Interaction Notes

Note 304

January 1977

Equivalent Lumped Parameters for a Bend in a Two-Wire Transmission Line: Part II. Capacitance

John Lam

The Dikewood Corporation
Westwood Research Branch
Los Angeles, California 90024

Abstract

The equivalent capacitance of a bend in a two-wire transmission line is calculated from a variational principle. The capacitance of an abrupt bend is obtained explicitly in simple form. The capacitance of a circular bend is expressed in terms of one-dimensional integrals to be computed numerically. It is concluded that the bend capacitance is strongly dependent on the bend radius.

CONTENTS

ILLUSTRATIONS

I. INTRODUCTION

The aircraft EMP internal-coupling study is concerned with the propagation of EMP energy along cable systems in an aircraft's interior. An important area of this study is the determination of the propagation characteristics of generic cable configurations. One very common type of aircraft internal cable geometry is that of a line parallel to a conducting ground. Due to the formation of the electrical image this cable is equivalent to a two-wire transmission line. Even though the propagation characteristics of the straight uniform two-wire line are well known, the presence of bends in the line may bring about significant corrections. This is because an electromagnetic signal in the cable incident on a bend is partially reflected by the bend. The reflection can be simulated at low frequencies by loading a lumped network circuit onto the uniform line at the location of the bend. The lumped network elements are related to the inductance and capacitance of the bend, and are calculable from the bend geometry by solving certain appropriate electromagnetic boundary-value problems.

Figure 1 shows a bend in a two-wire transmission line and its equivalent lumped network circuit. The two parallel wires are bent identically through an angle α. The bent section in each wire can be modeled by a circular arc of radius R and angle α connecting two semi-infinite straight sections. The geometry of the model is shown in Figure 2. The two wires are made up of two parallel, perfectly-conducting, solid cylinders of radius a and at a separation $2b$.

The objective of the present study is to calculate the lumped inductance L_d and the capacitance C_d in the equivalent circuit representation of the circular bend in Figures 1 and 2. The inductance calculation has been fully presented in Part I of this report [1]. Part II here deals with the capacitance calculation. The mathematical method employed will be based on the calculus of variations. In Section II the electrostatic boundary-value problem for the bend is formulated by means of an integral equation. From this equation a variational representation of the reciprocal of the capacitance is constructed in Section III. It is then applied to calculate the capacitances of an abrupt bend and of a circular bend in Sections IV and V.

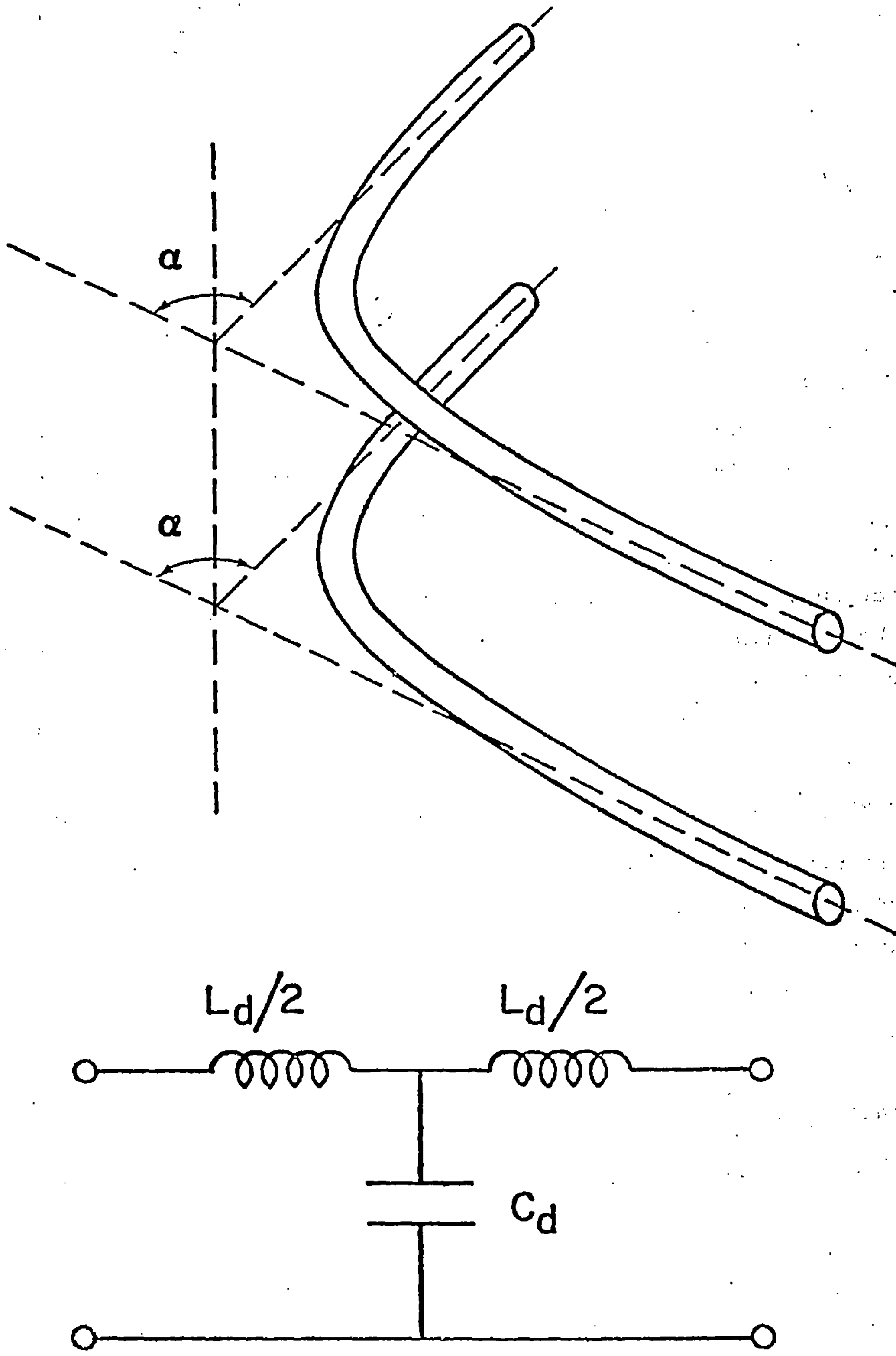

Figure 1. A bend in a parallel-wire transmission line and its equivalent circuit representation.

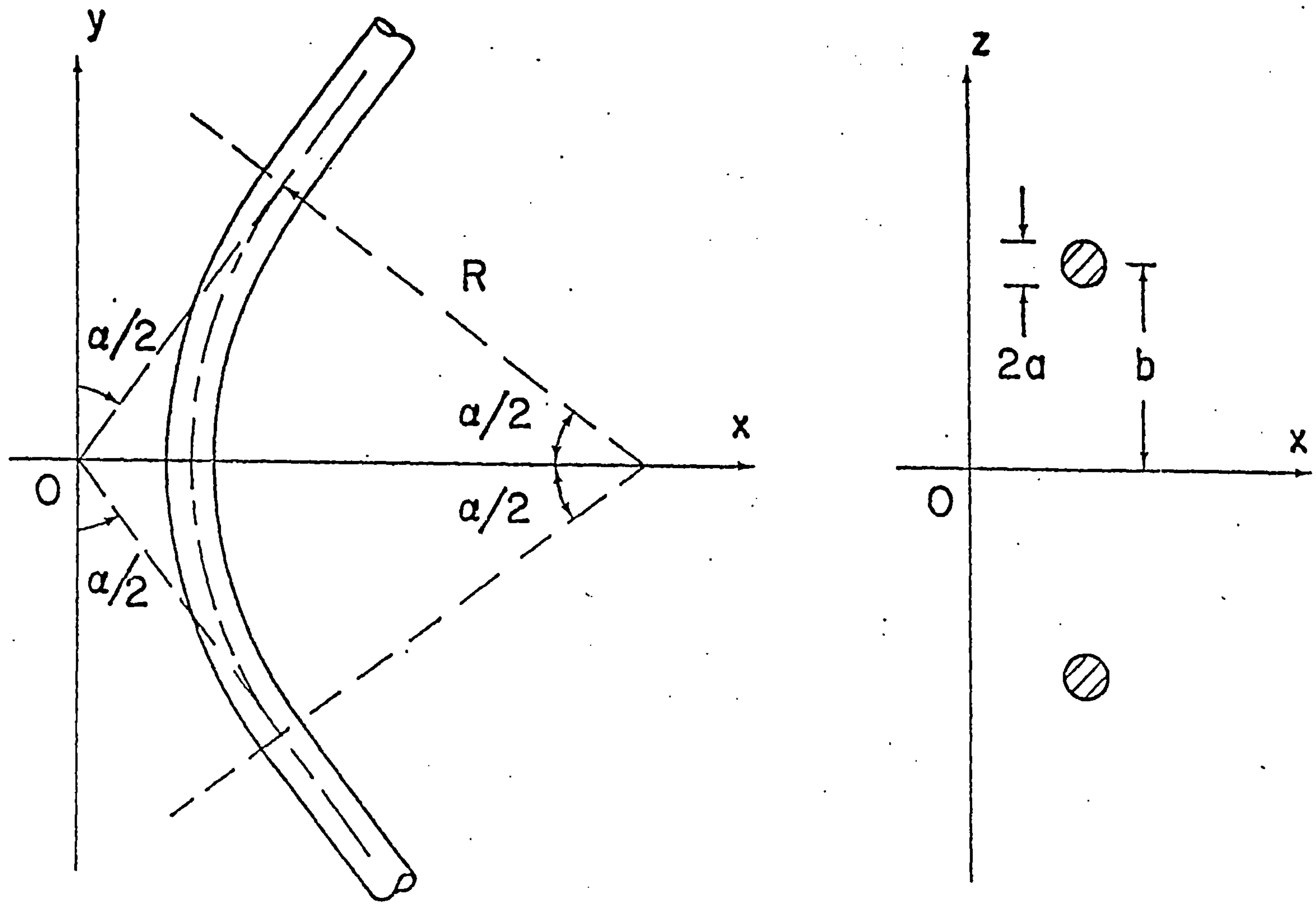

Figure 2. Geometry of a bend in a two-wire transmission line. The bend is modeled by a circular arc of radius R and angle α .

II. INTEGRAL-EQUATION FORMULATION

Consider the bent two-wire transmission line in the coordinate system in Figure 2. Let the upper wire at $z = b$ be charged to a potential $V_o/2$, and the lower wire at $z = -b$ to a potential $-V_o/2$. The potential difference between the two wires is therefore V_o. The charge densities per unit length on the upper and lower wires will be denoted by $\pm\sigma$, respectively. It will be assumed that the two wires are thin, so that the wire radius is much smaller than the wire separation. One can then consider the charges on the wires as being concentrated on the center lines of the wires; and σ can be expressed as a function of the y-coordinate alone.

At a general point (x,y,z) exterior to the two wires the total electrostatic potential due to the wires is obtained by summing up the contributions from all the charge elements along the center lines:

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_o}\int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-b)^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+b)^2}} \right) \tag{1}$$

In the formula s' is the arc length measured along the center line of a wire, so that

$$\frac{ds'}{dy'} = \sqrt{1 + \left(\frac{dx'}{dy'}\right)^2} \tag{2}$$

Strictly speaking, the charge density $\sigma(y')$ is to be determined by requiring that V reduce to $\pm V_o/2$ on the entire surfaces of the upper and lower wires, respectively. However, to be consistent with the thin-wire assumption inherent in formula (1), one can apply the boundary condition only along a line on the surface of each wire. This line will be chosen to be at $z = b - a$ on the upper wire, and at $z = -(b-a)$ on the lower wire. An integral equation for $\sigma(y')$ results:

$$V_o = \frac{1}{2\pi\varepsilon_o} \int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') \left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (2b-a)^2}} \right) \qquad -\infty < y < \infty \tag{3}$$

The integral equation (3) is to be solved under a specified functional relation between x and y, and similarly between x' and y'. This relation takes the form

$$x = f(y) , \qquad x' = f(y') \tag{4}$$

and describes the center line of a wire. Three forms of $f(y)$ will be considered in the following sections. For the line with a circular bend shown in Figure 2, one has

$$x = f_1(y) = \begin{cases} R \sec\left(\frac{\alpha}{2}\right) - \sqrt{R^2-y^2} & |y| < y_o \\ |y| \tan\left(\frac{\alpha}{2}\right) & |y| > y_o \end{cases} \tag{5}$$

where

$$y_o = R \sin\left(\frac{\alpha}{2}\right) \tag{6}$$

One also considers the abrupt V-shaped bend obtained in the limit as R tends to zero. The functional relation for this case is

$$x = f_2(y) = |y|\tan\left(\frac{\alpha}{2}\right) \qquad (7)$$

Finally, for a straight uniform line with no bend, one has

$$x = f_3(y) = 0 \qquad (8)$$

III. VARIATIONAL REPRESENTATION OF THE CAPACITANCE

It is possible to calculate the capacitance of the two-wire transmission line without solving explicitly the integral equation (3). The way to go about this is to phrase the capacitance calculation as an eigenvalue problem. From well-known results in the calculus of variations, a variational principle for the eigenvalue can be established. It can be applied to obtain an estimate of the capacitance with a judicious choice of the trial function.

Let Q denote the total charge on the upper wire. It is an infinite quantity and can be expressed as a line integral of the line charge density σ:

$$Q = \int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') \tag{9}$$

Q is directly proportional to the potential difference V_o, the constant of proportionality being the capacitance C:

$$Q = CV_o \tag{10}$$

Using equations (9) and (10), one can eliminate V_o from the integral equation (3), which then becomes

$$\frac{1}{C}\int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') = \int_{-\infty}^{\infty} dy' \, \frac{ds'}{dy'} \, \sigma(y') K(y,y') \qquad -\infty < y < \infty \tag{11}$$

The kernel $K(y,y')$ is defined by

$$K(y,y') = \frac{1}{2\pi\varepsilon_o}\left(\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{12}$$

where x and x' are related to y and y' through equation (4).

Equation (11) is linear and homogeneous, with the reciprocal of the capacitance $1/C$ playing the role of an eigenvalue. A variational representation of he eigenvalue for an integral equation of the type (11) is well known [2] and akes the form

$$\frac{1}{C} = \frac{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \frac{ds}{dy}\frac{ds'}{dy'} \sigma(y)K(y,y')\sigma(y')}{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \frac{ds}{dy}\frac{ds'}{dy'} \sigma(y)\sigma(y')} \tag{13}$$

By "variational representation" is meant that $1/C$ is a functional on the space of trial charge density functions σ, and that its value attains an absolute minimum at the exact solution of (11). This minimum corresponds to the exact value of the capacitance. If a trial charge density function differing from the exact solution by a small amount $\delta\sigma$ is inserted in (13), the error incurred in the approximate value of the capacitance so obtained is only of order $(\delta\sigma)^2$. Consequently an evaluation of expression (13), even with a very crude trial function, can yield a good estimate of the capacitance.

On physical grounds the charge density per unit length of the two-wire transmission line is uniform except in the vicinity of the bend. For a very long line with a length greatly exceeding the bend dimensions a good trial function is therefore

$$\sigma(y) = \text{constant} \tag{14}$$

With this simple choice expression (13) becomes

$$\frac{1}{C} = \frac{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \frac{ds}{dy}\frac{ds'}{dy'} K(y,y')}{\int_{-\infty}^{\infty} dy \frac{ds}{dy} \int_{-\infty}^{\infty} dy' \frac{ds'}{dy'}} \tag{15}$$

which depends only on the geometry of the bend. By applying to the kernel $K(y,y')$ in (12) the three functional relations f_1, f_2 and f_3 defined in equations (5), (7) and (8), one obtains three kernels K_1, K_2 and K_3. These, when inserted into formula (15), generate three capacitances C_1, C_2 and C_3. They are, respectively, the total capacitance of a two-wire transmission line with a circular bend, an abrupt bend and no bend.

In the following two sections the equivalent capacitances of an abrupt bend and a circular bend are calculated from formula (15). In Section IV the equivalent capacitance of an abrupt bend, denoted by C_d' and defined as the difference

$$C_d' = C_2 - C_3 \tag{16}$$

is evaluated in closed form. In Section V the difference C_d'' between the circular bend capacitance and the abrupt bend capacitance, given by

$$C_d'' = C_1 - C_2 \tag{17}$$

is expressed in the form of one-dimensional integrals ready for computation. The equivalent capacitance C_d of the circular bend is then obtained as

$$C_d = C_1 - C_3 = C_d' + C_d'' \tag{18}$$

IV. CAPACITANCE OF AN ABRUPT CABLE BEND

Consider a two-wire transmission line with an abrupt bend through an angle α. This bend can be regarded as the limit of the circular bend in Figure 2 when the bend radius R tends to zero. Let the transmission line be of finite length initially and stretch from $y = D$ to $y = -D$. Eventually the constant D will be allowed to tend to infinity. The total capacitance C_2 of this line is calculable from formula (15):

$$\frac{1}{C_2} = \frac{1}{S_2^2} \int_{-D}^{D} dy \int_{-D}^{D} dy' \, \frac{ds_2}{dy} \frac{ds_2'}{dy'} K_2(y,y') \tag{19}$$

The subscript 2 will everywhere refer to the line with an abrupt bend. The kernel K_2 is given by

$$K_2(y,y') = \frac{1}{2\pi\varepsilon_o}\left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}}\right) \tag{20}$$

with

$$\lambda = \tan\left(\frac{\alpha}{2}\right) \tag{21}$$

S_2 is the total length of the line between $y = D$ and $y = -D$:

$$S_2 = \int_{-D}^{D} dy' \, \frac{ds_2'}{dy'} = 2D\sqrt{1 + \lambda^2} \tag{22}$$

since, by (2) and (7),

$$\frac{ds_2'}{dy'} = \sqrt{1 + \lambda^2} \tag{23}$$

Now consider a straight two-wire transmission line of the same total length. That is, its length S_3 is given by

$$S_3 = S_2 = 2D\sqrt{1 + \lambda^2} \tag{24}$$

where the subscript 3 refers to the straight transmission line. This line can be taken to lie parallel to the y-axis in Figure 2, and to stretch between $y = D\sqrt{1 + \lambda^2}$ and $y = -D\sqrt{1 + \lambda^2}$. According to formula (15), its total capacitance C_3 is

$$\frac{1}{C_3} = \frac{1}{S_3^2} \int_{-D\sqrt{1+\lambda^2}}^{D\sqrt{1+\lambda^2}} dy \int_{-D\sqrt{1+\lambda^2}}^{D\sqrt{1+\lambda^2}} dy' \, K_3(y,y') \tag{25}$$

where

$$K_3(y,y') = \frac{1}{2\pi\varepsilon_o}\left(\frac{1}{\sqrt{(y-y')^2 + a^2}} - \frac{1}{\sqrt{(y-y')^2 + (2b-a)^2}}\right) \tag{26}$$

A simple change of variables reduces (25) to the form

$$\frac{1}{C_3} = \frac{1+\lambda^2}{S_3^2} \int_{-D}^{D} dy \int_{-D}^{D} dy' \, K_3\left(\sqrt{1+\lambda^2}\, y \, , \; \sqrt{1+\lambda^2}\, y'\right) \tag{27}$$

Substituting formulas (19) and (27) into the identity

$$C_2 - C_3 = C_2 C_3 \left(\frac{1}{C_3} - \frac{1}{C_2}\right) \tag{28}$$

and making use of relations (23) and (24), one obtains an expression for the equivalent capacitance C_d' of an abrupt cable bend defined in (16):

$$C'_d = \frac{C_2 C_3 (1+\lambda^2)}{S_3^2} \int_{-D}^{D} dy \int_{-D}^{D} dy' \left[K_3(\sqrt{1+\lambda^2}\; y\, ,\; \sqrt{1+\lambda^2}\; y') - K_2(y,y') \right] \tag{29}$$

When the total length of the line is allowed to increase, the effect of the bend on the total capacitance of the line becomes negligible. The total capacitance approaches the product of the line length and the constant capacitance per unit length of the uniform line. In mathematical terms, one has

$$\lim_{D \to \infty} C_2 = C_3 = \kappa S_3 \tag{30}$$

where

$$\kappa = \frac{\pi \varepsilon_o}{\ell n \left(\frac{2b}{a}\right)} \qquad b >> a \tag{31}$$

is the well-known capacitance per unit length of the uniform two-wire transmission line. Therefore in the limit $D \to \infty$, formula (29) goes over to the desired expression for the equivalent capacitance of an abrupt bend in an infinite two-wire transmission line:

$$C'_d = \kappa^2 (1+\lambda^2) \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left[K_3(\sqrt{1+\lambda^2}\; y\, ,\; \sqrt{1+\lambda^2}\; y') - K_2(y,y') \right] \tag{32}$$

It is easy to see from (20) and (26) that the integrand in (32) vanishes identically whenever y and y' are of the same sign. The nonzero contributions to C'_d can be rearranged as follows:

$$C'_d = F(0) - F(\lambda) \tag{33}$$

where

$$F(\lambda) = \frac{\kappa^2(1+\lambda^2)}{\pi\varepsilon_o} \int_0^{\infty} dy \int_{-\infty}^{0} dy' \left(\frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{34}$$

The integral is the same as that appearing in expression (18) of Part I [1], and can be evaluated analytically by the method described therein. The result is

$$F(\lambda) = \frac{2\kappa^2(b-a)}{\pi\varepsilon_o} \frac{1+\lambda^2}{\lambda} \tan^{-1}\lambda \tag{35}$$

Combining expressions (21), (31), (33) and (35), one obtains the equivalent capacitance of an abrupt bend in the simple formula

$$C_d' = \frac{2\pi\varepsilon_o(b-a)}{\left[\ell n\left(\frac{2b}{a}\right)\right]^2} (1 - \alpha \csc \alpha) \tag{36}$$

For completeness one quotes here the associated equivalent inductance L_d' of the abrupt bend derived in equation (22) of Part I:

$$L_d' = \frac{2\mu_o(b-a)}{\pi} (\alpha \cot \alpha - 1) \tag{37}$$

The expression (36) for the bend capacitance C_d' is evaluated for the case $b = 10a$, and plotted versus the bend angle α in Figure 3. The equivalent capacitance of an abrupt cable bend has previously been calculated by Tomiyasu [3] and King [4] using a different approximate method. Expression (36) agrees with their result to within a few percent. It is, however, simpler in form and applies to a wider range of the bend angle α.

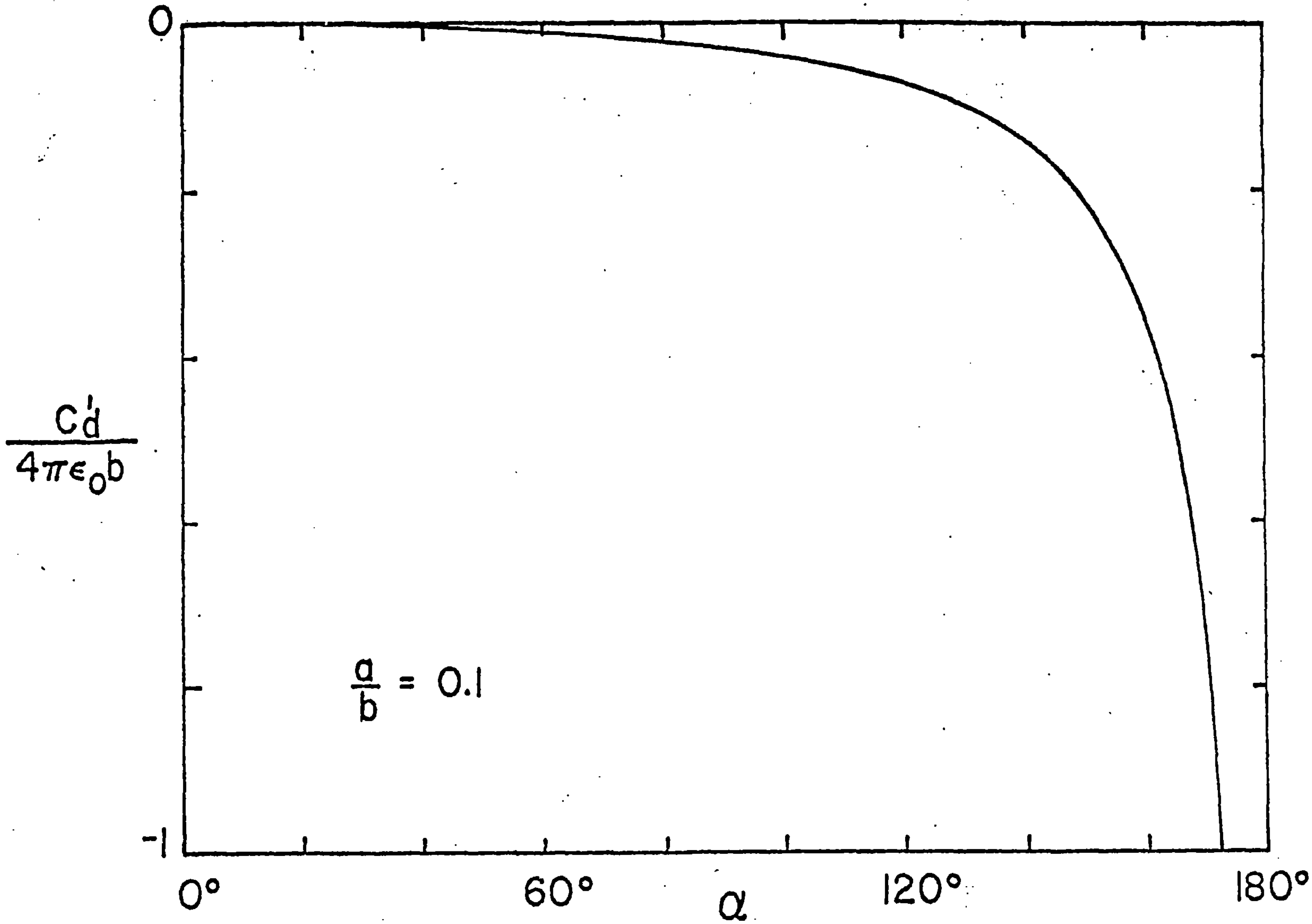

Figure 3. Plot of the equivalent capacitance C_d' of an abrupt cable bend versus the bend angle α .

V. CAPACITANCE OF A CIRCULAR CABLE BEND

The equivalent capacitance C_d of a circular cable bend is the sum of the equivalent capacitance C_d' of an abrupt bend calculated in Section IV and the correction term C_d'' defined in equation (17). The evaluation of C_d'' is undertaken in this section.

Applying formula (15) successively to the two transmission lines with the circular and abrupt bends and taking the difference of the two resulting expressions, one obtains

$$\frac{S_2^2}{C_2} - \frac{S_1^2}{C_1} = W \tag{38}$$

On the left-hand side, S_1 and S_2 are the total lengths of the two transmission lines:

$$S_1 = \int_{-\infty}^{\infty} dy' \, \frac{ds_1'}{dy'} , \qquad S_2 = \int_{-\infty}^{\infty} dy' \, \frac{ds_2'}{dy'} \tag{39}$$

They are both linearly divergent quantities. The right-hand side of (38) is a two-dimensional integral:

$$W = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left[\frac{ds_2}{dy} \frac{ds_2'}{dy'} K_2(y,y') - \frac{ds_1}{dy} \frac{ds_1'}{dy'} K_1(y,y') \right] \tag{40}$$

The kernel $K_2(y,y')$ has been written out explicitly in equation (20). The kernel $K_1(y,y')$ is given by

$$K_1(y,y') = \frac{1}{2\pi\varepsilon_o} \left(\frac{1}{\sqrt{[f_1(y) - f_1(y')]^2 + (y-y')^2 + a^2}} \right.$$

$$- \frac{1}{\sqrt{[f_1(y) - f_1(y')]^2 + (y-y')^2 + (2b-a)^2}} \Bigg) \qquad (41)$$

with $f_1(y)$ defined by (5). The two quantities s_1 and s_2 in (40) are arc lengths along the transmission lines. By (2) one obtains

$$\frac{ds_1'}{dy'} = \sqrt{1 + [f_1'(y')]^2} \, , \qquad \frac{ds_2'}{dy'} = \sqrt{1 + \lambda^2} \qquad (42)$$

The left-hand side of (38) is the difference of two infinite quantities. Its value is nevertheless finite, and is related to the capacitance difference $C_d'' = C_1 - C_2$. Introducing the relations

$$C_2 = C_1 - C_d'' \, , \qquad S_2 = S_1 + \Delta S \qquad (43)$$

where C_d'' and ΔS are finite and C_1 and S_1 are infinite, one finds that

$$\frac{S_2^2}{C_2} - \frac{S_1^2}{C_1} = \left(\frac{S_1}{C_1}\right)^2 C_d'' + 2\left(\frac{S_1}{C_1}\right)\Delta S \qquad (44)$$

The ratio C_1/S_1 in (44) is simply the capacitance per unit length of the infinite bent transmission line. It is equal to κ defined in (31) for the uniform line. Furthermore ΔS is the difference in the arc length between an abrupt bend and a circular bend. From Figure 2 one can immediately write down

$$\Delta S = R(2\lambda - \alpha) \qquad (45)$$

Substituting (44) and (45) into (38), one finally obtains an expression for the capacitance correction term:

$$C_d'' = \kappa^2 W - 2\kappa R(2\lambda - \alpha) \qquad (46)$$

It remains to evaluate the two-dimensional integral W appearing in (45) and defined in (40). One easily sees from the definition of $f_1(y)$ in (5) that the integrand in (40) vanishes identically whenever both y and y' lie outside the interval $(-y_o, y_o)$. This result is a reflection of the fact that the two differently-bent transmission lines coincide outside the bent sections. Consequently the nonzero contributions to the integral W come only from a certain cross-shaped region on the y-y' plane, as shown in Figure 4. The total contributions consist of a part from the central square of the cross, and a part from the four semi-infinite strips forming the four limbs:

$$W = W(\text{square}) + W(\text{strips}) \tag{47}$$

Each of the two parts can be further subdivided as follows:

$$\begin{aligned} W(\text{square}) &= G_1 - G_2 \\ W(\text{strips}) &= G_3 - G_4 \end{aligned} \tag{48}$$

The four G's are two-dimensional integrals defined explicitly as follows:

$$G_1 = \frac{1+\lambda^2}{2\pi\varepsilon_o} \int_{-y_o}^{y_o} dy \int_{-y_o}^{y_o} dy' \left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{49}$$

$$G_2 = \frac{R^2}{2\pi\varepsilon_o} \int_{-\alpha/2}^{\alpha/2} d\varphi \int_{-\alpha/2}^{\alpha/2} d\varphi' \left(\frac{1}{\sqrt{2R^2 + a^2 - 2R^2\cos(\varphi-\varphi')}} - \frac{1}{\sqrt{2R^2 + (2b-a)^2 - 2R^2\cos(\varphi-\varphi')}} \right) \tag{50}$$

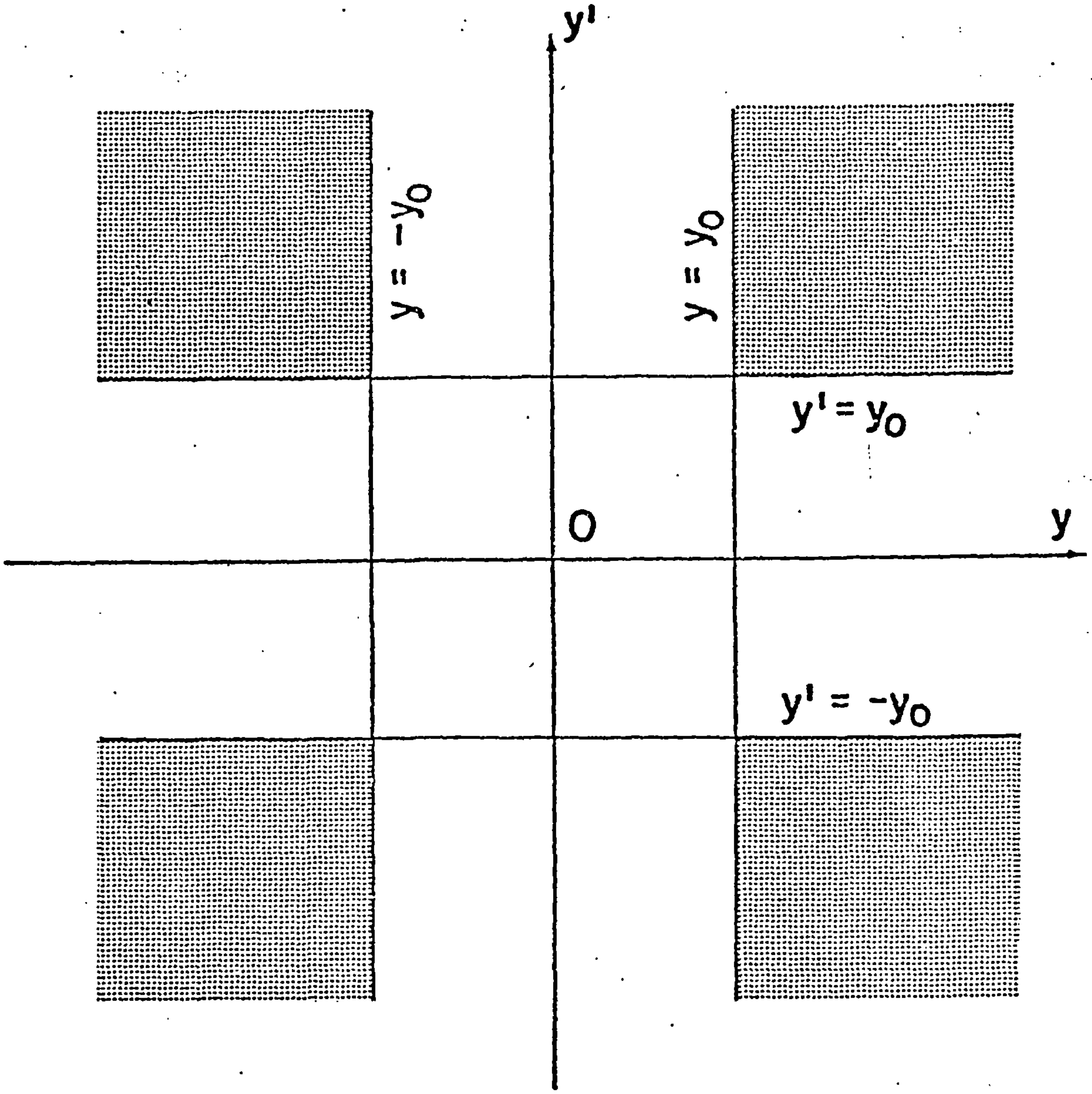

Figure 4. Domain of integration for calculating the two-dimensional integral W defined in equation (40). Nonzero contributions to W come from the undotted cross-shaped region.

$$G_3 = \frac{2(1+\lambda^2)}{\pi\varepsilon_o} \int_{y_o}^{\infty} dy \int_{-y_o}^{y_o} dy' \left(\frac{1}{\sqrt{\lambda^2(y-|y'|)^2 + (y-y')^2 + a^2}} \right.$$

$$\left. - \frac{1}{\sqrt{\lambda^2(y-|y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \tag{51}$$

$$G_4 = \frac{2R\sqrt{1+\lambda^2}}{\pi\varepsilon_o} \int_{y_o}^{\infty} dy \int_{-\alpha/2}^{\alpha/2} d\varphi' \left(\frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + R\cos\varphi')^2 + (y - R\sin\varphi')^2 + a^2}} \right.$$

$$\left. - \frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + R\cos\varphi')^2 + (y - R\sin\varphi')^2 + (2b-a)^2}} \right) \tag{52}$$

These four integrals are similar to those encountered in the inductance calculation. Using the method outlined in Section V of Part I, one can reduce the G's to one-dimensional integrals. The result of the reduction is

$$G_1 = \frac{2(1+\lambda^2)}{\pi\varepsilon_o} \int_0^{y_o} du(y_o - u) \left(\frac{1}{\sqrt{(1+\lambda^2)u^2 + a^2}} - \frac{1}{\sqrt{(1+\lambda^2)u^2 + (2b-a)^2}} \right)$$

$$+ \frac{1+\lambda^2}{\pi\varepsilon_o} \int_0^{y_o} dv[\Phi(2y_o - v, v) - \Phi(v, v)] \tag{53}$$

$$G_2 = \frac{R^2}{\pi\varepsilon_o} \int_0^{\alpha} du(\alpha - u) \left(\frac{1}{\sqrt{2R^2 + a^2 - 2R^2\cos u}} \right.$$

$$\left. - \frac{1}{\sqrt{2R^2 + (2b-a)^2 - 2R^2\cos u}} \right) \tag{54}$$

$$G_3 = \frac{2\sqrt{1+\lambda^2}}{\pi\varepsilon_o} \int_{-y_o}^{y_o} dy' \Psi(\lambda|y'|, y') \tag{55}$$

$$G_4 = \frac{2R}{\pi\varepsilon_o} \int_{-\alpha/2}^{\alpha/2} d\varphi' \Psi(R\sqrt{1+\lambda^2} - R\cos\varphi', R\sin\varphi') \tag{56}$$

with

$$\Phi(u,v) = \ell n \left(\frac{u + \sqrt{\lambda^2 v^2 + u^2 + a^2}}{u + \sqrt{\lambda^2 v^2 + u^2 + (2b-a)^2}} \right) \tag{57}$$

and

$$\Psi(x,y) = \ell n \left(\frac{\sqrt{1+\lambda^2}\sqrt{(\lambda y_o - x)^2 + (y_o - y)^2 + (2b-a)^2} + (1+\lambda^2)y_o - \lambda x - y}{\sqrt{1+\lambda^2}\sqrt{(\lambda y_o - x)^2 + (y_o - y)^2 + a^2} + (1+\lambda^2)y_o - \lambda x - y} \right) \tag{58}$$

For numerical purposes there is no advantage in trying to further reduce the integrals. They will therefore be left in the present form.

In summary the equivalent capacitance C_d of a circular cable bend of radius R and angle α is given by

$$C_d = C_d' - 2\kappa R(2\lambda - \alpha) + \kappa^2(G_1 - G_2 + G_3 - G_4) \tag{59}$$

The G's are evaluated numerically for the typical cases $b = 10a$ and $R = 2b$ and $4b$. The values of C_d are plotted versus α in Figure 5. The capacitance C_d' of an abrupt bend ($R = 0$) is also shown for comparison. It is obvious from the figure that the abrupt bend is not a good approximation to the smooth bend. The same conclusion was drawn from inductance consideration in Part I.

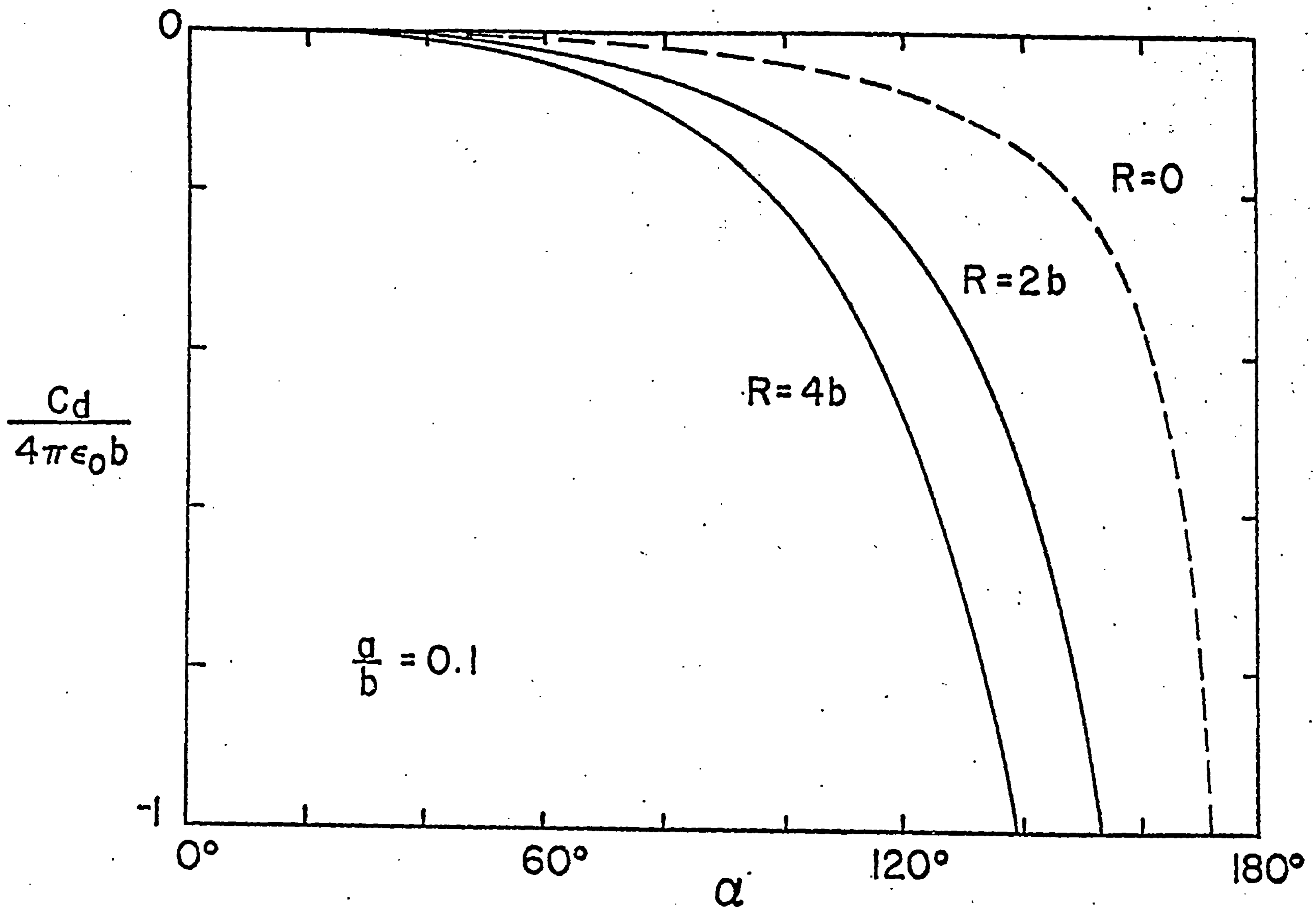

Figure 5. Plot of the equivalent capacitance C_d of a circular cable bend of radius R versus the bend angle α . The broken line is the capacitance C'_d of an abrupt bend (R = 0) .

VI. SUMMARY AND CONCLUSIONS

A calculation of the equivalent capacitance of a bend in an otherwise straight two-wire transmission line has been carried out based on the variational representation (13). The bend is modeled by a circular arc of radius R and angle α adjoining two semi-infinite straight sections. The detailed geometry is shown in Figures 1 and 2.

The bend capacitance is a function of four geometrical parameters R, α, a and b, where a and b are respectively the radius and one-half the separation of the two wires making up the transmission line. In Section IV the capacitance C_d' of an abrupt bend ($R = 0$) is obtained in the simple formula (36). In Section V the capacitance C_d of a circular bend is given in formula (59). The four quantities G_1, G_2, G_3 and G_4 appearing therein are one-dimensional integrals defined in expressions (53) through (56), and are meant to be evaluated on the computer. In the formulas quoted the three parameters y_o, λ and κ are defined in terms of R, α, a and b by equations (6), (21) and (31) respectively.

The bend capacitance C_d is found to be strongly dependent on the bend radius R, being directly proportional to R for large R. When R is comparable to the wire separation, the use of the simpler expression (36) for the abrupt bend capacitance to approximate the smooth bend capacitance is strongly discouraged.

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[1] J. Lam, "Equivalent lumped parameters for a bend in a two-wire transmission line: Part I. Inductance," Interaction Note 303, December 1976.

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