

Interaction Notes

Note 303

December 1976

Equivalent Lumped Parameters for a Bend
in a Two-Wire Transmission Line:
Part I. Inductance

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Abstract

The equivalent lumped inductance of a bend in a parallel-wire transmission line is calculated exactly within the thin-wire assumption. The inductance for an abrupt bend is obtained explicitly in simple closed form. The inductance for a smooth bend modeled by a circular arc is reduced to one-dimensional integrals that can readily be computed. Numerical studies show that the simpler abrupt bend is a poor approximation to the smooth bend.

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I. INTRODUCTION

Electromagnetic waves propagating along a transmission line are scattered by bends in the line. The amount of scattering depends on the shapes and sizes of the bends. At sufficiently low frequencies, when the wavelengths greatly exceed the bend dimensions, the scattering effect of each bend can be represented by equivalent lumped network elements loaded onto the transmission line at the location of the bend.

Figure 1 shows a bend in an otherwise straight parallel-wire transmission line and its representation by an equivalent symmetrical T section network. The bend is formed when the two parallel wires are deflected identically through an angle α . More precisely, the vertical plane in Figure 1 defined by the two parallel wires on one side of the bend intersects the vertical plane defined by the wires on the other side at an angle $\pi - \alpha$; the line of intersection is normal to the two parallel horizontal planes containing each of the wires individually.

The objective of the present effort is to determine the lumped inductance L_d and the capacitance C_d appearing in the equivalent circuit representation of the bend in Figure 1. These lumped elements can be calculated from a pair of quasi-electrostatic and quasi-magnetostatic boundary-value problems for the bend geometry.

The analysis of the bent two-wire transmission line is relevant to the EMP internal-coupling problem of an aircraft. A cable running parallel to a metallic wall or floor in the aircraft's interior is essentially a two-wire transmission line on account of the electrical image. A quantitative knowledge of the bend inductance and capacitance will enable one to estimate the effect of a bend on the EMP propagation characteristics of the cable.

The bent parallel-wire transmission line has previously been studied by Tomiyasu [1] and King [2]. These authors limited their investigations to the abrupt V-shaped bend. The present work improves on their results and, at the same time, extends their analysis to the more general case of a smooth gradual

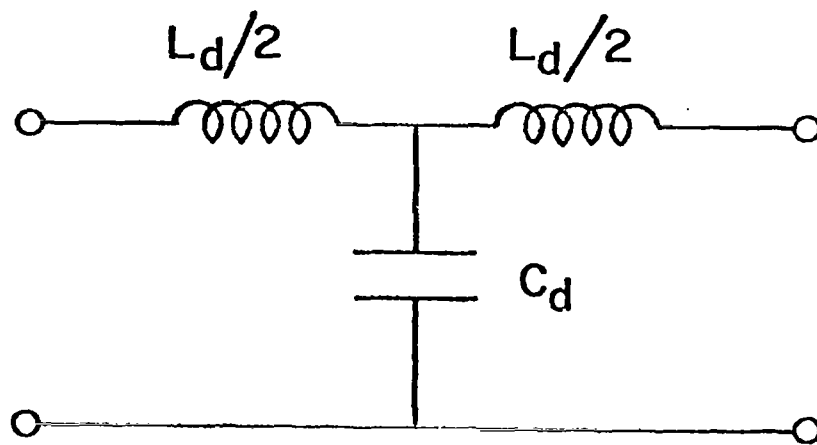
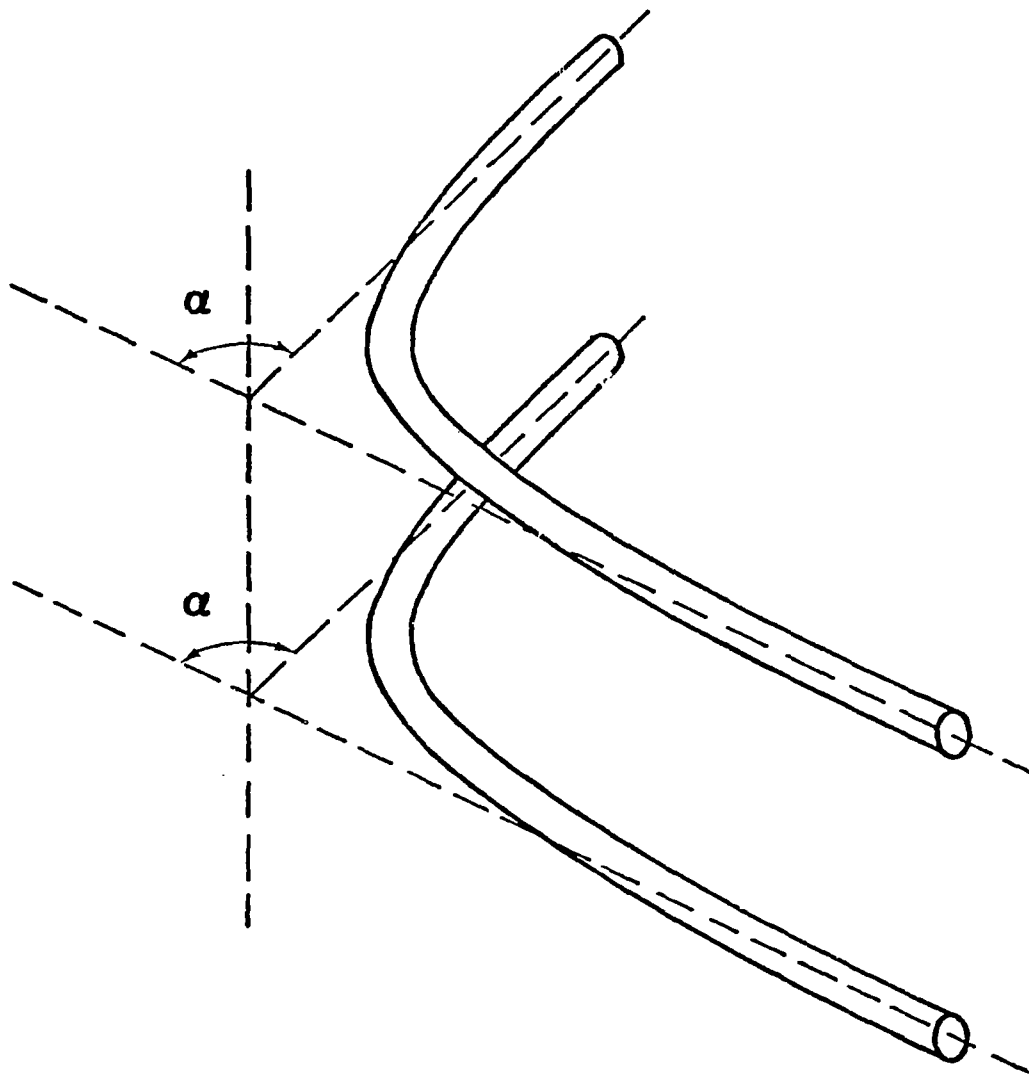


Figure 1. A bend in a parallel-wire transmission line and its equivalent circuit representation.

bend. Specifically, the bend is modeled here by a circular arc with a finite radius. The abrupt bend is recovered in the zero-radius limit. The circular arc is clearly a more realistic model of a cable bend. The calculation shows that the abrupt bend model becomes unreliable when the bend angle α is close to π .

In this report the inductance calculation of the cable bend is worked out. The capacitance calculation will be presented separately in a companion report.

II. GEOMETRY OF THE PROBLEM

Figure 2 shows a model of a bend in an infinitely-long two-wire transmission line. The line consists of two identical parallel conducting cylinders. The radius of each cylinder is a ; the separation of their center lines is $2b$. When the model is applied to the situation of a single conductor over a conducting ground, the parameter b becomes the height of the conductor center line above ground. In the following it will be assumed that the conductors are thin wires so that b is much greater than a .

The bend in each wire is modeled by a circular arc connecting the two semi-infinite straight sections of the wire. The radius of the arc is R ; the angle of the arc is the bend angle α . The two points on the center line at which the circular arc is joined to the straight sections are located at $(x_0, \pm y_0, \pm b)$ with

$$x_0 = R \sec\left(\frac{\alpha}{2}\right) - R \cos\left(\frac{\alpha}{2}\right) \quad (1)$$

$$y_0 = R \sin\left(\frac{\alpha}{2}\right) \quad (2)$$

The \pm sign of b refers to the upper and lower wires, respectively. In the limit of vanishingly small R , both x_0 and y_0 tend to zero. The bend geometry degenerates to that of an abrupt V-shaped bend at the coordinate origin.

The total inductance and capacitance of an infinite transmission line are infinite quantities. But the bend inductance L_d and the bend capacitance C_d are finite since they stem from localized deviations in the line geometry. They are functions of the geometrical parameters R , α , a and b .

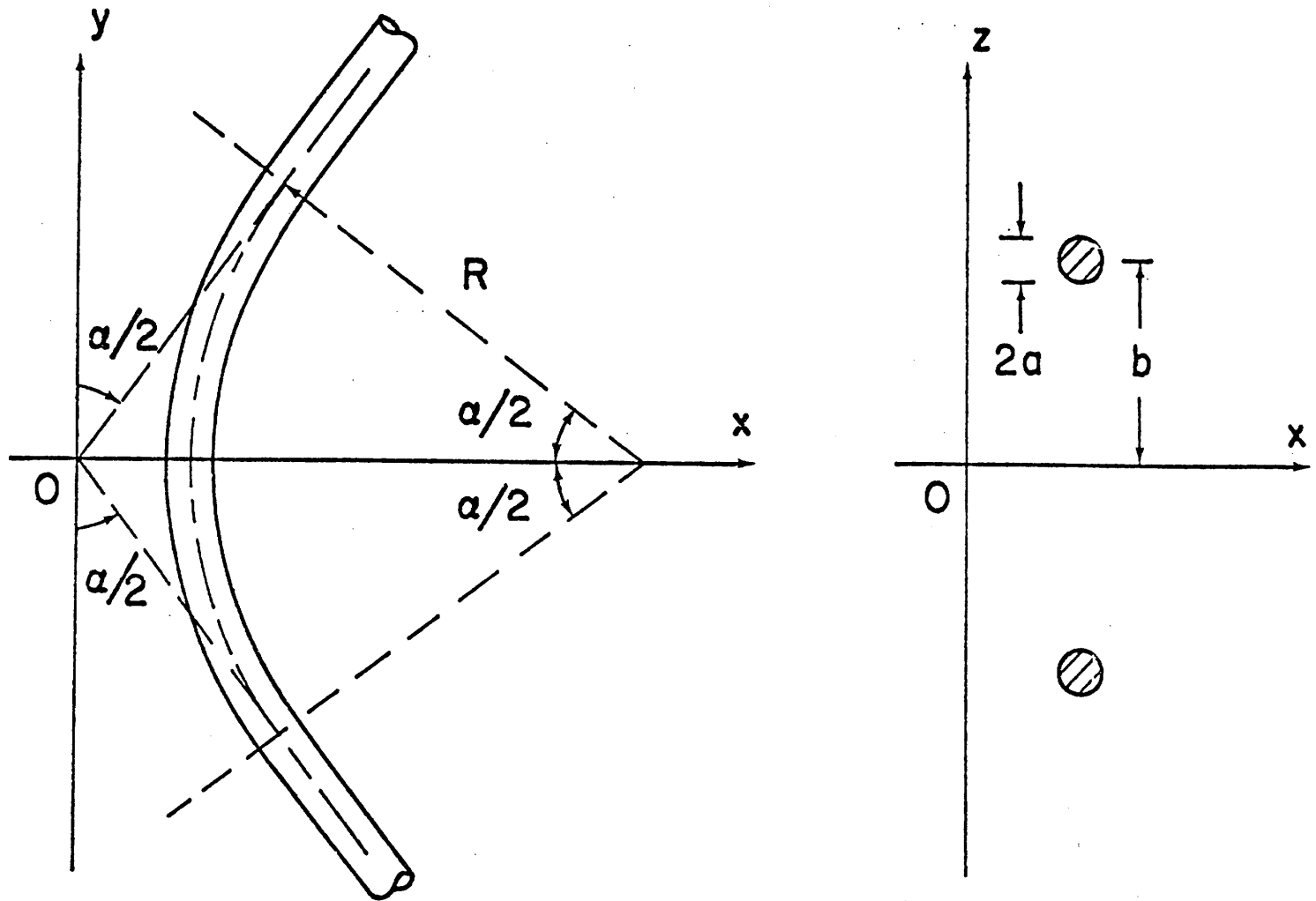


Figure 2. Geometry of a bend in a two-wire transmission line. The bend is modeled by a circular arc of radius R and angle α .

III. DERIVATION OF THE INDUCTANCE INTEGRAL

Let a total current I flow down one wire of the bent two-wire transmission line in Figure 2 and return by way of the other. If the wires are thin, the current can be assumed to distribute itself uniformly in the wire interior. Then the external inductance can be written down exactly in the form of a two-dimensional integral. The internal inductance is negligible.

One first writes down the vector potential at a general point (x,y,z) exterior to the wires

$$\underline{A}(x,y,z) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} dy' \left(\frac{dx'}{dy'} \hat{e}_x + \hat{e}_y \right) \times \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-b)^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+b)^2}} \right) \quad (3)$$

where x' and y' are the coordinates of the center line of a wire. The total magnetic flux passing between the two wires due to the currents in the wires is given by

$$\phi = \int \underline{B} \cdot d\underline{S} = \int (\nabla \times \underline{A}) \cdot d\underline{S} = \oint \underline{A} \cdot d\underline{r} \quad (4)$$

where the line integral goes around the perimeter of the area bounded by the two wires. A general expression for the total inductance L , defined as the ratio ϕ/I , can be obtained by combining equations (3) and (4):

$$L = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left(\frac{dx}{dy} \frac{dx'}{dy'} + 1 \right) \times \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+a^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(2b-a)^2}} \right) \quad (5)$$

which is symmetric in the pairs (x,y) and (x',y') .

To evaluate the integral (5) one must first supply from the bend geometry the functional relation between x and y , and similarly between x' and y' , in the form

$$x = f(y) , \quad x' = f(y') \quad (6)$$

The relation describes the locus of the center line of a wire. For the case of the circular bend in Figure 2, one has

$$x = f_1(y) = \begin{cases} R \sec\left(\frac{\alpha}{2}\right) - \sqrt{R^2 - y^2} & |y| < y_0 \\ |y| \tan\left(\frac{\alpha}{2}\right) & |y| > y_0 \end{cases} \quad (7)$$

where y_0 is defined by (2). For the case of the abrupt bend, one has

$$x = f_2(y) = |y| \tan\left(\frac{\alpha}{2}\right) \quad (8)$$

For the case of the straight transmission line, one has

$$x = f_3(y) = 0 \quad (9)$$

By evaluating the integral (5) with the three different functional expressions f_1 , f_2 and f_3 in (7), (8) and (9), one obtains three inductances L_1 , L_2 and L_3 . They are respectively the total inductance of a line with a circular bend, an abrupt bend, and no bend. All three are linearly divergent quantities. However, the equivalent inductance L_d of the circular bend given by the difference

$$L_d = L_1 - L_3 \quad (10)$$

is finite. In the following two sections the calculation of L_d will proceed in two steps. In Section IV one calculates the inductance difference

$$L'_d = L_2 - L_3 \quad (11)$$

This quantity is the equivalent inductance of an abrupt bend. In Section V one calculates the difference

$$L''_d = L_1 - L_2 \quad (12)$$

due to the deviation of a circular bend from an abrupt bend. The desired circular bend inductance L_d is then given by the sum

$$L_d = L'_d + L''_d . \quad (13)$$

IV. INDUCTANCE OF AN ABRUPT CABLE BEND

The equivalent inductance L'_d of an abrupt bend through an angle α in a two-wire transmission line is defined by expression (11). The two inductance integrals L_2 and L_3 are explicitly given by

$$L_2 = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left(1 + \lambda^2 \frac{yy'}{|yy'|} \right) \left(\frac{1}{\sqrt{\lambda^2 (|y| - |y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2 (|y| - |y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \quad (14)$$

and

$$L_3 = \frac{\mu_0}{2\pi} \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dy' \left(\frac{1}{\sqrt{(y-y')^2 + a^2}} - \frac{1}{\sqrt{(y-y')^2 + (2b-a)^2}} \right) \quad (15)$$

where

$$\lambda = \tan \left(\frac{\alpha}{2} \right) \quad (16)$$

One can show that the contribution to L_2 from those regions of integration in which y and y' are of the same sign exactly cancel the corresponding contribution to L_3 . The nonzero contribution to L'_d comes from the remaining regions in which y and y' are of opposite signs. In physical terms this result means that the self inductances of the two semi-infinite straight sections of the bent transmission line are unaffected by the bend; the bend inductance L'_d is entirely due to the change in the mutual inductance of the two semi-infinite sections. This change is given by

$$L'_d = F(\lambda) - F(0) \quad (17)$$

where

$$F(\lambda) = \frac{\mu_0}{\pi}(1-\lambda^2) \int_0^\infty dy \int_{-\infty}^0 dy' \left(\frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(y+y')^2 + (y-y')^2 + (2b-a)^2}} \right) \quad (18)$$

The integral $F(\lambda)$ can be worked out analytically by the following device. First perform a change of integration variables:

$$u = y - y' , \quad v = y + y' \quad (19)$$

The corresponding change in the integration is

$$\int_0^\infty dy \int_{-\infty}^0 dy' \rightarrow \frac{1}{2} \int_0^\infty du \int_{-u}^u dv \quad (20)$$

Next make the substitution $v = ut$ and interchange the order of integration. The two ensuing integrations, first over u and then over t , are both elementary. The result is

$$F(\lambda) = \frac{2\mu_0}{\pi}(b-a) \frac{1-\lambda^2}{\lambda} \tan^{-1} \lambda \quad (21)$$

Using definitions (16) and (17), one arrives at the simple explicit result

$$L'_d = \frac{2\mu_0(b-a)}{\pi} (\alpha \cot \alpha - 1) \quad (22)$$

This is the exact formula for the equivalent inductance of an abrupt cable bend within the thin-wire assumption. By contrast, the more complicated expression given by King [3] is only approximate.

A plot of L'_d versus α is shown in Figure 3. One sees that the bend inductance is a negative quantity. Its magnitude increases sharply as the bend approaches a hair-pin bend ($\alpha = \pi$).

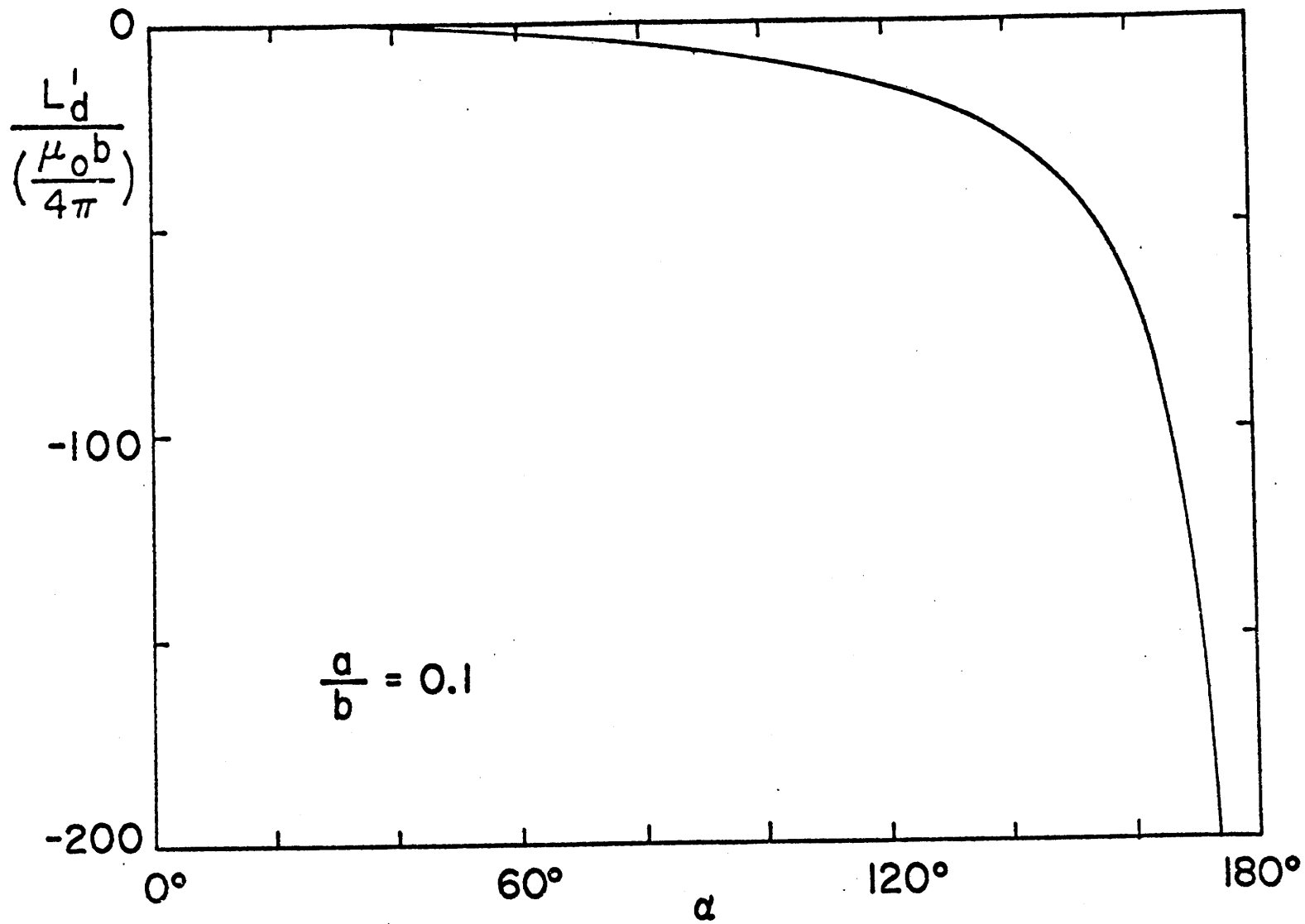


Figure 3. Plot of the equivalent inductance L'_d of an abrupt cable bend versus the bend angle α .

V. INDUCTANCE OF A CIRCULAR CABLE BEND

The charge L_d'' in the equivalent inductance when one goes from an abrupt cable bend to a smooth cable bend modeled by a circular arc is given by expression (12). The inductance L_1 is defined by (5) with $x = f_1(y)$ and $x' = f_1(y')$; L_2 is similarly defined with $x = f_2(y)$ and $x' = f_2(y')$. From equations (7) and (8) it is clear that $f_1(y)$ and $f_2(y)$ are identical for $|y| > y_0$. Consequently the nonzero contributions to L_d'' come from regions of integration in which either y or y' or both lie within the interval $(-y_0, y_0)$. On the y - y' plane these regions form a cross, as shown in Figure 4. The contributions to L_d'' consist of a part from the central square of the cross and a part from the four semi-infinite strips making up the four branches:

$$L_d'' = L_d''(\text{square}) + L_d''(\text{strips}) \quad (23)$$

These partial contributions are expressible as

$$L_d''(\text{square}) = G_1 - G_2 \quad (24)$$

$$L_d''(\text{strips}) = G_3 - G_4$$

where the quantities G_1 , G_2 , G_3 and G_4 are double integrals given explicitly by

$$G_1 = \frac{\mu_0}{2\pi} \int_{-y_0}^{y_0} dy \int_{-y_0}^{y_0} dy' \left(1 + \frac{yy'}{\sqrt{(R^2-y^2)(R^2-y'^2)}} \right) \left(\frac{1}{\sqrt{(\sqrt{R^2-y^2} - \sqrt{R^2-y'^2})^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{(\sqrt{R^2-y^2} - \sqrt{R^2-y'^2})^2 + (y-y')^2 + (2b-a)^2}} \right) \quad (25)$$

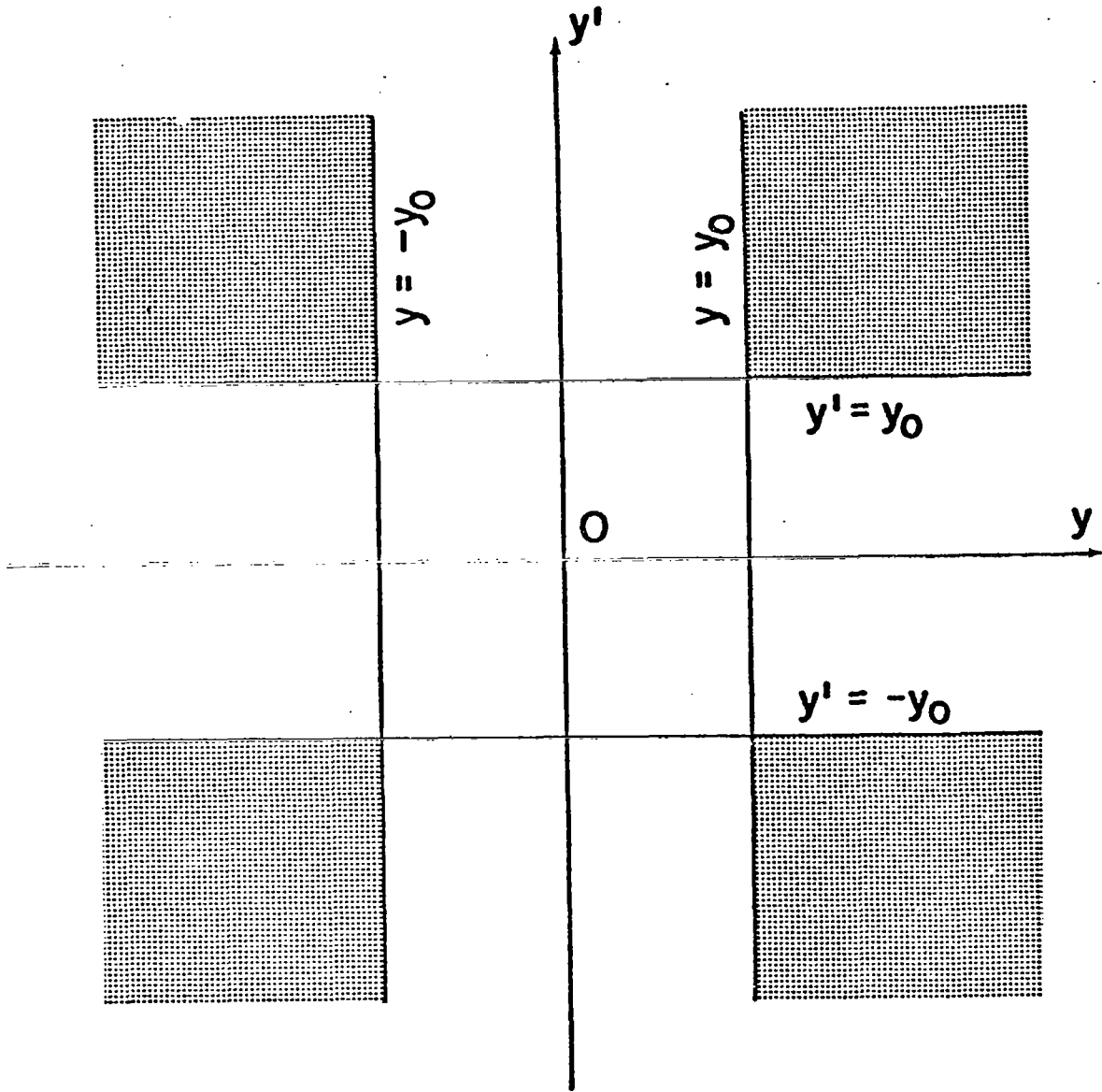


Figure 4. Domain of integration for calculating the inductance difference L_d'' between a circular bend and an abrupt bend. Nonzero contributions to L_d'' come from the undotted cross-shaped region.

$$G_2 = \frac{\mu_0}{2\pi} \int_{-y_0}^{y_0} dy \int_{-y_0}^{y_0} dy' \left(1 + \lambda^2 \frac{yy'}{|yy'|} \right) \left(\frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(|y|-|y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \quad (26)$$

$$G_3 = \frac{2\mu_0}{\pi} \int_{y_0}^{\infty} dy \int_{-y_0}^{y_0} dy' \left(1 + \frac{\lambda y'}{\sqrt{R^2 - y'^2}} \right) \left(\frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + \sqrt{R^2 - y'^2})^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{(\lambda y - R\sqrt{1+\lambda^2} + \sqrt{R^2 - y'^2})^2 + (y-y')^2 + (2b-a)^2}} \right) \quad (27)$$

$$G_4 = \frac{2\mu_0}{\pi} \int_{y_0}^{\infty} dy \int_{-y_0}^{y_0} dy' \left(1 + \frac{\lambda^2 y'}{|y'|} \right) \left(\frac{1}{\sqrt{\lambda^2(y-|y'|)^2 + (y-y')^2 + a^2}} - \frac{1}{\sqrt{\lambda^2(y-|y'|)^2 + (y-y')^2 + (2b-a)^2}} \right) \quad (28)$$

All four integrals are finite. The contribution L_d'' (square) is the shift in the self inductance of the bent section between y_0 and $-y_0$ when the bend geometry is changed from an abrupt bend to a smooth bend. The contribution L_d'' (strips) is the corresponding shift in the mutual inductance between the bent section and the two adjoining semi-infinite straight sections.

One integration of each of the four double integrals can be carried out. In the case of G_1 , it is convenient to first introduce angle variables φ and φ' such that

$$y = R \sin \varphi, \quad y' = R \sin \varphi' \quad (29)$$

as well as their difference u and sum v :

$$u = \varphi - \varphi' , \quad v = \varphi + \varphi' \quad (30)$$

The v -integration can at once be performed, with the following result:

$$G_1 = \frac{\mu_0 R^2}{\pi} \int_0^\alpha du (\alpha - u) \cos u \left(\frac{1}{\sqrt{2R^2 + a^2 - 2R^2 \cos u}} - \frac{1}{\sqrt{2R^2 + (2b-a)^2 - 2R^2 \cos u}} \right) \quad (31)$$

In the case of G_2 , one introduces the difference u and sum v :

$$u = y - y' , \quad v = y + y' \quad (32)$$

and then integrates over one of them. The result reads

$$G_2 = \frac{2\mu_0}{\pi} (1+\lambda^2) \int_0^{y_0} du (y_0 - u) \left(\frac{1}{\sqrt{(1+\lambda^2)u^2 + a^2}} - \frac{1}{\sqrt{(1+\lambda^2)u^2 + (2b-a)^2}} \right) + \frac{\mu_0}{\pi} (1-\lambda^2) \int_0^{y_0} dv [\phi(2y_0 - v, v) - \phi(v, v)] \quad (33)$$

where

$$\phi(u, v) = \lambda n \left(\frac{u + \sqrt{\lambda^2 v^2 + u^2 + a^2}}{u + \sqrt{\lambda^2 v^2 + u^2 + (2b-a)^2}} \right) \quad (34)$$

In the cases of G_3 and G_4 , one can carry out the y -integration directly without any change of variables, obtaining thereby the expressions

$$G_3 = \frac{2\mu_0}{\pi} \frac{1}{\sqrt{1+\lambda^2}} \int_{-y_0}^{y_0} dy' \left(1 + \frac{\lambda y'}{\sqrt{R^2 - y'^2}} \right) \Psi(R\sqrt{1+\lambda^2} - \sqrt{R^2 - y'^2}, y') \quad (35)$$

$$G_4 = \frac{2\mu_0}{\pi} \frac{1}{\sqrt{1+\lambda^2}} \int_{-y_0}^{y_0} dy' \left(1 + \frac{\lambda^2 y'}{|y'|} \right) \Psi(\lambda|y'|, y') \quad (36)$$

where

$$\Psi(x, y) = \ln \left(\frac{\sqrt{1+\lambda^2} \sqrt{(\lambda y_0 - x)^2 + (y_0 - y)^2 + (2b - a)^2} + (1+\lambda^2)y_0 - \lambda x - y}{\sqrt{1+\lambda^2} \sqrt{(\lambda y_0 - x)^2 + (y_0 - y)^2 + a^2} + (1+\lambda^2)y_0 - \lambda x - y} \right) \quad (37)$$

The remaining integration in G_2 and G_4 can be performed exactly. However, the explicit integration produces such a proliferation of terms that the results are practically useless. The integrals G_1 and G_3 in (31) and (35) contain parts which are essentially integrals of incomplete elliptic integrals. They are beyond the limits of the art of analytical integration. One must ultimately resort to numerical integration for their evaluation. For numerical purposes, it is more advantageous to retain all four integrals in the forms (31), (33), (35) and (36).

The equivalent inductance L_d of a smooth bend consisting of a circular arc of radius R and angle α is therefore given by

$$L_d = L'_d + G_1 - G_2 + G_3 - G_4 \quad (38)$$

where L'_d is the equivalent inductance of an abrupt bend of angle α as given in (22). The four G integrals are evaluated numerically for the cases $b = 10a$ and $R = 2b$ and $4b$. The values of L_d are plotted versus α in Figure 5. The inductance of an abrupt bend ($R = 0$) is also shown for comparison. It is seen that the dependence of L_d on the bend radius R is very pronounced

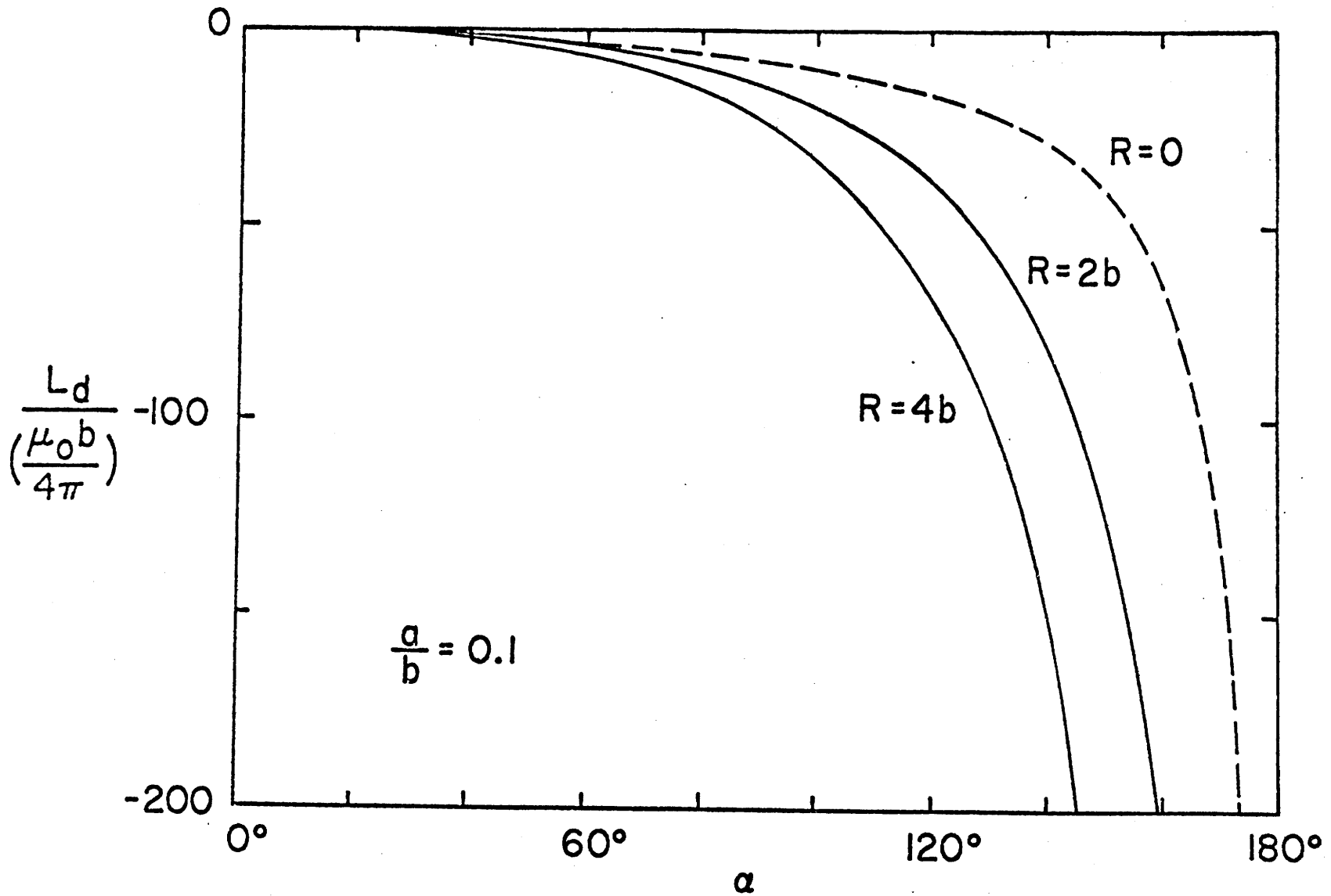


Figure 5. Plot of the equivalent inductance L_d of a circular cable bend of radius R versus the bend angle α . The broken line is the inductance L'_d of an abrupt bend ($R = 0$).

for large bend angle α . At the chosen values of the parameters the abrupt bend can be said to approximate the smooth bend only for α less than about 40° .

VI. SUMMARY AND CONCLUSIONS

The equivalent inductance L_d of a bend in a two-wire transmission line is calculated exactly within the thin-wire assumption. The bend is modeled by a circular arc of radius R and angle α as shown in Figures 1 and 2. The bend inductance L_d is a function of four geometrical parameters R , α , a and b , where a is the wire radius and b is one-half the wire separation.

In the limit as R tends to zero, one obtains the geometry of an abrupt bend. The bend inductance in this case, denoted by L_d' , is given by the simple formula (22). For arbitrary R the bend inductance L_d is given by formula (38) in terms of L_d' and four one-dimensional integrals G_1 , G_2 , G_3 and G_4 . These integrals are defined by expressions (31), (33), (35) and (36). The two parameters y_0 and λ appearing therein are related to R and α through expressions (2) and (16). The integrals can be easily evaluated on the computer.

It is concluded that the abrupt bend is a poor approximation to a smooth bend when the bend radius is comparable to the wire separation.

REFERENCES

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- [3] R.W.P. King, *op. cit.*, p. 384, formulas (18a) and (18b).