Interaction Notes

Note 316

February 1977

# Cavity Excitation via Apertures

K.S.H. Lee and F.C. Yang
The Dikewood Corporation, Westwood Research Branch
Los Angeles, California 90024

K.C. Chen
Air Force Weapons Laboratory
Albuquerque, New Mexico

## Abstract

In the analysis of EMP (electromagnetic pulse) penetration into cavities aboard an aircraft or other aeronautical systems, the low-frequency parts of the magnetic and electric fields are usually separated out for consideration not only because they are more analytically tractable but also because they are the dominant constituents of the EMP frequency spectrum. This report is concerned only with the calculation of the low-frequency penetrant magnetic field through apertures into cavities. Currently, there are two distinct methods available for this kind of calculation. These two methods are elucidated and applied to a rectangular (simply connected) cavity and to a coaxial (multiply connected) cavity. From the solutions of these two different types of cavities it is concluded that one method (the Eigenfunction Method) offers more flexibility in solution representation than the other (the Direct Method). Simple explicit engineering working formulas are found for estimating the low-frequency penetrant magnetic field into a rectangular cavity through an arbitrary hole.

AFWL-TR-77-147
Dec 77

ACKNOWLEDGMENT

The authors are indebted to Dr. J.P. Castillo and Dr. C.E. Baum of the Air Force Weapons Laboratory for enlightening discussions concerning the subject matter of the report.

Distribution Limited to U.S. Govt Agencies

Interaction Notes

Note 316

February 1977

# Cavity Excitation via Apertures

K.S.H. Lee and F.C. Yang
The Dikewood Corporation, Westwood Research Branch
Los Angeles, California 90024

K.C. Chen
Air Force Weapons Laboratory
Albuquerque, New Mexico

Abstract

In the analysis of EMP (electromagnetic pulse) penetration into cavities aboard an aircraft or other aeronautical systems, the low-frequency parts of the magnetic and electric fields are usually separated out for consideration not only because they are more analytically tractable but also because they are the dominant constituents of the EMP frequency spectrum. This report is concerned only with the calculation of the low-frequency penetrant magnetic field through apertures into cavities. Currently, there are two distinct methods available for this kind of calculation. These two methods are elucidated and applied to a rectangular (simply connected) cavity and to a coaxial (multiply connected) cavity. From the solutions of these two different types of cavities it is concluded that one method (the Eigenfunction Method) offers more flexibility in solution representation than the other (the Direct Method). Simple explicit engineering working formulas are found for estimating the low-frequency penetrant magnetic field into a rectangular cavity through an arbitrary hole.

ACKNOWLEDGMENT

The authors are indebted to Dr. J.P. Castillo and Dr. C.E. Baum of the Air Force Weapons Laboratory for enlightening discussions concerning the subject matter of the report.

TABLE OF CONTENTS

## LIST OF ILLUSTRATIONS

## I. INTRODUCTION

Many of the aircraft-EMP interaction problems are concerned with cavities coupled to the aircraft exterior through apertures. Typical examples are the cockpit cavity, the bomb bay, the avionics bay, and the wheel well, to mention just a few. These cavities are electromagnetically coupled to the exterior through windows or door slits. Although cavity excitation by apertures is an old topic, there seems to be certain persistent confusion among the techniques developed in the past for treating this problem. The confusion arises mainly from the need of a magnetic scalar potential $\psi$ , in addition to a electric vector potential $\underline{F}$ , to completely describe the cavity fields. This magnetic scalar potential $\psi$ is of particular importance in EMP interaction calculations because it yields the low-frequency penetrant magnetic field. There are, however, two distinct methods for the determination of $\psi$ each of which gives seemingly different solutions. It is one of the purposes of this note to end all the confusion in cavity excitation via apertures.

In Section II the two methods to calculate $\psi$ are described. In Section III a rectangular cavity (more precisely, a parallelepiped) is taken as an example of a simply connected cavity and explicit results for $\psi$ in terms of its normal derivative in the aperture are derived by the two methods. The problem of a multiply connected cavity is taken up in Section IV, where $\psi$ is worked out in detail for a coaxial cavity.

## II. THE TWO METHODS

It is well known that the cavity fields excited via an aperture can be described by the magnetic scalar potential $\psi$ and the electric vector potential $\underline{F}$, viz.,

$$\underline{H} = -\nabla\psi - s\underline{F}$$
$$\underline{E} = -\frac{1}{\epsilon}\nabla\times\underline{F} \qquad (1)$$

The counterparts of $\psi$ and $\underline{F}$ are the electric scalar potential $\varphi$ and the magnetic vector potential $\underline{A}$ by means of which one can write

$$\underline{E}' = -\nabla\varphi - s\underline{A}$$
$$\underline{H}' = \frac{1}{\mu}\nabla\times\underline{A} \qquad (2)$$

where $(\underline{E}', \underline{H}')$ is the field radiated by electric currents and electric charges such as those on a conductor, whereas $(\underline{E}, \underline{H})$ is the field radiated by magnetic currents and magnetic charges distributed in an aperture.

One of the possible ways to proceed with equation (1) is to express $\underline{F}$ in terms of cavity normal modes $(\underline{E}_p, \underline{H}_p)$ and to obtain

$$\underline{H} = -\nabla\psi + s\epsilon\sum(a_p/k_p)\underline{H}_p$$
$$\underline{E} = \sum a_p\underline{E}_p \qquad (3)$$

where the p-th normal modes satisfies

$$\nabla\times\underline{E}_p = k_p\underline{H}_p, \qquad \nabla\times\underline{H}_p = k_p\underline{E}_p \qquad (4)$$

and the boundary conditions that $\hat{n} \times \underline{E}_p$ and $\hat{n} \cdot \underline{H}_p$ vanish on the cavity walls with all apertures short circuited by "isotropic" perfect conductors. In the zero frequency limit $(s = 0)$, the magnetostatic field arises solely from $\nabla\psi$, while the electrostatic field is contributed by all the normal modes.

As mentioned in the Introduction, there are two distinct methods of finding $\psi$. In what follows, these two methods will be designated as the Direct Method and the Eigenfunction Method.

A. Direct Method [1,2]

It is customary to use the Coulomb gauge for $\psi$ and $\underline{F}$ in the calculation of cavity fields, so that $\underline{F}$ can be directly expressed in terms of the solenoidal normal modes. Thus, within the cavity volume V one has

$$\nabla^2 \psi = 0 \tag{5}$$

Assuming that the geometry of the cavity permits the separation of variables one writes the solution of (5) as

$$\psi = \sum A_\nu \psi_\nu \tag{6}$$

where $\psi_\nu$ satisfies (5) and

$$\frac{\partial}{\partial n} \psi_\nu = 0 , \qquad \text{on} \quad S - S_A \tag{7}$$

with the constraint that

$$\int_A \frac{\partial \psi}{\partial n} \, dS = 0$$

where $S - S_A$ represents the surfaces of all the cavity walls except the wall $S_A$ which contains the aperture A (Figure 1). To find the expansion coefficient $A_\nu$ one matches the normal derivative of (6) at the surface $S_A$, viz.,

$$\sum A_\nu \frac{\partial \psi_\nu}{\partial n} = \begin{cases} \dfrac{\partial \psi}{\partial n} , & \text{in } A \\ 0 , & \text{on } S_A - A \end{cases} \tag{8}$$

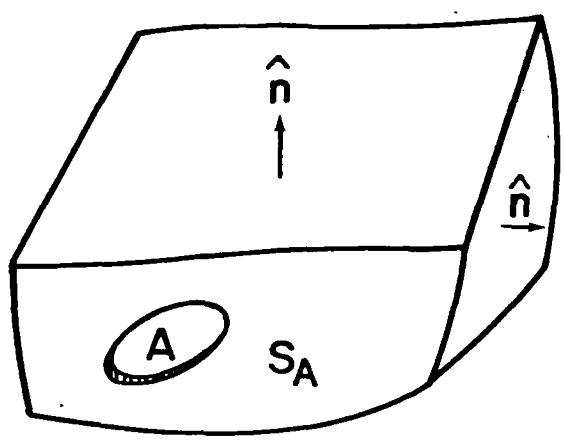


Figure 1. A cavity with an aperture in one of its walls.

By virtue of the orthogonality property of $(\partial/\partial n)\psi_\nu$ over the surface $S_A$ one has from (8)

$$A_\nu = \frac{\int_A \frac{\partial\psi_\nu}{\partial n}\frac{\partial\psi}{\partial n}\,dS}{\int_{S_A}\left(\frac{\partial\psi_\nu}{\partial n}\right)^2 dS} \tag{9}$$

where $(\partial/\partial n)\psi$ over $A$ is either given or matched to the field exterior to the cavity.

An alternative method to determine $A_\nu$ is to multiply both sides of (8) by $\psi_\nu$ and to make use of the orthogonality property of $(\partial/\partial n)\psi_\nu$ and $\psi_\nu$ over the surface $S_A$. Hence,

$$A_\nu = \frac{\int_A \psi_\nu \frac{\partial\psi}{\partial n}\,dS}{\int_V |\nabla\psi_\nu|^2 dV} \tag{9.a}$$

where one has used the following operations

$$\int_{S_A} \psi_\nu \frac{\partial\psi_\nu}{\partial n}\,dS = \int_S \psi_\nu \frac{\partial\psi_\nu}{\partial n}\,dS = \int_V \nabla\cdot(\psi_\nu\nabla\psi_\nu)dV = \int_V |\nabla\psi_\nu|^2 dV\ .$$

If one tries to solve (5) by the technique of Green's function, i.e., to represent $\psi$ within $V$ by an integral over $(\partial/\partial n)\psi$ in the aperture, one would encounter a serious difficulty, which is well known for the Neumann problem of a closed region.

B. Eigenfunction Method [3,4]

Instead of solving the Laplace equation (5) directly for $\psi$ one constructs $\psi$ from a complete set of eigenfunctions $\psi_q$, namely,

$$\psi = \sum B_q \psi_q \tag{10}$$

where $\psi_q$ satisfies

$$\begin{aligned} &\nabla^2 \psi_q + k_q^2 \psi_q = 0 , \quad \text{in} \quad V \\ &\frac{\partial}{\partial n} \psi_q = 0 , \quad \text{on} \quad S \end{aligned} \tag{11}$$

where $S$ is the surface of the closed cavity with all apertures short-circuited by isotropic perfect conductors. The determination of the expansion coefficient $B_q$ is not as simple as the determination of $A_\nu$ because the interchange of the order of differentiation and summation is not permissible on the surface $S$. To find $B_q$ one integrates $\psi_p \hat{n} \cdot \nabla \psi$ over $S$ and by means of the Gauss theorem one has

$$\begin{aligned} \int_S \psi_p \hat{n} \cdot \nabla \psi dS &= \int_S \hat{n} \cdot (\psi_p \nabla \psi) dS \\ &= \int_V \nabla \cdot (\psi_p \nabla \psi) dV \\ &= \int_V (\nabla \psi_p \cdot \nabla \psi + \psi_p \nabla^2 \psi) dV \\ &= \int_V \nabla \psi \cdot \nabla \psi_p dV \qquad (\nabla^2 \psi = 0 , \quad \text{in} \quad V) \end{aligned} \tag{12}$$

Using (10) to evaluate $\nabla\psi$ in (12) one gets

$$\sum B_q \int_V \nabla \psi_q \cdot \nabla \psi_p \, dV = \int_A \psi_p \frac{\partial \psi}{\partial n} \, dS \tag{13}$$

where the fact that $(\partial/\partial n)\psi = 0$ on $S - A$ has been used. To work out $\nabla\psi_p \cdot \nabla\psi_q$ one recalls that within $V$

$$\nabla^2\psi_p + k_p^2\psi_p = 0$$

$$\nabla^2\psi_q + k_q^2\psi_q = 0$$

from which

$$\nabla\cdot(\psi_p\nabla\psi_q + \psi_q\nabla\psi_p) - 2\nabla\psi_p\cdot\nabla\psi_q + (k_p^2 + k_q^2)\psi_p\psi_q = 0$$

which gives, when integrated over $V$ ,

$$\int_V \nabla\psi_p\cdot\nabla\psi_q dV = \frac{1}{2}(k_p^2 + k_q^2)\int_V \psi_p\psi_q dV$$
$$= \delta_{pq} k_p^2 \int_V \psi_p^2 dV \tag{14}$$

where $\delta_{pq}$ is the Kronecker delta and the orthogonality property of $\psi_q$ in $V$ has been used. Substitution of (14) in (13) gives

$$B_p = \frac{1}{k_p^2}\frac{\int_A \psi_p \frac{\partial\psi}{\partial n} dS}{\int_V \psi_p^2 dV} = \frac{\int_A \psi_p \frac{\partial\psi}{\partial n} dS}{\int_V |\nabla\psi_p|^2 dV} \tag{15}$$

Unlike the Direct Method which uses the fact that $\psi$ satisfies the Laplace equation in $V$ from the outset in the construction of the solution, the Eigenfunction Method makes use of that fact only in the process of determining the expansion coefficient $B_q$ .

In the next two sections expressions (6) and (10) will be evaluated for a rectangular cavity (a simply connected cavity) and for a coaxial cavity (a multiply connected cavity).

## III. SIMPLY CONNECTED CAVITY

The rectangular cavity shown in Figure 2 is an example of a simply connected cavity. The rectangular cavity is often a good approximation to many cavities on an aircraft, for example, the avionics bay, the wheel well, and the cargo bay. The magnetic scalar potential inside this cavity with an aperture in one of its walls will be calculated by the two methods described in the previous section.

### A. Direct Method

Referring to Figure 2 one immediately writes down

$$\psi = \sum A_\nu \psi_\nu \equiv \sum A_{mn}\psi_{mn}$$

$$\psi_{mn} = \cos(m\pi y/b)\cos(n\pi z/c)\cosh[\gamma_{mn}(a-x)]$$

$$\gamma_{mn}^2 = (m\pi/b)^2 + (n\pi/c)^2$$

Clearly, $\psi_{mn}$ satisfies the Laplace equation (5) and the boundary condition (7) on all the cavity walls except the wall $x=0$ which contains the aperture $A$ . With the help of (9) the coefficient $A_{mn}$ can be readily evaluated and the final result for $\psi$ is

$$\psi = -\frac{4}{bc}\sum_{m,n} \frac{\cos(m\pi y/b)\cos(n\pi z/c)\cosh[\gamma_{mn}(a-x)]}{\varepsilon_m \varepsilon_n \gamma_{mn} \sinh(\gamma_{mn} a)} \times \int_A \frac{\partial\psi}{\partial x'} \cos(m\pi y'/b)\cos(n\pi z'/c)dy'dz' \tag{16}$$

where $\varepsilon_m = 1 + \delta_{mo}$ , $\varepsilon_n = 1 + \delta_{no}$ . The important point about this series representation is that it contains oscillatory functions in the directions

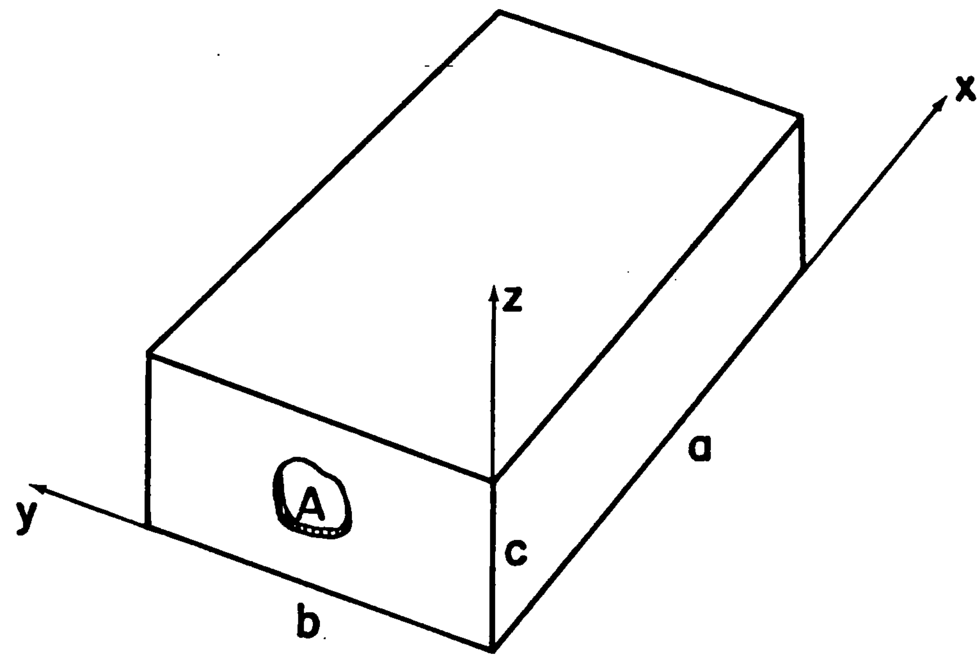


Figure 2. A rectangular cavity with an aperture in one of its walls.

parallel to and an non-oscillatory function in the direction perpendicular to the plane of the aperture. It is therefore not an efficient representation for shallow cavities, that is, for cavities whose depth $a \ll b$ or $c$.

B. Eigenfunction Method

According to this method the magnetic scalar potential $\psi$ for the geometry shown in Figure 2 is given by

$$\psi = \sum B_q \psi_q \equiv \sum_{\ell,m,n} B_{\ell mn} \psi_{\ell mn}$$

$$\psi_{\ell mn} = \cos(\ell\pi x/a)\cos(m\pi y/b)\cos(n\pi z/c)$$

$$k^2_{\ell mn} = (\ell\pi/a)^2 + (m\pi/b)^2 + (n\pi/c)^2$$

$$\equiv (\ell\pi/a)^2 + \gamma^2_{mn}$$

$$\equiv (m\pi/b)^2 + \gamma^2_{\ell n}$$

$$\equiv (n\pi/c)^2 + \gamma^2_{\ell m}$$

where $\psi_{\ell mn}$ clearly satisfies the Helmholtz equation and the boundary condition (11). Using (15) to evaluate $B_{\ell mn}$ and substituting the result into the expansion for $\psi$ one obtains

$$\psi = -\frac{8}{abc} \sum_{\ell,m,n} \frac{\cos(\ell\pi x/a)\cos(m\pi y/b)\cos(n\pi z/c)}{\varepsilon_\ell \varepsilon_m \varepsilon_n [(\ell\pi/a)^2 + \gamma^2_{mn}]} \times \int_A \frac{\partial\psi}{\partial x'} \cos(m\pi y'/b)\cos(n\pi z'/c)dy'dz' \tag{17}$$

$$= -\frac{4}{bc}\sum_{m,n}\frac{\cos(m\pi y/b)\cos(n\pi z/c)\cosh[\gamma_{mn}(a-x)]}{\varepsilon_m\varepsilon_n\gamma_{mn}\sinh(\gamma_{mn}a)}$$

(17.a)

$$\times\int_A \frac{\partial\psi}{\partial x'}\cos(m\pi y'/b)\cos(n\pi z'/c)dy'dz'$$

$$= -\frac{4}{ac}\sum_{\ell,n}\frac{\cos(\ell\pi x/a)\cos(n\pi z/c)}{\varepsilon_\ell\varepsilon_n\gamma_{\ell n}\sinh(\gamma_{\ell n}b)}$$

(17.b)

$$\times\int_A \frac{\partial\psi}{\partial x'}\cosh[\gamma_{\ell n}(b-y_>)]\cosh(\gamma_{\ell n}y_<)\cos(n\pi z'/c)dy'dz'$$

$$= -\frac{4}{ab}\sum_{\ell,m}\frac{\cos(\ell\pi x/a)\cos(m\pi y/b)}{\varepsilon_\ell\varepsilon_m\gamma_{\ell m}\sinh(\gamma_{\ell m}c)}$$

(17.c)

$$\times\int_A \frac{\partial\psi}{\partial x'}\cosh[\gamma_{\ell m}(c-z_>)]\cosh(\gamma_{\ell m}z_<)\cos(m\pi y'/b)dy'dz'$$

where one has used the formula [5]

$$\frac{2}{a}\sum_{\ell=0}^{\infty}\frac{1}{\varepsilon_\ell}\frac{\cos(\ell\pi x/a)}{(\ell\pi/a)^2+\gamma_{mn}^2} = \frac{\cosh[\gamma_{mn}(a-x)]}{\gamma_{mn}\sinh(\gamma_{mn}a)}$$

in going from (17) to (17.a), and the formula [5]

$$\frac{2}{d}\sum_{p=0}^{\infty}\frac{\cos(p\pi\zeta/d)\cos(p\pi\zeta'/d)}{\varepsilon_p[(p\pi/d)^2+\gamma_{eq}^2]} = \frac{\cosh[\gamma_{eq}(d-\zeta_>)]\cosh(\gamma_{eq}\zeta_<)}{\gamma_{eq}\sinh(\gamma_{eq}d)}$$

in going from (17) to either (17.b) or (17.c), and $\zeta_>(\zeta_<)$ is the larger (smaller) of $(\zeta,\zeta')$.

By now it should become clear that the Eigenfunction Method leads to four different representations of the solution, one of which, namely expression (17.a), is identical to (16) obtained via the Direct Method. While (17.a) is exponentially decaying in the x-direction, (17.b) and (17.c) decay respectively in the y- and z-directions. The latter two expressions are good representations of the solution for a shallow cavity in which $a \ll b , c$ , whereas (17.a) is useful for a deep cavity where $a \gg b , c$ .

Expressions (17.a), (17.b) and (17.c) are now applied to the calculation of the magnetic field of the dominant mode at the center of the cavity (Figure 2) for the case of a deep cavity $(a \gg b > c)$ and for the case of a shallow cavity $(b > c \gg a$ or $c > b \gg a)$ .

(i) <u>Deep Cavity</u> $(a \gg b > c)$

For this case expression (17.a) is most useful and it gives the dominant mode (that is, the mode with the smallest decay constant) as

$$\psi_d = \frac{-2}{\pi c}\left[e^{-\pi x/b} + e^{-\pi(2a/b - x/b)}\right]\cos(\pi y/b)\int_A \frac{\partial\psi}{\partial x'}\cos(\pi y'/b)dy'dz' \qquad (18.a)$$

from which the magnetic field at the cavity center is

$$\underline{H}_d = \hat{y}\,\frac{-2}{bc}\,e^{-\pi a/(2b)}\int_A \frac{\partial\psi}{\partial x'}\cos(\pi y'/b)dy'dz' \qquad (18.b)$$

If the maximum linear dimension of the aperture is much smaller than $b$ , the integral in (18.a) and (18.b) can be expressed in terms of the magnetic polarizability $\underline{\underline{\alpha}}_m$ of the aperture and the external short-circuited field $\underline{H}_{sc}$ via the relation

$$\int_A \underline{r}_s \frac{\partial\psi}{\partial n} dS = \underline{\underline{\alpha}}_m \cdot \underline{H}_{sc} = -\underline{m}$$

Let the "center" of the hole be located at $(y_o , z_o)$ in the $x = 0$ plane. Then

$$\int_A \frac{\partial\psi}{\partial x'} \cos(\pi y'/b)dy'dz' = \int_A \frac{\partial\psi}{\partial x'} \cos[\pi(y_o + y'')/b]dy''dz'$$

$$\simeq m_y(\pi/b)\sin(\pi y_o/b)$$

where A is the area of the hole. Substitution of this expression in (18.b) gives

$$\underline{H}_d = \hat{y}\,\frac{2\pi}{b^2 c}\, e^{-\pi a/(2b)} \sin(\pi y_o/b)\hat{y}\cdot\underline{\underline{\alpha}}_m\cdot\underline{H}_{sc} \qquad (18.c)$$

If the integral in (18.b) over an arbitrary-sized aperture is nonzero, one can still obtain an useful estimate of $\underline{H}_d$ relative to $\underline{H}_{sc}$ by taking $20\log_{10}$ of (18.b). Thus,

$$20\log_{10}\left|\frac{H_d}{H_{sc}}\right| \simeq -\ 8.686\ \pi a/(2b) = -\ 13.64\ a/b \quad (\text{db}) \qquad (18.d)$$

(ii) Shallow Cavity $(b > c >> a)$

For this case one uses (17.b) to obtain the dominant mode as

$$\psi_d = \frac{-2}{\pi a}\cos(\pi z/c)e^{-\pi b/c}$$

$$\times\int_A \frac{\partial\psi}{\partial x'} \cosh[(b - y_>)\pi/c]\cosh(\pi y_</c)\cos(\pi z'/c)dy'dz' \qquad (19.a)$$

from which one gets, at the center of the cavity,

$$\underline{H}_d = \hat{z}\,\frac{-2}{ac}\, e^{-\pi b/(2c)} \int_A \frac{\partial\psi}{\partial x'} \cosh(\pi y'_</c)\cos(\pi z'/c)dy'dz' \qquad (19.b)$$

where $y'_<$ is the smaller of $(y', b - y')$. For a small hole one has

$$\underline{H}_d = \hat{z}\,\frac{\pi}{ac^2}\,e^{-\pi|y_o - b/2|/c}\left[\sin(\pi z_o/c)\hat{z}\cdot\underline{\underline{\alpha}}_m\cdot\underline{H}_{sc} - \cos(\pi z_o/c)\hat{y}\cdot\underline{\underline{\alpha}}_m\cdot\underline{H}_{sc}\right] \tag{19.c}$$

If the center $(y_o, z_o)$ of the aperture does not lie on the center line $y = b/2$ in the $x = 0$ plane (that is, if $y_o \neq b/2$), then equation (19.b) gives the following useful formula for the field at the cavity center:

$$20 \log_{10}\left|\frac{H_d}{H_{sc}}\right| \approx -13.64\ b/c \quad \text{(db)} \tag{19.d}$$

Obviously, if $y_o = b/2$ and the observation point is at $y \neq b/2$ and $z = c/2$, equation (19.d) still applies. But if $y = y_o = b/2$, equation (19.d) is no longer true.

(iii) <u>Shallow Cavity</u> $(c > b >> a)$

For this case one uses (17.c) and gets

$$\psi_d = \frac{-2}{\pi a}\cos(\pi y/b)e^{-\pi c/b} \times \int_A \frac{\partial\psi}{\partial x'}\cosh[\pi(c - z_>)/b]\cosh(\pi z_</b)\cos(\pi y'/b)dy'dz' \tag{20.a}$$

where, as before, $z_>(z_<)$ is the larger (smaller) of $z$ and $z'$. At the center of the cavity one has

$$\underline{H}_d = \hat{y}\,\frac{-2}{ab}\,e^{-\pi c/(2b)}\int_A \frac{\partial\psi}{\partial x'}\cosh(\pi z_</b)\cos(\pi y'/b)dy'dz' \tag{20.b}$$

where $z_<$ is the smaller of $(z', c - z')$. For a small hole one has

$$\underline{H}_d = \hat{y}\,\frac{\pi}{ab^2}\,e^{-\pi|z_o - c/2|/b}\left[\sin(\pi y_o/b)\hat{y}\cdot\underline{\underline{\alpha}}_m\cdot\underline{H}_{sc} - \cos(\pi y_o/b)\hat{z}\cdot\underline{\underline{\alpha}}_m\cdot\underline{H}_{sc}\right] \tag{20.c}$$

For an arbitrary-sized hole one gets from (20.b)

$$20 \log_{10} \left| \frac{H_d}{H_{sc}} \right| \simeq - 13.64 \ c/b \quad \text{(db)} \tag{20.d}$$

at the center of the cavity. The same remarks on equation (19.d) apply to equation (20.d).

## IV. MULTIPLY CONNECTED CAVITY

An example of a multiply connected cavity is the coaxial cavity shown in Figure 3 which permits the method of separation of variables. The coaxial cavity can be used to model the weapons bay and, to some extent, the closed wheel well provided that an end capacitance is added at one end of the inner conductor to allow for the electrostatic interaction between the end of the wheel strut and the well's walls. As one will see shortly, the problem of a multiply connected cavity is much more difficult than that of a simply connected cavity from both the mathematical and conceptual viewpoint.

### A. Direct Method

An inspection of Figure 3 suggests that the solution of the Laplace equation takes the form

$$\psi = \sum_{m,n} A_{mn} F_m(n\pi\rho/c) \begin{pmatrix} \sin m\phi \\ \cos m\phi \end{pmatrix} \cos(n\pi z/c)$$

$$F_m = I_m(n\pi\rho/c)K_m'(n\pi b/c) - I_m'(n\pi b/c)K_m(n\pi\rho/c) \tag{21}$$

where $I_m$ and $K_m$ are the modified Bessel functions and the prime denotes differentiation with respect to the argument. Clearly, the representation of $\psi$ by (21) satisfies the boundary conditions

$$\frac{\partial\psi}{\partial\rho} = 0 , \quad \text{at} \quad \rho = b$$

$$\frac{\partial\psi}{\partial z} = 0 , \quad \text{at} \quad z = 0 , c$$

By means of (9) the expansion coefficient $A_{mn}$ can be readily evaluated. The final result is

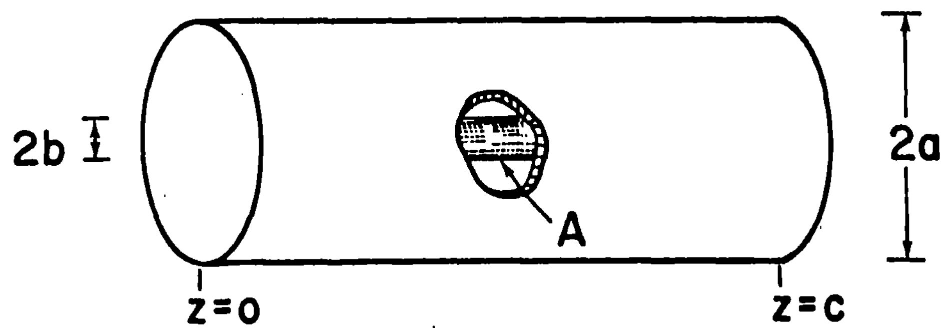


Figure 3. A coaxial cavity with an aperture in the outer conductor.

$$\psi = \frac{2}{\pi^2} \sum_{m,n} \frac{F_m(n\pi\rho/c\ , n\pi b/c)\cos(n\pi z/c)}{n\varepsilon_m\varepsilon_n F'_m(n\pi a/c\ , n\pi b/c)} \begin{pmatrix} \sin m\phi \\ \cos m\phi \end{pmatrix}$$

$$\times \int_A \frac{\partial\psi}{\partial\rho'} \cos(n\pi z'/c) \begin{pmatrix} \sin m\phi' \\ \cos m\phi' \end{pmatrix} d\phi' dz' \qquad (22)$$

The natural question to ask about (22) is whether or not the mode $\underline{H} = \hat{\phi}/\rho$ is contained in (22), since it is a legitimate solution of the magnetostatic equations

$$\left.\begin{aligned} \nabla \times \underline{H} &= 0 \\ \nabla \cdot \underline{H} &= 0 \end{aligned}\right\} \quad \text{in} \quad V$$

$$\hat{n} \cdot \underline{H} = 0 \quad \text{on} \quad S$$

Differentiating (22) and taking the limit of the resulting expression as $n \to 0$ and then $m \to 0$ one finds

$$H_\phi = -\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} \to \frac{-1}{2\pi c\ \ell n(a/b)} \frac{1}{\rho} \int_A \frac{\partial\psi}{\partial\rho'} \phi' a d\phi' dz' \qquad (23)$$

In terms of the magnetic polarizability $\underline{\underline{\alpha}}_m$ and the external short-circuited magnetic field $\underline{H}_{sc}$, equation (23) gives

$$H_\phi \to \frac{-1}{2\pi ac\ \ell n(a/b)} \frac{1}{\rho} \hat{\phi}\cdot\underline{\underline{\alpha}}_m\cdot\underline{H}_{sc} \qquad (24)$$

On the other hand, if one first takes the limit $m \to 0$ and then $n \to 0$ of the $\phi$-derivative of (22) one will end up with an indeterminate expression.

## B. Eigenfunction Method

Following the procedure described in Section II one can immediately write down the solution of $\psi$ as

$$\psi = \sum B_{mn,\alpha}\psi_{mn,\alpha}$$

$$\psi_{mn,\alpha} = G_m(\lambda_{m,\alpha}\rho)\cos(n\pi z/c)\begin{pmatrix}\sin m\phi \\ \cos m\phi\end{pmatrix} \tag{25}$$

$$G_m(\lambda_{m,\alpha}\rho) = J_m(\lambda_{m,\alpha}\rho)Y'_m(\lambda_{m,\alpha}b) - J'_m(\lambda_{m,\alpha}b)Y_m(\lambda_{m,\alpha}\rho)$$

where the $\lambda_{m,\alpha}$ are the roots of

$$\frac{\partial}{\partial\rho} G_m(\lambda_{m,\alpha}\rho)\bigg|_{\rho=a} = 0 \tag{26}$$

Clearly, $\psi_{mn,\alpha}$ satisfies

$$(\nabla^2 + k^2_{mn,\alpha})\psi_{mn,\alpha} = 0\,, \quad \text{in } V$$

$$\frac{\partial}{\partial n}\psi_{mn,\alpha} = 0\,, \quad \text{on } S$$

$$k^2_{mn,\alpha} = (n\pi/c)^2 + \lambda^2_{m,\alpha}$$

After evaluating $B_{mn,\alpha}$ by formula (15) one gets

$$\psi = \frac{\pi}{c}\sum_{\lambda_{m,\alpha}\neq 0}\frac{1}{\varepsilon_m\varepsilon_n}\frac{\lambda^2_{m,\alpha}}{(n\pi/c)^2 + \lambda^2_{m,\alpha}}\, b_{m,\alpha}\psi_{mn,\alpha}\int_A \psi_{mn,\alpha}\frac{\partial\psi}{\partial\rho'}\, a\,d\phi' dz'$$

$$+ \frac{2}{\pi c(a^2 - b^2)}\sum_{n\neq 0}\frac{\cos(n\pi z/c)}{(n\pi/c)^2}\int_A \frac{\partial\psi}{\partial\rho'}\cos(n\pi z'/c)\,a\,d\phi' dz' \tag{27}$$

$$+ \frac{1}{2\pi c\ \ell n(a/b)} \phi \int_A \frac{\partial\psi}{\partial\rho'} \phi' \, a d\phi' dz'$$

where

$$\frac{1}{b_{m,\alpha}} = \left[\frac{J_m'(\lambda_{m,\alpha} b)}{J_m'(\lambda_{m,\alpha} a)}\right]^2 \left[1 - \left(\frac{m}{\lambda_{m,\alpha} a}\right)^2\right] + \left(\frac{m}{\lambda_{m,\alpha} b}\right)^2 - 1, \quad \lambda_{m,a} \neq 0 \tag{28}$$

The second and third terms of (27) correspond to $\lambda_{m,\alpha} = 0$ and have been factored out for convenience. The third term should be made single-valued by introducing a cutting plane connecting the inner and outer conductors.

To show that equation (27) is identical to equation (22) one first makes use of the identity (see the Appendix)

$$\left. \frac{\partial}{\partial\zeta} G_m'(\zeta a , \zeta b) \right|_{\zeta=\lambda_{m,\alpha}} = \frac{-4}{a\pi^2 \lambda_{m,\alpha}^2} \left[G_m(\lambda_{m,\alpha} a , \lambda_{m,\alpha} b) b_{m,\alpha}\right]^{-1} \tag{29}$$

where

$$G_m'(\zeta a , \zeta b) = J_m'(\zeta a) Y_m'(\zeta b) - J_m'(\zeta b) Y_m'(\zeta a)$$

Then, the sum over $\lambda_{m,\alpha}$ in the first term $\psi_1$ of (27) can be written as a contour integral, viz.,

$$\psi_1 = \frac{-4}{\pi c} \sum_{m,n} \frac{1}{\varepsilon_m \varepsilon_n} \begin{pmatrix} \sin m\phi \\ \cos m\phi \end{pmatrix} \cos(n\pi z/c) \int_A \frac{\partial\psi}{\partial\rho'} \begin{pmatrix} \sin m\phi' \\ \cos m\phi' \end{pmatrix} \cos(n\pi z'/c) d\phi' dz'$$

$$\times \frac{1}{2\pi i} \int_{C_1+C_2+C_3} \frac{1}{(n\pi/c)^2 + \zeta^2} \frac{G_m(\zeta\rho , \zeta b)}{G_m'(\zeta a , \zeta b)} d\zeta \tag{30}$$

where the contour $C_1 + C_2 + C_3$ is shown in Figure 4. Since

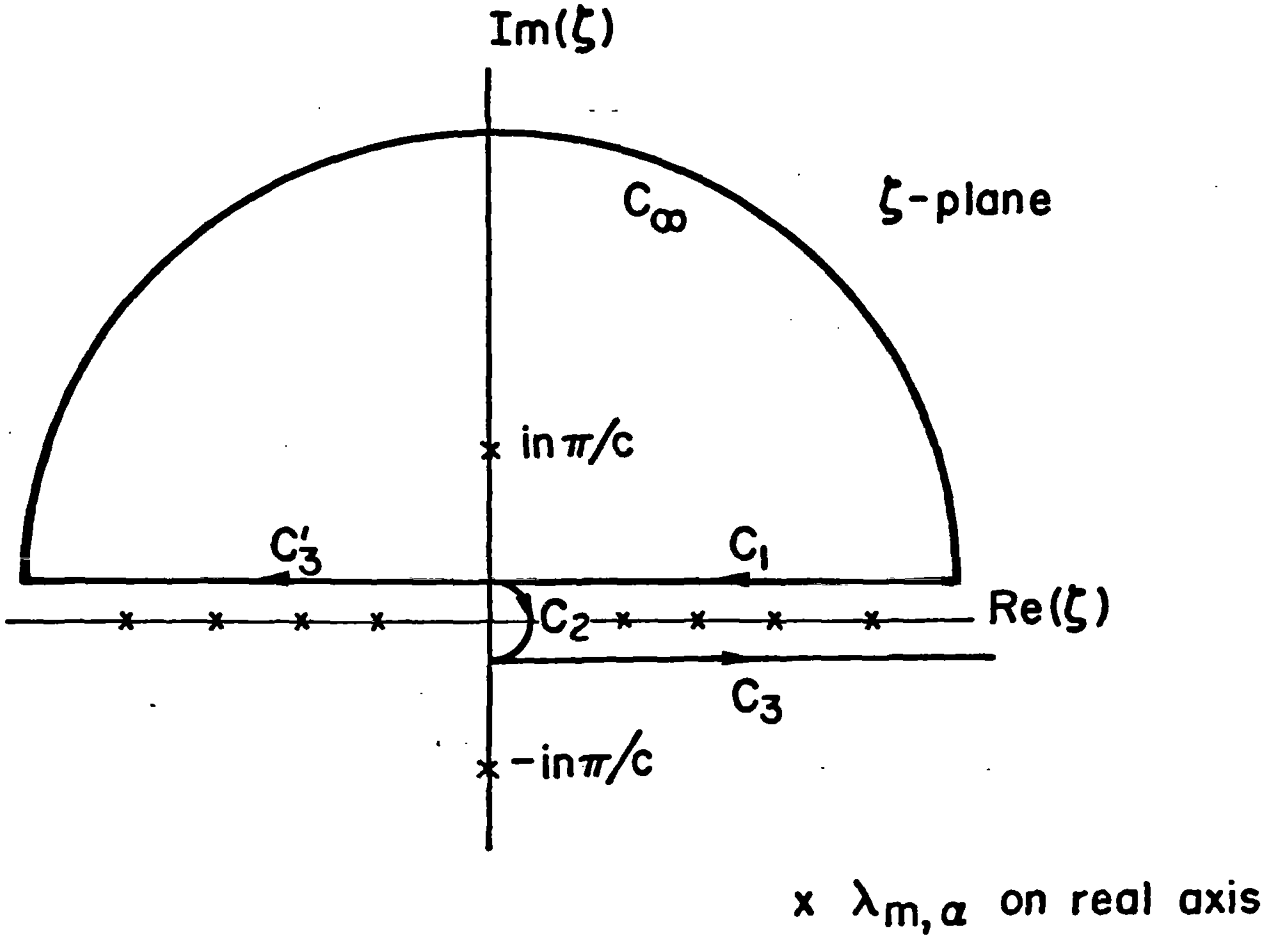


Figure 4. The contours in the complex ζ-plane.

$$G_m(\zeta\rho\ ,\ \zeta b) = -\ G_m(-\zeta\rho\ ,\ -\zeta b)$$

$$G'_m(\zeta a\ ,\ \zeta b) = G'_m(-\zeta a\ ,\ -\zeta b)$$

one has (see Figure 4)

$$\int_{C_1+C_2+C_3} = \int_{C_1+C'_3} + \int_{C_2} \qquad (31)$$

From the properties of the Bessel functions it can be shown that

(a) $G_m(\zeta)$ and $G'_m(\zeta)$ have no branch points in the entire $\zeta$-plane,

(b) $G_o(\rho)/G'_o(\zeta)$ has a pole at $\zeta = 0$,

(c) $G'_m(\zeta)$ has only real zeros, and

(d) $\dfrac{G_m(\zeta\rho\ ,\ \zeta b)}{G'_m(\zeta a\ ,\ \zeta b)} \sim e^{-|\zeta|(a-\rho)}$ on $C_\infty$ where $|\zeta| \to \infty$

Thus, the integral along $C_1+C'_3$ is equal to the residue at $\zeta = in\pi/c$ , $n > 0$ , namely,

$$\frac{1}{2\pi i}\int_{C_1+C'_3} \frac{1}{(n\pi/c)^2+\zeta^2}\,\frac{G_m(\zeta\rho\ ,\ \zeta b)}{G'_m(\zeta a\ ,\ \zeta b)}\,d\zeta = \frac{-c}{2n\pi}\,\frac{F_m(n\pi\rho/c\ ,\ n\pi b/c)}{F'_m(n\pi a/c\ ,\ n\pi b/c)}\ ,\quad n > 0 \qquad (32)$$

The integral over the contour $C_2$ will now be evaluated. Since the integrand of (32) has (a) no poles at $\zeta = 0$ for $m,n > 0$ , (b) a simple pole at $\zeta = 0$ for $m = 0$ , $n > 0$ , (c) a simple pole at $\zeta = 0$ for $m > 0$ , $n = 0$ , and (d) a triple pole at $\zeta = 0$ for $m = 0$ , $n = 0$ , one has

$$\frac{1}{2\pi i}\int_{C_2} \frac{1}{(n\pi/c)^2+\zeta^2}\,\frac{G_m(\zeta\rho\ ,\ \zeta b)}{G'_m(\zeta a\ ,\ \zeta b)}\,d\zeta = 0\ ,\qquad m > 0\ ,\ n > 0$$

$$= \frac{a}{(n\pi/c)^2(a^2-b^2)}\ ,\quad m = 0\ ,\ n > 0 \qquad (33)$$

$$= \frac{-c}{2\pi} \lim_{n\to 0} \frac{F_m(n\pi\rho/c\ , n\pi b/c)}{nF'_m(n\pi a/c\ , n\pi b/c)}\ , \quad m > 0, \quad n = 0$$

$$\neq 0, \quad m = 0, \quad n = 0$$

For the case where $m = 0$, $n = 0$ the first two terms of equation (27) vanish identically because

$$\int_A \frac{\partial\psi}{\partial\rho'}\, d\phi' dz' = 0$$

due to the solenoidal nature of $\underline{H}$, and only the third term remains. Hence, one need not evaluate the contour integral along $C_2$ in this case. Substituting (32), (33) and (30) in (27) and noting that the second term in (27) cancels the contribution from the contour integral along $C_2$, one sees that expression (27) is indeed identical to expression (22), if and only if one first takes the limit $n \to 0$ and then $m \to 0$ in (22) to obtain the third term of (27).

To get a representation exponentially decaying in the z-direction one can sum over $n$ in equation (27) and obtains, with $z_>(z_<)$ denoting the larger (smaller) of $(z,z')$,

$$\psi = \frac{\pi}{2} \sum_{\lambda_{m,\alpha}\neq 0} \frac{\lambda_{m,\alpha}}{\epsilon_m \sinh(\lambda_{m,\alpha} c)}\, b_{m,\alpha} G_m(\lambda_{m,\alpha} a\ , \lambda_{m,\alpha} b) G_m(\lambda_{m,\alpha}\rho\ , \lambda_{m,\alpha} b)$$

$$\times \begin{pmatrix} \sin m\phi \\ \cos m\phi \end{pmatrix} \int_A \frac{\partial\psi}{\partial\rho'} \cosh(\lambda_{m,\alpha} z_<) \cosh[\lambda_{m,\alpha}(c - z_>)] \begin{pmatrix} \sin m\phi' \\ \cos m\phi' \end{pmatrix} a d\phi' dz' \qquad (34)$$

$$- \frac{1}{\pi(a^2 - b^2)} \int_A \frac{\partial\psi}{\partial\rho'}\, z_> a d\phi' dz'$$

$$+ \frac{1}{2\pi c\ \ln(a/b)}\, \phi \int_A \frac{\partial\psi}{\partial\rho'}\, \phi' a d\phi' dz'$$

where some constant term has been left out, since a constant potential contributes nothing to the field. It is easy to see that the second term of (34) is zero if $z > z'$ and reduces to a constant if $z < z'$ . Thus, this term will contribute to the field only if $z$ and $z'$ belong to the same domain. The third term of (34) gives the field (23) which can be interpreted as the field of a magnetic dipole, as shown by equation (24).

A representation decaying exponentially in the $\phi$-direction can also be obtained by considering the following identity:

$$\frac{1}{\pi^2 i} \int_C \frac{F_\nu(n\pi\rho/c,\ n\pi b/c)}{F'_\nu(n\pi a/c,\ n\pi b/c)} \begin{pmatrix} \sin(\nu\phi_<)\sin[\nu(2\pi - \phi_>)] \\ \cos(\nu\phi_<)\cos[\nu(2\pi - \phi_>)] \end{pmatrix} \frac{d\nu}{\sin(2\pi\nu)} = 0 \qquad (35)$$

where the contour $C$ is an infinite circle in the complex $\nu$-plane and $\phi_>(\phi_<)$ denotes the larger (smaller) of $(\phi,\ \phi')$ . The identity is established from the fact that the integrand of the integral goes as $\nu^{-1}(\rho/a)^{|\nu|}\exp[-\ |\nu|(\phi_> - \phi_<)]$ as $|\nu| \to \infty$ . The integral is, of course, also equal to the total sum of the residues within the infinite contour $C$ . The residues can be separated into two parts, namely, (i) the part associated with the zeros of $\sin(2\pi\nu)$ and (ii) the part associated with the purely imaginary zeros of $F'_\nu(n\pi a/c,\ n\pi b/c)$ [6]. Thus equation (35) gives

$$\frac{2}{\pi^2} \sum_{m=0}^{\infty} \frac{F_m(n\pi\rho/c,\ n\pi b/c)}{\varepsilon_m F'_m(n\pi a/c,\ n\pi b/c)} \begin{pmatrix} -\ \sin(m\phi)\sin(m\phi') \\ \cos(m\phi)\cos(m\phi') \end{pmatrix}$$

$$+ \frac{2i}{\pi} \sum_{r=1}^{\infty} \frac{F_{i\nu_r}(n\pi\rho/c,\ n\pi b/c)}{\left[\frac{d}{d\nu} F'_\nu(n\pi a/c,\ n\pi b/c)\right]_{\nu = i\nu_r}} \qquad (36)$$

$$\times \begin{pmatrix} \sinh(\nu_r\phi_<)\sinh[\nu_r(2\pi - \phi_>)] \\ -\cosh(\nu_r\phi_<)\cosh[\nu_r(2\pi - \phi_>)] \end{pmatrix} \frac{1}{\sinh(2\pi\nu_r)} = 0$$

where the $\nu_r$ are the roots of

$$F'_{i\nu_r}(n\pi a/c,\ n\pi b/c) = 0$$

Applying equation (36) to equation (22) one finally obtains

$$\psi = \frac{2i}{\pi} \sum_{\substack{n,r \\ r\neq 0}} \frac{\cos(n\pi z/c) F_{i\nu_r}(n\pi\rho/c,\ n\pi b/c)}{n\varepsilon_n \sinh(2\pi\nu_r)\left[\frac{d}{d\nu} F'_{\nu_r}(n\pi a/c,\ n\pi b/c)\right]_{\nu=i\nu_r}}$$

$$\times \int_A \frac{\partial\psi}{\partial\rho'} \cos(n\pi z'/c) \begin{pmatrix} \sinh(\nu_r\phi_<)\sinh[\nu_r(2\pi-\phi_>)] \\ \cosh(\nu_r\phi_<)\cosh[\nu_r(2\pi-\phi_>)] \end{pmatrix} \tag{37}$$

To sum up, equation (27) gives the eigenfunction solution of the present problem, whereas the $\rho$-form solution is given by (22), the z-form solution by (34), and the $\phi$-form solution by (37).

## APPENDIX

## A MATHEMATICAL IDENTITY

To prove equation (29) one makes use of the recursion formula for Bessel's functions and obtains

$$\frac{\partial}{\partial\zeta} G'_m(\zeta a\, ,\, \zeta b)\Bigg|_{\zeta=\lambda_{m,\alpha}} = -\frac{-2}{\lambda_{m,\alpha}} G'_m(\lambda_{m,\alpha}a,\ \lambda_{m,\alpha}b)$$

$$- a[1 - (m/\lambda_{m,\alpha}a)^2]G_m(\lambda_{m,\alpha}a,\ \lambda_{m,\alpha}b)$$

$$+ b[1 - (m/\lambda_{m,\alpha}b)^2]G_m(\lambda_{m,\alpha}b,\ \lambda_{m,\alpha}a) \qquad \text{(A-1)}$$

Since $G'_m(\lambda_{m,\alpha}a,\ \lambda_{m,\alpha}b) = 0$ and $G_m(\xi,\xi) = 2/(\pi\xi)$ , one has

$$G_m(\lambda_{m,\alpha}a,\ \lambda_{m,\alpha}b) = \frac{J'_m(\lambda_{m,\alpha}b)}{J'_m(\lambda_{m,\alpha}a)}\,\frac{2}{\pi a\lambda_{m,\alpha}}$$

$$G_m(\lambda_{m,\alpha}b,\ \lambda_{m,\alpha}a) = \frac{J'_m(\lambda_{m,\alpha}a)}{J'_m(\lambda_{m,\alpha}b)}\,\frac{2}{\pi b\lambda_{m,\alpha}}$$

Recalling the definition of $b_{m,\alpha}$ one can easily see that equation (A-1) leads to equation (29).

## REFERENCES

[1] S.A. Schelkunoff, "On representation of electromagnetic fields in cavities in terms of natural modes of oscillation," Journal of Applied Physics, Vol. 26, No. 10, October 1955, p. 1231 - 1234.

[2] K.C. Chen and C.E. Baum, "On EMP excitations of cavities with small openings," Interaction Notes, Note 170, January 1974.

[3] T. Teichmann and E.P. Wigner, "Electromagnetic field expansions in loss-free cavities excited through holes," Journal of Applied Physics, Vol. 24, No. 3, March 1953, p. 262 - 267.

[4] F.E. Borgnis and C.H. Papas, Electromagnetic Waveguides and Resonators, Handbuch der Physik, Vol. XVI, 1958.

[5] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York, 1965.

[6] J. Dougall, "The determination of Green's Function by means of cylindrical or spherical harmonics," Proc. Edin. Math. Soc., Vol. XVIII, p. 33 - 83.