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SCATTERING OF ELECTROMAGNETIC RADIATION BY APERTURES VIII. THE NORMALLY SLOTTED CYLINDER THEORY

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ROBERT WILLIAMSON II
Captain, USN
Commander

D D Kerstetter

D. D. KERSTETTER
By direction

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INTRODUCTION

This publication is one of a series of theoretical and experimental reports of research conducted on the response of cylindrical conducting cavities to incident electromagnetic radiation from some distant external source. In these, still the earlier phases of our studies, attention is focused primarily on perfectly conducting, thin wall cylindrical cavities containing apertures of relatively simple geometry. The problems to be considered at first assume a vacant cavity. This choice of the initial configuration to be considered was made so as to simplify the theoretical problem to be solved and simultaneously to facilitate simple experimental checks on the validity of our various methods of analytic solution.

A spectrum of the effects of aperture parameters such as dimensions, geometry and number, and relative locations of perforations of the conducting shield are under investigation for the externally irradiated empty cavity. Included among the characteristics being studied are the efficiency of energy leakage into the cavity, the specific field distribution within the cavity and the external near and far-field distributions. In addition to these quantities and of obvious special importance are our studies of the surface currents on the conducting shields and the fields over the apertures. Particular interest is directed to the behavior of these quantities in the immediate vicinity of the edges of the apertures. It is presumed that possession and understanding of this considerably large body of information will permit us to develop some degree of control over the leakage into the cavity, the field distribution within and the division of the incident energy into that picked up by the target and that portion scattered away from the cavity.

Reports subsequent to these studies shall present the results of substantially the same research, again both experimental and theoretical for the same cylindrical cavities with apertures. These follow-on papers will be the results of studies wherein a sequence of configurations of increasing complexity are contained within the cavities. Among these situations we shall include for example one or more enclosed longitudinal cables, or concentric loops or helices. The results of the effects of positioning these within the aperture containing cavity will be discussed. This study will be continued to more complicated systems contained within the cavity to investigate the effects produced. Similar investigations are proceeding for a variety of simple cavity shapes with aperture systems. Complementary studies of further complicated effects of finite conductivity and, say, dielectric cladding are underway. These however will not be included in discussions in this report.

At this point a review of the more pertinent work related to the specific problem considered in this paper is appropriate. The earlier reports in this series by Bombardt and Libelo(1),(2) on the slotted conducting plane can be considered limiting cases of a very long circular cylinder of very large radius containing a slot parallel to the cylinder axis. The results were shown to be in excellent agreement with the early work of Morse and Rubenstein(3), Seshadri(4) and Skavlem(5). Next in the series of these reports were a number by Bombardt and Libelo(6),(7),(8) which were studies of the axially slotted, infinitely long, perfectly conducting, thin walled circular cylinder for symmetric incidence normal to the cylinder axis.

Measurements made by Macrakis(9) of the back scattering cross-section per unit length for large slot angles are in very close agreement with the analytic results predicted by Bombardt and Libelo(8). Also in the limit of very large aperture angle which corresponds to almost a flat strip we obtain back scattering cross-section results very nearly the same as those of the corresponding actual flat ribbon. The status of the problem for narrow slots for the infinite cylinder is not yet completely resolved to everyones satisfaction. In this case Macrakis has measured the back-scattering cross-sections for a fixed cylinder with a specific narrow slot at various frequencies. The data obtained show quite clearly resonance structures at certain frequencies. His analytic calculation of this quantity which coincides basically with that of Morse and Feshbach(10) and corresponds simply to assuming the static field distribution across the narrow aperture fails completely to predict any indication even of the observed structure in his experiments. On the other hand the calculations of Bombardt and Libelo predict the occurence of these resonances at almost precisely the experimentally observed frequencies. This total set of analytic results for the slotted cylinder which apparently agrees with experimental data, where available, leads us to presume that this scattering problem has been successfully resolved. It follows then that one of the scattering problems to be studied next is the perfectly conducting infinite circular cylinder with an annular slot normal to the cylinder axis. This problem will be merely formulated in this paper for normally incident radiation. In subsequent reports we shall present the results of solution by numerical techniques and some further analytic results as well.

Before proceeding to the explicit formulation of the scattering problem of the normally slotted cylinder we pause to inject some preliminary remarks which may lead to some information of considerable consequence. First, Pearson(11) has used the Wiener-Hopf technique to achieve formal solutions for the surface currents on a semi-infinite, open ended, circular, conducting, cylinder illuminated by an electromagnetic wave. In the special case of normal incidence Pearson's solution is seen to be that for $\vec{E}$ polarization. Kao[12] considered the same scattering problem and solved approximately for both $\vec{E}$ and for $\vec{H}$ polarization. It will be interesting to compare our results with those of Pearson and Kao for a pair of semi-infinite tubes so as to learn something of the correlation resulting from two scatterers. The second point relates to the fact that Kao[13],[14] has succeeded in approximately solving the problem of electromagnetic scattering by an open ended, perfectly conducting, circular cylindrical tube of finite length. These solutions are in numerical form and have been limited thus far to short cylinders. At present this limit is being removed via a joint program between the authors of this paper and Kao and R. W. P. King and the extended numerical results and experimental data will be published at a later date. Nevertheless, the fact that the results for the short finite tube are available is somewhat intriguing, in the sense that we are tempted to compare these to our results for the normally slotted infinite cylinder, the geometric configuration, to see to what extent, if any, the solution to either problem is related to that of the other problem. The results obtained and their discussion will be reserved for a subsequent report.

GEOMETRY OF THE PROBLEM AND THE BOUNDARY CONDITIONS

Figure 1 illustrates the geometry of the scattering problem under investigation. An infinite, perfectly conducting thin walled, right circular cylinder of radius a is aligned so that its axis coincides with the z-axis of the coordinate system. An aperture of width 2h is assumed in the cylinder. This aperture is produced by cutting the cylinder with a pair of parallel planes normal to the cylinder axis and removing the enclosed portion of the cylinder. For convenience the center of the aperture is taken coincident with the origin of the coordinate system. The incident radiation is a plane wave linearly polarized parallel to the cylinder axis and is assumed to be propagating along the positive direction of the x-axis and hence is falling normally on the slotted cylinder. Denoting the free space impedance by ζ_0 the incident electric and magnetic fields are respectively (suppressing the explicit $e^{-i\omega t}$ time dependence)

$$(1) \qquad \vec{E}^i_z(x) = e^{ikx}\vec{e}_z = e^{ikr\cos\theta}\vec{e}_z$$

$$(2) \qquad \vec{H}^i_y(x) = -\zeta_0^{-1}e^{ikx}\vec{e}_y = -\zeta_0^{-1}e^{ikr\cos\theta}\vec{e}_y$$

Note that we assume unit amplitude for the incident electric field.

Now the Maxwell equations for the scattered fields are in polar cylindrical coordinates:

$$(3) \qquad \vec{\nabla}\cdot\vec{B}^s(z,r,\theta) = 0$$

$$(4) \qquad \vec{\nabla}\times\vec{B}^s(z,r,\theta) = \mu_0\left[\vec{J}(z,\theta) - i\epsilon_0\omega\vec{E}^s(z,r,\theta)\right]$$

$$(5) \qquad \vec{\nabla}\cdot\vec{E}^s(z,r,\theta) = 0$$

$$(6) \qquad \vec{\nabla}\times\vec{E}^s(z,r,\theta) = i\omega\vec{B}^s(z,r,\theta)$$

where μ_0, ϵ_0 are the magnetic permeability and electric permitivity of free space. Utilizing the symmetry property in the variable θ we can express all surface currents and field quantities in Fourier series representation. This permits us to consider each Fourier component separately since they are all decoupled, so to speak. At the end these components can be reconstituted to give the final results for the fields and currents. Because we need only to consider one Fourier component at a time, or equivalently, one value of the integer index n at a time, we shall not insist on explicitly

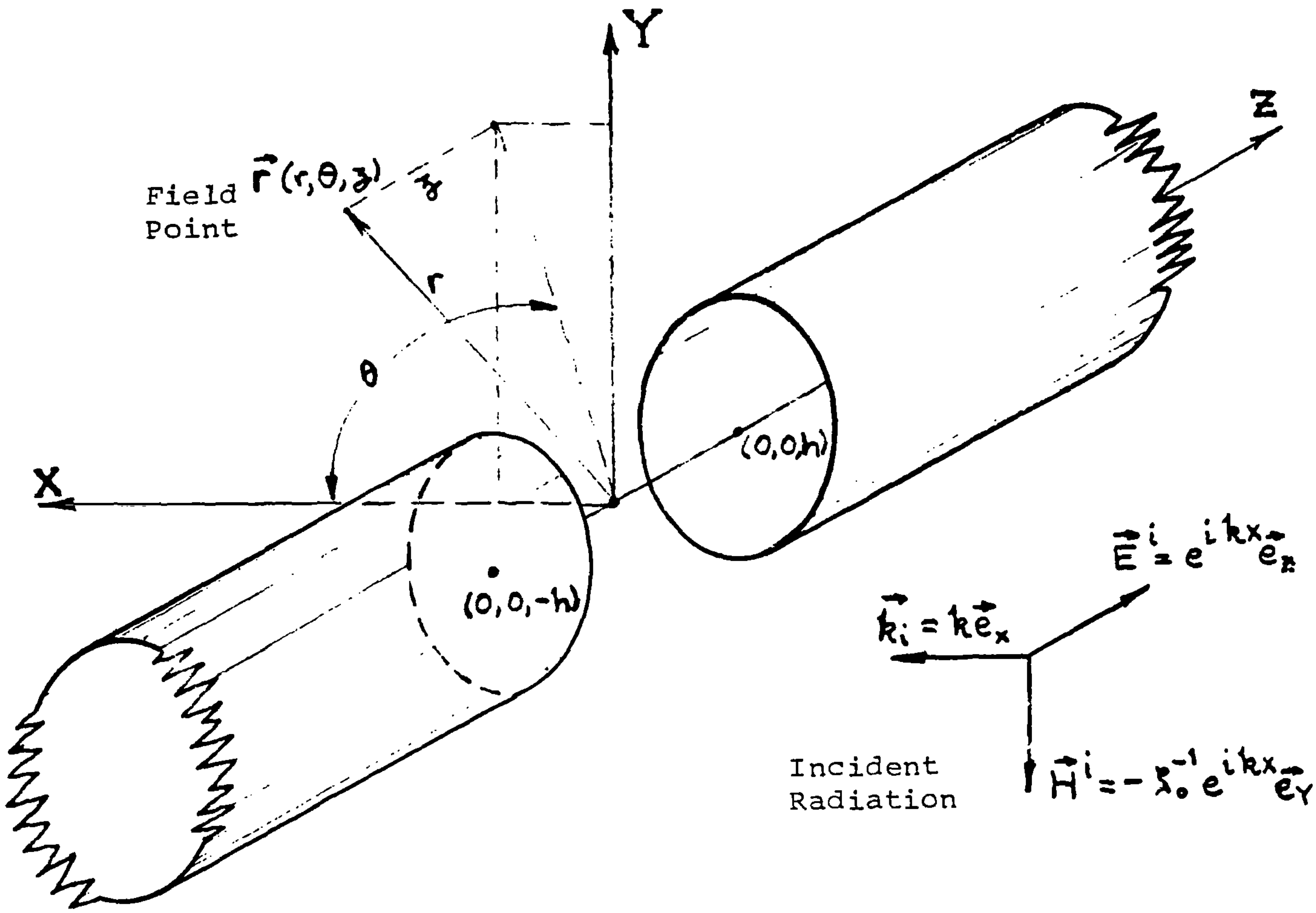


Figure 1. Geometry of the Scattering Problem

Infinite circular cylinder of radius a with axis along z-axis, containing a normal slot of width 2h centered about z-o and illuminated normally by a unit plane wave polarized parallel to the cylinder axis.

displaying this index except when needed for clarity. This is done merely for convenience. Furthermore, it is easier, as shall become evident later, to Fourier transform with respect to the z-coordinate and work with the transformed physical quantities and at the end to Fourier invert back to the z-coordinate. Thus in the intermediate stages of the development of the theory all physical quantities will be functions of (ζ, r, n) or more compactly (ζ, r). In the final stages they will revert back to dependence on the coordinates (z, r, θ).

To aid in clarifying notation we indicate here the Fourier transform of $f(z)$, some function of z, by:

(7) $$F_\zeta\{f(z)\} \equiv \bar{f}(\zeta) = \int_{-\infty}^{\infty} dz\, f(z)\, e^{i\zeta z}$$

and the inverse Fourier transform of $\bar{f}(\zeta)$ by

(8) $$f(z) = \frac{1}{2\pi} \int_c d\zeta\, \bar{f}(\zeta)\, e^{-i\zeta z}$$

where c indicates some appropriate contour.

Expressing the Maxwell equations, i.e. eqs. (3) through (6) in cylindrical coordinates and following the prescription above we readily find that the θ and r components of the scattered fields can be expressed in terms of the z-components as follows:

(9) $$\bar{E}_r^s(\zeta, r) = \xi^{-2}\left[-i\zeta \frac{\partial \bar{E}_z^s(\zeta, r)}{\partial r} - n\omega r^{-1} \bar{B}_z^s(\zeta, r)\right]$$

(10) $$\bar{E}_\theta^s(\zeta, r) = \xi^{-2}\left[n r^{-1} \zeta \bar{E}_z^s(\zeta, r) - i\omega \frac{\partial \bar{B}_z^s(\zeta, r)}{\partial r}\right]$$

(11) $$\bar{B}_r^s(\zeta, r) = \xi^{-2}\left[n\omega\mu_0 \epsilon_0 r^{-1} \bar{E}_z^s(\zeta, r) - i\zeta \frac{\partial \bar{B}_z^s(\zeta, r)}{\partial r}\right]$$

(12) $$\bar{B}_\theta^s(\zeta, r) = \xi^{-2}\left[i\omega\mu_0 \epsilon_0 \frac{\partial \bar{E}_z^s(\zeta, r)}{\partial r} - n r^{-1} \zeta \bar{B}_z^s(\zeta, r)\right]$$

where we have defined the quantity

(13) $$\xi^2 \equiv k^2 - \zeta^2$$

As a consequence of the expressions in eqs. (9) through (12) we observe that we have only to solve for the z-components of the scattered fields. From these we can then readily obtain all of the remaining components.

Next we shall consider the boundary condition relations for the problem. The conducting cylinder itself corresponds to setting r = a and allowing the z-coordinate to range over $|z| > h$, for all values of θ, the angular coordinate. Then the aperture is defined as r = a, all values of $|z| < h$ for all values of θ. If we denote the scattered electric field inside and outside of the cylindrical surface r = a by $\vec{E}^s(a_-,z)$ and $\vec{E}^s(a_+,z)$ respectively we can show directly that the Fourier transformed tangential components are also continuous as r = a is crossed upon passing from the inside to the outside. Thus

$$\bar{E}_z^s(a_-,\zeta) = \bar{E}_z^s(a_+,\zeta) \tag{14}$$

$$\bar{E}_\theta^s(a_-,\zeta) = \bar{E}_\theta^s(a_+,\zeta) \tag{15}$$

Furthermore these conditions hold on the conductor where $|z| > h$ and over the aperture where $|z| < h$. We pause in the development momentarily to note that we have suppressed the index n corresponding to the particular Fourier series component with respect to θ. Proceeding in the same notation we have for the scattered magnetic fields the boundary condition relations:

For $|z| > h$, i.e. on the conductor,

$$\bar{B}_z^s(a_+,\zeta) - \bar{B}_z^s(a_-,\zeta) = -\mu_o\bar{J}_\theta(\zeta) \tag{16}$$

$$\bar{B}_\theta^s(a_+,\zeta) - \bar{B}_\theta^s(a_-,\zeta) = \mu_o\bar{J}_z(\zeta) \tag{17}$$

where $\bar{J}_\theta(\zeta)$ is the Fourier transformed θ-component of the n-th component in the Fourier series representation of the surface current density, and $\bar{J}_z(\zeta)$ is the corresponding z-component of the current density.

For $|z| < h$, i.e. over the aperture,

$$\bar{B}_z^s(a_+,\zeta) = \bar{B}_z^s(a_-,\zeta) \tag{18}$$

$$\bar{B}_\theta^s(a_+,\zeta) = \bar{B}_\theta^s(a_-,\zeta). \tag{19}$$

Clearly both components are continuous across the aperture.

THE z-COMPONENT OF THE SCATTERED INTERNAL AND EXTERNAL FIELDS

Upon Fourier transforming the Maxwell equations with respect to the longitudinal coordinate z we obtain for each value of the index n the inhomogeneous Bessel differential equations for the z-components of the fields:

(20) $$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \xi^2 - \frac{n^2}{r^2}\right)\bar{E}_z^s(\zeta,r) = \frac{-i\xi^2}{\omega\epsilon_o r}\delta(r-a)\bar{J}_z(\zeta)$$

and

(21) $$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \xi^2 - \frac{n^2}{r^2}\right)\bar{B}_z^s(\zeta,r) = \frac{i\xi^2}{\omega^2\epsilon_o r}\delta(r-a)\bar{J}_\theta(\zeta)$$

Before continuing it is helpful to consider the surface current density components in some detail. For the z-component we have

(22) $$J_z(z) = \mathcal{J}_z(z)[\theta(z-h) + \theta(-z-h)]$$

where $\theta(x)$ is the unit step function

(23) $$\theta(x) \equiv \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Fourier transforming eq. (22) we obtain

(24) $$\bar{J}_z(\zeta) = \int_{-\infty}^{\infty} dz\, \mathcal{J}_z(z)[\theta(z-h) + \theta(-z-h)]\exp(i\zeta z)$$

Applying the convolution theorem we can rewrite this as

(25) $$\bar{J}_z(\zeta) = \frac{1}{2\pi}\int_C d\zeta' \bar{\mathcal{J}}_z(\zeta')F_{\zeta-\zeta'}\{\theta(z-h) + \theta(-z-h)\}$$

where for some arbitrary transformable function f(z) we used the notation in eq. (7)

(26) $$F_{\zeta-\zeta'}\{f(z)\} \equiv \int_{-\infty}^{\infty} dz f(z)\exp[i(\zeta-\zeta')z] \equiv \bar{f}(\zeta-\zeta')$$

Now if we introduce the function

(27) $$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

the step function of eq. (23) may also be expressed as

(28) $$\theta(\pm x) = \frac{1}{2}(1 \pm \operatorname{sgn} x)$$

Then we can write for the Fourier transform of the step functions in eq. (22)

(29) $$F_{\zeta}\{\theta(\pm z-h)\} = \frac{1}{2}\int_{-\infty}^{\infty} dz[1 \pm \operatorname{sgn}(z \mp h)]\exp(i\zeta z)$$

With the help of the identities

(30) $$I(\zeta) \equiv \int_{-\infty}^{\infty} dz \exp(i\zeta z)$$

and

(31) $$\frac{d}{dx}(\operatorname{sgn} x) = 2\delta(x)$$

we obtain after integration by parts

(32) $$F_{\zeta-\zeta}\{\theta(z-h) + \theta(-z-h)\} = I(\zeta-\zeta') - \frac{2\sin(\zeta-\zeta')h}{(\zeta-\zeta')}$$

Substituting this into eq. (25) we get in turn for the z-component of the current density

(33) $$\bar{J}_z(\zeta) = \bar{J}_z(\zeta) - \frac{1}{\pi}\int_C d\zeta' \bar{J}_z(\zeta') \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}$$

In precisely the same manner we find the θ-component of the current density to be

(34) $$\bar{J}_\theta(\zeta) = \bar{J}_\theta(\zeta) - \frac{1}{\pi}\int_C d\zeta' \bar{J}_\theta(\zeta') \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}$$

Using the appropriate Green's function we can now solve eqs. (20) and (21) for the scattered fields in terms of the current densities. The details of the solution are contained in Appendix A. We merely quote the results here. The scattered z-component of the electric field is given by

$$(35)\qquad \bar{E}_z^s(\zeta,r) = -\frac{\pi(\omega\varepsilon_o)^{-1}a\xi^2}{2} J_{|n|}(r_<\xi)H_{|n|}^{(1)}(r_>\xi)\{\bar{J}_z(\zeta) - \frac{1}{\pi}\int_c d\zeta'\bar{J}_z(\zeta')\frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\} - \frac{n\pi(\omega\varepsilon_o)^{-1}\zeta}{2} J_{|n|}(r_<\xi)H_{|n|}^{(1)}(r_>\xi) \times \{\bar{J}_\theta(\zeta) - \frac{1}{\pi}\int_c d\zeta'\bar{J}_\theta(\zeta')\frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\}$$

where $r_>$ denotes the larger of the two quantities (r,a) and $r_<$ denotes the smaller of the two. The scattered magnetic field inside the cylinder is

$$(36)\qquad \bar{B}_z^s(\zeta) = \frac{\pi\mu_o a\xi}{2i} J_{|n|}(r\xi)H_{|n|}^{(1)\prime}(a\xi)\{\bar{J}_\theta(\zeta) - \frac{1}{\pi}\int_c d\zeta'\bar{J}_\theta(\zeta') \times \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\}, \quad r < a$$

Outside the cylinder the scattered magnetic field is

$$(37)\qquad \bar{B}_z^s(\zeta) = \frac{\pi\mu_o a\xi}{2i} J'_{|n|}(a\xi)H_{|n|}^{(1)}(r\xi)\{\bar{J}_\theta(\zeta) - \frac{1}{\pi}\int_c d\zeta'\bar{J}_\theta(\zeta)\frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\}, \quad r > a.$$

In the next section using the boundary condition relations we derive the current integral equations.

THE INDUCED SURFACE CURRENT INTEGRAL EQUATIONS

On the conducting surface, i.e. for $|z| > h$ and $r = a$, the tangential components of the total electric field (which we denote by using a superscript t) are required to vanish. Thus we have

$$E_z^t(z,a;n) = E_z^i(z,a;n) + E_z^s(z,a;n) = 0 \tag{38}$$

$$E_\theta^t(z,a;n) = E_\theta^s(z,a;n) = 0 \tag{39}$$

or

$$E_z^s(z,a;n) = - E_z^i(z,a;n) \tag{40}$$

$$E_\theta^s(z,a;n) = 0 \tag{41}$$

Also for the Fourier transforms we have

$$\overline{E}_z^s(\zeta,a) = - \overline{E}_z^i(\zeta,a) \tag{42}$$

$$\overline{E}_\theta^s(\zeta,a) = 0 \tag{43}$$

If we divide eq. (35) through by ξ^2 and use the convolution theorem we obtain with the help of eq. (42) for $|z|>h$

$$\begin{aligned} - E_z^i(a)/k^2 = & \int_{-\infty}^{\infty} dz' \, M_z(z-z') \, J_z(z') - \\ & - \frac{1}{\pi}\int_{-\infty}^{\infty} dz' M_z(z-z') F_\zeta^{-1}\left\{\int_c d\zeta' \, \overline{J}_z(\zeta') \, \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\right\} \\ & + \int_{-\infty}^{\infty} dz' \, M_{z\theta} \, (z-z') \, J_\theta(z') \\ & - \frac{1}{\pi}\int_{-\infty}^{\infty} dz' M_{z\theta}(z-z') F_\zeta^{-1}\left\{\int_c d\zeta' \, \overline{J}_\theta(\zeta') \, \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\right\} \end{aligned} \tag{44}$$

where we have used the fact that

$$\frac{1}{\xi^2} = \frac{1}{k^2 - \zeta^2} \tag{45}$$

and where we have also defined the kernel functions

(46) $$\overline{M}_z(\zeta) \equiv -\tfrac{1}{2}\pi(\omega\varepsilon_o)^{-1}a\, J_{|n|}(a\xi)H^{(1)}_{|n|}(a\xi) \equiv F_\zeta\{M_z(z)\}$$

(47) $$\overline{M}_{z\theta}(\zeta) \equiv -\frac{n\pi}{2}(\omega\varepsilon_o)^{-1}\zeta\xi^{-2}J_{|n|}(a\xi)H^{(1)}_{|n|}(a\xi) \equiv F_\zeta\{M_{z\theta}(z)\}.$$

Now we can write the inverse transforms in eq. (44) more explicitly as

(48) $$F_\zeta^{-1}\{\int_c d\zeta'\ \overline{J}(\zeta')\ \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\} = \frac{1}{4\pi i}\int_{c'} d\zeta'\ \overline{J}(\zeta')\int_c d\zeta\ \times$$

$$\times\ \{\exp[i(\zeta-\zeta')h-i\zeta z] - \exp[-i(\zeta-\zeta')h-i\zeta z]\}/(\zeta-\zeta')$$

where c and c' are appropriate contours. It is quite evident from this that the inverse transform is dependent, in an essential way, on the values of the exponentials. In Appendix B we present the details of the calculation of this inverse Fourier transformation. The results of the inversion will merely be stated here. We find for the θ and the z components of current:

(49) $$\frac{1}{\pi}F_\zeta^{-1}\{\int_{c'} d\zeta'\ \overline{J}(\zeta')\ \frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}\} = -\frac{1}{2}J(z')\ \mathrm{sgn}(|z'|-h)$$

Note that we can also write the simple relation

(50) $$\mathrm{sgn}(|z'|-h) \equiv \tfrac{1}{2}[\mathrm{sgn}(z'-h) + \mathrm{sgn}(-z'-h)]$$

Substituting from eq. (49) into eq. (44) we obtain

(51) $$-E_z^i(a)/k^2 = \int_{-\infty}^{\infty} dz' M_z(z-z')J_z(z')[1 + \tfrac{1}{2}\,\mathrm{sgn}(|z'|-h)] +$$

$$+\int_{-\infty}^{\infty} dz' M_{z\theta}(z-z')J_\theta(z')[1 + \tfrac{1}{2}\,\mathrm{sgn}(|z'|-h],\quad |z|>h.$$

With the help of eq. (28) we can rewrite eq. (51) so that the integration extends only over the conductor:

(52) $$-E_z^i(a)/k^2 = \int_{-\infty}^{-h} dz' M_z(z-z')J_z(z') + \int_h^{\infty} dz' M_z(z-z')J_z(z') +$$

$$+\int_{-\infty}^{-h} dz' M_{z\theta}(z-z')J_\theta(z') + \int_h^{\infty} dz' M_{z\theta}(z-z')J_\theta(z'),\quad |z|>h$$

This is one of the pair of coupled integral equations for the θ and z-components of the current density on the slotted conducting cylinder. The remaining integral equation can be derived starting from eq. (10) for the Fourier transformed θ-component of the scattered electric field. Substituting into eq. (10) the transformed magnetic field in eq. (37) setting r = a and using eqs. (42) and (43) we obtain

$$(53) \quad na^{-1}\xi^2\, \bar{E}_z^i(\zeta,a) = \bar{M}_\theta[\bar{J}_\theta(\zeta) - \frac{1}{\pi}\int_C d\zeta'\bar{J}_\theta(\zeta')\,\frac{\sin(\zeta-\zeta')h}{(\zeta-\zeta')}$$

where we have now defined the additional kernel function

$$(54) \quad \bar{M}_\theta(\zeta) \equiv -\frac{1}{2}\,\frac{\pi\mu_o\tau a}{\zeta}\, J'_{|n|}(a\xi)H'^{(1)}_{|n|}(a\xi)$$

Fourier inverting eq. (53) we use eq. (49) and the convolution theorem to get the following

$$(55) \quad n(k^2a)^{-1}\, E^i_z(a) =$$

$$= \int_{-\infty}^{\infty} dz' M_\theta(z-z')\, J_\theta(z')[1 + \frac{1}{2}\,\mathrm{sgn}(|z'|-h)], \quad |z|>h$$

Equivalently this can be written as an integration over the conducting cylinder as

$$(56) \quad \frac{n}{(k^2a)}\, E_z^i(a) = \int_{-\infty}^{-h} dz' M_\theta(z-z')\, J_\theta(z') +$$

$$+ \int_h^{\infty} dz' M_\theta(z-z')\, J_\theta(z'), \quad |z|>h.$$

This is the remaining half of the pair of coupled integral equations for the components of the current density. Note it is easily seen in eqs. (52) and (56) that the surface currents are excited by the incident radiation.

Now the incident fields are completely known physical quantities. Then we can find the scattered electric field via eq. (35) and z-component of the scattered magnetic field from whichever of eqs. (36) or (37) are appropriate if we first solve the pair of coupled integral equations (52) and (56) and obtain the surface current densities $J_\theta(z)$ and $J_z(z)$. Note that the surface current densities are of course the only unknowns in eqs. (52) and (56).

As a final result in this section we shall derive the Fourier inverted expressions for the scattered fields E_z^s and B_z^s within and outside of the cylinder in terms of the current densities on the conductor. Analogous to the definitions in eqs. (46) and (47) of the Fourier transformed kernel functions on the cylinder r = a we introduce the following definitions which are expressed compactly for both r > a and r < a;

(57) $$\overline{M}_z^o(r,\zeta;n) \equiv -\frac{1}{2}\pi(\omega\varepsilon_o)^{-1}a\xi^2 J_{|n|}(r_<\xi)H_{|n|}^{(1)}(r_>\xi)$$

(58) $$\overline{M}_{z\theta}^o(r,\zeta;n) \equiv -\frac{n\pi}{2}(\omega\varepsilon_o)^{-1}\zeta\, J_{|n|}(r_<\xi)H_{|n|}^{(1)}(r_>\xi)$$

when again r< denotes the smaller of (r,a) and r> the larger of the two. Then by eqs. (33), (34) and (35) we can write the Fourier transformed z-component of the scattered electric field as follows

(59) $$\overline{E}_z^s(r,\zeta;n) = \overline{J}_z(\zeta;n)\overline{M}_z^o(r,\zeta;n) + \overline{J}_\theta(\zeta;n)\overline{M}_{z\theta}^o(r,\zeta;n)$$

Similarly if we define the following Fourier transformed kernel function

(60) $$\overline{N}_{z\theta}^o(r,\zeta;n) \equiv \begin{cases} -\frac{1}{2}\, i\pi\mu_o a\xi J_{|n|}(r\xi)H_{|n|}^{(1)\prime}(a\xi), & r<a \\ -\frac{1}{2}\, i\pi\mu_o a\xi J'_{|n|}(a\xi)H_{|n|}^{(1)}(r\xi), & r>a \end{cases}$$

we can write the Fourier transformed z-component of the scattered magnetic field from eqs. (36) and (37) as

(61) $$\overline{B}_z^s(\zeta) \equiv \overline{J}_\theta(\zeta;n)\overline{N}_{z\theta}^0(r,\zeta;n)$$

The details of Fourier inverting the kernels in eqs. (59) and (61) may be found in Appendix C. We merely state the results here. For the z-component of the scattered electric field we have, using the convolution theorem,

(62) $$E_z^s(r,z) = \int_{-\infty}^{-h} dz' J_z(z')M_z^o(r,z-z') + \int_h^{\infty} dz' J_z(z')M_z^o(r,z-z') + \int_{-\infty}^{-h} dz' J_\theta(z')M_{z\theta}^o(r,z-z') + \int_h^{\infty} dz' J_\theta(z')M_{z\theta}^o(r,z-z')$$

where as is shown in Appendix C the inverted kernel functions are

(63) $$M_z^o(r,z) = \frac{1}{2}\zeta_o k^2 a\{ i \int_o^\infty dx J_{|n|}(ka[1+x^2]^{\frac{1}{2}}) J_{|n|}(kr[1+x^2]^{\frac{1}{2}})(1+x^2)\exp(-k|z|x) -$$
$$- \int_o^1 dx J_{|n|}(ka[1-x^2]^{\frac{1}{2}}) J_{|n|}(kr[1-x^2]^{\frac{1}{2}})(1-x^2)\exp(ik|z|x)\}$$

and

(64) $$M_{z\theta}^o(r,z) = \frac{1}{2}nk\zeta_o \operatorname{sgn} z\{\int_o^\infty dx\, x\, J_{|n|}(ka[1+x^2]^{\frac{1}{2}}) J_{|n|}(kr[1+x^2]^{\frac{1}{2}})\exp(-k|z|x) +$$
$$+ \int_o^1 dx\, x\, J_{|n|}(ka[1-x^2]^{\frac{1}{2}}) J_{|n|}(kr[1-x^2]^{\frac{1}{2}})\exp(ik|z|x)\}$$

In the same manner inverting the transformed z-component of the scattered magnetic field in eq. (61) we find

(65) $$B_z^s(r,z) = \int_{-\infty}^{-h} dz' J_\theta(z') N_{z\theta}^o(r,z-z') + \int_h^\infty dz' J_\theta(z') N_{z\theta}^o(r,z-z')$$

where the Fourier inverted transform of the kernel is given by

(66) $$N_{z\theta}^o(r,z) = \frac{1}{2}a\mu_o k^2\{\int_o^1 dx\sqrt{1-x^2} J_{|n|}(kr[1-x^2]^{\frac{1}{2}}) N'_{|n|}(ka[1-x^2]^{\frac{1}{2}}) \cdot$$
$$\cdot \exp(ikx|z|) - i\int_o^\infty dx\sqrt{1+x^2} J_{|n|}(kr[1+x^2]^{\frac{1}{2}}) N'_{|n|}(ka[1+x^2]^{\frac{1}{2}})\exp(-k|z|x)\}$$

for $r < a$

(67) $$N_{z\theta}^o(r,z) = \frac{1}{2}a\mu_o k^2\{\int_o^1 dx\sqrt{1-x^2} J'_{|n|}(ka[1-x^2]^{\frac{1}{2}}) N_{|n|}(kr[1-x^2]^{\frac{1}{2}})\exp(ik|z|x) -$$
$$- i\int_o^\infty dx\sqrt{1+x^2} J'_{|n|}(ka[1+x^2]^{\frac{1}{2}}) N_{|n|}(kr[1+x^2]^{\frac{1}{2}})\exp(-k|z|x)\}$$

for $r > a$

Using eqs. (65), (66) and (67), we wish to verify that the boundary conditions on the cylinder (eqs. (16) and (17)) as well as those off the cylinder (eqs. (18) and (19)) are satisfied. We find

(68) $$B_z^s(a+,z) - B_z^s(a-,z) = \int_{-\infty}^{\infty} dz'\, J_\theta(z')\{N_{z\theta}^o(a+,z-z') - N_{z\theta}^o(a-,z-z')\} \cdot$$

$$\cdot\ (1 + \frac{1}{2}\,\mathrm{sgn}(|z'| - h))$$

where we have explicitly retained the sgn function. The Fourier inverted transform of the kernel will be evaluated along the real axis in the ζ plane, since the function $N_{z\theta}^o(a+,z) - N_{z\theta}^o(a-,z)$ will be shown to be single valued and regular everywhere.

(69) $$N_{z\theta}^o(a+,z) - N_{z\theta}^o(a-,z) = -\frac{1}{4}\, i\, \mu_o a \cdot$$

$$\cdot \int_{-\infty}^{\infty} d\zeta\ \xi\{J'_{|n|}(\xi a) H_{|n|}^{(1)}(\xi a) - J_{|n|}(\xi a) H_{|n|}^{(1)'}(\xi a)\} e^{-i\zeta z}$$

Using the Wronskian relation

(70) $$J_{|n|}(y) H_{|n|}^{(1)'}(y) - J'_{|n|}(y) H_{|n|}^{(1)}(y) = \frac{2i}{\pi y}$$

eq. (69) becomes

(71) $$N_{z\theta}^o(a+,z) - N_{z\theta}^o(a-,z) = -\frac{\mu_o}{2\pi}\int_{-\infty}^{\infty} d\zeta\ e^{-i\zeta z}$$

$$= -\mu_o\ \delta(z)$$

and eq. (68) becomes

(72) $$B_z^s(a+,z) - B_z^s(a-,z) = -\mu_o \int_{-\infty}^{\infty} dz'\, J_\theta(z')\,(1 + \tfrac{1}{2}\mathrm{sgn}\{|z'| - h\})\,\delta(z - z')$$

$$= -\mu_o J_\theta(z)\,(1 + \frac{1}{2}\,\mathrm{sgn}\{|z'| - h\})$$

$$= \begin{cases} -\mu_o J_\theta(z) & , \quad |z| > h \\ 0 & , \quad |z| < h \end{cases}$$

Thus, we see the boundary conditions for the z component of the scattered magnetic field are satisfied everywhere.

Using eqs. (9) through (12), (62) and (65) for generating the n-th component of the corresponding Fourier series for each scattered field we can reconstitute the fields to obtain the desired results

$$E_p^s(r,z,\theta) = \sum_{n=0}^{\infty} E_p^s(r,z;n)\cos n\theta$$

$$B_p^s(r,z,\theta) = \sum_{n=0}^{\infty} B_p^s(r,z;n)\cos n\theta$$

where p represents the indices r,θ,z. Note at this point we have formally determined the scattered fields in terms of the current densities on the conductor. In the next section we derive an equivalent formal solution for the scattered fields which this time are expressed in terms of the electric field over the slot region of the cylinder r=a.

APERTURE FIELD INTEGRAL EQUATIONS

We shall introduce the tangential components of the electric field over the aperture; $\mathcal{E}_z(z;n)$ and $\mathcal{E}_\theta(z;n)$, through the definitions

(73) $$E_z^s(a,z;n) = \begin{cases} -E_z^i(a;n), & |z| > h \\ \mathcal{E}_z(z;n) - E_z^i(a;n), & |z| < h \end{cases}$$

(74) $$E_\theta^s(a,z;n) = \begin{cases} 0, & |z| > h \\ \mathcal{E}_\theta(z;n), & |z| < h \end{cases}$$

In this section we shall derive expressions for the current densities in terms of the aperture fields and also integral equations for $\mathcal{E}_z(z;n)$ and $\mathcal{E}_\theta(z;n)$. Using eqs. (10), (59) and (61) we can solve for $\bar{J}_\theta^z$ and $\bar{J}_z$ in terms of the fields $\bar{E}_z^s$ and $\bar{E}_\theta$. These relations are

(75) $$\bar{J}_\theta(\zeta) = -(i/\omega)[na^{-1}\zeta\bar{E}_z^s(a,\zeta) - \xi^2\bar{E}_\theta^s(a,\zeta)]/\left(\frac{\partial\bar{N}_{z\theta}}{\partial r}\right)_{r=a}$$

(76) $$\bar{J}_z(\zeta) = \bar{E}_z^s(a,\zeta)/\bar{M}_z^o(a,\zeta) + (i/\omega)[na^{-1}\zeta\bar{E}_z^s(a,\zeta) - \xi^2\bar{E}_\theta^s(a,\zeta)]/\left(\frac{\partial\bar{N}_{z\theta}^o}{\partial r}\right)_{r=a} \times$$
$$\times [\bar{M}_{z\theta}^o(a,\zeta)/\bar{M}_z^o(a,\zeta)]$$

Fourier inverting $\bar{J}_\theta$ in eq. (75) we get formally

(77) $$J_\theta(z) = \frac{1}{2\pi}\int_C d\zeta\,(-i/\omega)\,[na^{-1}\zeta\bar{E}_z^s(a,\zeta) - \xi^2\bar{E}_\theta^s(a,\zeta)]/\left(\frac{\partial\bar{N}_{z\theta}^o}{\partial r}\right)_{r=a} e^{-i\zeta z}$$

To make the notation more compact let us define the quantities

(78) $$\bar{K}_{z\theta}^{(1)}(a,\zeta) \equiv -(in/a\omega)\zeta/\left(\frac{\partial\bar{N}_{z\theta}^o}{\partial r}\right)_{r=a}$$

(79) $$\bar{L}_{z\theta}^{(1)}(a,\zeta) \equiv (i/\omega)\xi^2/\left(\frac{\partial\bar{N}_{z\theta}^o}{\partial r}\right)_{r=a}$$

In the new notation we can, upon application of the convolution theorem, write the azimuthal current component as

(80) $$J_\theta(z) = \int_{-\infty}^{\infty} dz'[K_{z\theta}^{(1)}(z-z')E_z^s(a,z') + L_{z\theta}^{(1)}(z-z')E_\theta^s(a,z')]$$

where of course

$$K^{(1)}_{z\theta}(z) \equiv F^{-1}_{\zeta}\left\{\overline{K}^{(1)}_{z\theta}(a,\zeta)\right\} \tag{81}$$

and

$$L^{(1)}_{z\theta}(z) + F^{-1}_{\zeta}\left\{\overline{L}^{(1)}_{z\theta}(a,\zeta)\right\} \tag{82}$$

The details of the Fourier inversions in eqs. (81) and (82) are given in Appendix D. We merely write the results obtained there in what follows. Thus we have more explicitly

$$K^{(1)}_{z\theta}(z) = -\frac{2n(\operatorname{sgn} z)}{\pi^2\mu_o\omega a^2}\left\{\int_o^1 dx \frac{x}{1-x^2}\frac{e^{ikx|z|}}{[J'^{2}_{|n|}(ka[1-x^2]^{\frac{1}{2}})+N'^{2}_{|n|}(ka[1-x^2]^{\frac{1}{2}})]} + \right.$$
$$\left. + \int_o^\infty dx \frac{x}{1+x^2}\frac{e^{-kx|z|}}{[J'^{2}_{|n|}(ka[1+x^2]^{\frac{1}{2}})+N'^{2}_{|n|}(ka[1+x^2]^{\frac{1}{2}})]}\right\} \tag{83}$$

and

$$L^{(1)}_{z\theta}(z) = -\frac{2k}{\pi^2\mu_o\omega a}\left\{\int_o^1 dx \frac{e^{ikx|z|}}{[J'^{2}_{|n|}(ka[1-x^2]^{\frac{1}{2}})+N'^{2}_{|n|}(ka[1-x^2]^{\frac{1}{2}})]} - \right.$$
$$\left. - i\int_{o-\infty}^{\infty} dx \frac{e^{-kx|z|}}{[J'^{2}_{|n|}(ka[1+x^2]^{\frac{1}{2}})+N'^{2}_{|n|}(ka[1+x^2]^{\frac{1}{2}})]}\right\} \tag{84}$$

Proceeding in the same manner we can Fourier invert eq. (76) to obtain the longitudinal current component:

$$J_z(z) = \int_{-\infty}^{\infty} dz'[K^{(2)}_{z\theta}(z-z')E^s_z(a,z') + L^{(2)}_{z\theta}(z-z')E^s_\theta(a,z')] \tag{85}$$

where we have introduced the notation

$$\overline{K}^{(2)}_{z\theta}(a,\zeta) \equiv [1/\overline{M}^o_z(a,\zeta)]\left\{1+(in/\omega a)\zeta\overline{M}^o_{z\theta}(a,\zeta)/\left(\frac{\partial\overline{N}^o_{z\theta}}{\partial r}\right)_{r=a}\right\} \tag{86}$$

and

$$\overline{L}^{(2)}_{z\theta}(a,\zeta) \equiv (-i/\omega)[\zeta^2\overline{M}^o_{z\theta}(a,\zeta)/\left(\frac{\partial\overline{N}^o_{z\theta}}{\partial r}\right)_{r=a}\overline{M}^o_z(a,\zeta)] \tag{87}$$

and of course the Fourier inverses of these are formally just

$$K_{z\theta}^{(2)}(z) = \mathcal{F}_{\zeta}^{-1}\left\{\overline{K}_{z\theta}^{(2)}(a,\zeta)\right\} \tag{88}$$

$$L_{z\theta}^{(2)}(z) = \mathcal{F}_{\zeta}^{-1}\left\{\overline{L}_{z\theta}^{(2)}(a,\zeta)\right\} \tag{89}$$

The explicit details of the Fourier inversions are given in Appendix D. We merely state the results here

$$\begin{aligned} K_{z\theta}^{(2)}(z) = -(2\omega\epsilon_o/\pi^2 ka)\Bigg\{&\int_o^1 dx\, e^{ikx|z|}\Big[1/(1-x^2)\left\{J_{|n|}^2(ka[1-x^2]^{1/2}) + N_{|n|}^2(ka[1-x^2]^{1/2})\right\} + (n^2/k^2a^2)\,x^2/(1-x^2)^2\left\{J_{|n|}'^2(ka[1-x^2]^{1/2}) + N_{|n|}'^2(ka[1-x^2]^{1/2})\right\}\Big] \\ &- i\int_o^\infty dx\, e^{-kx|z|}\Big[1/(1+x^2)\left\{J_{|n|}^2(ka[1+x^2]^{1/2}) + N_{|n|}^2(ka[1+x^2]^{1/2})\right\} + (n^2/k^2a^2)\,x^2/(1+x^2)^2\left\{J_{|n|}'^2(ka[1+x^2]^{1/2}) + N_{|n|}'^2(ka[1+x^2]^{1/2})\right\}\Big]\Bigg\} \end{aligned} \tag{90}$$

and

$$\begin{aligned} L_{z\theta}^{(2)}(z) = [2n(\mathrm{sgn}\,z)/(\pi^2\mu_o\omega a^2)]\Bigg\{&\int_o^1 dx[x/(1-x^2)]\,e^{ikx|z|}\Big/\left[J_{|n|}'^2(ka[1-x^2]^{1/2}) + N_{|n|}'^2(ka[1-x^2]^{1/2})\right] \\ &+ \int_o^\infty dx[x/(1+x^2)]\,e^{-kx|z|}\Big/\left[J_{|n|}'^2(ka[1+x^2]^{1/2}) + N_{|n|}'^2(ka[1+x^2]^{1/2})\right]\Bigg\} \end{aligned} \tag{91}$$

Introducing the definitions of the aperture fields as given in eqs. (73) and (74) into the expressions for the current components, i.e. eqs. (80) and (85) we obtain:

$$\begin{aligned} J_\theta(z) = -E_z^i(a)\int_{-\infty}^{\infty} dz'\,K_{z\theta}^{(1)}(z-z') + \int_{-h}^{h} dz'\,K_{z\theta}^{(1)}(z-z')\,\mathcal{E}_z(z') \\ + \int_{-h}^{h} dz'\,L_{z\theta}^{(1)}(z-z')\,\mathcal{E}_\theta(z') \end{aligned} \tag{92}$$

and

(93) $$J_z(z) = -E_z^i(a)\int_{-\infty}^{\infty} dz'K_{z\theta}^{(2)}(z-z') + \int_{-h}^{h} dz'K_{z\theta}^{(2)}(z-z')\mathcal{E}_z(z') + + \int_{-h}^{h} dz'L_{z\theta}^{(2)}(z-z')\mathcal{E}_\theta(z')$$

Eqs. (92) and (93) can be processed further to yield the components of the surface current densities more explicitly in terms of the tangential components of the aperture electric fields. Further we can simultaneously obtain a pair of integral equations for the aperture fields themselves. Thus we can combine eqs. (16) and (25) to obtain the relation

(94) $$J_z(z) = J_z(z)[1 + \frac{1}{2} \operatorname{sgn}(|z| - h)]$$

Similarly we have

(95) $$J_\theta(z) = J_\theta(z)[1 + \frac{1}{2} \operatorname{sgn}(|z| - h)]$$

Next using the convolution theorem and the definition of the delta function we can write for j = 1 or 2

$$E_z^i(a)\int_{-\infty}^{\infty} dz'K_{z\theta}^{(j)}(z-z') = E_z^i(a)\int_{-\infty}^{\infty} d\zeta\ \delta(\zeta)\overline{K}_{z\theta}^{(j)}(a,\zeta)e^{-i\zeta z}$$
$$= E_z^i(a)\overline{K}_{z\theta}^{(j)}(a,o)$$

If we recall eqs. (78) and (60) we observe that

(96) $$\overline{K}_{z\theta}^{(1)}(a,0) = -(in/a\omega)[\zeta/(\frac{\partial\overline{N}_{z\theta}^{o}}{\partial r})_{r=a}]_{\zeta=0}$$

Similarly by eqs. (86), (57), (58) and (60) we obtain at $\zeta = 0$

(97) $$\overline{K}_{z\theta}^{(2)}(a,o) = \frac{1}{\overline{M}_z^o(a,o)} = -\frac{2\omega\varepsilon_o}{\pi k^2 a}/[J_{|n|}(ka)H_{|n|}^{(1)}(ka)]$$

Then by eqs. (96) and (97) we can write more compactly

(98) $$E_z^i(a)\int_{-\infty}^{\infty} dz'K_{z\theta}^{(1)}(z-z') = \begin{cases} 0 & , j = 1 \\ \dfrac{-2\omega\varepsilon}{\pi k^2 a}\ \dfrac{E_z^i(a)}{J_{|n|}(ka)H_{|n|}^{(1)}(ka)} & , j = 2 \end{cases}$$

Using this and the information in eqs. (94) and (95) in the current equations we obtain for the azimuthal current component on the metal the relation

$$(99) \qquad J_\theta(z) = \int_{-h}^{h} dz' K_{z\theta}^{(1)}(z-z')\mathcal{E}_z(z') + \int_{-h}^{h} dz' L_{z\theta}^{(1)}(z-z')\mathcal{E}_\theta(z') \qquad |z| > h$$

and for the longitudinal current component on the metal

$$(100) \qquad J_z(z) = \frac{2\omega\epsilon_o}{\pi k^2 a} \frac{E_z^i(a)}{J_{|n|}(ka) H_{|n|}^{(1)}(ka)} + \int_{-h}^{h} dz' K_{z\theta}^{(2)}(z-z')\mathcal{E}_z(z') + \int_{-h}^{h} dz' L_{z\theta}^{(2)}(z-z')\mathcal{E}_\theta(z') \qquad |z| > h$$

Note at this point that eqs. (99) and (100) give the current densities on the seminfinite conducting cylinders in terms of the tangential components of the aperture electric fields. All other quantities in the pair of equations are known quantities.

For $|z| < h$ i.e. in the gap between the semi-infinite conducting cylinders we have the very significant pair of simultaneous integral equations

$$(101) \qquad \int_{-h}^{h} dz' K_{z\theta}^{(1)}(z-z')\mathcal{E}_z(z') + \int_{-h}^{h} dz' L_{z\theta}^{(1)}(z-z')\mathcal{E}_\theta(z') = 0$$

and

$$(102) \qquad \int_{-h}^{h} dz' K_{z\theta}^{(2)}(z-z')\mathcal{E}_z(z') + \int_{-h}^{h} dz' L_{z\theta}^{(2)}(z-z')\mathcal{E}_\theta(z') = -\frac{2\omega\epsilon_o}{\pi k^2 a} \frac{E_z^i(a)}{J_{|n|}(ka) H_{|n|}^{(1)}(ka)}$$

This pair of equations represents the heart of the solution to this problem. Clearly if we solve for the tangential aperture electric fields these will in turn give us the currents on the conductors as can be seen in eqs. (99) and (100). The scattered fields can also be determined from these aperture fields as we shall show in the next section. It should then be obvious that the scattering problem can be completely solved if we find the aperture fields of eqs. (101) and (102).

THE SCATTERED FIELDS EXPRESSED IN TERMS OF THE APERTURE FIELDS AS SOURCES

In this section we shall demonstrate, by explicit derivation that, similar to the induced currents the scattered fields can be expressed solely in terms of the tangential components of the electric field over the gap region between the semi-infinite conducting cylinders.

Equation (62) gives us E_z^s for any r value in terms of the currents on the conductor. Then we can write for the aperture region

$$(103)\qquad \mathcal{E}_z(z) = E_z^i(a) + \int_h^{\infty} dz' J_z(z') M_z^o(a,z-z') + \int_{-\infty}^{-h} dz' J_z(z') M_z^o(a,z-z') + \int_h^{\infty} dz' J_\theta(z') M_{z\theta}^o(a,z-z') + \int_{\infty}^{-h} dz' J_\theta(z') M_{z\theta}^o(a,z-z'); \quad |z|<h,\ r=a.$$

With the help of this we can rearrange eq. (76) to obtain

$$(104)\qquad \mathcal{E}_\theta(z) = \int_h^{\infty} dz' J_z(z') P_z^o(a,z-z') + \int_{-\infty}^{-h} dz' J_z(z') P_z^o(a,z-z') + \int_h^{\infty} dz' J_\theta(z') P_{z\theta}^o(a,z-z') + \int_{-\infty}^{-h} dz' J_\theta(z') P_{z\theta}^o(a,z-z'); \quad |z|<h,\ r=a.$$

where merely for the sake of compactness we have defined the Fourier transformed functions

$$(105)\qquad \overline{P}_z^o(a,\zeta) \equiv na^{-1}\zeta\xi^{-2}\,\overline{M}_z^o(a,\zeta)$$

and

$$(106)\qquad \overline{P}_{z\theta}^o(a,\zeta) \equiv \xi^{-2}\left\{na^{-1}\zeta\,\overline{M}_{z\theta}^o(a,\zeta) - i\omega\left[\frac{\partial}{\partial r}\overline{N}_{z\theta}^o(r,\zeta)\right]_{r=a}\right\}$$

We already know that eq. (62) is valid for arbitrary values of the variable r. From eqs. (10) and (61) we can obtain the following corresponding equation for $\overline{E}_\theta^s(r,\zeta)$ which is valid for any r

$$(107)\qquad \overline{E}_\theta^s(r,\zeta) = \xi^{-2}\left\{nr^{-1}\zeta\,\overline{M}_{z\theta}^o(r,\zeta) - i\omega\frac{\partial}{\partial r}\overline{N}_{z\theta}^o(r,\zeta)\right\}\overline{J}_\theta(\zeta) + nr^{-1}\zeta\,\xi^{-2}\,\overline{M}_z^o(r,\zeta)\overline{J}_z(\zeta)$$

Now if we substitute from eqs. (75) and (76) into eqs. (62) and (107) eliminating $\bar{J}_\theta$ and $\bar{J}_z$ thereby we obtain the following relations after some simple manipulation

(108) $$\bar{E}_z^s(r,\zeta) = \bar{E}_z^s(a,\zeta)\left\{\frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)} + \frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)}\left(\frac{i}{\omega}\right)\frac{na^{-1}\zeta}{\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}}\bar{M}_{z\theta}^o(a,\zeta) - \right.$$
$$\left. - \bar{M}_{z\theta}^o(r,\zeta)(i/\omega)na^{-1}\zeta\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}^{-1}\right\} +$$
$$+ \bar{E}_\theta^s(a,\zeta)\left\{\bar{M}_{z\theta}^o(r,\zeta) - \frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)}\bar{M}_{z\theta}^o(a,\zeta)\right\}(i\xi^2/\omega)\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}^{-1}$$

and

(109) $$\bar{E}_\theta^s(r,\zeta) = \bar{E}_\theta^s(a,\zeta)\left\{\left[nr^{-1}\zeta\bar{M}_{z\theta}^o(r,\zeta) - i\omega\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right](i/\omega)\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}^{-1} - \right.$$
$$\left. - nr^{-1}\zeta\frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)}\bar{M}_{z\theta}^o(a,\zeta)(i/\omega)\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}^{-1}\right\} +$$
$$+ \bar{E}_z^s(a,\zeta)\left\{nr^{-1}\zeta\xi^{-2}\frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)} + \right.$$
$$+ (i/\omega)n^2(ra)^{-1}\zeta^2\xi^{-2}\frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)}\bar{M}_{z\theta}^o(a,\zeta)\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}^{-1} -$$
$$\left. - (i/\omega)na^{-1}\zeta\xi^{-2}\times\left[nr^{-1}\zeta\bar{M}_{z\theta}^o(r,\zeta) - i\omega\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]\left[\frac{\partial\bar{N}_{z\theta}^o(r,\zeta)}{\partial r}\right]_{r=a}^{-1}\right\}.$$

To condense this rather awkward pair of expressions is rather a simple task. Observation of eqs. (57), (58) and (59) reveals the following simple relations

(110) $$\frac{\bar{M}_z^o(r,\zeta)}{\bar{M}_z^o(a,\zeta)} \equiv \frac{J_{|n|}(r_<\xi)H_{|n|}^{(1)}(r_>\xi)}{J_{|n|}(a\xi)\;H_{|n|}^{(1)}(a\xi)}$$

(111) $$\overline{M}^{o}_{z\theta}(r,\zeta)\Big/\left[\frac{\partial \overline{N}^{o}_{z\theta}(r,\zeta)}{\partial r}\right]_{r=a} \equiv \frac{n(\omega\varepsilon_o)^{-1}\zeta^{-1} J_{|n|}(r_<\xi)H^{(1)}_{|n|}(r_>\xi)}{i\mu_o a\xi^2 J'_{|n|}(a\xi)H^{(1)\prime}_{|n|}(a\xi)}$$

With the aid of these identities the coefficient of $\overline{E}^{s}_{z}(a,\zeta)$ in eq. (108) can be abbreviated by introducing some new notation, namely

(112) $$\overline{Q}_z(r,\zeta) \equiv \begin{cases} H^{(1)}_{|n|}(r\xi)/H^{(1)}_{|n|}(a\xi) & \text{for } r > a \\ J_{|n|}(r\xi)/J_{|n|}(a\xi) & \text{for } r < a \end{cases}$$

The coefficient of $\overline{E}^{s}_{\theta}(a,\zeta)$ in eq. (108) can be easily shown to be identically zero. Thus eq. (108) can be rewritten quite simply as

(113) $$\overline{E}^{s}_{z}(r,\zeta = \overline{Q}_z(r,\zeta)\overline{E}^{s}_{z}(a,\zeta)$$

In the same manner further introduction of the following new notation will simply result in the coefficient of $\overline{E}^{s}_{\theta}(a,\zeta)$ in eq. (109) assuming the form

$$\overline{R}_{z\theta}(r,\zeta) = \frac{J'_{|n|}(r_<\xi)H^{(1)\prime}_{|n|}(r_>\xi)}{J'_{|n|}(a\xi)\ H^{(1)\prime}_{|n|}(a\xi)}$$

or more explicitly

(114) $$\overline{R}_{z\theta}(r,\zeta) = \begin{cases} H^{(1)\prime}_{|n|}(r\xi)/H^{(1)\prime}_{|n|}(a\xi) & \text{for } r > a \\ J'_{|n|}(r\xi)/J'_{|n|}(a\xi) & \text{for } r < a \end{cases}$$

and the corresponding coefficient of $\overline{E}^{s}_{z}(a,\zeta)$ in eq. (109) can be written

(115) $$\overline{R}_z(r,\zeta) = \begin{cases} n\zeta\xi^{-2}\left[\dfrac{H^{(1)}_{|n|}(r\xi)}{rH^{(1)}_{|n|}(a\xi)} + \dfrac{H^{(1)\prime}_{|n|}(r\xi)}{aH^{(1)\prime}_{|n|}(a\xi)}\right] & \text{for } r > a \\ n\zeta\xi^{-2}\left[\dfrac{J_{|n|}(r\xi)}{rJ_{|n|}(a\xi)} + \dfrac{J'_{|n|}(r\xi)}{aJ'_{|n|}(a\xi)}\right] & \text{for } r < a \end{cases}$$

In the new notation eq. (109) is thus

(116) $$\overline{E}^{s}_{\theta}(r,\zeta) = \overline{R}_{z\theta}(r,\zeta)\overline{E}^{s}_{\theta}(a,\zeta) + \overline{R}_z(r,\zeta)\overline{E}^{s}_{z}(a,\zeta)$$

If we Fourier invert the scattered z-component of electric field in eq. (113) we have

$$(117) \qquad E_z^s(r,z) = \int_{-\infty}^{\infty} dz' E_z^s(a,z') Q_z(r,z-z')$$

where

$$(118a) \qquad Q_z(r,z) = \frac{1}{2\pi} \int_c d\zeta e^{-i\zeta} H_{|n|}^{(1)}(r\,\xi)/H_{|n|}^{(1)}(a\,\xi), \quad \text{for } r > a \text{ i.e. external to the cylinder}$$

and

$$(118b) \qquad Q_z(r,z) = \frac{1}{2\pi} \int_c d\zeta e^{-i\zeta} J_{|n|}(r\,\xi)/J_{|n|}(a\,\xi), \quad \text{for } r < a \text{ i.e. inside the cylinder}$$

Recalling eq. (73) the defining relation for the z-component of electric field over the aperture, we can rewrite eq. (117) in the form

$$(119) \qquad E_z^s(r,z) = - E_z^i(a) \int_{-\infty}^{\infty} dz' Q_z(r,z-z') + \int_{-h}^{h} dz' \mathcal{E}_z(z') Q_z(r,z-z')$$

However the first integral in this relation is the Fourier integral of Q_z evaluated at $\zeta \equiv 0$ i.e.

$$(120) \qquad \int_{-\infty}^{\infty} dz' Q_z(r,z-z') = \bar{Q}_z(r,\zeta \equiv 0)$$

which according to eq. (112) tells us we have for this integral

$$(121) \qquad \bar{Q}_z(r,\zeta\equiv 0) = \begin{cases} H_{|n|}^{(1)}(kr)/H_{|n|}^{(1)}(ka), & \text{for } r > a, \text{ i.e. external to the cylinder} \\ J_{|n|}(kr)/J_{|n|}(ka), & \text{for } r < a, \text{ i.e. inside the cylinder} \end{cases}$$

Combining eq. (121) with (119) we obtain the following expressions for the z-component of the scattered electric field

$$(122) \qquad E_z^s(r,z) = -E_z^i(a) H_{|n|}^{(1)}(kr)/H_{|n|}^{(1)}(ka) + \int_{-h}^{h} dz' \mathcal{E}_z(z') Q_z(r,z-z')$$

for $r > a$ i.e. external to the normally slotted cylinder

and

$$(123) \qquad E_z^s(r,z) = -E_z^i(a) J_{|n|}(kr)/J_{|n|}(ka) + \int_{-h}^{h} dz' \mathcal{E}_z(z') Q_z(r,z-z')$$

which holds for $r < a$ i.e. within the slotted cylinder.

It should be brought out here for emphasis that the z-component of scattered electric field of eqs. (122) and (123) contain only the longitudinal aperture electric field as an unknown quantity. Thus solving the pair of simultaneous integral eqs. (101) and (102) for the tangential aperture electric fields will also give us the scattered electric field $E_z^s(r,z)$ everywhere. The kernel function $Q_z(r,z)$ which appears in eqs. (122) and (123) has been derived explicitly in Appendix E and is merely stated here for completeness:

For r > a

$$(124) \qquad Q_z(r,z) = \frac{k}{\pi}\Big[i\int_0^1 dx\, e^{ikx|z|}\Big\{J_{|n|}(ka\sqrt{1-x^2})N_{|n|}(kr\sqrt{1-x^2}) - J_{|n|}(kr\sqrt{1-x^2})N_{|n|}(ka\sqrt{1-x^2})\Big\}\Big/\Big\{J_{|n|}^2(ka\sqrt{1-x^2})+N_{|n|}^2(ka\sqrt{1-x^2})\Big\} + \int_0^\infty dx\, e^{-kx|z|}\Big\{J_{|n|}(ka\sqrt{1+x^2})N_{|n|}(kr\sqrt{1+x^2}) - J_{|n|}(kr\sqrt{1+x^2})N_{|n|}(kr\sqrt{1+x^2})\Big\}\Big/\Big\{J_{|n|}^2(ka\sqrt{1+x^2})+N_{|n|}^2(ka\sqrt{1+x^2})\Big\}\Big]$$

Whereas inside the normally slotted cylinder, r < a we have

$$(125) \qquad Q_z(r,z) = \frac{k}{\pi}\Big\{P\int_0^1 dx\cos kxz\,\frac{J_{|n|}(kr\sqrt{1-x^2})}{J_{|n|}(ka\sqrt{1-x^2})} + \int_1^\infty dx\cos kxz\,\frac{I_{|n|}(kr\sqrt{x^2-1})}{I_{|n|}(ka\sqrt{x^2-1})} + \frac{\pi i}{ka}\sum_\alpha \frac{\cos(kx_\alpha z)\sqrt{1-x_\alpha^2}\,J_{|n|}(kr\sqrt{1-x_\alpha^2})}{x_\alpha J'_{|n|}(ka\sqrt{1-x_\alpha^2})}\Big\}$$

where we have used the earlier notation $x \equiv \zeta/k$. Furthermore in eq. (125) we refer in the first term to the Cauchy principal value, in the second integral $I_{|n|}$ is the modified Bessel function and finally the sum in the last term is over the zeroes x_α of $J_{|n|}$ for $-1 < x < 1$ on the real axis. At this point note that we have expressed the rather important field $E_z^s(r,z)$ in terms of the tangential aperture fields.

As a final manipulation in this paper we shall show the derivation of the scattered field $E_\theta^s(r,z)$ for all values of the radial variable r. Again utilizing the defining relation, eq. (73) for the tangential aperture fields we can Fourier invert eq. (116) to the following expression:

(126) $$E_\theta^s(r,z) = - E_z^i(a)\int_{-\infty}^{\infty} dz' R_z(r,z-z') + \int_{-h}^{h} dz' \mathcal{E}_z(z') R_z(r,z-z') + + \int_{-h}^{h} dz' \mathcal{E}_\theta(z') R_{z\theta}(r,z-z')$$

Similar to the treatment of eqs. (120) and (121) we find for the first integral

(127) $$\int_{-\infty}^{\infty} dz' R_z(r,z-z') = \overline{R}_z(r,\zeta \equiv 0)$$

and consequently eq. (115) tells us quite simply that this first integral vanishes leaving us with

(128) $$E_\theta^s(r,z) = \int_{-h}^{h} dz' \mathcal{E}_\theta(z') R_{z\theta}(r,z-z') + \int_{-h}^{h} dz' \mathcal{E}_z(z') R_z(r,z-z')$$

for all values of the radius. Thus we have also expressed E_θ^s completely in terms of the tangential aperture electric fields. The remaining kernel functions appearing in eq. (128) are completely known. Their derivations are shown explicitly in Appendix E. For completeness we merely repeat them here.

Outside the normally slotted cylinder, $r > a$, the kernel functions are

(129) $$R_{z\theta}(r,z) = \frac{k}{\pi}\Big[i\int_0^1 dx\, e^{ikx|z|} \Big\{ J'_{|n|}(ka\sqrt{1-x^2}) N_{|n|}(kr\sqrt{1-x^2}) - - J_{|n|}(kr\sqrt{1-x^2}) N'_{|n|}(ka\sqrt{1-x^2}) \Big\} \Big/ \Big\{ J'^2_{|n|}(ka\sqrt{1-x^2}) + N'^2_{|n|}(ka\sqrt{1-x^2}) \Big\} + + \int_0^\infty dx\, e^{-kx|z|} \Big\{ \frac{J'_{|n|}(ka\sqrt{1+x^2}) N_{|n|}(kr\sqrt{1+x^2}) - J_{|n|}(kr\sqrt{1+x^2}) N'_{|n|}(ka\sqrt{1+x^2})}{J'^2_{|n|}(ka\sqrt{1+x^2}) + N'^2_{|n|}(ka\sqrt{1+x^2})} \Big\} \Big]$$

and

(130) $$R_z(r,z) = -\frac{in(\operatorname{sgn} z)}{\pi}\Big[\int_0^1 dx \frac{x}{1-x^2} e^{ikx|z|} \Big\{ \big[J_{|n|}(ka\sqrt{1-x^2}) N_{|n|}(kr\sqrt{1-x^2}) - - J_{|n|}(kr\sqrt{1-x^2}) N_{|n|}(ka\sqrt{1-x^2}) \big] \Big/ r\big[J^2_{|n|}(ka\sqrt{1-x^2}) + N^2_{|n|}(ka\sqrt{1-x^2}) \big] + + \big[J'_{|n|}(ka\sqrt{1-x^2}) N'_{|n|}(kr\sqrt{1-x^2}) - J'_{|n|}(kr\sqrt{1-x^2}) N'_{|n|}(ka\sqrt{1-x^2}) \big] \Big/ a\big[J'^2_{|n|}(ka\sqrt{1-x^2}) + + N'^2_{|n|}(ka\sqrt{1-x^2}) \big] \Big\} + \int_0^\infty dx \frac{x}{1+x^2} e^{-kx|z|} \Big\{ \big[J_{|n|}(ka\sqrt{1+x^2}) N_{|n|}(kr\sqrt{1+x^2}) - - J_{|n|}(kr\sqrt{1+x^2}) N_{|n|}(ka\sqrt{1+x^2}) \big] \Big/ r\big[J^2_{|n|}(ka\sqrt{1+x^2}) + N^2_{|n|}(ka\sqrt{1+x^2}) \big] + + \big[J'_{|n|}(ka\sqrt{1+x^2}) N'_{|n|}(kr\sqrt{1+x^2}) - J'_{|n|}(kr\sqrt{1+x^2}) N'_{|n|}(ka\sqrt{1+x^2}) \big] \Big/ a\big[J'^2_{|n|}(ka\sqrt{1+x^2}) + + N'^2_{|n|}(ka\sqrt{1+x^2}) \big] \Big\} \Big] - in\, r^{-1} e^{-ikz} (r/a)^{|n|} (\operatorname{sgn} z).$$

Within the normally slotted cylinder, r < a, the kernel functions in the field integral equation (128) are

(131) $$R_{z\theta}(r,z) = \frac{k}{\pi}\left[P\int_0^1 dx \cos kxz \left(\frac{J'_{|n|}(kr\sqrt{1-x^2})}{J'_{|n|}(ka\sqrt{1-x^2})} \right) + \right.$$
$$+ \int_1^\infty dx \cos kxz \left(\frac{I'_{|n|}(kr\sqrt{x^2-1})}{I'_{|n|}(ka\sqrt{x^2-1})} \right) +$$
$$\left. + \frac{\pi i}{ka} \sum_\alpha \frac{\cos(kx_\alpha z)\sqrt{1-x_\alpha^2}\; J'_{|n|}(kr\sqrt{1-x_\alpha^2})}{x_\alpha J''_{|n|}(ka\sqrt{1-x_\alpha^2})} \right]$$

and

(132) $$R_z(r,z) = \frac{-in}{\pi}\left[P\int_0^1 dx \left(\frac{x}{1-x^2}\right) \left(\frac{J_{|n|}(kr\sqrt{1-x^2})}{rJ_{|n|}(ka\sqrt{1-x^2})} + \frac{J'_{|n|}(kr\sqrt{1-x^2})}{aJ'_{|n|}(ka\sqrt{1-x^2})} \right) \sin kxz \right.$$
$$- \int_1^\infty dx \left(\frac{x}{x^2-1}\right) \left(\frac{I_{|n|}(kr\sqrt{x^2-1})}{rI_{|n|}(ka\sqrt{x^2-1})} + \frac{I'_{|n|}(kr\sqrt{x^2-1})}{aI'_{|n|}(ka\sqrt{x^2-1})} \right) \sin kxz +$$
$$+ \pi i \left\{ \frac{\sin kz}{r} \left(\frac{r}{a}\right)^{|n|} + \frac{1}{kra} \sum_\alpha \frac{\sin kx_\alpha z J_{|n|}(kr\sqrt{1-x_\alpha^2})}{x_\alpha \sqrt{1-x_\alpha^2}\; J'_{|n|}(ka\sqrt{1-x_\alpha^2})} + \right.$$
$$\left.\left. + \frac{1}{ka^2} \sum_\beta \frac{\sin kx_\beta z J'_{|n|}(kr\sqrt{1-x_\beta^2})}{x_\beta \sqrt{1-x_\beta^2}\; J''_{|n|}(ka\sqrt{1-x_\beta^2})} \right\} \right]$$

In eq. (132) x_β is a zero of $J'_{|n|}(ka\sqrt{1-x^2})$ on the real axis for $-1 < x < 1$.

In summary we have expressed J_θ and J_z on the conductor as well as E_z^s and E_θ^s external to and internal to the normally slotted cylinder completely in terms of the tangential components of the electric field in the gap between the semi-infinite cylinders. This information is contained in eqs. (99) and (100) for the current densities and eqs. (122) and (126) for the scattered field E_θ^s and E_z^s. Finally we have formulated the integral eqs. (101) and (102) which must be solved to obtain the tangential electric fields themselves over the aperture.

Of course we still must remember to reconstruct the whole solution of which we have only one general form in the Fourier series in θ.

CONCLUSIONS AND DISCUSSION

In concluding this formal report a number of points require clarification and others on the other hand need some emphasis. To begin with we should note that we have formulated the scattering problem quite rigorously. Not only has the problem been formulated, as is conventionally done, in terms of the induced currents on the slotted cylinder so that all the field components are derivable in terms of these currents, we have also derived the surface currents themselves. Eqs. (52) and (56) are the later relations referred to. In addition to this conventional approach to the situation via the induced surface currents we have recast the problem in a completely equivalent but different representation. In this case we have formulated the physical quantities of interest namely, the induced surface currents on the conductor and the scattered fields inside and outside the conductor in terms of the tangential components of the electric field over the aperture or gap between the semi-infinite conducting cylinders. Of course we have also derived the simultaneous integral equations which must be solved to yield these aperture fields, i.e. eqs. (101) and (102). It should be clear that this method could be a more convenient approach when one wishes to check against experimental measurements as only electric field measurements over the aperture need be made.

It also should be noted that we chose to explicitly display only the fields $E_z^s(r,z)$ and $E_\theta^s(r,z)$ in terms of the aperture fields. There is nothing of special significance about this choice although it would appear that $B_z^s(r,z)$ would have been a better candidate in the derivations. We shall be satisfied with merely pointing out that in a subsequent report involving the actual solutions of eqs. (101) and (102) we shall explicitly derive the scattered magnetic field. In that report we shall also solve for all fields using the solutions obtained for $\mathcal{E}_z(z)$ and $\mathcal{E}_\theta(z)$ over the gap. Furthermore all the related problems alluded to in the introductory remarks in this report will be elaborated on in considerable detail in ensuing reports. A preliminary paper[15] on one experimental aspect has already been presented namely measurements for circular cylinders for a large range of radius to wavelength ratio and also a broad range of length to wavelength ratio. The details of this preliminary paper will be spelled out more elaborately in a following report in this series.

APPENDIX A

DERIVATION OF THE NORMALLY-SLOTTED CYLINDER GREEN FUNCTIONS

Solution of the inhomogeneous differential equation, which we repeat here for convenience, will be effected by finding the corresponding appropriate Green's functions. Thus we have

$$(20),(21)\quad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\xi^2-\frac{n^2}{r^2}\right)\begin{Bmatrix}\overline{E}_z^s(r,\zeta,n)\\ \overline{B}_z^s(r,\zeta,n)\end{Bmatrix}=\begin{Bmatrix}-i\xi^2\delta(r-a)\,(\omega\epsilon_o r)^{-1}\overline{J}_z\,(\zeta,n)\\ -\xi^2\delta(r-a)\,(\omega^2\epsilon_o r)^{-1}\overline{J}_\theta\,(\zeta,n)\end{Bmatrix}$$

We consider the solution for the transformed magnetic field first since this can be carried through more easily. What we are looking for formally is a Green's function $G(y,y')$ that satisfies the inhomogeneous

$$(A1)\qquad L\,G(y,y') = -\,\delta(y-y')/y$$

where L is some Sturm-Liouville differential operator. The Green's function we are determining is constrained by the boundary condition on the magnetic field i.e.

$$(A2)\qquad \overline{B}_z^s(a_+,\zeta,n) - \overline{B}_z^s(a_-,\zeta,n) = -\,\mu_o\overline{J}_\theta(\zeta,n)$$

Now let $\psi_1(r,\xi,n)$ and $\psi_2(r,\xi,n)$ be two linearly independent solutions of Bessel's equation:

$$(A3a)\qquad \psi_1(r,\xi,n) = \begin{cases} C_1(\xi)\ H_{|n|}^{(1)\,'}(\xi r), & r < a \\ C_1(\xi)\ H_{|n|}^{(1)}(\xi r), & r > a \end{cases}$$

$$\text{(A3b)} \qquad \psi_2(r,\xi;n) = \begin{Bmatrix} C_2(\xi)\ J_{|n|}(\xi r), & r < a \\ C_2(\xi)\ J'_{|n|}(\xi r), & r > a \end{Bmatrix}$$

where the prime on the Bessel functions denotes differentiation with respect to the entire argument of the function. These two functions will be used to construct the desired Green's function. The quantities $C_1(\xi)$ and $C_2(\xi)$ shall be determined below from the boundary condition stated in eq. (A2). From Morse and Feshbach[15] we are then able to write down directly the Green's function which is identical with the magnetic field itself

$$\text{(A4)} \quad G_1(r,a) \equiv \bar{B}_z^s(r,\zeta;n) = \psi_1(r,\xi;n)\int_0^r dr'\psi_2(r',\xi;n)\delta(r'-a)/$$

$$[r'\cdot W(\psi_1,\psi_2)] + \psi_2(r,\xi;n)\int_r^\infty dr'\psi_1(r',\xi;n)\delta(r'-a)/[r'W(\psi_1,\psi_2)]$$

where $W(\psi_1,\psi_2)$ is the Wronskian of ψ_1 and ψ_2;

$$\text{(A5)} \qquad W(\psi_1,\psi_2) = 2i/\pi\xi a.$$

In eq. (A4) the first integral vanishes for $r < a$ and the second integral for $r > a$. Using eqs. (A3) and (A5) we can reduce the Fourier transformed magnetic field in eq. (A4) to

$$\text{(A6)} \qquad \bar{B}_z^s(r,\zeta,n) = \begin{cases} \tfrac{1}{2}i\pi\xi C_1(\xi)C_2(\xi)J_{|n|}(r\xi)H_{|n|}^{(1)'}(a\xi), & r < a \\ \tfrac{1}{2}i\pi\xi C_1(\xi)C_2(\xi)J'_{|n|}(a\xi)H_{|n|}^{(1)}(r\xi), & r > a \end{cases}$$

Next we evaluate $C_1(\xi)C_2(\xi)$ from eq. (A2). Substituting from eq. (A6) for $\bar{B}_z^s(a_\pm,\zeta;n)$ we obtain

$$\bar{B}_z^s(a_+,\zeta,n) - \bar{B}_z^s(a_-,\zeta,n) = -\frac{i\pi}{2}\xi C_1(\xi)C_2(\xi)\{J_{|n|}(a\xi)H_{|n|}^{(1)'}(a\xi) -$$

$$-J'_{|n|}(a\xi)H_{|n|}^{(1)}(a\xi)\} = -\frac{i\pi}{2}\xi\, C_1(\xi)C_2(\xi)\{2i/\pi\xi a\}$$

$$= +\ C_1(\xi)C_2(\xi)/a$$

where we have used the Wronskian of $J_{|n|}(a\xi)$ and $H_{|n|}^{(1)}(a\xi)$.

Clearly then we must have

$$\text{(A7)} \qquad C_1(\xi)C_2(\xi) = -\mu_o a\, \overline{J}_\theta(\zeta,n)$$

With this result we have the Fourier transformed field $\overline{B}_z^s(r,\zeta;n)$ completely determined and the results are given in eqs. (33) and (34).

What remains now is to determine the Fourier transformed electric field $\overline{E}_z^s(r,\zeta;n)$. This field must satisfy the boundary condition

$$\text{(A8)} \qquad \overline{E}_z^s(a_+,\zeta;n) = \overline{E}_z^s(a_-,\zeta;n)$$

If this were the sole boundary condition to be satisfied we could proceed just as we did immediately above for $\overline{B}_z^s$ and write down the answer at once. In this case however the situation is somewhat different. This is readily seen if we Fourier transform Maxwell's equations on circular cylindrical coordinates. One of the resulting equations of this process is

$$\text{(A9)} \qquad \overline{B}_\theta^s(r,\zeta;n) = \xi^{-2}\left[i\omega\mu_o\epsilon_o \frac{\partial \overline{E}_z^s(r,\zeta;n)}{\partial r} + nr^{-1}\zeta \overline{B}_z^s(r,\zeta;n)\right]$$

where this component of magnetic field satisfies the boundary condition

$$\text{(A10)} \qquad \overline{B}_\theta^s(a_+,\zeta;n) - \overline{B}_\theta^s(a_-,\zeta;n) = \mu_o \overline{J}_z(\zeta,n).$$

Since both $\overline{B}_\theta^s$ and $\overline{B}_z^s$ are discontinuous across the boundary at $r = a$ we conclude that $\dfrac{\partial \overline{E}_z^s}{\partial r}$ must also be discontinuous as we cross $r = a$. From eqs. (A2), (A9) and (A10) we obtain for the discontinuity in the radial component of the Fourier transformed electric field

$$\text{(A11)} \qquad \frac{\partial \overline{E}_z^s(a_+,\zeta;n)}{\partial r} - \frac{\partial \overline{E}_z^s(a_-,\zeta;n)}{\partial r} = -in(a\omega\epsilon_o)^{-1}\zeta \overline{J}_\theta(\zeta,n) - $$
$$-i(\omega\epsilon_o)^{-1}\xi^2 \overline{J}_z(\zeta,n)$$

We see from this that as we should have anticipated the discontinuity depends on both components of the current density. Now let us construct the corresponding Green's function for $\bar{E}_z^s$ from the following linearly independent pair of solutions of Bessel's equation:

(A12a) $$\psi_3(r,\xi;n) = C_3(\xi)J_{|n|}(\xi r)$$

(A12b) $$\psi_4(r,\xi;n) = C_4(\xi)H_{|n|}^{(1)}(\xi r)$$

These give rise to the following Green's function which also is identical to the field $\bar{E}_z^s$.

(A13) $$G_2(r,a) = \bar{E}_z^s(r,\zeta;n) = \psi_4(r,\xi;n)\int_0^r dr'\psi_3(r',\xi,n)\delta(r'-a)/[r'W(\psi_3,\psi_4)]$$

$$+ \psi_3(r,\xi;n)\int_r^\infty dr'\psi_4(r',\xi;n)\delta(r'-a)/[r'W(\psi_3,\psi_4)]$$

This reduces as in the earlier case to

(A14) $$\bar{E}_z^s\ (r,\zeta;n) = \begin{cases} \tfrac{1}{2}i\pi\xi C_3(\xi)C_4(\xi)J_{|n|}(\xi r)H_{|n|}^{(1)}(\xi a), & r < a \\ \tfrac{1}{2}i\pi\xi C_3(\xi)C_4(\xi)J_{|n|}(\xi a)H_{|n|}^{(1)}(\xi r), & r > a \end{cases}$$

Differentiating with respect to r and substituting the result into eq. (A11) we obtain with the help of the Wronskian for $J_{|n|}$ and $H_{|n|}^{(1)}$

(A15) $$C_3(\xi)C_4(\xi) = in(\omega\varepsilon_o)^{-1}\zeta\xi^{-1}\bar{J}_\theta(\zeta,n) + ia(\omega\varepsilon_o)^{-1}\xi\bar{J}_z(\zeta,n)$$

With this result $\bar{E}_z^s(r,\zeta;n)$ is completely determined in terms of the current densities $\bar{J}_\theta(\zeta,n)$ and $\bar{J}_z(\zeta,n)$. The final results are

(A16) $$\bar{E}_z^s(r,\zeta;n) = \begin{cases} -\frac{\pi}{2}(\omega\varepsilon_o)^{-1}[n\zeta\bar{J}_\theta(\zeta,n)+a\xi^2\bar{J}_z(\zeta,n)]J_{|n|}(\xi r)H_{|n|}^{(1)}(\xi a), r<a \\ -\frac{\pi}{2}(\omega\varepsilon_o)^{-1}[n\zeta\bar{J}_\theta(\zeta,n)+a\xi^2\bar{J}_z(\zeta,n)]J_{|n|}(\xi a)H_{|n|}^{(1)}(\xi r), r>a \end{cases}$$

APPENDIX B

THE INVERSE FOURIER TRANSFORM

$$F_{\zeta}^{-1}\{\int_C d\zeta'\bar{J}(\zeta')(\zeta-\zeta')^{-1}\sin(\zeta-\zeta')h\}$$

We showed in eq. (48) that we needed to evaluate the inverse Fourier transform

(B1) $$F_{\zeta}^{-1}\{\int_C d\zeta'\bar{J}(\zeta')(\zeta-\zeta')^{-1}\sin(\zeta-\zeta')h = \frac{1}{4\pi i}\int_C d\zeta'\bar{J}(\zeta')\int_C d\zeta \cdot$$

$$\cdot \{\exp[i\zeta(h-z)-i\zeta'h] - \exp[-i\zeta(h+z)+i\zeta'h]\}/(\zeta-\zeta')$$

For the first exponential we have two possible situations to consider namely

(i) $z < h$

(ii) $z > h$

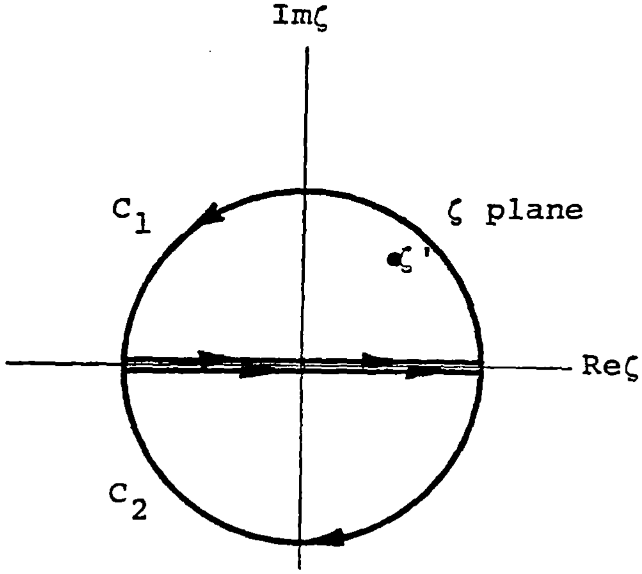
Likewise for the second exponential we have the two cases to consider

(iii) $z > -h$
(iv) $z < -h$.

Note that cases (i) and (iii) refer to the aperture region whereas cases (ii) and (iv) correspond to the conductor. Considering z values over the aperture region we observe that the first exponential gives rise to damping in the upper half ζ-plane. For this situation we must then close the contour in the upper half ζ-plane. This is accomplished by choosing the contour C_1 as shown in Figure 2(a). Note that in Figure 2, ζ' is arbitrarily assumed to be a point in the upper half plane. For this the first exponential then gives

(B2) $$\int_{C_1} d\zeta\, e^{i\zeta(h-z)}e^{-i\zeta'h}/(\zeta-\zeta') = 2\pi i\, e^{-i\zeta'z}$$

Damping occurs over the lower half ζ-plane for the second exponential in eq. (B1). Hence we integrate over the contour C_2 as shown in Figure 2(a). This gives no contribution since we have assumed



2(a) Aperture Region; $h - z > 0$, $h + z > 0$

2(b) Conductor Region; $h - z < 0$, $h + z < 0$

Figure 2. The Contours for the Inverse Fourier Transform Integrals:
$$\int_C d\zeta \; e^{\pm i\zeta(h\mp z)} (\zeta-\zeta')^{-1}$$

ζ' to be a point in the upper half ζ-plane. Thus

$$\text{(B3)} \qquad \int_{C_2} d\zeta \; e^{-i\zeta(h+z)} \; e^{i\zeta' h}/(\zeta-\zeta') = 0$$

since there are no poles in the lower half ζ-plane. If we had assumed ζ' to be a point in the lower half ζ-plane the first exponential would have given zero and the second exponential in (B1) would yield $2\pi i \; e^{-i\zeta' z}$. This clearly gives the same result. If ζ' lies on the real axis the contour for the first exponential would be the real axis from $\zeta' = -\infty$ to $\zeta' = +\infty$ closed in the upper half plane by a semi circle of infinite radius. We would then obtain the result from the Cauchy principal value, i.e.

$$\text{(B4)} \qquad \int_C d\zeta \; e^{+i\zeta(h-z)-i\zeta' h}/(\zeta-\zeta') = \pi i e^{-i\zeta' z}$$

The second exponential is taken care of by a contour along the entire real ζ-axis from $-\infty$ to $+\infty$ closed by a semi-circle of infinite radius in the lower half-plane. Again the Cauchy principal value gives

(B5) $$-\int_C d\zeta\ e^{-i\zeta(h+z)}\ e^{i\zeta' h}/(\zeta-\zeta') = \pi i e^{-i\zeta' h}$$

Thus we see that for values of z in the aperture region we obtain the same result from the integrations in the ζ-plane for all ζ', namely $2\pi i \exp(-i\zeta' h)$.

Finally we must consider values of z corresponding to the conductor. This means the remaining two situations i.e. cases (ii) and (iv). Now the first exponential in eq. (B1) possesses damping in the lower half ζ-plane and the second exponential in the upper half-plane. Thus we use contour C_3 for the first exponential and contour C_4 for the second where Figure 2(b) illustrates the contours. The result we obtain is

(B6) $$\int_{C_3} d\zeta\ \frac{e^{i\zeta(h-z)} e^{-i\zeta' h}}{(\zeta-\zeta')} - \int_{C_4} d\zeta \frac{e^{-i\zeta(h+z)} e^{+i\zeta h}}{(\zeta-\zeta')} = -2\pi i e^{-i\zeta' z}$$

Again if ζ' is on the real ζ axis the same choice of contours as for z in the aperture will give the same result as in eq. (B6). In summary then we have

(B7) $$\int d\zeta \{\exp[i\zeta(h-z)-i\zeta' h] - \exp[-i\zeta(h+z)+i\zeta' h]\}/(\zeta-\zeta') =$$

$$= \begin{cases} 2\pi i \exp(-i\zeta' z) & \text{for } |z| < h \\ -\ 2\pi i \exp(-i\zeta' z) & \text{for } |z| > h \end{cases}$$

If we use the relation in eq. (47) we find the result we are seeking, namely the Fourier inversion

(B8) $$\frac{1}{\pi} F_{\zeta}^{-1}\{\int_C d\zeta' \tilde{J}(\zeta')(\zeta-\zeta')^{-1}\sin(\zeta-\zeta')h\} = -\frac{1}{2}\text{sgn}(|z|-h)\frac{1}{2\pi}\int_C d\zeta' e^{-i\zeta' z}\tilde{J}(\zeta')$$

$$= -\frac{1}{2}\text{sgn}(|z|-h)\ J(z).$$

APPENDIX C

DERIVATION OF THE KERNEL FUNCTIONS $M_z^o(r,z;n)$, $M_{z\theta}^o(r,z;n)$ and $N_{z\theta}^o(r,z;n)$

The scattered field integral equation, i.e., eq. (62) for $E_z^s(r,z;n)$ and eq. (65) for $B_z^s(r,z;n)$, in terms of the surface current densities contain the kernel functions $M_z^o(r,z;n)$, $M_{z\theta}^o(r,z;n)$ and $N_{z\theta}^o(r,z;n)$. To obtain eqs. (62) and (65) more explicitly then requires that we find the inverse Fourier transforms of $\overline{M}_z^o(r,\zeta;n)$, $\overline{M}_{z\theta}^o(r,\zeta;n)$ and $\overline{N}_{z\theta}^o(r,\zeta;n)$. In this appendix we present the details of determining these inverse transforms. Consider first the kernel M_z^o. Formally we have

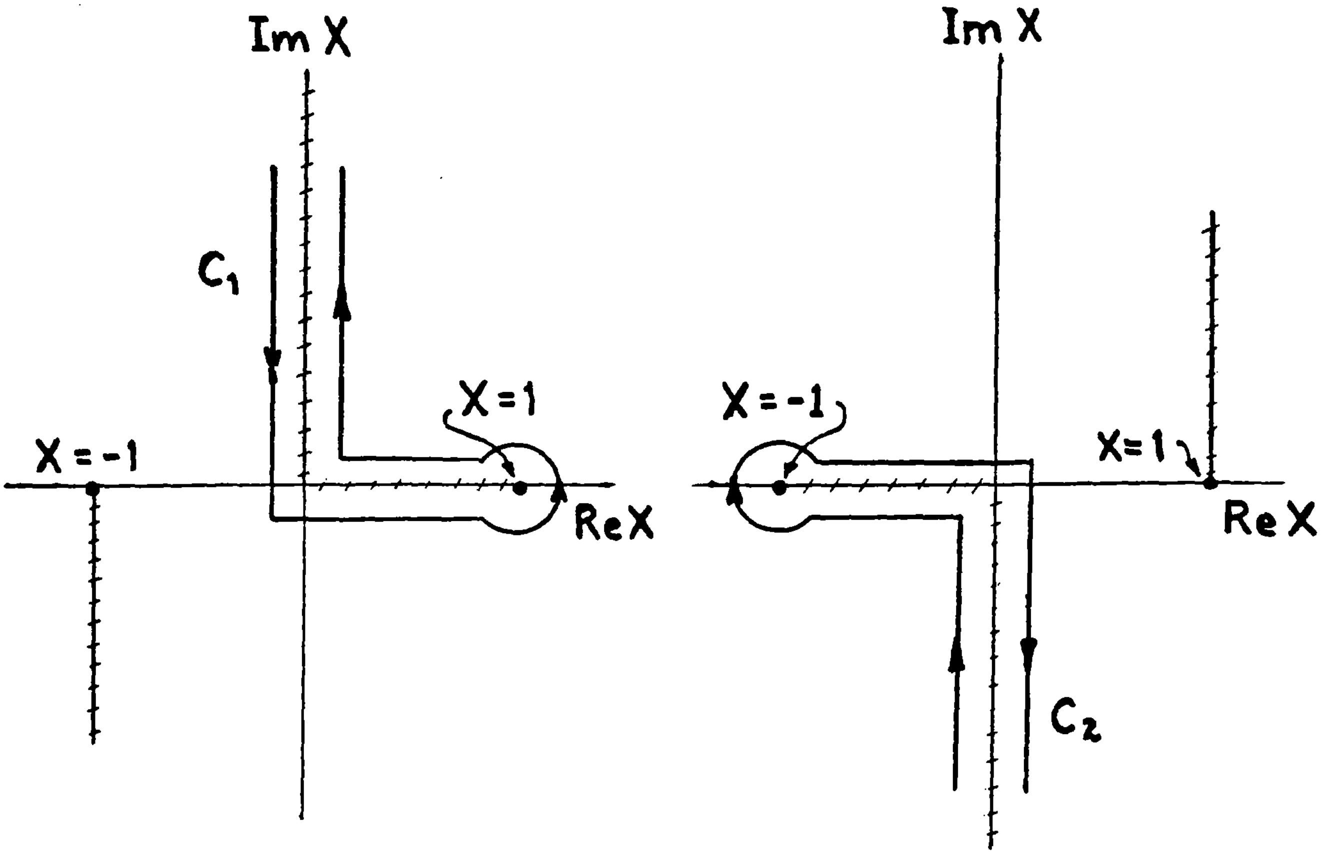
$$\text{(C1)}\qquad M_z^o(r,z;n) = \frac{1}{2\pi}\int_C d\zeta\ \overline{M}_z^o(r,\zeta;n)\exp(-i\zeta z)$$

where the contour C chosen in the $x \equiv \zeta/k$ plane is illustrated in Figure 3. Repeating eq. (54) defining the kernel we have

$$\text{(C2)}\qquad \overline{M}_z^o(r,\zeta;n) \equiv -\frac{1}{2}\pi(\omega\epsilon_o)^{-1}a\xi^2 J_{|n|}(r_<\xi)H_{|n|}^{(1)}(r_>\xi)$$

where $r_<(r_>)$ is the lesser (larger) of the two quantities (r,a).

Since the arguments of the Bessel and Hankel functions are multivalued functions of $x \equiv \zeta/k$ we cut the complex x plane so as to connect the branch points $x = \pm 1$ as shown in figure 3 where, for emphasis only, we have placed short slash marks along the cut. For $z < 0$ we use the integration contour C_1 and for $z > 0$ the integration contour C_2 as indicated in Figure 3. First we deal with the $z < 0$ case. Then in the complex $x \equiv \zeta/k$ plane we can write eq. (C2) in more detail as



a) Branch Cut and Contour for Z < 0

b) Branch Cut and Contour for Z > 0

Figure 3. Branch Cuts and Integration Contours for $\bar{M}^{\circ}_{Z}$, $\bar{M}^{\circ}_{Z\theta}$ and $\bar{N}^{\circ}_{Z\theta}$

(C3) $$- 4\zeta_0 (k^2 a)^{-1} M_z^{\circ}(r,z;n) =$$

$$= \int_{+i\infty}^{0} dx\ J_{|n|}(+kr_< \sqrt{1-x^2})\, H_{|n|}^{(1)}(+kr_> \sqrt{1-x^2})\,(1-x^2)\exp(+ik|z|x)$$

$$+ \int_{0}^{+i\infty} dx\ J_{|n|}(-kr_< \sqrt{1-x^2})\, H_{|n|}^{(1)}(-kr_> \sqrt{1-x^2})\,(1-x^2)\exp(+ik|z|x)$$

$$+ \int_{0}^{+1} dx\ J_{|n|}(+kr_< \sqrt{1-x^2})\, H_{|n|}^{(1)}(+kr_> \sqrt{1-x^2})\,(1-x^2)\exp(+ik|z|x)$$

$$+ \int_{+1}^{0} dx\ J_{|n|}(-kr_< \sqrt{1-x^2})\, H_{|n|}^{(1)}(-kr_> \sqrt{1-x^2})\,(1-x^2)\exp(+ik|z|x)$$

$$+ \oint dx\ J_{|n|}(kr_< \sqrt{1-x^2})\, H_{|n|}^{(1)}(kr_> \sqrt{1-x^2})\,(1-x^2)\exp(+ik|z|x)$$

Note that above the cut the negative root has been taken and below the cut the positive root is used.

The behavior of the integrand in the neighborhood of $x = +1$ will next be investigated so as to evaluate the last integral in eq. (C3). To do this we need the small argument asymptotic relations

(C4) $$J_{|n|}(y) \approx \frac{1}{n!}\left(\frac{y}{2}\right)^n \qquad \text{for } |y| << 1$$

$$\text{(C5)}\qquad H_n^{(1)}(y) \approx \begin{cases} \dfrac{-i(n-1)!}{\pi}\left(\dfrac{2}{y}\right)^n + \dfrac{1}{n!}\left(\dfrac{y}{2}\right)^n; & n \neq 0 \\ \dfrac{2i}{\pi}\left[\ln\left(\dfrac{y}{2}\right) + \gamma\right]; & n = 0 \end{cases} \qquad \text{for } |y| << 1$$

where γ is the Euler constant

$$\gamma = 0.577215 \ldots.$$

From eqs. (C4) and (C5) we can find the asymptotic form of the product in the integrand, i.e. for $|x| \approx 1$

$$\text{(C6)}\qquad J_{|n|}(kr_<\sqrt{1-x^2})H_{|n|}^{(1)}(kr_>\sqrt{1-x^2}) \approx \begin{cases} -\dfrac{i}{n\pi}\left(\dfrac{r_<}{r_>}\right)^n, & n \neq 0 \\ \dfrac{2i}{\pi}\left[\ln\left(\dfrac{kr_>\sqrt{1-x^2}}{2}\right)+\gamma\right], & n = 0 \end{cases}$$

Clearly for $n \neq 0$ the integrand is analytic along the circular portion of the contour at $x = +1$. For $n = 0$ we have to consider instead

$$(1-x^2)J_{|n|}(kr_<\sqrt{1-x^2})H_{|n|}^{(1)}(kr_>\sqrt{1-x^2} \approx$$

$$\frac{2i}{\pi}(1-x^2)\ (\gamma + \ln\frac{kr_>}{2}) \quad + \quad \frac{i}{\pi}(1-x^2)\ln(1-x^2)$$

which is also analytic over the circular portion of C_1. We thus note that the last integral in eq. (C3) gives zero contribution.

Now consider the first two integrals in eq. (C3). These run parallel to and just off the positive half of the imaginary axis. By simple change of variable these become

$$i\int_\infty^0 dx\ J_{|n|}(+kr_<\sqrt{1+x^2})H_{|n|}^{(1)}(+kr_>\sqrt{1+x^2})(1+x^2)\exp(-k|z|x)\ +$$

$$+\ i\int_0^\infty dx\ J_{|n|}(-kr_<\sqrt{1+x^2})H_{|n|}^{(1)}(-kr_>\sqrt{1+x^2})(1+x^2)\exp(-k|z|x)$$

Using the identities[16]

(C7a) $$J_{|n|}(e^{m\pi i}y) = e^{m\pi|n|i}\, J_{|n|}(y)$$

(C7b) $$N_{|n|}(e^{m\pi i}y) = e^{-m\pi|n|i}N_{|n|}(y) + 2i\ \sin(m|n|\pi)\ \cot(|n|\pi)J_{|n|}(y)$$

we can reduce this pair of integrals. The case $m = 1$ represents the branch of these functions above the cut. Then we have

(C8) $$J_{|n|}(-kr_<\sqrt{1\pm x^2})\, H^{(1)}_{|n|}(-kr_>\sqrt{1\pm x^2}) =$$

$$= -J_{|n|}(kr_<\sqrt{1\pm x^2})J_{|n|}(kr_>\sqrt{1\pm x^2}) + iJ_{|n|}(kr_<\sqrt{1\pm x^2})N_{|n|}(kr_>\sqrt{1\pm x^2})$$

$$= -J_{|n|}(kr_<\sqrt{1\pm x^2})H^{(2)}_{|n|}(kr_>\sqrt{1\pm x^2})$$

Thus with the result of eq. (C8) the two integrals reduce to

(C9) $$-2i\int_0^\infty dx J_{|n|}(kr_<\sqrt{1+x^2})J_{|n|}(kr_>\sqrt{1+x^2})(1+x^2)\exp(-k|z|x)$$

Again using the results in eq. (C8) the third and fourth integrals in eq. (C3) can be combined so that we have finally

(C10) $$M_z^o(r,z;n) = \frac{1}{2}\zeta_0^{-1}(k^2a)\{-\int_0^1 dx\ J_{|n|}(kr_<\sqrt{1-x^2})J_{|n|}(kr_>\sqrt{1-x^2}) \cdot$$

$$\cdot (1-x^2)\exp(ik|z|x) + i\int_0^\infty dx\ J_{|n|}(kr_<\sqrt{1+x^2})J_{|n|}(kr_>\sqrt{1+x^2}) \cdot$$

$$\cdot (1+x^2)\exp(-k|z|x)\} \qquad \text{for } z < 0$$

We shall merely note, without going through the details, that using contour C_2 for $z > 0$ we obtain precisely the same result in eq. (C10). This result is eq. (63) in the body of the text.

Furthermore the results quoted in eq. (66) and (67) for $\overset{\circ}{N}_{z\theta}(r,z;n)$ are calculated in the same manner and again, for brevity, we omit the details here. The result in eq. (64) in the text for $M^{\circ}_{z\theta}(r,z;n)$ is also determined in the same manner. Note however that a sign prefix occurs for this kernel function that is related to the sign of the coordinate variable z. Since the derivation of this kernel is carried out in the same manner we also omit the details here.

APPENDIX D

EVALUATION OF THE KERNELS $K^{(1)}_{z\theta}(z)$, $L^{(1)}_{z\theta}(z)$, $K^{(2)}_{z\theta}$ and $L^{(2)}_{z\theta}(z)$

(D1)
$$K^{(1)}_{z\theta}(z) = \frac{1}{2\pi}\left(-\frac{in}{\omega a}\right)\int_c d\zeta \, \frac{\zeta\, e^{-i\zeta z}}{\left(-\frac{i\pi\mu_o a\xi^2}{2}\, J'_{|n|}(a\xi)\, H^{(1)\prime}_{|n|}(a\xi)\right)}$$

$$= \frac{n}{\pi^2 \mu_o \omega a}\int_c d\zeta \left(\frac{\zeta}{\xi^2}\right) \frac{e^{-i\zeta z}}{J'_{|n|}(a\xi)\, H^{(1)\prime}_{|n|}(a\xi)}$$

Since the integrand is multivalued, we may use the contours shown in Figure (3). We see

(D2)
$$K^{(1)}_{z\theta}(z) = \frac{n}{\pi^2 \mu_o \omega a^2}\left[\int_o^1 dx\, \frac{x}{1-x^2}\, \frac{e^{ikx|z|}}{J'_{|n|}(ka[1-x^2]^{\frac{1}{2}})\, H^{(1)\prime}_{|n|}(ka[1-x^2]^{\frac{1}{2}})} -\right.$$

$$- \int_o^\infty dx\, \frac{x}{1+x^2}\, e^{-kx|z|} / J'_{|n|}(-ka[1+x^2]^{\frac{1}{2}})\, H^{(1)\prime}_{|n|}(-ka[1+x^2]^{\frac{1}{2}}) +$$

$$+ \int_1^o dx\, \frac{x e^{-kx|z|}}{1-x^2}\, e^{-ikx|z|} / J'_{|n|}(-ka[1-x^2]^{\frac{1}{2}})\, H^{(1)\prime}_{|n|}(-ka[1-x^2]^{\frac{1}{2}}) -$$

$$\left. - \int_\infty^o dx\, \frac{x e^{-kx|z|}}{1+x^2} / J'_{|n|}(ka[1+x^2]^{\frac{1}{2}})\, H^{(1)\prime}_{|n|}(ka[1+x^2]^{\frac{1}{2}})\right] \quad \text{for } z<0$$

Making use of the identities

$$H^{(1)}_{|n|}(-y) = (-1)^{n+1} H^{(2)}_{|n|}(y)$$

and

$$J_{|n|}(-y) = (-1)^n J_{|n|}(y)$$

we may write (D2) more compactly as

$$\text{(D3)} \qquad K_{z\theta}^{(1)}(z) = \frac{2n}{\pi^2 \mu_o \omega a^2}\left[\int_o^1 dx \frac{x}{1-x^2}\, e^{ikx|z|} \Big/ \left\{J_{|n|}'^{2}(ka[1-x^2]^{\frac{1}{2}}) + N_{|n|}'^{2}(ka[1-x^2]^{\frac{1}{2}})\right\} + \right.$$

$$\left. + \int_o^\infty dx \frac{x}{1+x^2}\, e^{-kx|z|} \Big/ \left\{J_{|n|}'^{2}(ka[1+x^2]^{\frac{1}{2}}) + N_{|n|}'^{2}(ka[1+x^2]^{\frac{1}{2}})\right\}\right], \quad z<0$$

we get the negative of eq. (D3) for z>0

Thus

$$\text{(D4)} \qquad K_{z\theta}^{(1)}(z) = -\frac{2n \operatorname{sgn} z}{\pi^2 \mu_o \omega a^2}\left[\int_o^1 dx \frac{x}{1-x^2}\, e^{ikx|z|} \Big/ \left\{J_{|n|}'^{2}(ka[1-x^2]^{\frac{1}{2}}) + N_{|n|}'^{2}(ka[1-x^2]^{\frac{1}{2}})\right\} + \right.$$

$$\left. + \int_o^\infty dx \frac{x}{1+x^2}\, e^{-kx|z|} \Big/ \left\{J_{|n|}'^{2}(ka[1+x^2]^{\frac{1}{2}}) + N_{|n|}'^{2}(ka[1+x^2]^{\frac{1}{2}})\right\}\right] \quad \text{for all } z.$$

For the kernel $L_{z\theta}^{(1)}(z)$ we see

$$\text{(D5)} \qquad L_{z\theta}^{(1)}(z) = -\frac{1}{\pi^2 \mu_o \omega a} \int_c d\zeta\, e^{-i\zeta z} / J_{|n|}'(a\xi) H_{|n|}^{(1)\prime}(a\xi)$$

Using the same contours as in the first case

$$\text{(D6)} \qquad L_{z\theta}^{(1)}(z) = -\frac{k}{\pi^2 \mu_o \omega a}\left[\int_o^1 dx\, e^{ikx|z|} / J_{|n|}'(ka[1-x^2]^{\frac{1}{2}}) H_{|n|}^{(1)\prime}(ka[1-x^2]^{\frac{1}{2}}) + \right.$$

$$+ i\int_o^\infty dx\, e^{-kx|z|} / J_{|n|}'(-ka[1+x^2]^{\frac{1}{2}}) H_{|n|}^{(1)\prime}(-ka[1+x^2]^{\frac{1}{2}}) +$$

$$+ \int_1^o dx\, e^{ikx|z|} / J_{|n|}'(-ka[1-x^2]^{\frac{1}{2}}) H_{|n|}^{(1)\prime}(-ka[1-x^2]^{\frac{1}{2}}) +$$

$$\left. + i\int_\infty^o dx\, e^{-kx|z|} / J_{|n|}'(ka[1+x^2]^{\frac{1}{2}}) H_{|n|}^{(1)\prime}(ka[1+x^2]^{\frac{1}{2}})\right], \quad z<0$$

Using the identities following eq. (D2) we get

(D7)
$$L_{z\theta}^{(1)}(z) = -\frac{2k}{\pi^2 \mu_o \omega a}\left[\int_o^1 dx\, e^{ikx|z|} \Big/ \left\{ J_{|n|}'^{\,2}(ka[1-x^2]^{\frac{1}{2}}) + N_{|n|}'^{\,2}(ka[1-x^2]^{\frac{1}{2}}) \right\} - \right.$$
$$\left. - i\int_o^\infty dx\; e^{-kx|z|} \Big/ \left\{ J_{|n|}'^{\,2}(ka[1+x^2]^{\frac{1}{2}}) + N_{|n|}'^{\,2}(ka[1+x^2]^{\frac{1}{2}}) \right\} \right]$$

For z>0 we use the other contour in Fig. (3) which yields the identical result.

Thus, eq. (D7) is valid for all z.

For the kernel $K_{z\theta}^{(2)}(z)$ we may write

(D8)
$$K_{z\theta}^{(2)}(z) = \frac{1}{2\pi}\left(-\frac{2\omega\epsilon_o}{\pi a}\right)\int_c d\zeta \frac{1}{\xi^2 J_{|n|}(\xi a) H_{|n|}^{(1)}(\xi a)} \left[1 + \frac{\frac{in}{\omega a}\left(\frac{-n\pi}{2\omega\epsilon_o}\right)\zeta^2 J_{|n|}(a\xi) H_{|n|}^{(1)}(a\xi)}{-\frac{i\pi\mu_o a}{2}\xi^2 J_{|n|}'(a\xi) H_{|n|}^{(1)\prime}(a\xi)} \right] \times$$
$$\times\, e^{-i\zeta z}$$
$$= -\frac{\omega\epsilon_o}{\pi^2 a}\int_c d\zeta\, e^{-i\zeta z}\left[\frac{1}{\xi^2 J_{|n|}(a\xi) H_{|n|}^{(1)}(a\xi)} + \frac{n^2}{k^2 a^2}\left(\frac{\zeta^2}{\xi^4 J_{|n|}'(a\xi) H_{|n|}^{(1)\prime}(a\xi)} \right) \right]$$

Using the contours shown in Figure (3) we get

(D9)
$$K_{z\theta}^{(2)}(z) = -\frac{\omega\epsilon_o}{\pi^2 ka}\left[\int_o^1 dx\, e^{ikx|z|}\left(\frac{1}{(1-x^2) J_{|n|}(ka[1-x^2]^{\frac{1}{2}}) H_{|n|}^{(1)}(ka[1-x^2]^{\frac{1}{2}})} + \right.\right.$$
$$\left. + \frac{n^2}{k^2a^2} \cdot \frac{x^2}{(1-x^2)^2 J_{|n|}'(ka[1-x^2]^{\frac{1}{2}}) H_{|n|}^{(1)\prime}(ka[1-x^2]^{\frac{1}{2}})} \right) +$$
$$+ i\int_o^\infty dx\; e^{-kx|z|}\left[\frac{1}{(1+x^2) J_{|n|}(-ka[1+x^2]^{\frac{1}{2}}) H_{|n|}^{(1)}(-ka[1+x^2]^{\frac{1}{2}})} - \right.$$
$$\left. - \frac{n^2}{k^2a^2}\, \frac{x^2}{(1+x^2)^2 J_{|n|}'(-ka[1+x^2]^{\frac{1}{2}}) H_{|n|}^{(1)\prime}(-ka[1+x^2]^{\frac{1}{2}})} \right] +$$

$$+\int_1^0 dx\, e^{ikx|z|}\left[\frac{1}{(1-x^2)J_{|n|}(-ka[1-x^2]^{\frac{1}{2}})H_{|n|}^{(1)}(-ka[1-x^2]^{\frac{1}{2}})}\right. +$$

$$\left.+\frac{n^2}{k^2a^2}\cdot\frac{x^2}{(1-x^2)^2 J'_{|n|}(-ka[1-x^2]^{\frac{1}{2}})H_{|n|}^{(1)\prime}(-ka[1-x^2]^{\frac{1}{2}})}\right] +$$

$$+ i\int_\infty^0 dx\, e^{-kx|z|}\left[\frac{1}{(1+x^2)J_{|n|}(ka[1+x^2]^{\frac{1}{2}})H_{|n|}^{(1)}(ka[1+x^2]^{\frac{1}{2}})}\right. -$$

$$\left.\left.-\frac{n^2}{k^2a^2}\cdot\frac{x^2}{(1+x)^2 J'_{|n|}(ka[1+x^2]^{\frac{1}{2}})H_{|n|}^{(1)\prime}(ka[1+x^2]^{\frac{1}{2}})}\right]\right] \quad \text{for } z<0$$

Again using the identities following eq. (D2) we may simplify the above to

$$\text{(D10)}\qquad K_{z\theta}^{(2)}(z) = -\frac{2\omega\varepsilon_o}{\pi^2 ka}\left[\int_0^1 dx\, e^{ikx|z|}\left[\frac{1}{(1-x^2)\left\{J_{|n|}^2(ka[1-x^2]^{\frac{1}{2}})+N_{|n|}^2(ka[1-x^2]^{\frac{1}{2}})\right\}}\right.\right. +$$

$$\left.+\frac{n^2}{k^2a^2}\cdot\frac{x^2}{(1-x^2)\left\{J'^2_{|n|}(ka[1-x^2]^{\frac{1}{2}})+N'^2_{|n|}(ka[1-x^2]^{\frac{1}{2}})\right\}}\right] -$$

$$- i\int_0^\infty dx\, e^{-kx|z|}\left[\frac{1}{(1+x^2)\left\{J_{|n|}^2(ka[1+x^2]^{\frac{1}{2}})+N_{|n|}^2(ka[1+x^2]^{\frac{1}{2}})\right\}}\right. +$$

$$\left.\left.+\frac{n^2}{k^2a^2}\cdot\frac{x^2}{(1+x^2)^2\left\{J'^2_{|n|}(ka[1+x^2]^{\frac{1}{2}})+N'^2_{|n|}(ka[1+x^2]^{\frac{1}{2}})\right\}}\right]\right], \quad z<0$$

An identical result follows for $z>0$, eq. (D10) is valid for all z.

We now shall consider the kernel $L_{z\theta}^{(2)}(z)$

(D11) $$L_{z\theta}^{(2)}(z) = \frac{1}{2\pi} \frac{\left(-\frac{i}{\omega}\frac{n\pi}{2\omega\epsilon_o}\right)}{\left(-\frac{i\pi\mu_o a}{2}\right)\left(-\frac{\pi a}{2\omega\epsilon_o}\right)} \int_c d\zeta \frac{\zeta e^{-i\zeta z}\xi^2 J_{|n|}(a\xi) H_{|n|}^{(1)}(a\xi)}{\xi^2 J'_{|n|}(a\xi) H_{|n|}^{(1)\prime}(a\xi)\, \xi^2 J_{|n|}(a\xi) H_{|n|}^{(1)}(a\xi)}$$

$$= -\frac{n}{\pi^2 \mu_o a^2} \int_c d\zeta\; e^{-i\zeta z} \left(\frac{\zeta}{\xi^2}\right) \frac{1}{J'_{|n|}(a\xi) H'_{|n|}(a\xi)}$$

$$\equiv -K_{z\theta}^{(1)}(z)$$

Then we may just write down the negative of eq. (D4)

(D12) $$L_{z\theta}^{(2)}(z) = \frac{2n \operatorname{sgn} z}{\pi^2 \mu_o \omega a^2} \left[\int_o^1 dx \frac{x}{1-x^2} e^{ikx|z|} \Big/ \left\{ J'^{\,2}_{|n|}(ka[1-x^2]^{\frac{1}{2}}) + N'^{\,2}_{|n|}(ka[1-x^2]^{\frac{1}{2}}) \right\} + \right.$$

$$\left. + \int_o^\infty dx \frac{x}{1+x^2} e^{-kx|z|} \Big/ \left\{ J'^{\,2}_{|n|}(ka[1+x^2]^{\frac{1}{2}}) + N'^{\,2}_{|n|}(ka[1+x^2]^{\frac{1}{2}}) \right\} \right]$$

APPENDIX E

EVALUATION OF THE KERNELS $Q_z(r,z)$, $R_{z\theta}(r,z)$ and $R_z(r,z)$

The zeroes of the Hankel function are in the interior of the complex plane so we may use the contours shown in Figure 3.

(E1)
$$Q_z(r,z) = \frac{k}{2\pi}\left[\int_0^1 dx\, \frac{H^{(1)}_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{H^{(1)}_{|n|}(ka[1-x^2]^{\frac{1}{2}})}\, e^{ikx|z|} + \right.$$
$$+ i\int_0^\infty dx\, \frac{H^{(1)}_{|n|}(-kr[1+x^2]^{\frac{1}{2}})}{H^{(1)}_{|n|}(-ka[1+x^2]^{\frac{1}{2}})}\, e^{-kx|z|} + \int_1^0 dx\, \frac{H^{(1)}_{|n|}(-kr[1-x^2]^{\frac{1}{2}})}{H^{(1)}_{|n|}(-ka[1-x^2]^{\frac{1}{2}})}\, e^{ikx|z|} +$$
$$\left. + i\int_\infty^0 dx\, \frac{H^{(1)}_{|n|}(kr[1+x^2]^{\frac{1}{2}})}{H^{(1)}_{|n|}(ka[1+x^2]^{\frac{1}{2}})}\, e^{-kx|z|}\right] \quad , \quad z < 0,\ r > a$$

Using the identity

(E2)
$$H^{(1)}_{|n|}(-y) = (-1)^{|n|+1} H^{(2)}_{|n|}(y)$$

eq. (D1) may be more compactly expressed as

(E3)
$$Q_z(r,z) = \frac{k}{\pi}\left[i\int_0^1 dx\, e^{ikx|z|}\left[J_{|n|}(ka[1-x^2]^{\frac{1}{2}})N_{|n|}(kr[1-x^2]^{\frac{1}{2}}) - \right.\right.$$
$$\left. - J_{|n|}(kr[1-x^2]^{\frac{1}{2}})N_{|n|}(ka[1-x^2]^{\frac{1}{2}})\right] / \left[J^2_{|n|}(ka[1-x^2]^{\frac{1}{2}}) + N^2_{|n|}(ka[1-x^2]^{\frac{1}{2}})\right] +$$
$$\left. + \int_0^\infty dx\, e^{-kx|z|}\left[\frac{J_{|n|}(ka[1+x^2]^{\frac{1}{2}})N_{|n|}(kr[1+x^2]^{\frac{1}{2}}) - J_{|n|}(kr[1+x^2]^{\frac{1}{2}})N_{|n|}(ka[1+x^2]^{\frac{1}{2}})}{J^2_{|n|}(ka[1+x^2]^{\frac{1}{2}}) + N^2_{|n|}(ka[1+x^2]^{\frac{1}{2}})}\right]\right]$$

For $z > 0$ we choose the alternative contour in Figure 3b which, in this case, yields the same result, so eq. (D3) is valid for all z and for $r > a$.

For the case r < a we must consider the Bessel functions. Since we know

(E4) $$J_{|n|}(-\xi r) = (-1)^{|n|} J_{|n|}(\xi r)$$

the integrand in eq. (85), for r < a, is single valued. We may use as a contour the real x axis but we must take care. The zeroes of the Bessel function lie along the real axis. We must detour around all such singularities, thus obtaining the principal value of the integrand plus πi times the residues at the singular points. The residues are found by expanding the denominator of the integrand in a Taylor series about its zeroes x_α, retaining the linear term and ignoring the higher order ones, i.e.

(E5) $$J_{|n|}(ka[1-x^2]^{\frac{1}{2}}) \approx - \frac{k_a x_\alpha}{(1-x_\alpha^2)^{\frac{1}{2}}} J'_{|n|}(ka[1-x_\alpha^2]^{\frac{1}{2}})(x-x_\alpha)$$

All such points x_α lie between -1 and +1 on the real x axis since for x > 1,

(E6) $$J_{|n|}(kai[x^2-1]^{\frac{1}{2}}) = i^{|n|} I_{|n|}(ka[x^2-1]^{\frac{1}{2}})$$

where $I_{|n|}$ is the modified Bessel function which possesses no zeroes.

(E7) $$Q_z(r,z) = \frac{k}{\pi}\left[P\int_0^1 dx \cos kxz \frac{J_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{J_{|n|}(ka[1-x^2]^{\frac{1}{2}})} + \right.$$

$$+ \int_1^\infty dx \ \cos kxz \frac{I_{|n|}(kr[x^2-1]^{\frac{1}{2}})}{I_{|n|}(ka[x^2-1]^{\frac{1}{2}})} +$$

$$\left. + \frac{\pi i}{ka} \sum_\alpha \cos kx_\alpha z \frac{(1-x_\alpha^2)^{\frac{1}{2}} J_{|n|}(kr[1-x_\alpha^2]^{\frac{1}{2}})}{x_\alpha J'_{|n|}(ka[1-x_\alpha^2]^{\frac{1}{2}})}\right] \qquad r < a$$

where P denotes the principal value of the integral.

In the neighborhood of x_α, $x_\alpha \pm \frac{\Delta x}{2}$, the integral

(E8) $$P\int_{\Delta x} dx\ \cos kxz\ \frac{J_{|n|}(kr[1-x^2]^{1/2})}{J_{|n|}(ka[1-x^2]^{1/2})} \longrightarrow$$

$$\longrightarrow P\int_{\Delta x} dx\ \cos kxz\ \frac{J_{|n|}(kr[1-x^2]^{1/2})(1-x_\alpha{}^2)^{1/2}}{kax_\alpha J'_{|n|}(ka[1-x_\alpha{}^2]^{1/2})(x-x_\alpha)}$$

which is the form of the integral whose principal value must be calculated.

The kernel $R_{z\theta}(r,z)$ differs from $Q_z(r,z)$ only slightly. If we replace $J_{|n|}(y)$ by $J'_{|n|}(y)$ and $H_{|n|}(y)$ by $H'_{|n|}(y)$, then $Q_z(r,z) \rightarrow R_{z\theta}(r,z)$.

Since

(E9) $$H^{(1)\prime}_{|n|}(-y) = (-1)^{|n|+1} H^{(2)\prime}_{|n|}(y)$$

(E10) $$J'_{|n|}(-y) = (-1)^{|n|} J'_{|n|}(y)$$

the procedure for simplifying the integral representation of $R_{z\theta}(r,z)$ is identical to that shown for $Q_z(r,z)$ via eqs. (E1) through (E7), replacing $J_{|n|}(y)$ and $H^{(1)}_{|n|}(y)$ with their derivatives, everywhere they appear.

The result is

(E11) $$R_{z\theta}(r,z) = \frac{k}{\pi}\Bigg[i\int_0^1 dx\, e^{ikx|z|}\Big[J'_{|n|}(ka[1-x^2]^{1/2})N'_{|n|}(kr[1-x^2]^{1/2}) -$$

$$- J'_{|n|}(kr[1-x^2]^{1/2})N'_{|n|}(ka[1-x^2]^{1/2})\Big]\Big/\Big\{J'^2_{|n|}(ka[1-x^2]^{1/2}) + N'^2_{|n|}(ka[1-x^2]^{1/2})\Big\} +$$

$$+ \int_0^\infty dx\, e^{-kx|z|}\frac{\left[J'_{|n|}(ka[1+x^2]^{1/2})N'_{|n|}(kr[1+x^2]^{1/2}) - J'_{|n|}(kr[1+x^2]^{1/2})N'_{|n|}(ka[1+x^2]^{1/2})\right]}{J'^2_{|n|}(ka[1+x^2]^{1/2}) + N'^2_{|n|}(ka[1-x^2]^{1/2})}\Bigg]$$

for $r > a$,

$$\text{(E12)} \qquad R_{z\theta}(r,z) = \frac{k}{\pi}\left[P\int_0^1 dx \cos kxz \,\frac{J'_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{J_{|n|}(ka[1-x^2]^{\frac{1}{2}})} + \right.$$

$$\left. + \int_1^\infty dx \cos kxz \,\frac{I'_{|n|}(kr[x^2-1]^{\frac{1}{2}})}{I'_{|n|}(ka[x^2-1]^{\frac{1}{2}})} + \frac{\pi i}{ka}\sum_\alpha \frac{\cos(kx_\alpha z)(1-x_\alpha^2)^{\frac{1}{2}} J'_{|n|}(kr[1-x_\alpha^2]^{\frac{1}{2}})}{x_\alpha J''_{|n|}(ka[1-x_\alpha^2]^{\frac{1}{2}})}\right]$$

for $r < a$.

Next consider the function $R_z(r,z)$ which is a weighted combination of $Q_z(r,z)$ and $R_{z\theta}(r,z)$.

For the case $r > a$ we look at the singular points of the integrand. Near the point $x = 1$, we may use the small argument expansion of the Bessel function to find

$$\frac{J_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{J_{|n|}(ka[1-x^2]^{\frac{1}{2}})} \approx \frac{\left\{\frac{kr[1-x^2]^{\frac{1}{2}}}{2}\right\}^{|n|}}{\left\{\frac{ka[1-x^2]^{\frac{1}{2}}}{2}\right\}^{|n|}} = \left(\frac{r}{a}\right)^{|n|}$$

and
$$\frac{J'_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{J'_{|n|}(ka[1-x^2]^{\frac{1}{2}})} \approx \left(\frac{r}{a}\right)^{|n|-1}$$

Since the integrand for $r<a$ is single valued we choose the real x axis as the integration path. The point $x = -1$ is also a singular point of the integrand as well.

The residue, $R(1)$ at $x = 1$ is

$$\text{(E13)} \qquad R(1) = \frac{e^{-ikxz}x}{(x+1)}\left[\frac{1}{r}\left(\frac{r}{a}\right)^{|n|} + \frac{1}{a}\left(\frac{r}{a}\right)^{|n|-1}\right]\Bigg|_{x=1} = \frac{e^{-ikz}}{r}\left(\frac{r}{a}\right)^{|n|}$$

The residue $R(-1)$ at $x = -1$ is

$$\text{(E14)} \qquad R(-1) = \frac{e^{-ikxz}x}{(1-x)}\left[\frac{1}{r}\left(\frac{r}{a}\right)^{|n|} + \frac{1}{a}\left(\frac{r}{a}\right)^{|n|-1}\right]\Bigg|_{x=-1} = -\frac{e^{+ikz}}{r}\left(\frac{r}{a}\right)^{|n|}$$

There are also the singularities of the integrand that correspond to the zeroes of $J_{|n|}(ka[1-x^2]^{\frac{1}{2}})$ and $J'_{|n|}(ka[1-x^2]^{\frac{1}{2}})$. These are treated in the same way as those associated with the kernel $Q_z(r,z)$

(E15)
$$R_z(r,z) = -\frac{in}{\pi}\Bigg[P\int_0^1 dx\frac{x}{1-x^2}\left\{\frac{J_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{rJ_{|n|}(ka[1-x^2]^{\frac{1}{2}})} + \frac{J'_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{aJ'_{|n|}(ka[1-x^2]^{\frac{1}{2}})}\right\}\sin kxz -$$

$$- \int_1^\infty dx\frac{x}{x^2-1}\left\{\frac{I_{|n|}(kr[x^2-1]^{\frac{1}{2}})}{rI_{|n|}(ka[x^2-1]^{\frac{1}{2}})} + \frac{I'_{|n|}(kr\ x^2-1]^{\frac{1}{2}})}{aI'_{|n|}(ka[x^2-1]^{\frac{1}{2}})}\right\}\sin kxz +$$

$$+ \pi i\left\{\frac{\sin kz}{r}\left(\frac{r}{a}\right)^{|n|} + \frac{1}{kra}\sum_\alpha \frac{\sin(kx_\alpha z)J_{|n|}(kr[1-x_\alpha{}^2]^{\frac{1}{2}})}{x_\alpha(1-x_\alpha{}^2)^{\frac{1}{2}}J'_{|n|}(ka[1-x_\alpha{}^2]^{\frac{1}{2}})} +\right.$$

$$\left.+ \frac{1}{ka^2}\sum_\beta \frac{\sin(kx_\beta z)J'_{|n|}(kr[1-x_\beta{}^2]^{\frac{1}{2}})}{x_\beta(1-x_\beta{}^2)^{\frac{1}{2}}J''_{|n|}(ka[1-x_\beta{}^2]^{\frac{1}{2}})}\right\}\Bigg]$$

for $r < a$

Consider the case for $r > a$.

The zeroes of the Hankel function are in the interior of the complex plane; furthermore the integrand in this case is multi= valued, so we use the integration contours shown in Figure 3, but we still have the singularities at $x = \pm 1$

(E16)
$$R_z(r,z) = \frac{n}{2\pi}\Bigg[\int_0^1 dx\frac{x}{1-x^2}\left\{\frac{H^{(1)}_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{rH^{(1)}_{|n|}(ka[1-x^2]^{\frac{1}{2}})} + \frac{H^{(1)\prime}_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{aH^{(1)\prime}_{|n|}(ka[1-x^2]^{\frac{1}{2}})}\right\}e^{ikx|z|} -$$

$$- \int_0^\infty dx\ \frac{x}{1+x^2}\left\{\frac{H_{|n|}(-kr[1+x^2]^{\frac{1}{2}})}{rH^{(1)}_{|n|}(-ka[1+x^2]^{\frac{1}{2}})} + \frac{H^{(1)\prime}_{|n|}(kr[1-x^2]^{\frac{1}{2}})}{aH^{(1)\prime}_{|n|}(-ka[1+x^2]^{\frac{1}{2}})}\right\}e^{-kx|z|} +$$

$$+ \int_1^0 dx \frac{x}{1-x^2} \left\{ \frac{H_{|n|}^{(1)}(-kr[1-x^2]^{\frac{1}{2}})}{rH_{|n|}^{(1)}(-ka[1-x^2]^{\frac{1}{2}})} + \frac{H_{|n|}^{(1)'}(-kr[1-x^2]^{\frac{1}{2}})}{aH_{|n|}^{(1)'}(-ka[1-x^2]^{\frac{1}{2}})} \right. e^{ikx|z|} +$$

$$+ \int_\infty^0 dx \frac{x}{1+x^2} \left\{ \frac{H_{|n|}^{(1)}(kr[1+x^2]^{\frac{1}{2}})}{rH_{|n|}^{(1)}(ka[1+x^2]^{\frac{1}{2}})} + \frac{H_{|n|}^{(1)'}(kr[1+x^2]^{\frac{1}{2}})}{aH_{|n|}^{(1)'}(ka[1+x^2]^{\frac{1}{2}})} \right\} e^{-kx|z|} \Bigg] + in \frac{e^{-ikz}}{r} \left(\frac{r}{a}\right)^{|n|}$$

for $z < 0$

Using eq. (E2) we may write eq. (E15) compactly as

(E17)
$$R_z(r,z) = \frac{n}{\pi}\Bigg[i \int_0^1 dx \frac{x}{1-x^2} e^{-ikx|z|} \Bigg\{ \Big[J_{|n|}(ka[1-x^2]^{\frac{1}{2}}) N_{|n|}(kr[1-x^2]^{\frac{1}{2}}) -$$

$$- J_{|n|}(kr[1-x^2]^{\frac{1}{2}}) N_{|n|}(ka[1-x^2]^{\frac{1}{2}}) \Big] \Big/ r\left[J_{|n|}^2(ka[1-x^2]^{\frac{1}{2}}) + N_{|n|}^2(ka[1-x^2]^{\frac{1}{2}}) \right] +$$

$$+ \frac{\left[J'_{|n|}(ka[1-x^2]^{\frac{1}{2}}) N'_{|n|}(kr[1-x^2]^{\frac{1}{2}}) - J'_{|n|}(kr[1-x^2]^{\frac{1}{2}}) N'_{|n|}(ka[1-x^2]^{\frac{1}{2}}) \right]}{a\left[J'^2_{|n|}(ka[1-x^2]^{\frac{1}{2}}) + N'^2_{|n|}(ka[1-x^2]^{\frac{1}{2}}) \right]} \Bigg\} +$$

$$+ i \int_0^\infty dx \frac{x}{1+x^2} e^{-kx|z|} \Bigg\{ \Big[J_{|n|}(ka[1+x^2]^{\frac{1}{2}}) N_{|n|}(kr[1+x^2]^{\frac{1}{2}}) -$$

$$- J_{|n|}(kr[1+x^2]^{\frac{1}{2}}) N_{|n|}(ka[1+x^2]^{\frac{1}{2}}) \Big] \Big/ r\left[J_{|n|}^2(ka[1+x^2]^{\frac{1}{2}}) + N_{|n|}^2(ka[1+x^2]^{\frac{1}{2}}) \right] +$$

$$+ \frac{\left[J'_{|n|}(ka[1+x^2]^{\frac{1}{2}}) N'_{|n|}(kr[1+x^2]^{\frac{1}{2}}) - J'_{|n|}(kr[1+x^2]^{\frac{1}{2}}) N'_{|n|}(ka[1+x^2]^{\frac{1}{2}}) \right]}{a\left[J'^2_{|n|}(ka[1+x^2]^{\frac{1}{2}}) + N'^2_{|n|}(ka[1+x^2]^{\frac{1}{2}}) \right]} \Bigg\} \Bigg] +$$

$$+ in \frac{e^{-ikz}}{r} \left(\frac{r}{a}\right)^{|n|}$$

For $z > 0$ we choose the alternative contour in Figure (3b) which in this case yields the negative of the result found for $z < 0$. ∴ The result for all z is just the result of eq. (E16) times (- sgn z).

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