi+a^2)(\xi-\eta)}$$

5.3 FIELDS ON APERTURE

$$\vec{E} = \frac{\vec{E}^0}{2} + \frac{E_z\vec{\rho}}{\pi\sqrt{a^2-\rho^2}}$$

or

$$E_x = \frac{xE_z^0}{\pi\sqrt{a^2-x^2-y^2}}$$

$$E_y = \frac{yE_z^0}{\pi\sqrt{a^2-x^2-y^2}}$$

$$E_z = \frac{E_z^0}{2}$$

$$H_z = -\frac{H_x^0 a^3 e^2 xz\sqrt{\xi+b^2}}{2(\xi-\eta)(\xi-\zeta)\sqrt{\xi(\xi+a^2)}\,[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)]}$$

$$-\frac{H_y^0 a^3 b^2 e^2 yz\sqrt{\xi+a^2}}{2(\xi-\eta)(\xi-\zeta)\sqrt{\xi(\xi+b^2)}\,[a^2E(\frac{\pi}{2},e) - b^2F(\frac{\pi}{2},e)]}$$

5.3 FIELDS ON APERTURE

$$E_x = \frac{E_z^0 bx}{2a^2E(\frac{\pi}{2},e)\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

$$E_y = \frac{E_z^0 y}{2bE(\frac{\pi}{2},e)\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

$$E_z = \frac{E_z^0}{2}$$

$$\vec{H} = \frac{\vec{H}^0}{2} - \frac{2(\vec{H}^0 \cdot \vec{\rho})}{\pi\sqrt{a^2-\rho^2}} \hat{z}$$

or

$$H_x = \frac{H_x^0}{2}$$

$$H_y = \frac{H_y^0}{2}$$

$$H_z = -\frac{2(xH_x^0 + yH_y^0)}{\pi\sqrt{a^2-x^2-y^2}}$$

5.4 FIELDS ON SCREEN

$$\vec{E} = \frac{\vec{E}^0}{\pi}\left(\cot^{-1}\sqrt{\rho^2/a^2 - 1}\right) - \frac{1}{\rho^2/a^2 - 1} = \frac{\vec{E}^0}{\pi}\left(\alpha - \frac{a}{\sqrt{\xi}}\right)$$

$$H_x = \frac{H_x^0}{2}$$

$$H_y = \frac{H_y^0}{2}$$

$$H_z = -\frac{H_x^0 e^2 x}{2b[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)]\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

$$- \frac{H_y^0 e^2 y a^2}{2b[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)]\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

5.4 FIELDS ON SCREEN

$$\vec{E} = \frac{\vec{E}^0}{2E(\frac{\pi}{2},e)} [E(\alpha,e) - \frac{a\sqrt{\xi+b^2}}{\sqrt{\xi(\xi+a^2)}}]$$

$$\vec{H} = \frac{\vec{H}^0}{\pi}\left(\cot^{-1}\sqrt{\rho^2/a^2 - 1} - \frac{\sqrt{\rho^2/a^2 - 1}}{\rho^2/a^2}\right) - \frac{2(\vec{H}^0 \cdot \hat{\rho})\,\hat{\rho}}{\pi(\rho^2/a^2)\sqrt{\rho^2/a^2 - 1}}$$

$$= \frac{\vec{H}^0}{\pi}\left(\alpha - \frac{a\sqrt{\xi}}{\xi + a^2}\right) - \frac{2a^3(\vec{H}^0 \cdot \hat{\rho})\,\hat{\rho}}{\pi(\xi + a^2)\sqrt{\xi}}$$

or

$$H_x = \frac{H_x^0}{\pi}\left(\alpha - \frac{a\sqrt{\xi}}{\xi + a^2}\right) - \frac{2a^3 x(xH_x^0 + yH_y^0)}{\pi(\xi + a^2)^2\sqrt{\xi}}$$

$$H_y = \frac{H_y^0}{\pi}\left(\alpha - \frac{a\sqrt{\xi}}{\xi + a^2}\right) - \frac{2a^3 y(xH_x^0 + yH_y^0)}{\pi(\xi + a^2)^2\sqrt{\xi}}$$

$$H_x = \frac{H_x^0}{2}[F(\tfrac{\pi}{2},e) - E(\tfrac{\pi}{2},e)]^{-1}[F(\alpha,e) - E(\alpha,e) - \frac{a(\xi+a^2)\sqrt{\xi+b^2}}{(\xi-\zeta)\sqrt{\xi(\xi+a^2)}}$$

$$- \frac{H_y^0 a^3 b^2 e^2 xy}{4(\xi-\zeta)\sqrt{\xi(\xi+a^2)(\xi+b^2)}[a^2E(\tfrac{\pi}{2},e) - b^2F(\tfrac{\pi}{2},e)]}$$

$$H_Y = - \frac{H_x^0 a^3 e^2 xy}{2(\xi-\zeta)\sqrt{\xi(\xi+a^2)(\xi+b^2)}[F(\tfrac{\pi}{2},e) - E(\tfrac{\pi}{2},e)]}$$

$$- \frac{H_y^0 a^3 b^2 e^2 y^2\sqrt{\xi+a^2}}{2(\xi-\zeta)\sqrt{\xi(\xi+b^2)^3}[a^2E(\tfrac{\pi}{2},e) - b^2F(\tfrac{\pi}{2},e)]}$$

$$+ \frac{H_Y^0}{2}[a^2E(\tfrac{\pi}{2},e) - b^2F(\tfrac{\pi}{2},e)]^{-1}[a^2E(\alpha,e) - b^2F(\alpha,e)$$

$$- \frac{a^3e^2\sqrt{\xi}}{\sqrt{(\xi+a^2)(\xi+b^2)}}]$$

$H_z = 0$

5.5 FIELDS ON AXIS

$$\vec{E} = \frac{\vec{E}^0}{\pi} \left(\cot^{-1} \frac{z}{a} - \frac{az}{a^2 + z^2}\right) = \frac{\vec{E}^0}{\pi} \left(\alpha - \frac{az}{\xi + a^2}\right)$$

$$\vec{H} = \frac{\vec{H}^0}{\pi} \left(\cot^{-1} \frac{z}{a} - \frac{az}{a^2 + z^2}\right) = \frac{\vec{H}^0}{\pi} \left(\alpha - \frac{az}{\xi + a^2}\right)$$

$H_z = 0$

5.5 FIELDS ON AXIS

$$\vec{E} = \frac{\vec{E}^0}{2E(\frac{\pi}{2},e)} [E(\alpha,e) - \frac{az}{\sqrt{(z^2+a^2)(z^2+b^2)}}$$

$$H_x = \frac{H_x^0}{2} \frac{F(\alpha,e) - E(\alpha,e)}{F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)}$$

$$H_y = \frac{H_x^0}{2} [a^2E(\frac{\pi}{2},e) - b^2F(\frac{\pi}{2},e)]^{-1}[a^2E(\alpha,e) - b^2F(\alpha,e)$$

$$- \frac{a^3e^2z}{\sqrt{(z^2+a^2)(z^2+b^2)}}]$$

$H_z = 0$

5.6 DIPOLE EQUIVALENTS AT LARGE DISTANCES

$$\vec{P}_E = \frac{4}{3}\varepsilon_0 a^3 \vec{E}^0$$

$$\vec{P}_M = -\frac{8}{3} a^3 \vec{H}^0$$

5.6 DIPOLE EQUIVALENTS AT LARGE DISTANCES

$$\vec{P}_E = \frac{2\pi\varepsilon_0 ab^2 \vec{E}^0}{3E(\frac{\pi}{2},e)}$$

$$\vec{P}_M = -\frac{2\pi H_x^0 a^3 e^2}{3[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)]}\hat{x} - \frac{2\pi H_y^0 a^3 b^2 e^2}{3[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)]}\hat{y}$$

For either aperture the far fields are given by:

$$\vec{E} = \frac{3\hat{r}(\hat{r}\cdot\vec{P}_E) - \vec{P}_E}{4\pi\varepsilon_0 r^3}$$

$$\vec{H} = \frac{3\hat{r}(\hat{r}\cdot\vec{P}_M) - \vec{P}_M}{4\pi r^3}$$

5.7 FAR FIELDS

$$E_r = \frac{2a^3 E_z^0 \cos\theta}{3\pi r^3}$$

$$E_\theta = \frac{a^3 E_z^0 \sin\theta}{3\pi r^3}$$

5.7 FAR FIELDS

$$E_r = \frac{ab^2 E_z^0 \cos\theta}{3r^3 E(\frac{\pi}{2},e)}$$

$$E_\theta = \frac{ab^2 E_z^0 \sin\theta}{6r^3 E(\frac{\pi}{2},e)}$$

$$E_\phi = 0$$

$$H_r = -\frac{4a^3 \sin\theta}{3\pi r^3}\left(H_x^0 \cos\phi + H_y^0 \sin\phi\right)$$

$$H_\theta = \frac{2a^3 \cos\theta}{3\pi r^3}\left(H_x^0 \cos\theta + H_y^0 \sin\phi\right)$$

$$H_\phi = \frac{2a^3}{3\pi r^3}\left(-H_x^0 \sin\phi + H_y^0 \cos\phi\right)$$

$$E_\phi = 0$$

$$H_r = -\frac{H_x^0 a^3 e^2 \sin\theta \cos\phi}{3r^3[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)} - \frac{H_z^0 a^3 b^2 e^2 \sin\theta \sin\phi}{3r^3[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)]}$$

$$H_\theta = \frac{H_x^0 a^3 e^2 \cos\theta \cos\phi}{6r^3[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)]} + \frac{H_y^0 a^3 b^2 e^2 \cos\theta \sin\phi}{6r^3[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)]}$$

$$H_\phi = -\frac{H_x^0 a^3 e^2 \sin\phi}{6r^3[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)]} + \frac{H_y^0 a^3 b^2 e^2 \cos\phi}{6r^3[a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)]}$$

6.0 GENERAL SHAPE CAVITY

Jones [6] gives the field in a cavity produced by an aperture field as

$$\vec{E} = \Sigma A_M \vec{E}_M$$

$$\vec{H} = \nabla\phi + j\omega_0\varepsilon_0\Sigma \frac{A_M}{k_M^2} \nabla x\vec{E}_M$$

where

$$A_M = (k_0^2 - k_M^2)^{-1} \int_S (\hat{n}x\vec{E})\cdot(\nabla x\vec{E}_M) \; dS$$

and $\vec{E}_M$ are the natural modes of the cavity normalized such that

$$\int_V |\vec{E}_M|^2 \, d\tau = 1$$

V is the volume of the cavity. S is the surface of the cavity.

$$k_M^2 = \omega_M^2/c_0^2, \; k_0^2 = \frac{\omega_0^2}{c_0^2}$$

$$\omega_M = 2\pi f_M, \; \omega_0 = 2\pi f_0$$

f_M = a natural frequency, f_o = frequency of the impressed aperture field corresponding to the field $\vec{E}_M$

c_0 = the speed of light

$$\phi = \int_{S_a} \hat{n}\cdot\vec{H}\ G(\vec{r},\vec{r}\,')\ dS$$

(not the same ϕ as in previous sections)

where $\nabla^2 G = -\delta(\vec{r}-\vec{r}\,')$

$$\left.\frac{\partial G}{\partial n}\right|_S = 0$$

G = Green's function

S_a = aperture

For a perfectly conducting cavity $\hat{n}\times\vec{E}=0$ so that

$$A_M = [k_o{}^2 - k_M{}^2]^{-1} \int_{S_a} \hat{n}\times\vec{E}\cdot\nabla\times\vec{E}_M\ dS$$

$\hat{n}$ is the unit normal outward from the cavity.

If we used Stokes' differentiation, we could obtain $\vec{H}$ by $\vec{H} = -\nabla\times\vec{E}/j\omega\mu_0$. The introduction of ϕ circumvents the need to differentiate an infinite series. It can be seen from these equations that after the natural modes and the Green's function have been determined, the field in the cavity is determined by the normal $\vec{H}$ and the tangential $\vec{E}$ at the aperture.

7.0 THE ELECTRIC FIELD OF A RECTANGULAR CAVITY

The electric field is given in Section 6.0 where $\hat{n} = -\hat{z}$. The normal modes $\vec{E}_M$ of a rectangular cavity are found by solving the wave equation in rectangular coordinates subject to the constraints that $\partial E_n/\partial n$ and tangential $\vec{E}$ are both zero on the boundary of the shorted cavity. They are given by Chen (1) and Waldron (8) as:

$$\text{TE modes:}\quad \vec{E}_M^{(1)} = \frac{k_2 N^{1/2}\hat{x}}{k_c} \cos k_1 x' \sin k_2 y' \sin k_3 z' - \frac{k_1 N^{1/2}\hat{y}}{k_c} \sin k_1 x' \cos k_2 y' \sin k_3 z'$$

$m=0,1,2,\ldots;\ n=0,1,2,\ldots;\ p=1,2,3,\ldots;\ m^2+n^2 \neq 0$

$$\text{TM Modes:}\quad \vec{E}_M^{(2)} = \frac{-k_1 k_3 N^{1/2}\hat{x}}{k_M k_c} \cos k_1 x' \sin k_2 y' \sin k_3 z' - \frac{k_2 k_3 N^{1/2}\hat{y}}{k_M k_c} \sin k_1 x' \cos k_2 y' \sin k_3 z' + \frac{k_c N^{1/2}\hat{z}}{k_M} \sin k_1 x' \sin k_2 y' \cos k_3 z'$$

$m=1,2,\ldots;\ n=1,2,\ldots;\ p=0,1,2,\ldots$

TE Modes: $$\vec{H}_M^{(1)} = \frac{-k_1 k_3 N^{1/2}\hat{x}}{j\omega_M \mu_0 k_c} \sin k_1 x' \cos k_2 y' \cos k_3 z'$$

$$- \frac{k_2 k_3 N^{1/2}\hat{y}}{j\omega_M \mu_0 k_c} \cos k_1 x' \sin k_2 y' \cos k_3 z'$$

$$+ \frac{k_c N^{1/2}\hat{z}}{j\omega_M \mu_0} \cos k_1 x' \cos k_2 y' \sin k_3 z'$$

$m=0,1,2,...;\ n=0,1,2,...;\ p=1,2,...;\ m^2+n^2 \neq 0$

TM Modes: $$\vec{H}_M^{(2)} = \frac{jk_2 N^{1/2}\hat{x}}{Z_o k_c} \sin k_1 x' \cos k_2 y' \cos k_3 z'$$

$$- \frac{jk_1 N^{1/2}\hat{y}}{Z_o k_c} \cos k_1 x' \sin k_2 y' \cos k_3 z'$$

$m=1,2,...;\ n=1,2,...;\ p=0,1,2,...$

where the index M corresponds to the triplet (m,n,p), and

$$k_1 = \frac{m\pi}{a_0},\ k_2 = \frac{n\pi}{b_o},\ k_3 = \frac{p\pi}{h_0},\ k_c = (k_1^2+k_2^2)^{1/2},\ k_M = (k_c^2+k_3^2)^{1/2}$$

a_0, b_0, and h_0 are the dimensions of the cavity, and x', y' and z' are measured from the walls of the cavity.

$$N = \frac{8}{a_0 b_0 h_0} \varepsilon_m \varepsilon_n \varepsilon_p$$

where

$$\varepsilon_\ell = 1 \text{ if } \ell \neq 0$$

$$\varepsilon_\ell = 1/2 \text{ if } \ell = 0$$

For an elliptical aperture in the z'=0 face of the cavity, let the center of the ellipse be at x_0, y_0 from the corner of the box.

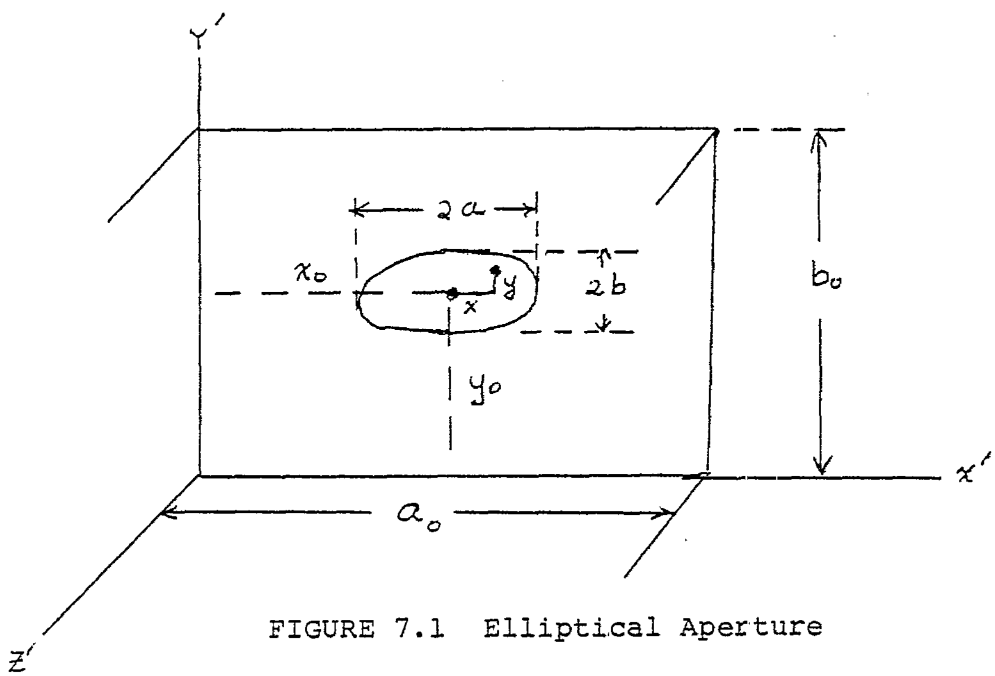

FIGURE 7.1 Elliptical Aperture

Since the values of x and y used in the expressions for the aperture fields are measured from the center of the aperture, we have:

$$x' = x_0 + x$$

$$y' = y_0 + y$$

$$z' = z = 0$$

The expression for A_M becomes

$$A_M^{(i)} = R_M^{-1} \int_{S_a} \hat{n}\times\vec{E} \cdot \nabla\times\vec{E}_M^{(i)} \, dS$$

$$= -j\omega_M \mu_0 R_M^{-1} \int_{S_a} (E_y H_{Mx}^{(i)} - E_x H_{My}^{(i)}) \, dS \qquad i=1,2$$

where

$$R_M = k_o^2 - k_M^2$$

and now

$$\vec{E} = \sum_{i=1}^{2} \sum_{M} A_M^{(i)} \vec{E}_M^{(i)}$$

and E_x and E_y are the fields for an aperture of an infinite conducting screen.

For a general aperture and general fields at the aperture, the electric field in a rectangular cavity is given by:

$$E_x = \sum_M \frac{-k_3 N \cos k_1 x' \sin k_2 y' \sin k_3 z'}{k^2 - k_M^2} \int_A E_x \cos k_1 x' \sin k_1 y' \, dS$$

$$E_y = \sum_M \frac{-k_3 N \sin k_1 x' \cos k_2 y' \sin k_3 z'}{k^2 - k_M^2} \int_A E_y \sin k_1 x' \cos k_2 y' \, dS$$

$$E_z = \sum_M \frac{N \sin k_1x' \sin k_2y' \cos k_3z'}{k^2-k_M{}^2} \int_A (k_1E_x \cos k_1x' \sin k_1y'$$

$$+ k_2E_y \sin k_1x' \cos k_1y')\, dS$$

For an elliptical aperture, the aperture electric field (assuming no reflections from the cavity) is given by:

$$E_x = \frac{E_z^o bx}{2a^2E(\frac{\pi}{2},e)\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$E_y = \frac{E_z^o y}{2bE(\frac{\pi}{2},e)\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$H_{Mx}{}^{(1)} = \frac{-k_1k_3N^{1/2}}{j\omega_M\mu_0k_c} \sin k_1x' \cos k_2y'$$

$$H_{Mx}{}^{(2)} = \frac{j\omega_M\varepsilon_0k_2N^{1/2}}{k_M k_c} \sin k_1x' \cos k_2y'$$

$$H_{My}{}^{(1)} = \frac{-k_2k_3N^{1/2}}{j\omega_M\mu_0k_c} \cos k_1x' \sin k_2y'$$

$$H_{My}{}^{(2)} = \frac{-j\omega_M\varepsilon_0k_1N^{1/2}}{k_M k_c} \cos k_1x' \sin k_2y'$$

$$A_M(1) = \frac{-k_3 N^{1/2} E_z^o}{2k_c R_M a^2 b E(\frac{\pi}{2}, e)} [k_2 b^2 \int_{S_a} \frac{x \cos k_1 x' \sin k_2 y' \, dS}{(1-x^2/a^2-y^2/b^2)^{1/2}}$$

$$- k_1 a^2 \int_{S_a} \frac{y \sin k_1 x' \cos k_2 y' dS}{(1-x^2/a^2-y^2/b^2)^{1/2}}]$$

$$A_M(2) = \frac{k_M N^{1/2} E_z^o}{2k_c R_M a^2 b \; E(\frac{\pi}{2}, \ell)} [k_1 b^2 \int_{S_a} \frac{x \cos k_1 x' \sin k_2 y' dS}{(1-x^2/a^2-y^2/b^2)^{1/2}}$$

$$+ k_2 a^2 \int_{S_a} \frac{y \sin k_1 x' \cos k_2 y' dS}{(1-x^2/a^2-y^2/b^2)^{1/2}}]$$

These integrals can be integrated in terms of elementary functions

$$I_1 = \int_{S_a} \frac{x \cos k_1 x' \sin k_2 y' \, dS}{\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$= \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} \frac{x \cos[\frac{m\pi}{a_0} (x_0+x)] \sin[\frac{n\pi}{b_0} (y_0+y)] \, dy \, dx}{\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$= -4 \sin \frac{m\pi x_0}{a} \sin \frac{n\pi y_0}{b_0} \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \frac{x \cos \frac{n\pi y}{b_0} \sin \frac{m\pi x}{a_0} \, dy \, dx}{\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$= -2 \sin k_1 x_0 \sin k_2 y_0 \int_0^a x \sin \frac{m\pi x}{a_0} J_0 \left(\frac{n\pi b}{b_0} \sqrt{1-x^2/a^2}\right) dx$$

(See appendix.) Let $v = \frac{n\pi b}{b_0}\sqrt{1-x^2/a^2}$; $\alpha = \frac{n\pi b}{b_0}$; and $\beta = \frac{mb_0 a}{na_0 b}$

$$I_1 = \frac{-2a^2 b_0^2 \sin k_1 x_0 \sin k_2 y_0}{n^2 \pi b} \int_0^\alpha vJ_0(v) \sin \beta\sqrt{\alpha^2 - v^2}\, dv$$

$$I_1 = -\pi\sqrt{2\pi}\, a^3 b k_1 \sin k_1 x_0 \sin k_2 y_0\, J_{3/2}(s)/s^{3/2}$$

where $s = [(k_1 a)^2 + (k_2 b)^2]^{1/2}$ (see appendix)

Similarly:

$$I_2 = \int_{S_a} \frac{y \sin k_1 x' \cos k_2 y'\, dS}{\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$I_2 = -\pi\sqrt{2\pi}\, ab^3 k_2 \sin k_1 x_o \sin k_2 y_o\, J_{3/2}(s)/s^{3/2}$$

After substituting these expressions, one obtains:

$$A_M^{(1)} = 0$$

$$A_M^{(2)} = \frac{-\pi ab^2 k_c k_M E_z^o \sin \frac{m\pi x_0}{a_0} \sin \frac{n\pi y_0}{b_0} N^{1/2}(\sin s - s \cos s)}{E(\frac{\pi}{2}, e)\; s^3 R_M}$$

$$\vec{E} = \frac{8\pi ab^2 E_z^0}{a_0 b_0 h_0 E(\frac{\pi}{2}, e)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{\varepsilon_p \sin \frac{m\pi x_0}{a_0} \sin \frac{n\pi y_0}{b_0}}{[k_o^2 - k_M^2]\; s^3}$$

$$[k_1 k_3 \cos\frac{m\pi x'}{a_0} \sin\frac{n\pi y'}{b_0} \sin\frac{p\pi z'}{h_0}\,\hat{x}$$

$$+ k_2 k_3 \sin\frac{m\pi x'}{a_0} \cos\frac{n\pi y'}{b_0} \sin\frac{p\pi z'}{h_0}\,\hat{y}$$

$$- k_c^2 \sin\frac{m\pi x'}{a_0} \sin\frac{n\pi y'}{b_0} \cos\frac{p\pi z'}{h_0}\,\hat{z}]\ (\sin s - s\cos s)$$

For a circular aperture $s = k_c a$ and the factor in front of the summation sign becomes $16\, a^3 E_z^o / a_0 b_0 h_0$ where a is the radius of the aperture.

8.0 THE MAGNETIC FIELD OF A RECTANGULAR CAVITY

If we could find $\vec{E}$ as a power series in k_o (i.e., ω_o/c_o) as $\vec{E} = \vec{E}^{(0)} + k_o\vec{E}^{(1)} + k_o^2\,\vec{E}^{(2)} + \ldots$; then $\vec{H}$ could be calculated from

$$\vec{H} = \frac{\nabla x\vec{E}}{-j\omega_o\mu_o} = \frac{\nabla x\vec{E}^{(1)}}{-jZ_o} + 0(k_o) = \vec{H}^{(0)} + k_oH^{(1)} + \ldots$$

$$\vec{H}^{(0)} = j\,\frac{\nabla x\vec{E}^{(1)}}{Z_o}$$

In deriving the aperture field, we assumed a low frequency incoming wave so that $\nabla^2\phi=0$ and first order terms in k_o were neglected. This was equivalent to finding $\vec{E}^{(0)}$ and not $\vec{E}^{(1)}$. Hence, $\vec{H}^{(0)}$ cannot be calculated from the $\vec{E}$ that was given in the previous chapter by using

$$\vec{H} = \frac{\nabla xE}{-j\omega\mu_o}$$

Instead, let us use the method of Jones [6]. He gives

$$\vec{H} = \nabla\phi + k_0\sum_M \frac{A_M\vec{H}_M}{k_M} = \nabla\phi + k_o\sum_{i=1}^{2}\sum_M \frac{A_M^{(i)}\vec{H}_M^{(i)}}{k_M}$$

where

$$\phi(\vec{r}\,') = \int_{S_a} \hat{n}\cdot\vec{H}\, G(\vec{\bar{r}},\vec{r}\,')\, dS$$

where $\vec{\bar{r}}$ is the variable of integration, measured from the corner of the cavity; and G, the Green's function, is a solution of the boundary value problem

$$\nabla^2 G = -\delta(\vec{r}-\vec{r}\,')$$

$$\left.\frac{\partial G}{\partial n}\right|_S = 0$$

Using finite transforms it can be shown that

$$G(\vec{r},\vec{r}\,') = \sum_{mnp} \frac{N}{k_M^2} \cos k_1x' \cos k_2y' \cos k_3z' \cos k_1\bar{x} \cos k_2\bar{y} \cos k_3\bar{z}$$

where $\bar{x} = x_o + x$, $\bar{y} = y_o + y$, $\bar{z} = z = 0$ on the aperture. $m^2+n^2+p^2 \neq 0$ and $\hat{n}\cdot\vec{H} = -H_z$ at z=0 and all sums go from 0 to ∞ and $N = 8\,\varepsilon_m\varepsilon_n\varepsilon_p/a_oh_ob_o$

On an elliptical aperture, we have found that to the zero order

$$H_z = \frac{-H_x^0 e^2 x}{2b[F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)]\sqrt{1-x^2/a^2-y^2/b^2}}$$

$$- \frac{H_y^0 e^2 y\, a^2}{2b[a^2E(\frac{\pi}{2},e) - b^2F(\frac{\pi}{2},e)]\sqrt{1-x^2/a^2-y^2/b^2}}$$

The integrals can be determined from the table in the appendix. The expression for $\vec{H}$ is

$$\vec{H}(x',y',z') = \frac{8\pi a^3 e^2}{a_0 b_0 h_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \varepsilon_m \varepsilon_n \frac{\varepsilon_p}{k_M^2 s^3} (\sin s - s \cos s)$$

$$(k_1 \hat{x} \sin k_1 x' \cos k_2 y' \cos k_3 z' + k_2 \hat{y} \cos k_1 x' \sin k_2 y' \cos k_3 z'$$

$$+ k_3 \hat{z} \cos k_1 x' \cos k_2 y' \sin k_3 z')$$

$$\left[\frac{H_x^0 k_1 \sin k_1 x_0 \cos k_2 y_0}{F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)} + \frac{b^2 H_y^0 k_2 \cos k_1 x_0 \sin k_2 y_0}{a^2 E(\frac{\pi}{2},e) - b^2 F(\frac{\pi}{2},e)}\right]$$

$$- \frac{8\pi a b^2 k_0 E_{\hat{z}}^0}{a_0 b_0 h_0 Z_0 E(\frac{\pi}{2},e)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=0}^{\infty} \frac{\varepsilon_p \sin k_1 x_0 \sin k_2 y_0}{s^3 k_M}$$

$$(\sin s - s \cos s)\ (k_2 \hat{x} \sin k_1 x' \cos k_2 y' \cos k_3 z' - k_1 \hat{y} \cos k_1 x'$$

$$\sin k_2 y' \cos k_3 z')$$

where $m^2 + n^2 \neq 0$ in the first triple summation and, as before,

$\varepsilon_p = 1$ if $p \neq 0$ and $\varepsilon_p = \frac{1}{2}$ if $p = 0$

$s = (a^2k_1{}^2+b^2k_2{}^2)^{1/2}$, $k_1 = \frac{m\pi}{a_0}$, $k_2 = \frac{n\pi}{b_0}$, $k_3 = \frac{p\pi}{h_0}$, $k_c = (k_1{}^2+k_2{}^2)^{1/2}$

$k_M = (k_c{}^2+k_3{}^2)^{1/2}$, $k_0 = \frac{\omega_0}{c}$, $\omega_0 = 2\pi f_0$, $e = (1-b^2/a^2)^{1/2}$,

$F(\frac{\pi}{2},e) = \int_0^{\pi/2} (1-e^2 \sin^2\phi)^{1/2}\, d\phi$, $E(\frac{\pi}{2},e) = \int_0^{\pi/2} (1-e^2\sin^2\phi)^{-1/2}\, d\phi$

$R_M = k_o{}^2-k_m{}^2$

$E_z^o = 2E_o^i \sin\psi \sin\theta$, $H_x^0 = -2H_o^i(\sin\psi \cos\theta + \cos\theta \sin\phi \cos\psi)$

$H_y{}^0 = -2H_o^i(\sin\phi \sin\psi - \cos\theta \cos\phi \cos\psi)$, $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$, $H_o^i = \frac{E_o^i}{Z_0}$

The input variables are $\theta, \phi, \psi, E_o^i, f_o, a, b, x_0, y_0, a_0, b_0, h_0, x', y', z'$.

Note that the k_0 terms were not included in the aperture fields since $k_0a << 1$ but are included in the cavity fields since k_o is of the same order as k_M.

9.0 GENERAL ORIENTATION OF AN ELLIPTIC APERTURE

Let us generalize the case of an elliptical aperture in a rectangular cavity, by rotating the axes of the ellipse an angle θ_o.

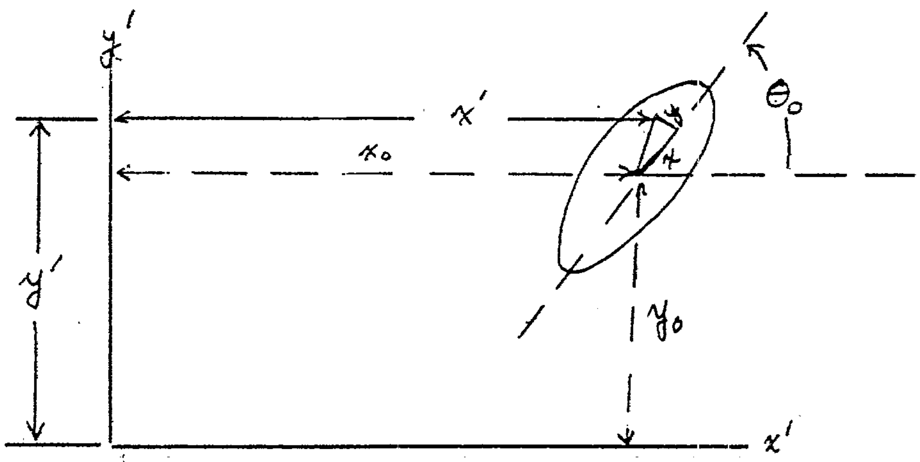

Figure 1

GENERAL ORIENTATION OF ELLIPTICAL APERTURE

From the geometry it follows that

$$\begin{cases} x' = x+\rho\cos(\theta+\theta_o) = x_o+\rho(\cos\theta\cos\theta_o-\sin\theta\sin\theta_o) \\ \quad = x_o+x\cos\theta_o - y\sin\theta_o \\ \\ y' = y_o+\rho\sin(\theta+\theta_o) = y_o+\rho(\sin\theta\cos\theta_o+\sin\theta_o\cos\theta) \\ \quad = y_o+y\cos\theta_o + x\sin\theta_o \end{cases}$$

The expansion coefficients are given by

$$A_M^{(1)} = \frac{k_3 N^{1/2} E_z^0 (-k_2 b^2 I_1 + k_1 a^2 I_2)}{2k_c R_M a^2 b E(\frac{\pi}{2}, e)}$$

$$A_M^{(2)} = \frac{k_M N^{1/2} E_z^o (k_1 b^2 I_1 + k_2 a^2 I_2)}{2k_c R_M a^2 b E(\frac{\pi}{2}, e)}$$

where

$$I_1 = \int_{S_a} \frac{x \cos k_1 x' \sin k_2 y' \, dS}{\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

$$I_2 = \int_{S_a} \frac{y \sin k_1 x' \cos k_2 y' \, dS}{\sqrt{1 - x^2/a^2 - y^2/b^2}}$$

The trigonometric factors are

$$\cos k_1 x' = \cos k_1 x_o [\cos(k_1 x \cos \theta_o) \cos (k_1 y \sin \theta_o]$$

$$+ \sin(k_1 x \cos \theta_o) \sin(k_1 y \sin \theta_o)]$$

$$- \sin k_1 x_o [\sin(k_1 x \cos \theta_o) \cos(k_1 y \sin \theta_o)$$

$$- \sin(k_1 y \sin \theta_o) \cos(k_1 x \cos \theta_o)]$$

$$\sin k_1x' = \sin k_1x_o[\cos(k_1x \cos \theta_o)\cos(k_1y \sin \theta_o)$$

$$+ \sin (k_1x \cos \theta_o)\sin(k_1y \sin \theta_o)]$$

$$+ \cos k_1x_o[\sin(k_1x \cos \theta_o)\cos(k_1y \sin \theta_o)$$

$$- \sin (k_1y \sin \theta_o)\cos(k_1x \cos \theta_o)]$$

$$\cos k_2y' = \cos k_2y_o[\cos(k_2x \sin \theta_o)\cos(k_2y \cos \theta_o)$$

$$- \sin(k_2x \sin \theta_o)\sin(k_2y \cos \theta_o)]$$

$$- \sin k_2y_o[\sin(k_2x \sin \theta_o)\cos(k_2y \cos \theta_o)$$

$$+ \cos(k_2x \sin \theta_o)\sin(k_2y \cos \theta_o)]$$

$$\sin k_2y' = \sin k_2y_o[\cos(k_2x \sin \theta_o)\cos(k_2y \cos \theta_o)$$

$$- \sin(k_2x \sin \theta_o)\sin(k_2y \cos \theta_o)]$$

$$+ \cos k_2y_o[\sin(k_2x \sin \theta_o)\cos(k_2y \cos \theta_o)$$

$$+ \cos(k_2x \sin \theta_o)\sin(k_2y \cos \theta_o)]$$

Each of the above integrals involves 16 terms. However, because of symmetry, all but four are zero. With the use of trigonometric identifies, these can be evaluated.

$$A_M^{(1)} = \frac{\pi k_3 k_c N^{1/2} E_z^o ab^2 \sin\theta_o}{2R_M E(\frac{\pi}{2},e)}[\cos(k_1x_o+k_2y_o)(s_1 \cos s_1-\sin s_1)/s_1^3$$

$$+ \cos (k_1x_o-k_2y_o)(s_2 \cos s_2-\sin s_2)/s_2^3]$$

$$A_M^{(2)} = \frac{\pi k_M k_c N^{1/2} E_z^o ab^2 \cos\theta_o}{2R_M E(\frac{\pi}{2},e)}[\cos(k_1x_o-k_2y_o)(s_2 \cos s_2 - \sin s_2)/s_2^3$$

$$- \cos (k_1x_o+k_2y_o)(s_1 \cos s_1)/s_1^3]$$

where

$$s_1 = [a^2(k_1 \cos\theta_o+k_2 \sin\theta_o)^2+b^2(k_1 \sin\theta_o-k_2 \cos\theta_o)^2]^{1/2}$$

$$s_2 = [a^2(-k_1 \cos\theta_o+k_2 \sin\theta_o)^2+b^2(k_1 \sin\theta_o+k_2 \cos\theta_o)^2]^{1/2}$$

As $\theta_o \to 0$, $A_M^{(1)}$ goes to zero and $A_M^{(2)}$ goes to the value previously calculated. As $\theta_o \to \frac{\pi}{2}$, $A_M^{(2)}$ goes to zero and

$$A_M^{(1)} \to \frac{\pi k_3 k_c N^{1/2} E_z^o\, ab^2}{s_1{}^3 R_M E(\frac{\pi}{2},e)} \quad (s' \cos s'-\sin s')\cos k_1y_o \cos k_2y_o$$

where $s' = (a^2k_2^2+b^2k_1^2)^{1/2}$

In any case, the electric field is

$$\vec{E} = \sum_M (A_M^{(1)}\vec{E}_M^{(1)}+A_M^{(2)}\vec{E}_M^{(2)})$$

In order to find $\vec{H}$, $\nabla\phi$ can be determined from the same relations used previously.

$$\nabla\phi = \frac{4\pi a^3 e^2}{a_o b_o h_o} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\varepsilon_m \varepsilon_n \varepsilon_p}{k_M^2} (k_1 \hat{x} \sin k_1 x' \cos k_2 y' \cos k_3 z'$$

$$+ k_2 \hat{y} \cos k_1 x' \sin k_2 y' \cos k_3 z' + k_3 \hat{z} \cos k_1 x' \cos k_2 y' \sin k_3 z') \cdot$$

$$\Bigg\{ s_1^{-3} (\sin s_1 - s_1 \cos s_1) \sin(k_1 x_o + k_2 y_o) \cdot$$

$$\left[\frac{H_x^o (k_1 \cos\theta_o + k_2 \sin\theta_o)}{F(\frac{\pi}{2}, e) - E(\frac{\pi}{2}, e)} - \frac{H_y^o b^2 (k_1 \sin\theta_o - k_2 \cos\theta_o)}{a^2 E(\frac{\pi}{2}, e) - b^2 F(\frac{\pi}{2}, e)} \right]$$

$$+ s_2^{-3} (\sin s_2 - s_2 \cos s_2) \sin(k_1 x_o - k_2 y_o) \cdot$$

$$\left[\frac{H_x^o (k_1 \cos\theta_o - k_2 \sin\theta_o)}{F(\frac{\pi}{2}, e) - E(\frac{\pi}{2}, e)} - \frac{H_y^o b^2 (k_1 \sin\theta_o + k_2 \cos\theta_o)}{a^2 E(\frac{\pi}{2}, e) - b^2 F(\frac{\pi}{2}, e)} \right] \Bigg\}$$

When $\theta_o = 0$ this reduces to the result derived earlier, which is the first half of the expression on page (8-3). When $\theta_o = \frac{\pi}{2}$,

$$\nabla\phi = \frac{8\pi a^3 e^2}{a_o b_o h_o} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\varepsilon_m \varepsilon_n \varepsilon_p}{k_M^2} (k_1 \hat{x} \sin k_1 x' \cos k_2 y' \cos k_3 z' +$$

$$k_2 \hat{y} \cos k_1 x' \sin k_2 y' \cos k_3 z' + k_3 \hat{z} \cos k_1 x' \cos k_2 y' \sin k_3 z')$$

$$(\sin s' - s' \cos s')/s'^3$$

$$\cdot \left[\frac{H_x^o k_2 \cos k_1 x_o \sin k_2 y_o}{F(\frac{\pi}{2}, e) - E(\frac{\pi}{2}, e)} - \frac{H_y^o b^2 k_1 \sin k_1 x_o \cos k_2 y_o}{a^2 E(\frac{\pi}{2}, e) - b^2 F(\frac{\pi}{2}, e)} \right].$$

where

$$s' = \sqrt{k_1{}^2a^2 + k_2{}^2b^2}$$

10.0 CAVITY EXCITATION NEAR RESONANCE

The boundary condition for a metal which is not perfectly conducting is derived in the appendix as

$$\hat{n}x\vec{E} = \sqrt{\omega\mu_o/2\sigma}(1+j)\vec{H}$$

(see also Reference 9). If the resonance is to be damped, we must use the above formula in evaluating the coefficients A_M. $\nabla\phi$ is unchanged because $\hat{n}\cdot\vec{H} \propto \hat{n}\cdot\hat{n}x\vec{E} = 0$ on the metal. Following the method of Chen,(3) we obtain

$$\vec{E} = \sum_{M,i} A_M^{(i)}\vec{E}_M^{(i)}$$

(see appendix). Where

$$A_M^{(i)} = \frac{1}{k_o^2-k_M^2}\int_S \hat{n}x\vec{E}\cdot\nabla x\vec{E}_M^{(i)}dS$$

$$= \frac{1}{k_o^2-k_M^2}\left[\int_{S_a} \hat{n}x\vec{E}\cdot\nabla x\vec{E}_M^{(i)}\,dS + \int_{S_1} \hat{n}x\vec{E}\cdot\nabla x\vec{E}_M^{(i)}\,dS\right]$$

$$= \frac{1}{k_o^2-k_M^2}\left[\int_{S_a} \hat{n}x\vec{E}\cdot\nabla x\vec{E}_M^{(i)}\,dS + \sqrt{\omega\mu_o/2\sigma}(1+j)\int_{S_1} \vec{H}\cdot\nabla x\vec{E}_M^{(i)}dS\right]$$

At this point, we should note that the series for $\vec{E}$ is not uniformly convergent since $\hat{n}x\vec{E}_M^{(i)}$ is zero on the boundary whereas $\hat{n}x\vec{E}$ is not. Hence, we will use the expression for $\vec{H}$ which has a uniformly convergent series

$$\vec{H} = \nabla\phi + k_o \sum_{M,i} \frac{A_M^{(i)}\vec{H}_M^{(i)}}{k_M}$$

Substituting this in the above surface integral, we obtain

$$(k_o^2-k_M^2)A_M^{(i)} = \int_{S_a} \hat{n}x\vec{E}\cdot\nabla x\vec{E}_M^{(i)}\, dS + \sqrt{\omega\mu/2\sigma}\ (1+j)$$

$$\cdot\ [\int_{S_1} \nabla\phi \cdot \nabla x\vec{E}_M^{(i)}\, dS + k_o \sum_{\bar{M},\ell} \frac{A_{\bar{M}}^{(\ell)}}{k_{\bar{M}}} \int_{S_1} \vec{H}_{\bar{M}}^{(\ell)}\cdot\nabla x\vec{E}_M^{(i)}\, dS]$$

$$= I_M^{(i)} + \sqrt{\omega\mu/2\sigma}\ (1+j)\ [J_M^{(i)} - jk_o Z_o k_M \sum_{\bar{M},\ell} \frac{A_{\bar{M}}^{(\ell)} K_{M\bar{M}}^{(i\ell)}}{k_{\bar{M}}}\ ]$$

These integrals can be evaluated for particular apertures. Let us consider a rotated elliptic aperture.

$$\nabla\phi = \sum_{\bar{M}} \frac{\bar{N}}{k_{\bar{M}}^2} [-\hat{x}\bar{k}_1 \sin \bar{k}_1 x' \cos \bar{k}_2 y' \cos \bar{k}_3 z' - \hat{y}\bar{k}_2 \cos \bar{k}_1 x'$$

$$\sin \bar{k}_2 y' \cos \bar{k}_3 z' - \hat{z}\bar{k}_3 \cos \bar{k}_1 x' \cos \bar{k}_2 y' \sin \bar{k}_3 z'] P_{\bar{M}}$$

where $P_M = \int_{S_a} \hat{n}\cdot\vec{H} \cos k_1\bar{x} \cos k_2\bar{y}\ dS$

$$I_M^{(i)} = \int_{S_a} \hat{n}x\vec{E}\cdot\vec{H}_M^{(i)}\, dS\ (-jk_M z_o)$$

$$K_{M\bar{M}}^{(i\ell)} = \int_{S_1} \vec{H}_{\bar{M}}^{(\ell)} \cdot\vec{H}_M^{(i)}\, dS$$

$$J_M^{(i)} = \int_{S_1} \nabla\phi \cdot \nabla x E_M^{(i)} \, dS = -jk_M z_o \int_{S_1} \nabla\phi \cdot \vec{H}_M^{(i)} \, dS$$

$$= -jk_M z_o \sum_{\bar{K}_M} \frac{\bar{N}P_{\bar{M}}}{k_{\bar{M}}^2} \int_{S_1} \vec{H}_M^{(i)} \cdot [\,-\bar{k}_1 x \sin \bar{k}_1 \hat{x}' \cos \bar{k}_2 y' \cos \bar{k}_3 z'$$

$$-\bar{k}_2 \hat{y} \cos \bar{k}_1 x' \sin \bar{k}_2 y' \cos \bar{k}_3 z$$

$$-\bar{k}_3 \hat{z} \cos \bar{k}_1 x' \cos \bar{k}_2 y' \sin \bar{k}_3 z'] \, dS$$

$$P_M = \frac{-\pi e^2 a^3}{2} \{ \frac{H_x^o}{F(\frac{\pi}{2},e) - E(\frac{\pi}{2},e)} [\,\sin (k_1 x_o + k_2 y_o) C_1 (\sin s_1 - s_1 \cos s_1)/s_1^3$$

$$-\sin (k_1 x_o - k_2 y_o) C_2 (\sin s_2 - s_2 \cos s_1)/s_1^3]$$

$$+ \frac{H_y^o b^2}{b^2 F(\frac{\pi}{2},e) - a^2 E(\frac{\pi}{2},e)} [\sin(k_1 x_o - k_2 y_o) c_3 (\sin s_2 - s_2 \cos s_2)/s_2^3$$

$$- \sin(k_1 x_o + k_2 y_o) c_4 (\sin s_1 - s_1 \cos s_1)/s_1^3]$$

where

$$C_1 = k_1 \cos \theta_o + k_2 \sin \theta_o$$

$$C_2 = -k_1 \cos \theta_o + k_2 \sin \theta_o$$

$$C_3 = k_1 \sin \theta_o + k_2 \cos \theta_o$$

$$C_4 = -k_1 \sin \theta_o + k_2 \cos \theta_o$$

$$s_1 = (a^2C_1{}^2+b^2C_4^2)^{1/2}$$

$$s_2 = (a^2C_2{}^2+b^2C_3{}^2)^{1/2}$$

$$K_{M\bar{M}}^{(11)} = \frac{-2}{k_M\ k_{\bar{M}}\ Z_o{}^2 k_c \bar{k}_c} \{\frac{(\varepsilon_1\bar{\varepsilon}_1)^{1/2}}{a_o}\ \delta_n^{\bar{n}}\ \delta_p^{\bar{p}}[1+(-1)^{m+m}]$$

$$(k_2{}^2k_3{}^2+k_c{}^2\bar{k}_c{}^2) + \frac{(\varepsilon_2\bar{\varepsilon}_2)^{1/2}}{b_o}\ \delta_m^{\bar{m}}\ \delta_p^{\bar{p}}[1+(-1)^{n+n}]$$

$$(k_1{}^2k_3{}^2+k_c{}^2\bar{k}_c{}^2) + \frac{1}{h_o}\ \delta_m^{\bar{m}}\ \delta_n^{\bar{n}}[1+(-1)^{p+\bar{p}}]k_2{}^2k_3\bar{k}_3\}$$

$$K_{M\bar{M}}^{(12)} = \frac{2k_3}{k_M k_c \bar{k}_c Z_o{}^2} \{\frac{-\varepsilon_2{}^{1/2}k_1\bar{k}_2}{b_o}\ \delta_m^{\bar{m}}\ \delta_p^{\bar{p}}[1+(-1)^{n+\bar{n}}]$$

$$\frac{+\varepsilon_1{}^{1/2}\bar{k}_1k_2}{a_o}\ \delta_n^{\bar{n}}\ \delta_p^{\bar{p}}[1+(-1)^{m+\bar{m}}]\}$$

$$K_{M\bar{M}}^{(21)} = \frac{2k_3}{k_{\bar{M}} k_c \bar{k}_c Z_o{}^2} \{- \frac{\bar{\varepsilon}_2{}^{1/2}k_1k_2}{b_o}\ \delta_m^{\bar{m}}\ \delta_p^{\bar{p}}[1+(-1)^{n+\bar{n}}]$$

$$+ \frac{\bar{\varepsilon}_1{}^{1/2}k_1k_2}{a_o}\ \delta_n^{\bar{n}}\ \delta_p^{\bar{p}}[1+(-1)^{m+\bar{m}}]\}$$

$$K_{MM}^{(22)} = -\frac{2k_1\bar{k}_1\delta_n^{\bar{n}}\delta_p^{\bar{p}}}{a_o k_c\bar{k}_c Z_o^2}\,(1+(-1)^{m+\bar{m}}] - \frac{2k_2\bar{k}_2\delta_m^{\bar{m}}\delta_p^{\bar{p}}}{b_o k_c\bar{k}_c Z_o^2}\,[1+(-1)^{m+\bar{m}}]$$

$$-\frac{2(\varepsilon_3\bar{\varepsilon}_3)^{1/2}\delta_m^{\bar{m}}\delta_n^{\bar{n}}}{h_o Z_o^2}\,[1+(-1)^{p+\bar{p}}]$$

$$J_M^{(1)} = -jk_3\sqrt{2}\,\{-\frac{k_2^2}{\sqrt{a_o}}\sum_n \frac{[1+(-1)^{n+\bar{n}}]\cdot p_{m\bar{n}p}}{k^2_{m\bar{n}p}}$$

$$-\frac{k_1^2}{\sqrt{b_o}}\sum_m \frac{[1+(-1)^{m+\bar{m}}]\,p_{\bar{m}np}}{k^2_{\bar{m}np}} + \frac{k_c^2}{\sqrt{h_o}}\sum_p \frac{[1+(-1)^{p+\bar{p}}]\,p_{mn\bar{p}}}{k^2_{mn\bar{p}}}\}$$

$$J_M^{(2)} = -\frac{k_1k_2k_M\sqrt{2}}{k_c}\,\{-\frac{1}{\sqrt{b_o}}\sum_n \frac{[1+(-1)^{n+\bar{n}}]\,p_{m\bar{n}p}}{k^2_{m\bar{n}p}}$$

$$+\frac{1}{\sqrt{a_o}}\sum_m \frac{[1+(-1)^{m+\bar{m}}]\,p_{\bar{m}np}}{k^2_{\bar{m}np}}\}$$

$$I_M^{(1)} = \frac{\pi k_3 k_c N^{1/2} E_z^o\, ab^2 \sin\theta_o}{2E(\frac{\pi}{2},e)}\,[\cos(k_1x_o+k_2y_o)(S_1\cos S_1-\sin S_1)/S_1^3$$

$$+\cos(k_1x_o-k_2y_o)(S_2\cos S_2-\sin S_2)/S_2^3]$$

$$I_M^{(2)} = \frac{\pi k_M k_c N^{1/2} E_z^o\, ab^2 \cos\theta_o}{2E(\frac{\pi}{2},e)}\,[\cos(k_1x_o-k_2y_o)(S_2\cos S_2-\sin S_2)/S_2^3$$

$$-\cos(k_1x_o+k_2y_o)(S_1\cos S_1-\sin S_1)/S_1^3]$$

To solve the doubly infinite set of equations for A_M, separate out the diagonal terms.

$$[k_o^2-k_M^2+jk_oZ_oK_{MM}^{(11)}\sqrt{\omega\mu_o/2\sigma}(1+j)]\,A_M^{(1)}$$

$$+\,jk_oZ_o\sqrt{\omega\mu_o/2\sigma}(1+j)K_{MM}^{(12)}\,A_M^{(2)}$$

$$=I_M^{(1)}+\sqrt{\omega\mu_o/2\sigma}(1+j)[J_M^{(1)}-jk_oZ_ok_M\cdot\sum_{\bar{M}\neq M,\ell}\frac{A_{\bar{M}}^{(\ell)}K_{M\bar{M}}^{(1\ell)}}{k_{\bar{M}}}]$$

$$[k_o^2-k_M^2+jk_oZ_oK_{MM}^{(22)}\sqrt{\omega\mu_o/2\sigma}(1+j)]\,A_M^{(2)}+jk_oZ_o\sqrt{\omega\mu_o/2\sigma}(1+j)$$

$$A_M^{(1)}K_{MM}^{(21)}=I_M^{(2)}+\sqrt{\omega\mu_o/2\sigma}(1+j)[J_M^{(2)}-jk_ok_MZ_o\sum_{\bar{M}\neq M,\ell}\frac{A_{\bar{M}}^{(\ell)}K_{M\bar{M}}^{(2\ell)}}{k_{\bar{M}}}]$$

If k_o is close to some k_M, say k_N, then the above equations are approximately

$$A_M^{(1)}=\frac{I_M^{(1)}}{k_o^2-k_M^2}\qquad M\neq N$$

$$A_M^{(2)}=\frac{I_M^{(2)}}{k_o^2-k_M^2}\qquad M\neq N$$

where the terms involving $1/\sqrt{\sigma}$ have been neglected. These can be substituted into the resonant equation to obtain

$$[k_o^{\,2}-k_N^{\,2}-Z_o k_o K_{NN}^{(11)} \sqrt{\omega\mu_o/2\sigma}(1-j)]\, A_N^{(1)} - k_o Z_o K_{NN}^{(12)} A_N^{(2)} (1-j)$$

$$\sqrt{\omega\mu_o/2\sigma} = I_N^{(1)} + \sqrt{\omega\mu_o/2\sigma}(1+j)\ J_N^{(1)} + k_o Z_o k_N \sqrt{\omega\mu_o/2\sigma}(1-j)$$

$$\sum_{\bar{M}\neq N} \frac{K_{N\bar{M}}^{(11)} I_{\bar{M}}^{(1)} + K_{N\bar{M}}^{(12)} I_{\bar{M}}^{(2)}}{k_{\bar{M}}(k_o^{\,2}-k_{\bar{M}}^{\,2})} = F_1$$

$$[k_o^{\,2}-k_N^{\,2}-Z_o k_o K_{NN}^{(22)} \sqrt{\omega\mu_o/2\sigma}(1-j)]\, A_N^{(2)} - k_o Z_o K_{NN}^{(21)} A_N^{(1)}$$

$$\sqrt{\omega\mu_o/2\sigma}(1-j) = I_N^{(2)} + \sqrt{\omega\mu_o/2\sigma}(1+j)\ J_N^{(2)} + k_o Z_o k_N \sqrt{\omega\mu_o/2\sigma}(1-j)$$

$$\sum_{\bar{M}\neq N} \frac{K_{N\bar{M}}^{(21)} I_{\bar{M}}^{(1)} + K_{N\bar{M}}^{(22)} I_{\bar{M}}^{(2)}}{k_{\bar{M}}(k_o^{\,2}-k_{\bar{M}}^{\,2})} = F_2$$

This set of two equations and two unknowns can be solved for $A_N^{(1)}$ and $A_N^{(2)}$

$$A_N^{(1)} = \frac{1}{\Delta}(F_1 D - F_2 B)$$

$$A_N^{(2)} = \frac{1}{\Delta}(F_2 A - F_1 C)$$

where $\Delta = AD-BC$

$$A = k_o^2 - k_N^2 - Z_o k_o K_{NN}^{(11)} \sqrt{\omega\mu_o/2\sigma}(1-j)$$

$$B = -Z_o k_o K_{NN}^{(12)} \sqrt{\omega\mu_o/2\sigma}(1-j)$$

$$C = -Z_o k_o K_{NN}^{(21)} \sqrt{\omega\mu_o/2\sigma}(1-j)$$

$$D = k_o^2 - k_N^2 - Z_o k_o K_{NN}^{(2)} \sqrt{\omega\mu_o/2\sigma}(1-j)$$

If k_o is close to more than one k_M, say k_{N_1}, $k_{N_2}, \ldots k_{N_e}$, then we still have

$$A_M^{(i)} = \frac{I_M^{(i)}}{k_o^2 - k_M^2} \quad M \neq N_1, N_2, \ldots N_e$$

and we can proceed as above to find $A_{N_1}^{(i)}$, $A_{N_2}^{(i)}, \ldots A_{N_e}^{(i)}$ by solving 2e equations in 2e unknowns.

If this approximation is not made, then the equation for $A_M^{(i)}$ must be solved by numerical techniques. If $m,n,p < 10$, this leads to eight independent sets of equations with a maximum of 250 equations in each set. There would be 1701 unknowns which would be a laborious calculation.

It is important to note that $A_N^{(i)}$ does not go to infinity as $\omega \to \omega_N$.

11.0 PULSE INPUT

Let an incoming wave incident on the aperture be an electromagnetic pulse (EMP) instead of a sinusoidal wave. The Fourier transform of Maxwell's Equations are of the same form as if we had assumed a sinusoidal input. If the incoming field is of the form $\vec{E}^i = \vec{E}^i_0 f(t)$ then the Fourier transform is $\tilde{\vec{E}}^i = \vec{E}^i F(\omega)$.

Where $F(\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} f(t)dt$

The inverse transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} F(\omega)d\omega$$

To find the response of a cavity to EMP we must multiply the result of a sinusoidal input by $F(\omega)$ and then take the inverse transform of the result. As an example consider the function illustrated in Figure 3.

$$f(t) = (e^{-\alpha t} - e^{-\beta t})u(t) \quad \text{where} \quad 0<\alpha<\beta$$

and $u(t)$ is the unit step function.

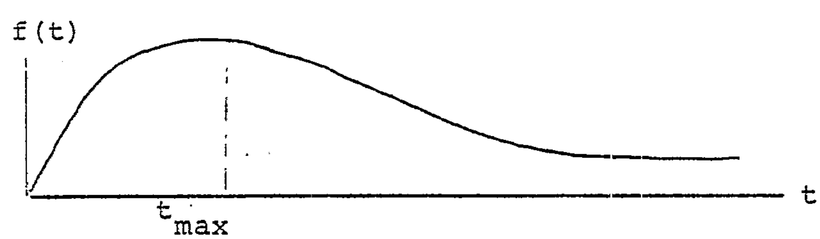

Figure 3. EMP

The maximum occurs at $t_{max} = \frac{1}{\beta-\alpha} \ln \beta/\alpha$

The transform of f(t) is

$$F(\omega) = \frac{1}{\alpha+j\omega} - \frac{1}{\beta+j\omega}$$

The transformed field in a perfectly conducting cavity of arbitrary shape and arbitrary aperture is given by

$$\tilde{\vec{E}} = c_o^2 \sum_{M,i} \frac{\vec{E}_M(i) \int_{s_a} \hat{n} \times \tilde{\vec{E}} \cdot \nabla \times \vec{E}_M^{(i)} \, ds}{\omega^2 - \omega_M^2}$$

Let $\tilde{\vec{E}}$ on the aperture be $F(\omega)\vec{G}(\vec{r})$. It follows that

$$\vec{E}(\vec{r},t) = \sum_M \frac{\vec{E}_M}{k_M^2} \int_{s_a} \hat{n} \times \vec{G}(\vec{r}) \cdot \nabla \times \vec{E}_M \, ds \; B_M(t)$$

where $B_M(t)$ = inverse Fourier transform of $\frac{\omega_M^2 F(\omega)}{\omega^2 - \omega_M^2}$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega_M^2 \frac{e^{j\omega t}}{\omega^2 - \omega_M^2} \left(\frac{1}{\alpha+j\omega} - \frac{1}{\beta+j\omega} \right) d\omega$$

$B_M(t)$ gives the time dependence of each mode.

This integral has a singularity but the principal value exists.

It can be verified by direct integration that

$$F.T.\left[-\omega_M \sin\omega_M t e^{-\gamma t} u(t)\right] = \frac{-\omega_M^2}{(\gamma+j\omega)^2+\omega_M^2} \quad (\gamma>0)$$

As $\gamma \to 0+$ this is $\frac{\omega_M^2}{\omega^2-\omega_M^2}$

Using the convolution theorem, the inverse Fourier Transform of the product $F_1(\omega)F_2(\omega)$ is given by

$\int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)d\tau$. Let $F_1(\omega) = \frac{-\omega_M^2}{(\gamma+j\omega)^2+\omega_M^2}$

and $F_2(\omega) = \frac{1}{\alpha+j\omega} - \frac{1}{\beta+j\omega}$. It follows that

$$f_1(t) = -\omega_M \sin\omega_M t e^{-\gamma t} u(t)$$

$$f_2(t) = (e^{-\alpha t} - e^{-\beta t})u(t)$$

$$B_M(t) = -\lim_{\gamma\to 0+} \int_{-\infty}^{\infty} \omega_M \sin\omega_M(t-\tau)e^{-\gamma(t-\tau)}$$

$$\times\ u(t-\tau)(e^{-\alpha\tau} - e^{-\beta\tau})u(\tau)d\tau$$

$$= -\lim_{\gamma\to 0+} u(t)\omega_M \int_0^t \sin\omega_M(t-\tau)\ (e^{-\gamma t+\gamma\tau-\alpha\tau} - e^{-\gamma t+\gamma\tau-\beta\tau})d\tau$$

$$= -\lim_{\gamma \to 0+} \omega_M \, u(t) \left\{ \frac{\omega_M e^{-\alpha t} - e^{-\gamma t}[(\gamma-\alpha)\sin \omega_M t + \omega_M \cos \omega_M t]}{(\gamma-\alpha)^2 + \omega_M^2} \right.$$

$$\left. - \frac{\omega_M e^{-\beta t} - e^{-\gamma t}[(\gamma-\beta)\sin\omega_M t + \omega_M \cos \omega_M t]}{(\gamma-\beta)^2 + \omega_M^2} \right\}$$

$$B_M(t) = -u(t)\omega_M \left\{ \frac{\omega_M e^{-\alpha t} + \alpha \sin \omega_M t - \omega_M \cos \omega_M t}{\alpha^2+\omega_M^2} - \frac{\omega_M e^{-\beta t} + \beta \sin \omega_M t - \omega_M \cos \omega_M t}{\beta^2 + \omega_M^2} \right\}$$

This is made up of a distorted EMP

$$- u(t) \left[\frac{\omega_M^2 e^{-\alpha t}}{\omega_M^2 + \alpha^2} - \frac{\omega_M^2 e^{-\beta t}}{\beta^2+\omega_M^2} \right]$$

plus the undamped natural modes. The distortion goes to zero as $\omega_M \to \infty$. The maximum value of the distorted wave as well as the undamped wave both have an upper bound of one. The maximum of $B_M(t)$ has an upper bound of two. This can be obtained for intermediate ω, i.e., when $\alpha \ll \omega_M \ll \beta$ so that

$$B_M(t) \approx -u(t) \; (1- \cos \omega_M t)$$

$$t_{max} = \frac{\pi}{\omega_M}$$

For large ω_M $(\alpha<<\beta<<\omega_M)$, $B_M(t) = -u(t)\ (e^{-\alpha t} - e^{-\beta t})$ (EMP shape)

and for small ω_M $(\omega_M<<\alpha<<\beta)$

$$B_M(t) = -u(t)\ \frac{\omega_M}{\alpha} \sin \omega_M t \quad \text{(no EMP shape)}$$

For the case of very large ω_M,

$$\vec{E}(\vec{r},t) = -u(t)(e^{-\alpha t} - e^{-\beta t}) \sum_M \frac{\vec{E}_M}{k_M^2} \int_{s_a} \hat{n} \times \vec{G}(\vec{r}) \cdot \nabla \times \vec{E}_M ds$$

$$= - f(t) \sum_M \frac{\vec{E}_M}{k_M{}^2} \int_{s_a} \hat{n} \times \vec{G}(\vec{r}) \cdot \nabla \times \vec{E}_M ds$$

where the input on the aperture is $G(\vec{r})f(t)$.

We note that the plane-wave expressions

$$\vec{E}(\vec{r},t) = f(t) \sum_M A_M \vec{E}_M \quad , \quad f(t) = e^{j\omega t}$$

$$A_M = \frac{\int_{s_a} \hat{n} \times \vec{G}(\vec{r}) \cdot \nabla \times \vec{E}_M ds}{k_o^2 - k_M{}^2}$$

yield the same expression when $\omega_m >> \omega_o$.

The wave length of the natural modes of a rectangular cavity are

$$\lambda_M = \frac{2}{\sqrt{\frac{m^2}{a_o{}^2} + \frac{n^2}{b_o{}^2} + \frac{p^2}{h_o{}^2}}}$$

To satisfy the boundary conditions only one of m,n,p can be zero at one time.

If $a_o < b_o < h_o$ then the largest wave length occurs when m=0, n=1, p=1. Hence

$$\lambda_M \leq \frac{2}{\sqrt{\frac{1}{b_o^2} + \frac{1}{h_o^2}}} < 2b_o$$

The largest wave length is less than twice the intermediate dimension of the cavity.

If the rise time of EMP is 3.5 ns and the fall time is 275 ns, then

$$\alpha = \frac{1}{\text{fall time}} = 3.6 \text{ MHz}$$

$$\beta = \frac{1}{\text{rise time}} = 286 \text{ MHz}$$

If a cavity has an intermediate length of 3 meters, its lowest natural frequency is greater than 50 MHz. From the above analysis we see that this lowest mode could have twice the amplitude of the incoming EMP, and its peak would occur at 10 ns after the pulse hit the aperture.

12.0 REFERENCES

1. Leigh-Wei Chen, "On Cavity Excitation Through Small Apertures," IN 45, AFWL Interaction Notes.

2. I.S. Gradshteyn, Table of Integrals Series and Products, Academic Press, 1965.

3. Handbook of Mathematical Functions, Ed. by M. Abramowitz Dover, 1964.

4. Standard Mathematical Tables, Ed. by S.M. Selby, The Chemical Rubber Company, 1967.

5. H. Goldstein, Classical Mechanics, Addison-Wesley, 1950.

6. D.S. Jones, The Theory of Electromagnetism, Macmillan, 1964, pages 222-223.

7. N.J. Rudie, Principles and Techniques of Radiation Hardening, Volume 3.

8. R.A. Waldron, Theory of Guided Electromagnetic Waves, Van Nostrand, 1969.

9. J.D. Jackson, Classical Electrodynamics, John Wiley & Sons, New York, 1962.

10. H.A. Bethe, "Theory of Diffraction of Small Holes," The Physical Review, Vol. 66, #7,8, page 163, October 1, 1944.

11. D.K. Cheng, and C.A. Chen, "On Transient Electromagnetic Excitation of a Rectangular Cavity Through an Aperture," IN 237, AFWL Interaction Notes.

12. C.M. Butler, Y. Rahmat-Samii, and R. Mittra, "Electromagnetic Penetration Through Apertures in Conducting Surfaces," IEEE Transactions on Antennas and Propagation, Vol. AP-26, #1, January, 1978, p. 82.

13.0 APPENDIX

13.1 ELLIPSOIDAL COORDINATES ξ, η, ζ

1. $$x^2 = \frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{a^2(a^2-b^2)}, \quad y^2 = \frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{-b^2(a^2-b^2)},$$

$$z^2 = \frac{\xi\eta\zeta}{a^2b^2}$$

$$a > b, \quad 0 \leq \xi < \infty, \quad -b^2 \leq \eta \leq 0, \quad -a^2 \leq \zeta \leq -b^2$$

Let ϕ be defined as $\zeta = -a^2 + (a^2-b^2)\cos^2\phi$

If $\xi \to \infty$, then $\xi/r^2 \to 1$

If $\eta \to -b^2$, then the surface of constant η is y=0 and $\frac{x^2}{a^2-b^2} - \frac{z^2}{b^2} \leq 1$

If $\zeta \to -b^2$, then the surface of constant ζ is y=0 and $\frac{x^2}{a^2-b^2} - \frac{z^2}{b^2} \geq 1$

If $\zeta \to -a^2$, then the surface of constant ζ is x=0

If $\eta = -b^2$ and $\zeta = -a^2$, the locus is the z axis

If $b \to a$, then the coordinates ξ, η, ζ are related to the oblate coordinates ξ_{ob}, η_{ob}, ϕ_{ob} as $\phi \to \phi_{ob}$, $\xi \to \xi_{ob}$, and $\eta \to \eta_{ob}$

If $\eta \to 0$, then the surface of constant η is z=0, $x^2/a^2+y^2/b^2 \geq 1$

If $\xi \to 0$, then the surface of constant ξ is z=0, $x^2/a^2+y^2/b^2 \leq 1$

2. $$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{\mu} = 1 \text{ where } \mu = \xi, \eta, \text{ or } \zeta$$

On the z-axis, $\eta = -b^2$, $\zeta = -a^2$, and $\xi = z^2$.

On the aperture, $\xi = 0$ and $\eta,\zeta = \frac{\rho^2-a^2-b^2}{2} \pm [(\frac{\rho^2-a^2-b^2}{2})^2$
$+ a^2b^2 \; (-1 + \frac{x^2}{a^2} + \frac{y^2}{b^2})]^{1/2}$

On the screen, $\eta = 0$ and $\xi,\zeta = \frac{\rho^2-a^2-b^2}{2} \pm [(\frac{\rho^2-a^2-b^2}{2})^2$
$+ a^2b^2 \; (-1 + \frac{x^2}{a^2} + \frac{y^2}{b^2})]^{1/2}$

For the far field, then $\xi/r^2 \rightarrow 1$ and $\eta,\zeta \rightarrow -\frac{1}{2r^2}[b^2x^2+a^2y^2+(a^2+b^2)z^2]$

$\pm \frac{1}{2r^2}\{[b^2x^2+a^2y^2+(a^2+b^2)z^2]^2 - 4a^2b^2z^2r^2\}^{1/2}$

If $\xi >> a^2$ and $z=0$, then $\xi \rightarrow \rho^2$, $\eta=0$ and $\zeta \rightarrow -a^2+(a^2-b^2)\cos^2\phi$
where $\cos\phi = x/\rho$

3. The inverse transformation involves the cubic equation

$$\mu^3+\mu^2[a^2+b^2-r^2]+\mu[-(a^2+b^2)z^2+a^2b^2(1-x^2/a^2-y^2/b^2)]-a^2b^2z^2 = 0$$

The solutions give ξ,η,ζ in terms of x,y,z as follows[4]:

$$\xi = -\frac{p}{3} + m\cos\theta_1$$

$$\eta = -\frac{p}{3} + m\cos(\theta_1 + \frac{4\pi}{3})$$

$$\zeta = -\frac{p}{3} + m\cos(\theta_1 + \frac{2\pi}{3})$$

where

$$\theta_1 \equiv \frac{1}{3} \cos^{-1} \frac{3b}{Am}$$

$$m \equiv 2\sqrt{-A/3}$$

$$A = Q - P^2/3$$

$$B = \frac{1}{27} (2P^3 - 9PQ + 27R)$$

$$P = a^2+b^2-x^2-y^2-z^2$$

$$Q = a^2b^2-a^2z^2-b^2z^2-a^2y^2-b^2x^2$$

$$R = -a^2b^2z^2$$

The following argument shows which solution of the cubic corresponds to each of the coordinates ξ,η,ζ. It is seen from the cubic equation written in the form

$$\frac{x^2}{a^2+\mu} + \frac{y^2}{b^2+\mu} + \frac{z^2}{\mu} = 1$$

that the surfaces of constant μ are

ellipsoids if $0 \leq \mu < +\infty$ $(\mu=\xi)$
hyperboloids of one sheet if $-b^2 \leq \mu \leq 0$ $(\mu=\eta)$
hyperboloids of two sheets if $-a^2 \leq \mu \leq -b^2$ $(\mu=\zeta)$

and that any point in space is on exactly one surface from each of these three families. Thus, the equation always has three real solutions for μ, one in each of the intervals just cited. If the three roots are distinct,(4) then

$$\frac{A^3}{27} + \frac{B^2}{4} < 0$$

Therefore, $A < 0$ and $m = 2\sqrt{-A/3}$ in real and positive. Also,

$$\frac{\frac{B^2}{4}}{-\frac{A^3}{27}} < 1$$

$$\frac{9B^2}{A^2m^2} < 1$$

$$\left|\frac{3B}{Am}\right| < 1$$

so that $\theta_1 = 1/3 \cos^{-1} \frac{3B}{Am}$ is real.

$$0 \leq \theta_1 \leq \frac{\pi}{3}$$

$$\frac{1}{2} \leq \cos \theta_1 \leq 1$$

$$-1 \leq \cos \left(\theta_1 + \frac{2\pi}{3}\right) \leq -\frac{1}{2}$$

$$-\frac{1}{2} \leq \cos \left(\theta_1 + \frac{4\pi}{3}\right) \leq \frac{1}{2}$$

Thus,

$$\cos(\theta_1 + \frac{2\pi}{3}) \leq \cos(\theta_1 + \frac{4\pi}{3}) \leq \cos\theta_1$$

$$-\frac{P}{3} + m\cos(\theta_1 + \frac{2\pi}{3}) \leq -\frac{P}{3} + m\cos(\theta_1 + \frac{4\pi}{3}) \leq -\frac{P}{3} + m\cos\theta_1$$

Since these expressions are the solutions of the cubic equation, it is now apparent which corresponds to each of the ellipsoidal coordinates:

$$\xi = -\frac{P}{3} + \cos\theta_1$$

$$\eta = -\frac{P}{3} + \cos(\theta_1 + \frac{4\pi}{3})$$

$$\zeta = -\frac{P}{3} + \cos(\theta_1 + \frac{2\pi}{3})$$

There are two curves on which not all the roots are distinct: The ellipse $x^2/a^2 + y^2/b^2 = 1$, $z=0$, where $\xi=\eta=0$; and the hyperbola $x^2/a^2-b^2 - z^2/b^2 = 1$, $y=0$, where $\eta=\zeta=-b^2$.

In both cases, it may be shown that $A < 0$ and $A^3/27 + B^2/4 = 0$, whence $m > 0$, $|3B/Am| = 1$, and θ_1 is real. Then the proof follows as before.

13.2 OBLATE SPHEROIDAL COORDINATES ξ,η,ϕ

(a=b)

$$\frac{x^2}{\mu+a^2} + \frac{y^2}{\mu+a^2} + \frac{z^2}{\mu} = 1 \quad \text{where } \mu = \xi \text{ or } \eta$$

$$\rho^2 = x^2+y^2 = (\xi+a^2)(\eta+a^2)/a^2, \quad z^2 = -\xi\eta/a^2$$

$$0 \le \xi < \infty, \quad -a^2 \le \eta \le 0$$

$$\xi,\eta = \frac{r^2-a^2}{2} \pm \left[\left(\frac{r^2-a^2}{2}\right)^2+a^2z^2\right]^{1/2}$$

$$\phi = \tan^{-1} y/x$$

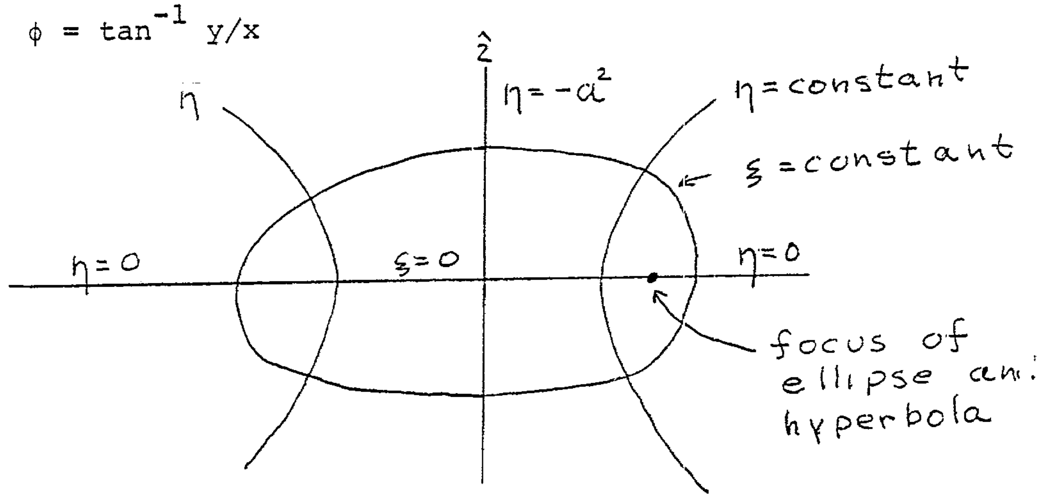

FIGURE 10.1 Oblate Spheroidal Coordinates

On the z axis, $\xi = z^2$, $\eta = -a^2$

On the aperture, $\xi = 0$ and $\eta = -a^2+\rho^2$

On the screen, $\eta = 0$ and $\xi = -a^2+\rho^2$

For the far field, $\xi/r^2 \to 1$ and $\eta = \frac{-a^2z^2}{r^2} = -a^2 \cos^2\theta$ where θ is the angle between the z axis and $\vec{r}$.

13.3 INTEGRALS AND FORMULAE

$\alpha = \cot^{-1}(\xi^{1/2}/a)$

$e = (1-b^2/a^2)^{1/2}$

$a > b$

1. $$\int_\xi^\infty \frac{d\xi}{\xi^{1/2}(\xi+a^2)^{3/2}(\xi+b^2)^{1/2}} = \frac{2[F(\alpha,e) - E(\alpha,e)]}{a(a^2-b^2)}$$ (grad[(2)] 222/18)

2. $$\int_\xi^\infty \frac{d\xi}{\xi^{1/2}(\xi+a^2)^{1/2}(\xi+b^2)^{3/2}} = \frac{2[a^2E(\alpha,e) - b^2F(\alpha,e)]}{ab^2(a^2-b^2)}$$

$$- \frac{2\xi^{1/2}}{b^2(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}}$$ (Grad [(2)] 222/12)

3. $$\int_\xi^\infty \frac{d\xi}{\xi^{3/2}(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}} = \frac{2}{b^2\xi^{1/2}}\frac{(\xi+b^2)^{1/2}}{(\xi+a^2)^{1/2}} - \frac{2}{ab^2}E(\alpha,e)$$

(Grad [(2)] 221/6)

4. $$\int_0^\infty \frac{d\xi}{\xi^{1/2}(\xi+a^2)^{3/2}(\xi+b^2)^{1/2}} = \frac{2}{a(a^2-b^2)}[F(\tfrac{\pi}{2},e) - E(\tfrac{\pi}{2},e)]$$

5. $$\int_0^\infty \frac{d\xi}{\xi^{1/2}(\xi+a^2)^{1/2}(\xi+b^2)^{3/2}} = \frac{2}{ab^2(a^2-b^2)}[a^2E(\tfrac{\pi}{2},e) - b^2F(\tfrac{\pi}{2},e)]$$

6. $$\lim_{\varepsilon\to 0} \varepsilon^{1/2}\int_\varepsilon^\infty \frac{d\xi}{\xi^{3/2}(\xi+a^2)^{1/2}(\xi+b^2)^{1/2}} = \frac{2}{ab}$$

7. $E(\alpha,e) = \int_0^{\alpha} \sqrt{1-e^2\sin^2\phi}\, d\phi$

8. $F(\alpha,e) = \int_0^{\alpha} \frac{d\phi}{\sqrt{1-e^2\sin^2\phi}}$

9. $\lim_{b\to a} \frac{\zeta+a^2}{F(\frac{\pi}{2},e)-E(\frac{\pi}{2},e)} = \frac{4a^2}{\pi} \cos^2\phi$

10. $\lim_{b\to a} \frac{e^2}{F(\frac{\pi}{2},e)-E(\frac{\pi}{2},e)} = \frac{4}{\pi}$

11. $\lim_{b\to a} \frac{a^2-b^2}{a^2E(\frac{\pi}{2},e)-b^2F(\frac{\pi}{2},e)} = \frac{4}{\pi}$

12. $\lim_{b\to a} \frac{F(\alpha,e)-E(\alpha,e)}{F(\frac{\pi}{2},e)-E(\frac{\pi}{2},e)} = \frac{2}{\pi} (\alpha-\frac{a\xi^{1/2}}{\xi+a^2})$

13. $\lim_{b\to a} \frac{a^2E(\alpha,e)-b^2F(\alpha,e)}{a^2E(\frac{\pi}{2},e)-b^2F(\frac{\pi}{2},e)} = \frac{2}{\pi} (\alpha+\frac{a\xi^{1/2}}{\xi+a^2})$

14. $\int_0^1 \frac{\cos \alpha x\, dx}{\sqrt{1-x^2}} = \frac{\pi}{2} J_0(\alpha)$ (grad [(2)] 419/3.753/2)

15. $\int_0^a x^{\nu+1}\sin(b\sqrt{a^2-x^2})\, J_\nu(x)\, dx = \sqrt{\pi/2}\; a^{\nu+3/2} b(1+b^2)^{-\nu/2-3/4}$

$J_{\nu+3/2}\,(a\sqrt{1+b^2})$ (grad [(2)] 761/1)

16. $j_n(z) = \sqrt{\pi/2z}\, J_{n+1/2}(z)$ (Abram [3] 437/10.1.1)

17. $j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}$ (Abram [3] 438/10.1.11)

18. $E(\frac{\pi}{2},e) = \frac{\pi}{2}[1-(\frac{1}{2})^2 \frac{e^2}{1} - (\frac{1\cdot 3}{2\cdot 4})^2 \frac{e^4}{3} - (\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6})^2 \frac{e^6}{5} - \ldots]$

(Abram [3] 591/17.3.12)

19. $F(\frac{\pi}{2},e) = \frac{\pi}{2}[1+(\frac{1}{2})^2 e^2 + (\frac{1\cdot 3}{2\cdot 4})^2 e^4 + (\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6})^2 e^6 + \ldots]$

(Abram [3] 591/17.3.11)

13.4 LAPLACE EQUATION

1. $\nabla^2 S = \frac{1}{h_1 h_2 h_3} [\frac{\partial}{\partial\mu_1}(\frac{h_2 h_3}{h_1} \frac{\partial S}{\partial\mu_1}) + \frac{\partial}{\partial\mu_2}(\frac{h_1 h_3}{h_2} \frac{\partial S}{\partial\mu_2}) + \frac{\partial}{\partial\mu_3}(\frac{h_1 h_2}{h_3} \frac{\partial S}{\partial\mu_3})]$

where

$h_j^2 = \sum_{i=1}^{3} (\frac{\partial x_i}{\partial\mu_j})^2$

$x_i = x_i(\mu_1,\mu_2,\mu_3)$ (μ_j is an orthogonal system) (CRC [4] p. 494)

2. $\hat{e}_{\mu_i} = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial \mu_i}$

where $\vec{r} = \sum_{i=1}^{3} \hat{x}_i x_i = x\hat{x} + y\hat{y} + z\hat{z}$

3. For ellipsoidal coordinates

$$h_1 = \frac{1}{2} \left[\frac{(\xi-\eta)(\xi-\zeta)}{(\xi+a^2)(\xi+b^2)\xi}\right]^{1/2}$$

$$h_2 = \frac{1}{2} \left[\frac{(\xi-\eta)(\zeta-\eta)}{(\eta+a^2)(\eta+b^2)\eta}\right]^{1/2}$$

$$h_3 = \frac{1}{2} \left[\frac{(\zeta-\xi)(\zeta-\eta)}{(\zeta+a^2)(\zeta+b^2)(\zeta)}\right]^{1/2}$$

$$\nabla^2 S = \frac{-4}{(\xi-\eta)(\zeta-\xi)(\eta-\zeta)} \Big[(\eta-\zeta) R_\xi \frac{\partial}{\partial \xi} R_\xi \frac{\partial S}{\partial \xi}$$

$$+ (\xi-\zeta) R_\eta \frac{\partial}{\partial \eta} R_\eta \frac{\partial S}{\partial \eta} + (\xi-\eta) R_\zeta \frac{\partial}{\partial \zeta} R_\zeta \frac{\partial S}{\partial \zeta}\Big]$$

where $R_\xi = [\xi(\xi+a^2)(\xi+b^2)]^{1/2}$

$R_\eta = [-\eta(\eta+a^2)(\eta+b^2)]^{1/2}$

$R_\zeta = [\zeta(\zeta+a^2)(\zeta+b^2)]^{1/2}$

$$x_\xi = \frac{(\eta+a^2)(\zeta+a^2)}{2xa^2(a^2-b^2)}, \quad y_\xi = \frac{(\eta+b^2)(\zeta+b^2)}{-2yb^2(a^2-b^2)}, \quad z_\xi = \frac{\eta\zeta}{2za^2b^2}$$

$$x_\eta = \frac{(\xi+a^2)(\zeta+a^2)}{2xa^2(a^2-b^2)}, \quad y_\eta = \frac{(\xi+b^2)(\zeta+b^2)}{-2yb^2(a^2-b^2)}, \quad z_\eta = \frac{\xi\zeta}{2za^2b^2}$$

$$x_\zeta = \frac{(\xi+a^2)(\eta+a^2)}{2xa^2(a^2-b^2)}, \quad y_\zeta = \frac{(\xi+b^2)(\eta+b^2)}{-2yb^2(a^2-b^2)}, \quad z_\zeta = \frac{\xi\eta}{2za^2b^2}$$

$$\hat{e}_\xi = \frac{1}{h_\xi}\frac{\partial\vec{r}}{\partial\xi} = \frac{1}{h_\xi}[\hat{x}\ x_\xi + \hat{y}\ y_\xi + \hat{z}\ z_\xi]$$

$$\hat{e}_\xi = [\frac{(\xi-\eta)(\xi-\zeta)}{\xi(\xi+a^2)(\xi+b^2)}]^{-1/2}[\hat{x}\ \frac{(\eta+a^2)(\zeta+a^2)}{xa^2(a^2-b^2)} - \hat{y}\ \frac{(\eta+b^2)(\zeta+b^2)}{yb^2(a^2-b^2)}$$

$$+ \frac{\hat{z}\eta\zeta}{za^2b^2}]$$

$$\hat{e}_\eta = [\frac{\eta(\eta+a^2)(\eta+b^2)}{(\xi-\eta)(\zeta-\eta)}]^{1/2}[\hat{x}\ \frac{(\xi+a^2)(\zeta+a^2)}{xa^2(a^2-b^2)} - \hat{y}\ \frac{(\xi+b^2)(\zeta+b^2)}{yb^2(a^2-b^2)}$$

$$+ \frac{\hat{z}\xi\zeta}{za^2b^2}]$$

$$\hat{e}_\zeta = \left[\frac{\zeta(\zeta+a^2)(\zeta+b^2)}{(\zeta-\xi)(\zeta-\eta)}\right]^{1/2}\left[\hat{x}\ \frac{(\xi+a^2)(\eta+a^2)}{xa^2(a^2-b^2)} - \hat{y}\ \frac{(\xi+b^2)(\eta+b^2)}{yb^2(a^2-b^2)} + \frac{\hat{z}\xi\eta}{za^2b^2}\right]$$

$$\xi_x = \frac{2x\xi(\xi+b^2)}{(\xi-\zeta)(\xi-\eta)},\quad \xi_y = \frac{2y\xi(\xi+a^2)}{(\xi-\eta)(\xi-\zeta)},\quad \xi_z = \frac{2z(\xi+a^2)(\xi+b^2)}{(\xi-\eta)(\xi-\zeta)}$$

$$\eta_x = \frac{2x\eta(\eta+b^2)}{(\eta-\xi)(\eta-\zeta)},\quad \eta_y = \frac{2y\eta(\eta+a^2)}{(\eta-\xi)(\eta-\zeta)},\quad \eta_z = \frac{2z(\eta+a^2)(\eta+b^2)}{(\eta-\xi)(\eta-\zeta)}$$

$$\zeta_x = \frac{2x\zeta(\zeta+b^2)}{(\zeta-\xi)(\zeta-\eta)},\quad \zeta_y = \frac{2y\zeta(\zeta+a^2)}{(\zeta-\xi)(\zeta-\eta)},\quad \zeta_z = \frac{2z(\zeta+a^2)(\zeta+b^2)}{(\zeta-\xi)(\zeta-\eta)}$$

$$\Delta = \left[\begin{vmatrix} \xi_x & \eta_x & \zeta_x \\ \xi_y & \eta_y & \zeta_y \\ \xi_z & \eta_z & \zeta_z \end{vmatrix}\right]^{-1} = \begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{vmatrix} = \frac{(\xi-\eta)(\eta-\zeta)(\zeta-\xi)}{8a^2b^2(a^2-b^2)xyz}$$

13.5 DERIVATION OF BOUNDARY CONDITIONS

To get a boundary condition for an EMP plane wave, consider an incident wave which has a transform

$$\hat{\vec{E}}^{i} = \vec{E}_o{}^{i}(\omega)\exp\left(-\frac{j\omega\hat{k}_i}{c_o}\cdot\vec{r}\right)$$

If we neglect the reflected waves on the metal, then the transmitted wave will have fields related by

$$\hat{\vec{H}}^{t} = \hat{k}_t \times \vec{E}^{t}\sqrt{\varepsilon/\mu - j\sigma/\omega\mu} = \hat{k}_t \times \vec{E}^{t}\sqrt{-\frac{j\sigma}{\omega\mu}\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}$$

For EMP the highest "practical" frequency is 10^8 Hz. For copper $\omega\varepsilon/\sigma = 2\pi(10^8)(10)(8.834\times10^{-12})/5.8\times10^7 = 10^{-9}$ so that the correction term can be neglected even for the highest frequency. Hence

$$\hat{k}_t \times \vec{E}^{t} = \sqrt{\omega\mu/2\sigma}\ \vec{H}^{t}(1+j)$$

At the boundary $\hat{n}\cdot\vec{H}$ is approximately zero so that $\hat{k}_t = \hat{n}$ = the normal into the metal. Since $\hat{n}\times\vec{E}^{t}$ is continuous across a boundary, then this must be the boundary condition inside the cavity, at the cavity wall. This is the boundary condition of Jackson.(2)

13.6 DERIVATION OF COEFFICIENTS

$$\nabla\times\vec{E} = -j\omega\mu_o\,\vec{H} \qquad \nabla\times\vec{E}_M = -j\omega_M\mu_o\,\vec{H}_M$$

$$\nabla\times\vec{H} = j\omega\varepsilon_o\,\vec{E} \qquad \nabla\times\vec{H}_M = j\omega_M\varepsilon_o\,\vec{E}_M$$

$$\begin{cases} \vec{E} = \Sigma\, A_M\,\vec{E}_M \\ \vec{H} = \nabla\phi + k_o\Sigma\,\dfrac{A_M\,\vec{H}_M}{k_M} \end{cases}$$

$$\phi = \int_S \hat{n}\cdot\vec{H}\, G(\vec{r},\vec{r}\,')\, dS \qquad \nabla^2 G = \delta(\vec{r},\vec{r}\,')$$

$$\frac{\partial G}{\partial n} = 0 \text{ on } S$$

$$A_M = \int \vec{E}\cdot\vec{E}_M\, d\tau \qquad \nabla x \nabla x E_M = +k_M{}^2 E_M$$

$$k_M{}^2 A_M = \int \vec{E}\cdot\nabla x \nabla x \vec{E}_M\, d\tau$$

$$k_o{}^2 A_M = \int \vec{E}_M\cdot\nabla x \nabla x \vec{E}\, d\tau$$

$$(k_o{}^2 - k_M{}^2)A_M = \int (\vec{E}_M\cdot\nabla x\nabla x\vec{E} - \vec{E}\cdot\nabla x\nabla x\vec{E}_M)\, d\tau$$

$$\nabla\cdot(\vec{E}_M x\nabla x\vec{E} - \vec{E}x\nabla x\vec{E}_M) = \nabla x\vec{E}\cdot\nabla x\vec{E}_M - \vec{E}_M\cdot\nabla x\nabla x\vec{E} - \nabla x\vec{E}_M\cdot\nabla x\vec{E} + \vec{E}\cdot\nabla x\nabla x\vec{E}_M$$

$$(k_o{}^2 - k_M{}^2)A_M = \int \nabla\cdot(\vec{E}x\nabla x\vec{E}_M - \vec{E}_M x\nabla x\vec{E})\, d\tau$$

$$= \int_S \hat{n}\cdot(\vec{E}x\nabla x\vec{E}_M - \vec{E}_M x\nabla x\vec{E})\, dS$$

$$= \int_S (\hat{n}x\vec{E}\cdot\nabla x\vec{E}_M - \hat{n}x\vec{E}_M\cdot\nabla x\vec{E})\, dS$$

But $\hat{n}x\vec{E}_M = 0$ on S

$$(k_o{}^2 - k_M{}^2)A_M = \int_S \hat{n}\cdot\vec{E}x\nabla x\vec{E}_M\, dS$$