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A theory of characteristic modes for material bodies is developed using equivalent surface currents. This is in contrast to the alternative approach using induced volume currents. The mode currents form a weighted orthogonal set over the material body surface, and the mode fields form an orthogonal set over the sphere at infinity. The characteristic modes of material bodies have most of the properties of those for perfectly conducting bodies. Formulas for the use of these modes in electromagnetic scattering problems are given. A procedure for computing the characteristic modes is developed, and applied to two-dimensional bodies. Illustrative examples of the computation of characteristic currents and scattering cross sections are given for cylinders of different material constants.

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CHAPTER 1

INTRODUCTION

1.1 Background

Characteristic modes have long been used in the analysis of radiation and scattering by dielectric and/or magnetic bodies whose surfaces coincide with coordinate surfaces in coordinate systems for which the Helmholtz equation is separable. From consideration of the scattering matrix, Garbacz [1] has shown that similar modes must exist for any material body. An extensive theory for perfectly conducting bodies was given in reference [1], but the dielectric and magnetic body case was not developed. An alternative treatment of the characteristic modes for perfectly conducting bodies, starting from the impedance operator for the conducting surface, has been given by Harrington and Mautz [2]. The computation of such modes has also been considered by Harrington and Mautz [3]. A theory of characteristic modes for dielectric bodies, magnetic bodies, and for bodies both dielectric and magnetic, has been developed by Harrington, Mautz, and Chang [4]. In this work, a theory of characteristic modes for material bodies is developed using equivalent surface currents. This is in contrast to the approach used in [4], which used the induced volume currents.

The modes are defined by a weighted eigenvalue equation in such a way that both the generalized network matrix [5] and the scattering matrix [1], [2] for the body are diagonalized. The presentation given in this work leads to explicit formulas for determining the mode currents and fields of two-dimensional objects. The formulas remain the same for dielectric bodies, magnetic bodies, and for bodies both dielectric and magnetic. In particular, the scattering problem of a two dimensional material cylinder will be presented. This formulation of the problem is applicable to any general material body. Details are worked out only for two-dimensional problems.

1-2 The Fundamental Operator Equation

Let the material body be represented as in Figure 1-1.

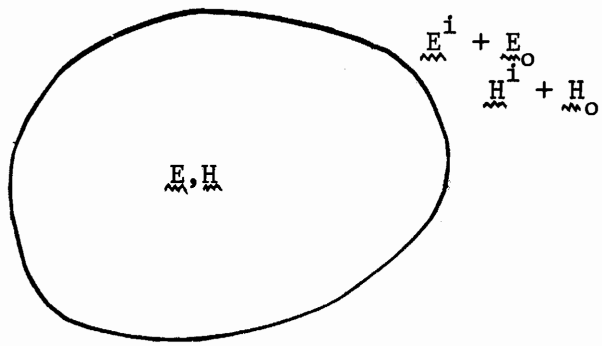


Figure 1-1. A general material body.

H^i, E^i = incident fields (Wavy underline denotes vector quantitics.)

E, H = inside fields

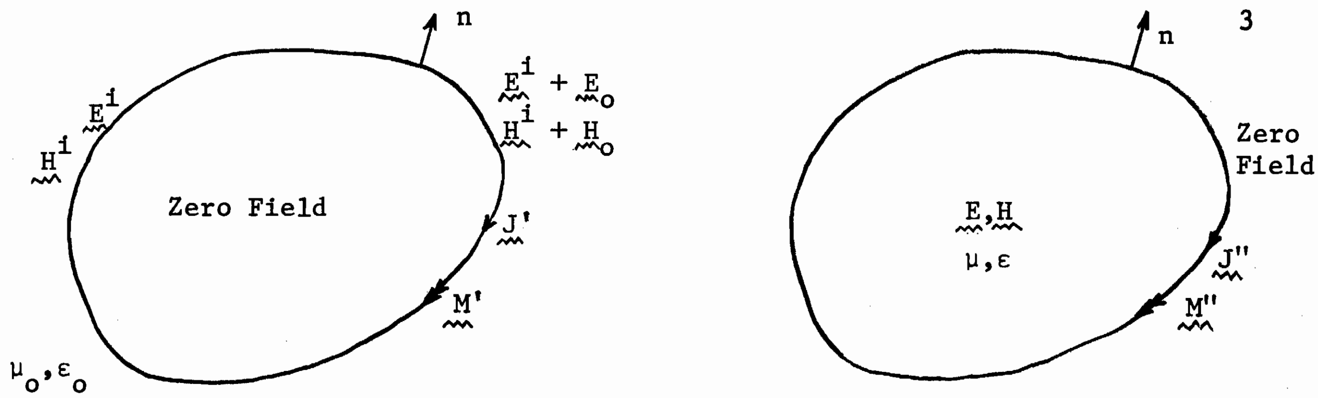
E_o, H_o = outside fields

The problem of Figure 1-1 can be viewed as a linear superposition of two cases,

(I) zero field inside

(II) zero field outside.

These two cases are illustrated in Figure 1-2.



Case (I) Case (II)

Figure 1-2. A decomposition of the original problem.

In case (I), let μ_o, ε_o be the material constants in the zero-field region, and similarly in case (II), let μ, ε be the material constants in the zero-field region. Having done so, radiation formulas [6] for unbounded space can be employed. The $\mathbf{J}$ and $\mathbf{M}$ are equivalent surface currents [6]. Since there are no actual surface currents, the following conditions should be satisfied by the equivalent currents

$$\mathbf{J}' + \mathbf{J}'' = 0 \tag{1-1}$$

$$\mathbf{M}' + \mathbf{M}'' = 0 \tag{1-2}$$

Equations (1-1) and (1-2) come from the fact that tangential components of fields are continuous across the interface in the original problem. Note that

$$\mathbf{E}_o = -j\omega\mathbf{A}' - \nabla\phi_e' - \frac{1}{\varepsilon_o}\nabla\times\mathbf{F}' + \mathbf{E}^i \tag{1-3}$$

$$\mathbf{H}_o = -j\omega\mathbf{F}' - \nabla\phi_m' + \frac{1}{\mu_o}\nabla\times\mathbf{A}' + \mathbf{H}^i \tag{1-4}$$

and

$$\underset{\sim}{E} = -j\omega\underset{\sim}{A}'' - \nabla\underset{\sim}{\phi}''_e - \frac{1}{\varepsilon}\underset{\sim}{\nabla} \times \underset{\sim}{F}'' \tag{1-5}$$

$$\underset{\sim}{H} = -j\omega\underset{\sim}{F}'' - \nabla\underset{\sim}{\phi}''_m + \frac{1}{\mu}\underset{\sim}{\nabla} \times \underset{\sim}{A}'' \tag{1-6}$$

and

$$\underset{\sim}{E}^s = -j\omega\underset{\sim}{A}' - \nabla\underset{\sim}{\phi}'_e - \frac{1}{\varepsilon_o}\underset{\sim}{\nabla} \times \underset{\sim}{F}' \tag{1-7}$$

$$\underset{\sim}{H}^s = -j\omega\underset{\sim}{F}' - \nabla\underset{\sim}{\phi}'_m + \frac{1}{\mu_o}\underset{\sim}{\nabla} \times \underset{\sim}{A}' \tag{1-8}$$

where $\underset{\sim}{E}^s$ and $\underset{\sim}{H}^s$ are scattered fields, and

$\underset{\sim}{A}$ = vector potential due to electric current

$\underset{\sim}{F}$ = vector potential due to magnetic current

ϕ_e = scalar potential due to electric charge

ϕ_m = scalar potential due to magnetic charge

Primed quantities refer to case (I)

doubly primed quantities refer to case (II)

In terms of general operator notations, the following equations are obtained

$$\begin{bmatrix} L'_{11} & L'_{12} \\ L'_{21} & L'_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{J}' \\ \underset{\sim}{M}' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{E}^s \\ \underset{\sim}{H}^s \end{bmatrix} \tag{1-9}$$

$$\begin{bmatrix} L''_{11} & L''_{12} \\ L''_{21} & L''_{22} \end{bmatrix} \begin{bmatrix} \underset{\sim}{J}'' \\ \underset{\sim}{M}'' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{E} \\ \underset{\sim}{H} \end{bmatrix} \tag{1-10}$$

where the definitions of the operators are obvious when comparing equations (1-9) and (1-10) with (1-5) to (1-8). Note that the tangential fields are continuous at the boundary surface, i.e.

$$\mathbf{n} \times (\mathbf{E}^s + \mathbf{E}^i - \mathbf{E}) = 0 \tag{1-11}$$

$$\mathbf{n} \times (\mathbf{H}^s + \mathbf{H}^i - \mathbf{H}) = 0 \tag{1-12}$$

where n = unit normal pointing outward, or

$$-\mathbf{n} \times \mathbf{E}^s + \mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{E}^i \tag{1-13}$$

$$-\mathbf{n} \times \mathbf{H}^s + \mathbf{n} \times \mathbf{H} = \mathbf{n} \times \mathbf{H}^i \tag{1-14}$$

The following equations are obtained by substituting equations (1-9) and (1-10) into (1-13) and (1-14).

$$-\mathbf{n} \times L'_{11}(\mathbf{J}') - \mathbf{n} \times L'_{12}(\mathbf{M}') + \mathbf{n} \times L''_{11}(\mathbf{J}'') + \mathbf{n} \times L''_{12}(\mathbf{M}'') = \mathbf{n}\times\mathbf{E}^i \tag{1-15}$$

$$-\mathbf{n} \times L'_{21}(\mathbf{J}') - \mathbf{n} \times L'_{22}(\mathbf{M}') + \mathbf{n} \times L''_{21}(\mathbf{J}'') + \mathbf{n} \times L''_{22}(\mathbf{M}'') = \mathbf{n}\times\mathbf{H}^i \tag{1-16}$$

By equations (1-1) and (1-2), it follows that

$$-\mathbf{n} \times [(L'_{11} + L''_{11})\mathbf{J}'] - \mathbf{n} \times [(L'_{12} + L''_{12})\mathbf{M}'] = \mathbf{n} \times \mathbf{E}^i \tag{1-17}$$

$$-\mathbf{n} \times [(L'_{21} + L''_{21})\mathbf{J}'] - \mathbf{n} \times [(L'_{22} + L''_{22})\mathbf{M}'] = \mathbf{n} \times \mathbf{H}^i \tag{1-18}$$

Define all the operators to be tangential operators; the above two equations can be put into standard matrix form as:

$$\begin{bmatrix} -(L'_{11} + L''_{11}) & -(L'_{12} + L''_{12}) \\ -(L'_{21} + L''_{21}) & -(L'_{22} + L''_{22}) \end{bmatrix} \begin{bmatrix} \underset{\sim}{J}' \\ \underset{\sim}{M}' \end{bmatrix}_{tan} = \begin{bmatrix} \underset{\sim}{E}^{i} \\ \underset{\sim}{H}^{i} \end{bmatrix}_{tan} \tag{1-19}$$

The $[\ ]_{tan}$ means the tangential components of the bracketed quantity on the boundary surface. Let

$$Le' = -L'_{11}, \quad Le'' = -L''_{11}, \quad L'_{m} = -L'_{22}, \quad L''_{m} = -L''_{22}$$

and define

$$-C'(\underset{\sim}{M}') = -\frac{1}{\varepsilon_o} \nabla \times \underset{\sim}{F}' = L'_{12}(\underset{\sim}{M}')$$

$$-C''(\underset{\sim}{M}') = -\frac{1}{\varepsilon} \nabla \times \underset{\sim}{F}'' = L''_{12}(\underset{\sim}{M}')$$

$$C'(\underset{\sim}{J}') = \frac{1}{\mu_o} \nabla \times \underset{\sim}{A}' = L'_{21}(\underset{\sim}{J}')$$

$$C''(\underset{\sim}{J}') = \frac{1}{\mu} \nabla \times \underset{\sim}{A}'' = L''_{21}(\underset{\sim}{J}')$$

Hence equation (1-19) becomes

$$\begin{bmatrix} Le' + Le'' & C' + C'' \\ -(C' + C'') & Lm' + Lm'' \end{bmatrix} \begin{bmatrix} \underset{\sim}{J}' \\ \underset{\sim}{M}' \end{bmatrix}_{tan} = \begin{bmatrix} \underset{\sim}{E}^{i} \\ \underset{\sim}{H}^{i} \end{bmatrix}_{tan} \tag{1-20}$$

It is convenient to rearrange equation (1-20) into the form

$$\begin{bmatrix} Le & -jC \\ -jC & Lm \end{bmatrix} \begin{bmatrix} \underset{\sim}{J} \\ j\underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{E}^i \\ j\underset{\sim}{H}^i \end{bmatrix} \tag{1-21}$$

where

$$Le = Le' + Le'' \qquad \underset{\sim}{J} = \underset{\sim}{J}'$$

$$Lm = Lm' + Lm'' \qquad \underset{\sim}{M} = \underset{\sim}{M}'$$

$$C = C' + C''$$

and the subscript "tan" has been dropped for brevity. Equation (1-21) is simply the familiar operator equation expressed below.

$$L(\underset{\sim}{f}) = \underset{\sim}{g} \tag{1-22}$$

L is a tangential operator on the surface of the material body, and

$$L = \begin{bmatrix} Le & -jC \\ -jC & Lm \end{bmatrix} \quad ; \quad \underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{J} \\ j\underset{\sim}{M} \end{bmatrix} \quad ; \quad \underset{\sim}{g} = \begin{bmatrix} -\underset{\sim}{E}^i \\ j\underset{\sim}{H}^i \end{bmatrix}$$

1-3 Format

In this work the impressed magnetic field is assumed to be axially-directed (perpendicular polarization). The derivation of all the formulas for an impressed axially-directed electric field (parallel polarization) will not be given in the main body of this work, in order to conserve space, however, explicit formulas will be provided in Appendix A. A list of computer programs will be given in Appendix B.

The content of this work is as follows. In Chapter 2, the operator equation is reduced to a matrix equation suitable for numerical computation.

The reduction is accomplished by using the method of moments for perpendicular polarization. The equivalent surface currents can be obtained by matrix inversion. A concise theory of characteristic modes for material bodies, based on the surface formulation, and explicit formulas for obtaining the modal solutions are given in Chapter 3. Chapter 4 is a presentation of calculations made for cylinders of different material constants using both the matrix inversion method and the modal method. Chapter 5 is a discussion of the results. The computations presented in this work were performed on an IBM System 370, Model 155 digital computer. The computer programs are written in FORTRAN IV.

CHAPTER 2

MATRIX FORMULATION

The determination of equivalent surface currents requires the solution of the following inhomogeneous equation

$$L(\underset{\sim}{f}) = \underset{\sim}{g} \tag{2-1}$$

where L is the matrix of operators

$$L = \begin{bmatrix} Le & -jC \\ -jC & Lm \end{bmatrix} \tag{2-2}$$

and

$$\underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{J} \\ j\underset{\sim}{M} \end{bmatrix} \qquad \underset{\sim}{g} = \begin{bmatrix} \underset{\sim}{E}^i \\ j\underset{\sim}{H}^i \end{bmatrix} \tag{2-3}$$

This chapter presents the reduction of equation (2-1) to matrix form by the method of moments.

2.1 Method of Moments

To apply the method of moments, an appropriate inner product for the problem is ($\sim$ indicates transpose)

$$\begin{aligned} \langle \underset{\sim}{f}, \underset{\sim}{g} \rangle &= \iint_s \tilde{\underset{\sim}{f}}\, \underset{\sim}{g}\, ds \\ &= \iint_s (\underset{\sim}{J} \cdot \underset{\sim}{E} - \underset{\sim}{H} \cdot \underset{\sim}{M}) ds \end{aligned} \tag{2-4}$$

A solution by the method of moments is obtained as follows. Define electric expansion and testing functions as

$$\underset{\sim}{f}_n^e = \begin{bmatrix} \underset{\sim}{J}_n \\ 0 \end{bmatrix} \qquad \underset{\sim}{W}_n^e = \begin{bmatrix} \underset{\sim}{W}_n^e \\ 0 \end{bmatrix} \tag{2-5}$$

and magnetic expansion and testing functions as

$$\underset{\sim}{f}_n^m = \begin{bmatrix} 0 \\ \underset{\sim}{M}_n \end{bmatrix} \qquad \underset{\sim}{W}_n^m = \begin{bmatrix} 0 \\ \underset{\sim}{W}_n^m \end{bmatrix} \tag{2-6}$$

The expansion for f is then of the form

$$\underset{\sim}{f} = \sum_n (I_n \underset{\sim}{f}_n^e + V_n \underset{\sim}{f}_n^m) \tag{2-7}$$

where the I_n and V_n are constants to be determined.

The inner product of each $\{\underset{\sim}{W}_m^e\}$ with equation (2-1) yields

$$\langle \underset{\sim}{W}_m^e, L(\underset{\sim}{f}_n)\rangle = \langle \underset{\sim}{W}_m^e, \underset{\sim}{g}\rangle \tag{2-8}$$

where

$$\begin{aligned} \langle \underset{\sim}{W}_m^e, L(\underset{\sim}{f}_n)\rangle &= \langle \underset{\sim}{W}_m^e, \sum_n (I_n L(\underset{\sim}{f}_n^e) + V_n L(\underset{\sim}{f}_n^m)\rangle \\ &= \sum_n I_n \langle \underset{\sim}{W}_m^e, L(\underset{\sim}{f}_n^e)\rangle \\ &\quad + \sum_n V_n \langle \underset{\sim}{W}_m^e, L(\underset{\sim}{f}_n^m)\rangle \\ &= \sum_n I_n \iint_s \underset{\sim}{W}_m^e \cdot Le(\underset{\sim}{J}_n)ds + \sum_n V_n \iint_s \underset{\sim}{W}_m^e \cdot (-jC)(\underset{\sim}{M}_n)ds \end{aligned} \tag{2-9}$$

and

$$\langle \underset{\sim}{W}_m^e, \underset{\sim}{g}\rangle = \iint_s \underset{\sim}{W}_m^e \cdot \underset{\sim}{E}^i ds \tag{2-10}$$

Similarly, the inner product of each $\{\underset{\sim}{W}_m^m\}$ with equation (2-1) yields

$$\langle \underset{\sim}{W}_m^m, \underset{\sim}{L}(f_n)\rangle = \langle \underset{\sim}{W}_m^m, \underset{\sim}{g}\rangle \tag{2-11}$$

and

$$\langle \underset{\sim}{W}_m^m, \underset{\sim}{L}(f_n)\rangle = \sum_n I_n \iint_s \underset{\sim}{W}_m^m \cdot (-jC)(\underset{\sim}{J}_n)ds + \sum_n V_n \iint_s \underset{\sim}{W}_m^m \cdot L_m(\underset{\sim}{M}_n)ds \tag{2-12}$$

$$\langle \underset{\sim}{W}_m^m, \underset{\sim}{g}\rangle = \iint_s \underset{\sim}{W}_m^m \cdot \underset{\sim}{jH}^i ds \tag{2-13}$$

Equation (2-8) and Equation (2-11) can be placed in matrix form

$$\begin{bmatrix} [Z] & [B] \\ [C] & [Y] \end{bmatrix} \begin{bmatrix} [I] \\ [V] \end{bmatrix} = \begin{bmatrix} [V^i] \\ [I^i] \end{bmatrix} \tag{2-14}$$

where

$$Z_{mn} = \iint_s \underset{\sim}{W}_m^e \cdot Le(\underset{\sim}{J}_n)ds \tag{2-15}$$

$$B_{mn} = \iint_s \underset{\sim}{W}_m^e \cdot (-jC)(\underset{\sim}{M}_n)ds \tag{2-16}$$

$$C_{mn} = \iint_s \underset{\sim}{W}_m^m \cdot (-jC)(\underset{\sim}{J}_n)ds \tag{2-17}$$

$$Y_{mn} = \iint_s \underset{\sim}{W}_m^m \cdot L_m(\underset{\sim}{M}_n)ds \tag{2-18}$$

$$V_m^i = \iint_s \mathbf{W}_m^e \cdot \mathbf{E}^i ds \tag{2-19}$$

$$I_m^i = \iint_s \mathbf{W}_m^m \cdot j\mathbf{H}^i ds \tag{2-20}$$

Choose $\mathbf{J}_n = \mathbf{W}_n^e$. Note that [Z] is obviously symmetric, already shown by Harrington and Mautz. With the choice $\mathbf{M}_n = \mathbf{W}_n^m$, [Y] is the dual of [Z], magnetic instead of electric, so the symmetric nature of [Y] can be easily established. It is known that $C(\mathbf{M}_n)$ gives rise to an electric field and $C(\mathbf{J}_n)$ will produce a magnetic field. Observe that by reciprocity

$$\iint_s (\mathbf{E}^a \cdot \mathbf{J}^b - \mathbf{H}^a \cdot \mathbf{M}^b) = \iint_s (\mathbf{E}^b \cdot \mathbf{J}^a - \mathbf{H}^b \cdot \mathbf{M}^a) ds \tag{2-21}$$

Now, consider

(i) In situation "a" only electric sources

(ii) In situation "b" only magnetic sources.

It follows that

$$\iint_s (-\mathbf{H}^a \cdot \mathbf{M}^b) ds = \iint_s (\mathbf{E}^b \cdot \mathbf{J}^a) ds \tag{2-22}$$

Hence

$$B_{mn} = C_{nm} \tag{2-23}$$

Consequently, [C] is the transpose of [B], or

$$[B] = [\widetilde{C}] \quad \text{with} \quad \mathbf{J}_n = \mathbf{W}_n^e, \quad \mathbf{M}_n = \mathbf{W}_n^m \tag{2-24}$$

To this point the matrix formulation is completely general and has been achieved without reference to specific excitation, expansion functions, and testing functions. Note that every one of the operators Le, C, and Lm is composed of two parts as indicated by equation (1-21), and consequently every matrix element in [Z], [B], [C] and [Y] has two parts; one is due to the primed operator, and the other is due to the doubly primed operator.

2.2 Expansion Functions and the Evaluation of [Z] Matrix Element

In this section, the incident field is the axially directed magnetic field, $\underset{\sim}{H}_z^i$. Before going into any specific excitation for the scattering problem, some general considerations about the evaluation of different types of matrix elements will be presented as follows.

Note that the original problem, the scattering from material bodies, has been decomposed into two cases, and their associated operators are of the same functional form. For instance, the expression for Le"$(\underset{\sim}{J})$ will be identical to Le'$(\underset{\sim}{J})$ except for the constitutive constants, μ and ε. For the sake of brevity, only Le' will be considered. Once Le'$(\underset{\sim}{J})$ is known, Le"$(\underset{\sim}{J})$ is obtained by replacing ε_o and μ_o by ε and μ, respectively.

By equation (2-15)

$$Z_{mn} = \int \underset{\sim}{W}_m \cdot (j\omega \underset{\sim}{A}_n + \nabla \underset{\sim}{\phi}_n) d\ell \qquad (2\text{-}25)$$

where

$\underset{\sim}{A}_n$ = magnetic vector potential due to $\underset{\sim}{J}_n$

ϕ_n = scalar potential due to σ_n, surface charge

Applying the one dimensional divergence theorem to the vector $\phi_n \underset{\sim}{W}_m$, and noting that

$$\underset{\sim}{\nabla} \cdot \phi_n \underset{\sim}{W}_m = \phi_n \underset{\sim}{\nabla} \cdot \underset{\sim}{W}_m + \underset{\sim}{W}_m \cdot \underset{\sim}{\nabla} \phi_n \tag{2-26}$$

the following relationship is obtained

$$\int_c \underset{\sim}{\nabla} \phi_n \cdot \underset{\sim}{W}_m d\ell = - \int_c \phi_n \underset{\sim}{\nabla} \cdot \underset{\sim}{W}_m d\ell \tag{2-27}$$

Define σ_m such that

$$\sigma_m = - \frac{1}{j\omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{W}_m \tag{2-28}$$

Observe that equation (2-28) has the form of the continuity equation if $\underset{\sim}{W}_m$ and σ_m are interpreted as current and charge, respectively. An alternative form for the [Z] matrix element can be obtained by substituting equations (2-27) and (2-28) into equation (2-25). The new form is computationally more attractive.

$$Z_{mn} = j\omega \int (\underset{\sim}{W}_m \cdot \underset{\sim}{A}_n + \sigma_m \phi_n) d\ell \tag{2-29}$$

Define the two-dimensional Green's function G(r,r').

$$G(r,r') = \frac{1}{4j} H_o^{(2)}(k|r - r'|) \tag{2-30}$$

and

$$\underset{\sim}{A}(r) = \mu \int \underset{\sim}{J}(r') \, G(r,r') d\ell' \tag{2-31}$$

$$\phi(r) = \frac{1}{\varepsilon} \int_c q(r') \, G(r,r') d\ell' \tag{2-32}$$

where q is related to $\underset{\sim}{J}$ by the equation of continuity

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{J} = - j\omega q \tag{2-33}$$

After substituting equations (2-31), (2-32), and (2-33) into equation (2-29), the expression for Z_{mn} becomes

$$Z_{mn} = j\omega \int_C \{ \int_C [\underset{\sim}{W}_m \cdot \mu \underset{\sim}{J}_n(r') + \frac{\sigma_m}{\varepsilon} q_n(r')] \, G(r,r')d\ell' \} d\ell$$

$$= \int_C \int_C [j\omega\mu \underset{\sim}{W}_m \cdot \underset{\sim}{J}_n + \frac{1}{j\omega\varepsilon}(\underset{\sim}{\nabla} \cdot \underset{\sim}{W}_m)(\underset{\sim}{\nabla}' \cdot \underset{\sim}{J}_n)] G(r,r')d\ell' d\ell \tag{2-34}$$

Note that the primed symbols refer to source location variation, while the unprimed symbols relate to variation in field point location.

The specific formulation proceeds by dividing the contour C into N segments, not necessarily equal in length. There are N segments and N+1 points,

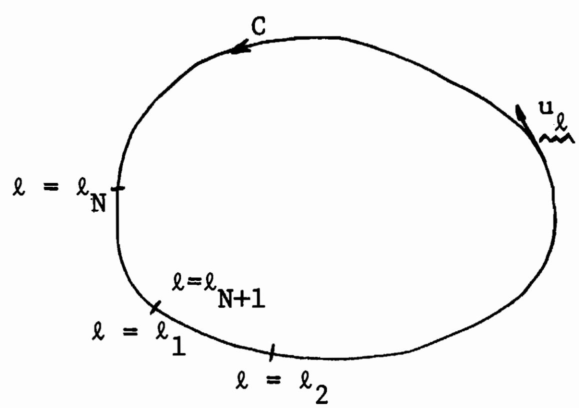


Fig. 2-1. A cross sectional contour.

and ℓ is the path length proceeding counterclockwise around contour C.

The sets of expansion and testing functions are chosen as triangle functions for both electric and magnetic surface currents.

$$\underset{\sim}{W}_k = T(\ell - \ell_k)\underset{\sim}{u}_\ell \qquad k=1,2,...,N \tag{2-35}$$

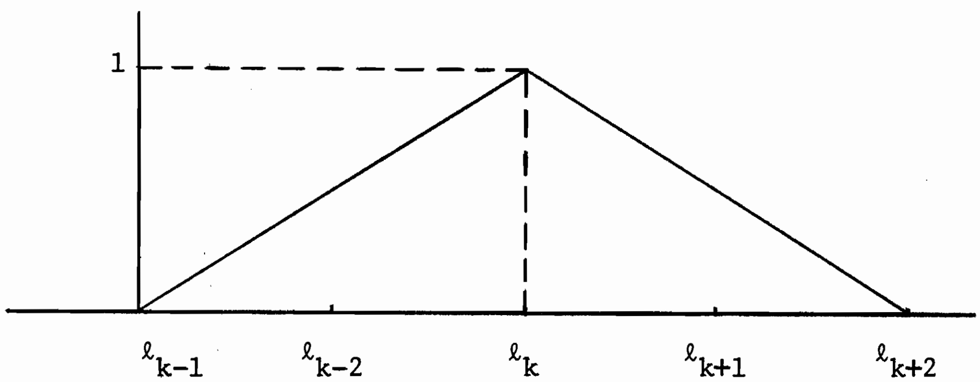


Fig. 2-2. A triangle function.

where $\underset{\sim}{u}_\ell$ is the unit vector tangent to C path length value ℓ, and T is the triangle function defined by

$$T(\ell - \ell_k) = \begin{cases} 1 - \dfrac{\ell - \ell_k}{\ell_{k+2} - \ell_k} & \ell_k \leq \ell \leq \ell_{k+2} \\[2ex] 1 + \dfrac{\ell - \ell_k}{\ell_k - \ell_{k-2}} & \ell_{k-2} \leq \ell \leq \ell_k \\[2ex] 0 & \ell \geq \ell_{k+2} \\[1ex] 0 & \ell_{k-2} \geq \ell \end{cases} \qquad (2\text{-}36)$$

or

$$T(\ell - \ell_k) = \begin{cases} 1 - \dfrac{\ell - \ell_k}{\ell_{k+2} - \ell_k} & 0 \leq \ell - \ell_k \leq \ell_{k+2} - \ell_k \\[2ex] 1 + \dfrac{\ell - \ell_k}{\ell_k - \ell_{k-2}} & -(\ell_k - \ell_{k-2}) \leq \ell - \ell_k \leq 0 \\[2ex] 0 & \ell - \ell_k \geq \ell_{k+2} - \ell_k \\[1ex] 0 & -(\ell_k - \ell_{k-2}) \geq \ell - \ell_k \end{cases} \qquad (2\text{-}37)$$

Let $\Delta\ell_k = \ell_{k+1} - \ell_k$ then

$$T(\ell - \ell_k) = \begin{cases} 1 - \dfrac{\ell - \ell_k}{\Delta\ell_k + \Delta\ell_{k+1}} & \sigma \leq \ell - \ell_k \leq \Delta\ell_k + \Delta\ell_{k+1} \\[2ex] 1 + \dfrac{\ell - \ell_k}{\Delta\ell_{k-2} + \Delta\ell_{k-1}} & -(\Delta\ell_{k-2} + \Delta\ell_{k-1}) \leq \ell - \ell_k \leq 0 \\[2ex] 0 & \ell - \ell_k \geq \Delta\ell_k + \Delta\ell_{k+1} \\[2ex] 0 & -(\Delta\ell_{k-2} + \Delta\ell_{k-1}) \geq \ell - \ell_k \end{cases} \qquad (2\text{-}38)$$

and

$$\Delta\ell_0 = \Delta\ell_N$$

Now, Z_{mn} can be written as

$$Z_{mn} = \int\int [j\omega\mu \underset{\sim}{W}_m \cdot \underset{\sim}{J}_n + \frac{1}{j\omega\varepsilon} (\underset{\sim}{\nabla} \cdot \underset{\sim}{W}_m)(\underset{\sim}{\nabla}' \cdot \underset{\sim}{J}_n)] G \, d\ell' d\ell$$

$$= \int_{\ell_{m-2}}^{\ell_{m+2}} \int_{\ell_{n-2}}^{\ell_{n+2}} [j\omega\mu \; T(\ell-\ell_m) \; T(\ell'-\ell_n)(\underset{\sim}{u}_\ell \cdot \underset{\sim}{u}'_\ell)$$

$$+ \frac{1}{j\omega\varepsilon} T'(\ell' - \ell_n) \; T'(\ell - \ell_n)] G d\ell' d\ell \qquad (2\text{-}39)$$

The subscript m indicates the mth triangle testing function and the subscript n indicates the nth triangle expansion function. T' is the derivative of the triangle function.

$$T'(\ell-\ell_k) = \begin{cases} -\dfrac{1}{\ell_{k+2} - \ell_k} & \ell_k \le \ell \le \ell_{k+2} \\ \dfrac{1}{\ell_k - \ell_{k-2}} & \ell \ge \ell_{k+2} \\ 0 & \ell \ge \ell_{k+2} \\ 0 & \ell_{k-2} \ge \ell \end{cases} \qquad (2\text{-}40)$$

or

$$T'(\ell-\ell_k) = \begin{cases} \dfrac{1}{\Delta\ell_k + \Delta\ell_{k+1}} & 0 \le \ell-\ell_k \le 0 \; \ell_k + \Delta\ell_{k+1} \\ \dfrac{1}{\Delta\ell_{k-2} + \Delta\ell_{k-1}} & -(\Delta\ell_{k-2}+\Delta\ell_{k-1}) \le \ell-\ell_k \le 0 \\ 0 & \ell-\ell_k \ge \Delta\ell_k + \Delta\ell_{k+1} \\ 0 & -(\Delta\ell_{k-2} + \Delta\ell_{k-1}) \ge \ell-\ell_k \end{cases} \qquad (2\text{-}41)$$

The triangle function is approximated by four pulses with amplitudes h_{k-2}, h_{k-1}, h_k, and h_{k+1} as shown below

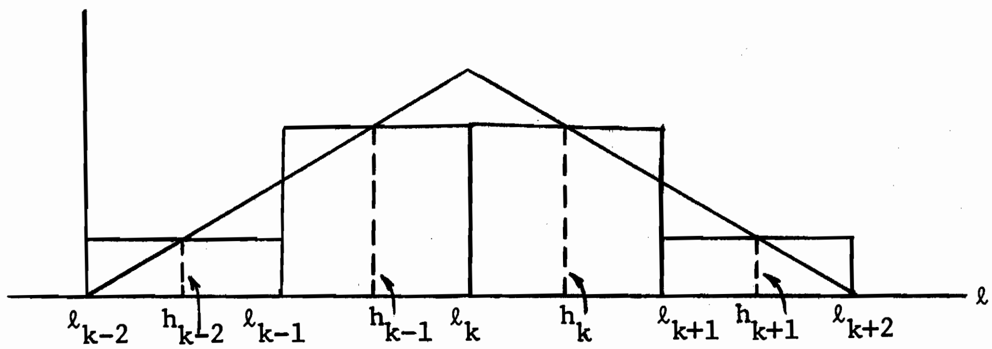


Fig. 2-3. A four-pulse approximation

where

$$h_{k-2} = \frac{\frac{1}{2}\,\Delta\ell_{k-2}}{\Delta\ell_{k-2} + \Delta\ell_{k-1}} \qquad h_{k-1} = \frac{\Delta\ell_{k-2} + \frac{1}{2}\,\Delta\ell_{k-1}}{\Delta\ell_{k-2} + \Delta\ell_{k-1}}$$

$$h_k = \frac{\frac{1}{2}\,\Delta\ell_k + \Delta\ell_{k+1}}{\Delta\ell_k + \Delta\ell_{k+1}} \qquad h_{k+1} + \frac{\frac{1}{2}\,\Delta\ell_{k+1}}{\Delta\ell_k + \Delta\ell_{k+1}} \tag{2-42}$$

The derivative T' of the Triangle function can be represented graphically as

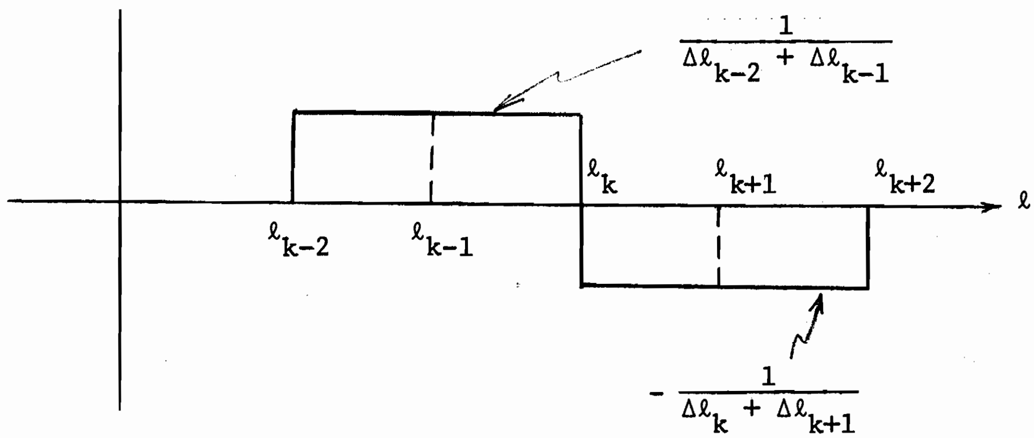


Fig. 2-4. The derivative of a triangle function.

Consider the contour interval spanned by the expansion or testing triangle function as shown in Fig. 2-5.

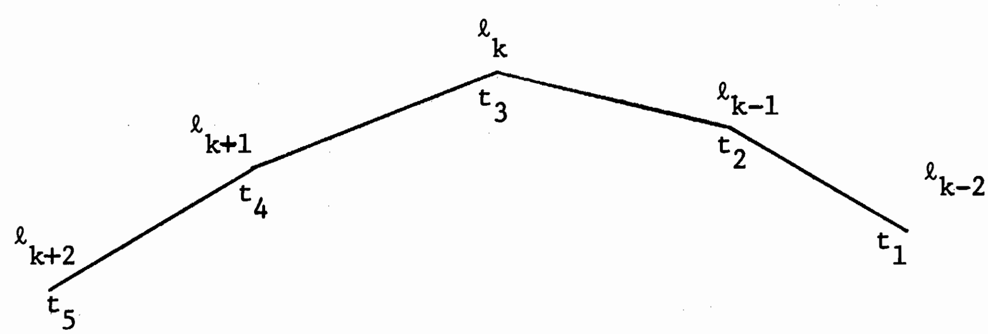


Fig. 2-5. Straight line representation of the contour.

To evaluate the integrals in equation (2-39) each such portion of the contour is replaced by straight line segments drawn between the points on the actual contour defined by $\ell_1, \ell_2, \ldots, \ell_{N+1}$. The integration variables in equation (2-39) are taken along these straight line segments. For the nth expansion function, the index p=1,2,3,4 is associated with the four pulse intervals for increasing path length. Similarly, the index q=1,2,3,4 is defined for the mth testing function. Equation (2-39) can be put in the following form

$$Z_{mn} = \sum_{q=1}^{4} \sum_{p=1}^{4} \int_{t_q}^{t_{q+1}} \int_{t_p}^{t_{p+1}} [j\omega\mu \, T_p T_q (u_p \cdot u_q) + \frac{1}{j\omega\varepsilon} T'_q \, T'_p] \, G d\ell_p d\ell_q \qquad (2\text{-}43)$$

Note that T_p and T_q are the amplitudes of the pth and qth pulses, respectively. The unit vectors u_p and u_q are parallel to the straight lines of the pth and qth intervals in the direction of increasing path length. Observe that each of the sixteen terms on the right hand side of equation (2-43) results from one of two situations. Either the pth and qth intervals coincide, or they do not. These two situations will be considered separately.

(i) Noncoincident intervals

In this situation each integral in equation (2-44) is approximated by the product of its integrand evaluated at the interval midpoint times the interval length. Hence equation (2-44) becomes

$$Z_{mn} = \sum_{q=1}^{4} \sum_{p=1}^{4} \Delta t_p \; \Delta t_q \; [j\omega\mu \; T_p T_q (\underset{\sim}{u}_p \cdot \underset{\sim}{u}_q) + \frac{1}{j\omega\varepsilon} T_p' \; T_q'] \; \frac{H_o^{(2)}(kR_{pq})}{4j} \tag{2-44}$$

Δt_p and Δt_q are determined by

$$\Delta t_k = t_{k+1} - t_k \qquad (k = 1,2,3,4)$$

and R_{pq} is the distance between the midpoints of the pth and qth pulses.

(ii) Coincident intervals

For coincident pth and qth intervals the integral evaluations proceed as follows. The q integral is approximated by the product of the integrand, sampled at the midpoint of the interval, times the interval length. The p integral is then evaluated as an improper integral.

The small argument approximation for $H_o^{(2)}(kR)$ is

$$H_o^{(2)}(kR) \approx 1 - j \frac{2}{\pi} \log \left(\frac{\gamma kR}{2}\right) \tag{2-46}$$

where log γ is Euler's constant. Then for coincident pulse intervals

$$\int_{t_q}^{t_{q+1}} \int_{t_p}^{t_{p+1}} [j\omega\mu \; T_p T_q (\underset{\sim}{u}_p \cdot \underset{\sim}{u}_q) + \frac{1}{j\omega\varepsilon} T_p' \; T_q'] \; \frac{H_o^{(2)}(kR)}{4} \, d\ell_p d\ell_q$$

$$= \frac{1}{4} \Delta t_q [j\omega\mu \; T_p T_q + \frac{1}{j\omega\varepsilon} T_p' \; T_q'] \int_{t_p}^{t_{p+1}} H_o^{(2)}(kR) d\ell_p \tag{2-47}$$

Note that the integrand is singular at the midpoint. The improper integral can be treated as:

$$\int_{t_p}^{t_{p+1}} H_o^{(2)}(kR)d \;=\; \int_{-\Delta t_p/2}^{\Delta t_p/2} [1 - j\,\frac{2}{\pi}\,\log\,(\frac{\gamma k|x|}{2})]dx$$

$$= \lim_{\varepsilon\to 0} \int_{-\Delta t_p/2}^{-\varepsilon} [1 - j\,\frac{2}{\pi}\,\log\,(-\,\frac{\gamma kx}{2})]dx$$

$$+ \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\Delta t_p/2} [1 - j\,\frac{2}{\pi}\,\log\,(\frac{\gamma kx}{2})]dx$$

$$= \Delta t_p[1 - j\,\frac{2}{\pi}\,\log\,\frac{\gamma k\Delta t_p}{4e}] \tag{2-48}$$

Therefore Z_{mn} can be expressed as

$$Z_{mn} = \frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4} \Delta t_p \Delta t_q [\omega\mu\; T_p T_q (u_p \cdot u_q) - \frac{1}{\omega\varepsilon}\, T'_p\; T'_q]Z \tag{2-49}$$

where

$$Z = \begin{cases} H_o^{(2)}(kR_{pq}) & \text{noncoincident intervals} \\ 1-j\,\frac{2}{\pi}\,\log\,\frac{\gamma k\Delta t_p}{4e} & \text{coincident intervals} \end{cases}$$

Equation (2-49) is used to compute the two parts of each matrix element, with ε_o, μ_o, and k_o in one part and ε,μ, and k in the other. It can be readily observed that the use of equation (2-49) will lead to a symmetric [Z] matrix.

2.3 Evaluation of [B] and [C] Matrix Elements

Matrix elements for [B] are expressed by equation (2-16)

$$B_{mn} = \int_c \underset{\sim}{W}^e_m \cdot (-jC)(\underset{\sim}{M}_n)d\ell \tag{2-50}$$

Because of the discontinuity of the curl operator at the boundary, care should be exercised in evaluating equation (2-50). The Green's function is singular, and a simple interchange of differentiation and integration is not always possible. Note that the operator C consists of two kinds of operators, namely, C' and C". C' is for the outside field, and C" for the inside field, with respect to the material body. In the following development, the symbol C can be either C' or C" unless stated otherwise. Since the incident field, $\underset{\sim}{H}^i$ in the present case, is considered to be axially directed, there will be a circumferentially directed electric current and an axially directed magnetic current. Let

$$\underset{\sim}{W}^e_m = T(\ell - \ell_k)\underset{\sim}{u}_\ell$$

and (2-51)

$$\underset{\sim}{M}_n = T(\ell - \ell_k)\underset{\sim}{u}_z$$

Hence, equation (2-51) takes the form

$$B_{mn} = -\frac{1}{4}\int T(\ell-\ell_m)\underset{\sim}{u}_\ell \cdot \underset{\sim}{\nabla} \times \underset{\sim}{u}_z \int T(\ell'-\ell_n)\, H_o^{(2)}(kR)d\ell'd\ell$$

$$= -\frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4}\int_{t_q}^{t_{q+1}} T_q\underset{\sim}{u}_\ell \cdot \underset{\sim}{\nabla} \times \underset{\sim}{u}_z \int_{t_p}^{t_{p+1}} T_p H_o^{(2)}(kR)d\ell'd\ell \tag{2-52}$$

Since T_p and T_q are constant between t_p and t_{p+1}, t_q and t_{q+1}, respectively, hence they can be taken outside the integral signs, so

$$B_{mn} = -\frac{1}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} T_p T_q \int_{t_q}^{t_{q+1}} \underset{\sim}{u}_\ell \cdot \underset{\sim}{\nabla} \times \underset{\sim}{u}_z \int_{t_p}^{t_{p+1}} H_o^{(2)}(kR)\, d\ell' d\ell \qquad (2\text{-}53)$$

Again, two situations will be treated separately.

(i) Non-coincident p and q intervals

The Hankel function is continuous and differentiable. After performing the indicated curl operation in equation (2-53), and noting that u_ℓ refers to q coordinates, then

$$B_{mn} = -\frac{1}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} \Delta t_p \left\{ \frac{\partial H_o^{(2)}(kR_{pq})}{\partial y_q} \Delta x_q - \frac{\partial H_o^{(2)}(kR_{pq})}{\partial x_q} \Delta y_q \right\} T_p T_q \qquad (2\text{-}54)$$

where

$$R_{pq} = [(x_q - x_p)^2 + (y_q - y_p)^2]^{1/2} \qquad (2\text{-}55)$$

and

$$\frac{\partial H_o^{(2)}(kR)}{\partial x} = -\, kH_1^{(2)}(kR) \frac{x - x_p}{R} \qquad (2\text{-}56)$$

$$\frac{\partial H_o^{(2)}(kR)}{\partial y} = -\, kH_1^{(2)}(kR) \frac{y - y_p}{R} \qquad (2\text{-}57)$$

Hence

$$B_{mn} = -\frac{1}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} \Delta t_p \frac{-\, T_p T_q kH_1^{(2)}(kR)}{R} [-\,(y_p - y_q)\Delta x_q + (x_p - x_q)\Delta y_q] \qquad (2\text{-}58)$$

Equation (2-58) is obtained through the application of triangle expansion and testing function employing a four-pulse approximation to the triangle, and the integrand was evaluated at the midpoint of each pulse interval.

(ii) Coincident p and q

Note that in this case, the method used in evaluating the improper integral for [Z] can not be applied here because the curl operator is not continuous across the boundary, for instance

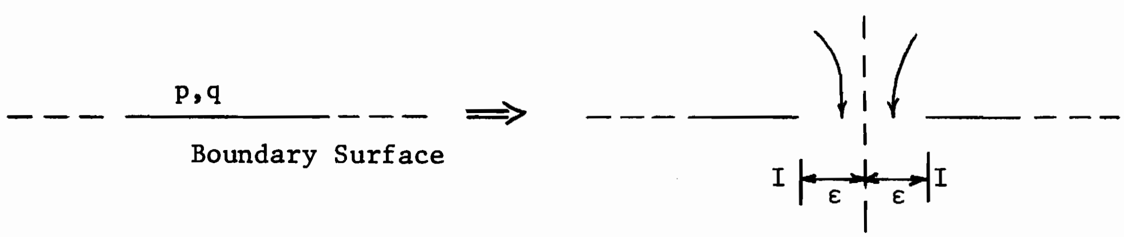


Fig. 2-6 Boundary Surface

By visualizing a current sheet that flows into the paper as shown in Fig. 2-6, it is evident that the tangential field component will decidedly be zero as $\varepsilon \to 0$.

A better way is to find the field at a point above the boundary surface, then find the limit as it approaches the boundary surface from above. Before performing the limiting process, the integrals in equation (2-53) becomes

$$\int_{t_q}^{t_{q+1}} \underset{\sim}{u}_\ell \cdot \underset{\sim}{\nabla} \times \underset{\sim}{u}_z \int_{t_p}^{t_{p+1}} H_o^{(2)}(kR)\,d\ell'\,d\ell$$

$$= \Delta t_p \int_{t_q}^{t_{q+1}} [\underset{\sim}{u}_x \frac{\partial H_o^{(2)}(kR)}{\partial y_q} - \underset{\sim}{u}_y \frac{\partial H_o^{(2)}(kR)}{\partial x_q}] \cdot [\underset{\sim}{u}_x dx_q + \underset{\sim}{u}_y dy_q]$$

Note that in equation (2-58) $\underset{\sim}{u}_\ell$ refers to the q-coordinates and the p-integral is evaluated as the product of the integrand sampled at the midpoint of the p-interval times the interval length.

A local coordinate system is constructed for the evaluation of the improper integral as $R \to 0$. A local coordinate system is shown in Fig. 2-7.

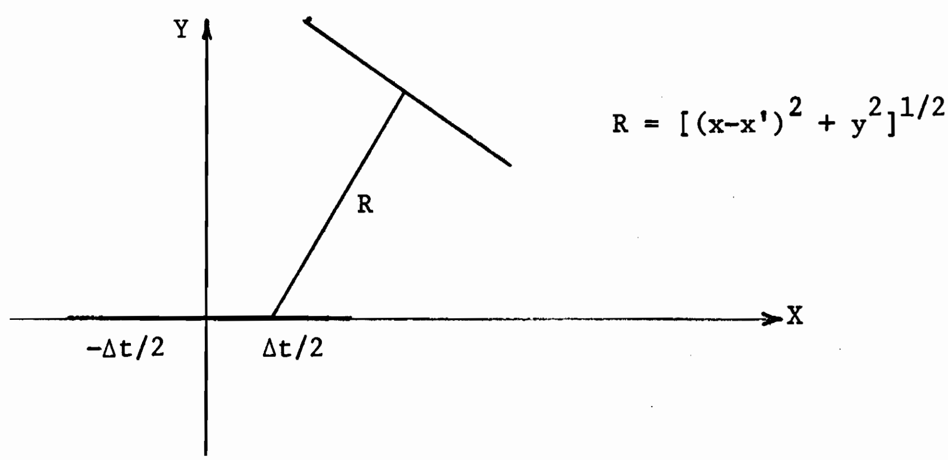


Fig. 2-7. A local coordinate system.

Next, consider the term

$$\int_{-\Delta t/2}^{\Delta t/2} \frac{\partial H_o^{(2)}(kR)}{\partial y} dx' \tag{2-59}$$

and noting that

$$\frac{\partial H_o^{(2)}(kR)}{\partial y} = -kH_1^{(2)}(kR)\frac{\partial R}{\partial y} \tag{2-60}$$

By using small argument approximation for $H_1^{(2)}(kR)$

$$\begin{aligned} H_1^{(2)}(kR) &= J_1(kR) - jN_1(kR) \\ &\approx \frac{kR}{2} + j\frac{2}{\pi}\frac{1}{kR} \end{aligned} \tag{2-61}$$

equation (2-59) becomes

$$\int_{-\Delta t/2}^{\Delta t/2} \frac{\partial H_o^{(2)}(kR)}{\partial y} dx' = -k\int_{-\Delta t/2}^{\Delta t/2} [\frac{kR}{2} + j\frac{2}{\pi kR}]\frac{\partial R}{\partial y} dx'$$

$$= -\frac{k^2}{2}y\Delta t - j\frac{2}{\pi}\left[\tan^{-1}\left(\frac{\frac{\Delta t}{2}-x}{y}\right) - \tan^{-1}\left(\frac{-\frac{\Delta t}{2}-x}{y}\right)\right] \tag{2-62}$$

Note that as $x \to 0$, $y \to 0$, the improper integral approaches $-j2$.

Finally the expression for B_{mn} can be stated as follows.

$$B_{mn} = -\frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4} \Delta t_p T_p T_q B \tag{2-63}$$

where

$$B = -\frac{kH_1^{(2)}(kR_{pq})}{R_{pq}}[-(y_p - y_q)\Delta x_q + (x_p - x_q)\Delta y_q]$$

(non-coincident)

$$B = -j2 \qquad \text{(coincident)}$$

Equation (2-63) is used to compute the two parts of each [B] matrix element, R in one part (due to primed operator) and k in the other (due to doubly primed operator). Since C' is an outside operator and C" is an inside operator, the values of B for the coincidental case will have opposite signs. Hence the coincidental-pulse-interval situation contributes nothing to the values of the matrix elements.

In spite of the fact that $[C] = [\tilde{B}]$, it is advantageous to evaluate C_{mn} explicitly. The procedures involved will be essentially the same as in evaluating B_{mn}. Recall equation (2-17), and it can be expressed in greater detail as

$$C_{mn} = \int_C \underset{\sim}{W}_m^m \cdot (-jC)(\underset{\sim}{J}_n)d\ell$$

A specific form, suitable for computational purposes is developed in a manner similar to that used for B_{mn}. Considerations governing the choices of expansion and testing functions are the same as those discussed at the beginning of this section. Note that the electric surface current, in the present case, is circumferentially directed.

$$\underset{\sim}{W}_m^m = T(\ell - \ell_k)\underset{\sim}{u}_z$$
$$\underset{\sim}{J}_n = T(\ell - \ell_k)\underset{\sim}{u}_\ell \qquad (2\text{-}65)$$

where T is defined by equation (2-38) and $\underset{\sim}{u}_z$ is the unit vector in the direction of the axis; $\underset{\sim}{u}_\ell$ is the unit vector along the cross-sectional contour. C_{mn} can be expressed as

$$C_{mn} = -\frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4}\int_{t_q}^{t_{q+1}} T_q \underset{\sim}{u}_z \cdot \underset{\sim}{\nabla} \times \underset{\sim}{u}_\ell \int_{t_p}^{t_{p+1}} T_p H_o^{(2)}(kR)\,d\ell' d\ell \tag{2-66}$$

The evaluation of the integrals appearing in equation (2-66) is facilitated by approximating the triangle function by four pulses. The index p = 1,2,3,4 is associated with each pulse, respectively, for the nth expansion function, while the index q = 1,2,3,4 is similarly defined for the mth testing function. Since T_p is constant between t_p and t_{p+1} and T_q is constant between t_q and t_{q+1}, they can be taken outside the integral signs. $\underset{\sim}{u}_\ell$ is the unit vector along the contour with respect to p coordinates. Hence

$$\begin{aligned} C_{mn} &= -\frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4} T_p T_q \Delta t_q \int_{t_p}^{t_{p+1}} \underset{\sim}{u}_z \cdot \underset{\sim}{\nabla} \times [\underset{\sim}{u}_x dx_p H_o^{(2)}(kR) \\ &\qquad + \underset{\sim}{u}_y dy_p H_o^{(2)}(kR)] \\ &= -\frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4} T_p T_q \Delta t_q \int_{t_p}^{t_{p+1}} [\frac{\partial H_o^{(2)}(kR)}{\partial x_q} dy_p - \frac{\partial H_o^{(2)}(kR)}{\partial y_q} dx_p] \end{aligned} \tag{2-67}$$

Note that the integral

$$\int_{t_p}^{t_{p+1}} [\frac{\partial H_o^{(2)}(kR)}{\partial x_q} dy_p - \frac{\partial H_o^{(2)}(kR)}{\partial y_q} dx_p] \tag{2-68}$$

can be evaluated just like in the previous B_{mn} case. It follows that equation (2-68) becomes

$$-kH_1^{(2)}(kR_{pq})[-\frac{x_p - x_q}{R_{pq}}\Delta y_p + \frac{y_p - y_q}{R_{pq}}\Delta x_p] \quad (2\text{-}69)$$

For coincident p-pulse and q-pulse intervals, the evaluation of the improper integral is identical to that developed for B_{mn}. To this point, the matrix elements for [C] can be conveniently specified as

$$C_{mn} = -\frac{1}{4}\sum_{q=1}^{4}\sum_{p=1}^{4}\Delta t_q\, T_p T_q C \quad (2\text{-}70)$$

where

$$C = -\frac{kH_1^{(2)}(kR)}{R_{pq}}[-(x_p - x_q)\Delta y_p + (y_p - y_q)\Delta x_p]$$

(noncoincident intervals)

$$C = -j2$$

(coincident intervals)

Equation (2-70) is used to compute the two parts of each [C] matrix elements; one part is due to the outside operator C', and the other is due to the inside operator C". Again, for the coincidental-pulse-interval situation, the net contribution, to the value of each matrix element, is zero, because the two values of C in equation (2-70) have opposite signs.

2.4 Evaluation of [Y] Matrix Elements

The only expression left to be developed is that for the [Y] matrix elements. By equation (2-18)

$$Y_{mn} = \int_C \underset{\sim}{W}_m^m \cdot L_m(\underset{\sim}{M}_n)d\ell$$

$$= \int \underset{\sim}{W}_m^m \cdot (j\omega\underset{\sim}{F}_n + \nabla\underset{\sim}{\phi}_n^m)d\ell \quad (2\text{-}71)$$

The superscript, m, indicates magnetic quantities. By equations (2-26) to (2-28), the following is obtained

$$Y_{mn} = j\omega \int_C (\underset{\sim}{W}_m \cdot \underset{\sim}{F}_n + \sigma_m \phi_n^m) d\ell \tag{2-72}$$

In this case all the currents are axially directed. The expansion and testing functions are chosen as

$$\underset{\sim}{W}_k = \underset{\sim}{M}_k = T(\ell - \ell_k) \underset{\sim}{u}_z \tag{2-73}$$

where T is the triangle function defined by equation (2-38), and $\underset{\sim}{u}_z$ is the unit vector in the axial direction. The continuity equation in this case is

$$\sigma_n = -\frac{1}{j\omega} \underset{\sim}{\nabla} \cdot \underset{\sim}{M}_n \tag{2-74}$$

Note that $\underset{\sim}{M}_n$ is $\underset{\sim}{u}_z$ directed, so

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{M}_n = 0 \tag{2-75}$$

and it follows that

$$\sigma_n = 0 \quad \text{and} \quad \phi_n^m = 0 \tag{2-76}$$

Therefore

$$Y_{mn} = j\omega \int_c \underset{\sim}{W}_m^m \cdot \underset{\sim}{A}_n^m \, d\ell$$

$$= \frac{\omega\varepsilon}{4} \int_c \int_c (\underset{\sim}{W}_m \cdot \underset{\sim}{M}_n) H_o^{(2)}(kR) d\ell' d\ell \tag{2-77}$$

where the unprimed integration is taken over field points and the primed integration over the source points. A specific form is developed in a manner similar to that used for Z_{mn}. Equation (2-77) can be expressed in greater detail as

$$Y_{mn} = \frac{\omega\varepsilon}{4} \int_{\ell_{m-2}}^{\ell_{m+2}} \int_{\ell_{n-2}}^{\ell_{n+2}} [T(\ell-\ell_m)\ T(\ell'-\ell_m')\ H_o^{(2)}(kR)]d\ell' d\ell \qquad (2\text{-}78)$$

The evaluation of the integrals appearing in equation (2-78) is carried out by approximating the triangle function by four pulses as shown in Fig. 2-3. Equation (2-78) can be written as

$$Y_{mn} = \frac{\omega\varepsilon}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} \int_{t_q}^{t_{q+1}} \int_{t_p}^{t_{p+1}} T_p T_q H_o^{(2)}(kR) d\ell' d\ell \qquad (2\text{-}79)$$

The indices p and q have the usual meaning. T_p and T_q have already been defined by equation (2-42). Each of the sixteen terms contributing to Y_{mn} falls into one of two categories. Either the p-pulse and q-pulse intervals coincide, or they do not. If the latter is true, each integral is approximated by the product of its integrand sampled at the interval midpoint times the interval length. The expression for Y_{mn} becomes

$$Y_{mn} = \frac{\omega\varepsilon}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} T_p T_q\ \Delta t_p \Delta t_q\ H_o^{(2)}(kR_{pq}) \qquad (2\text{-}80)$$

As previously stated, Δt_p and Δt_q are defined by equation (2-45) and R_{pq} is the distance between the midpoints of the p-pulse and q-pulse intervals. For coincident p-pulse and q-pulse intervals, the improper integral and its evaluation are the same as those in Z_{mn}. Hence

$$\int_{t_q}^{t_{q+1}} \int_{t_p}^{t_{p+1}} T_p T_q \; H_o^{(2)}(kR)\, d\ell' d\ell$$

$$= T_p T_q \; \Delta t_p \Delta t_q \; [1 - j \frac{2}{\pi} \log (\frac{\gamma k \, \Delta t_p}{4e})] \qquad (2\text{-}81)$$

where $\log \gamma$ = Euler's constant

$e = 2.718 \ldots.$

The final expression for Y_{mn} is

$$Y_{mn} = \frac{\omega\varepsilon}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} T_p T_q \; \Delta t_p \Delta t_q Y \qquad (2\text{-}82)$$

where

$$Y = H_o^{(2)}(kR_{pq}) \quad \text{(non-coincident intervals)}$$

$$= [1 - j \frac{2}{\pi} \log (\frac{\gamma k \Delta t_p}{4e})] \quad \text{(coincident intervals)}$$

2.5 Excitation Matrix, Measurement Matrix, and Scattering Cross Section

The matrix elements of the excitation matrix are represented by two expressions, equations (2-19) and (2-20). It is important to realize that the transformation of (2-19) and (2-20) into computable forms depends on the type and polarization of the impressed field. In the case under consideration, the excitation is assumed to be a z-directed magnetic field of unit magnitude. The incident field is given by

$$\underset{\sim}{H_z^i}(\rho) = e^{-j\underset{\sim}{k} \cdot \underset{\sim}{\rho}} \qquad (2\text{-}83)$$

The wave number vector k points in the direction of travel of the incident wave. A coordinate system for the evaluation of the excitation matrix elements is shown in Fig. 2-8.

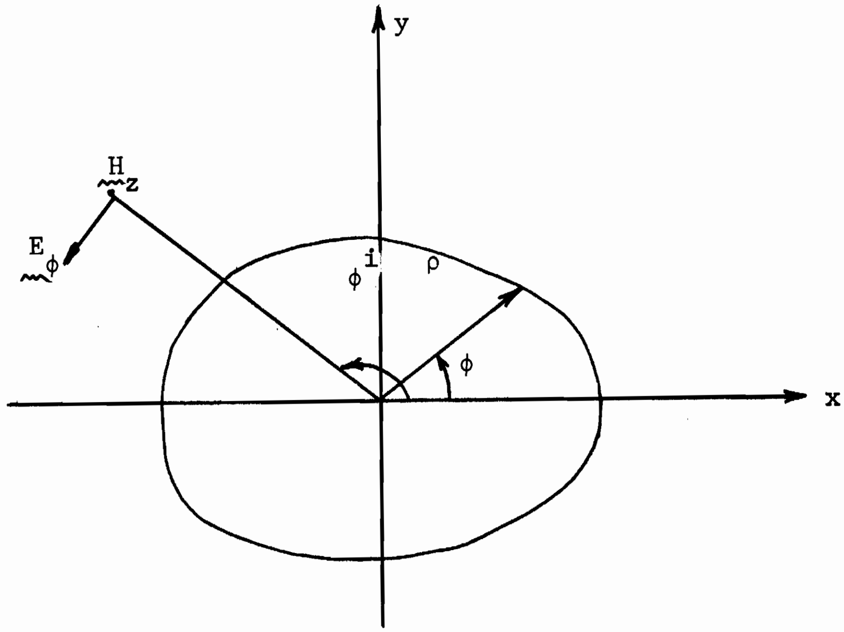


Fig. 2-8. Incident field.

Equation (2-19) will be considered first. The testing function is

$$\underset{\sim}{W}_m^e = T(\ell - \ell_k)\underset{\sim}{u}_\ell \tag{2-84}$$

For plane wave excitation the ϕ-directed electric field, associated with the z-directed magnetic field defined by equation (2-83), is

$$\begin{aligned} \underset{\sim}{E}^i &= -\eta \underset{\sim}{u}_\phi e^{-j\underset{\sim}{k}\cdot\underset{\sim}{\rho}} \\ &= -\eta(-\underset{\sim}{u}_x \sin\phi^i + \underset{\sim}{u}_y \cos\phi^i)e^{jk(x_{mp}\cos\phi^i + y_{mp}\sin\phi^i)} \end{aligned} \tag{2-85}$$

where $\underset{\sim}{u}_\phi$ = ϕ-directed unit vector

$\underset{\sim}{u}_x$ = x-directed unit vector

$\underset{\sim}{u}_y$ = y-directed unit vector

ϕ^i = incident angle

x_{mp}, y_{mp} = midpoint coordinates of each straight line segment

η = intrinsic wave impedance

By using four-pulse approximations for the triangle testing functions, equation (2-19) can be expressed as

$$V_m^i = \sum_{p=1}^{4} \int_{t_p}^{t_{p+1}} \underset{\sim}{u_\ell}\, T_p \cdot E^i(\rho)d\ell \tag{2-86}$$

Note that the excitation matrix element V_m^i is given as the component of $E^i(\rho)$ tangent to the contour for the mth triangle. The integral in equation (2-86) is evaluated as the tangential field component of E^i sampled at the midpoint of each p-pulse interval. Hence

$$V_m^i = - \eta \sum_{p=1}^{4} T_p \{ e^{jk(x_{mp}\cos\phi^i + y_{mp}\sin\phi^i)} [- \Delta x \sin\phi^i + \Delta y \cos\phi^i] \tag{2-87}$$

where Δx and Δy are the rectangular components of the pulse interval. A portion of the contour is shown in Fig. 2-9 which illustrates how equation (2-87) is obtained.

Equation (2-20) can be evaluated in a similar manner. The testing functions are triangle functions, and each triangle function is represented by equation (2-39) and chosen to be z-directed.

$$\underset{\sim}{W_k} = T(\ell - \ell_k)\underset{\sim}{u_z} \tag{2-88}$$

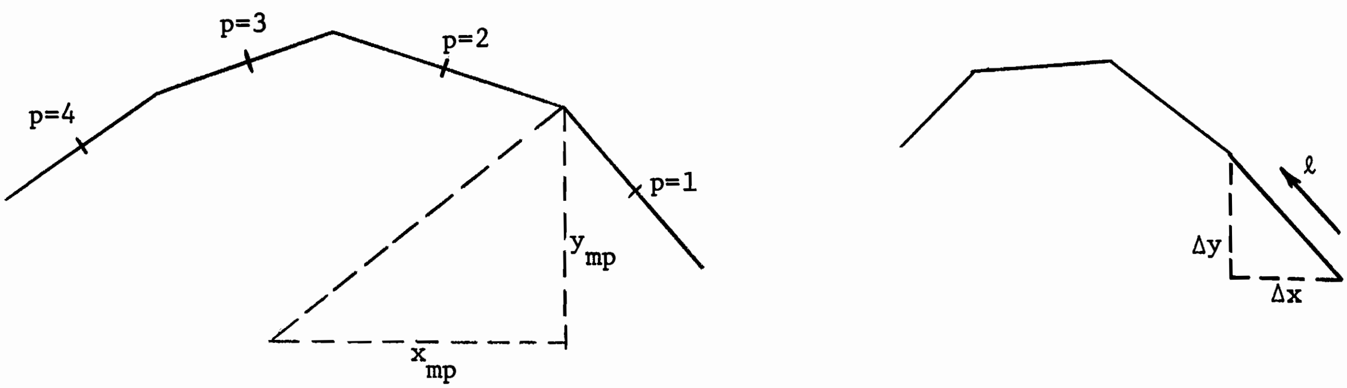


Fig. 2-9. A partial contour.

The evaluation of (2-20) is quite straightforward and the procedures are identical to those used in the evaluation of V_m^i. Hence

$$I_m^i = j \sum_{p=1}^{4} T_p \, \Delta t_p \, e^{jk(x_{mp} \cos \phi^i + y_{mp} \sin \phi^i)} \tag{2-89}$$

The distant scattered field can be evaluated by reciprocity. A z-directed magnetic current filament at ρ_o of strength M_p is adjusted to produce the unit plane wave incident on the material body

$$\underset{\sim}{H}^i = \underset{\sim}{u}_z \, e^{jk(x_n \cos \phi^s + y_n \sin \phi^s)} \tag{2-90}$$

Note that $\underset{\sim}{M}_p$ produces a ϕ-directed $\underset{\sim}{E}^i$ and a z-directed $\underset{\sim}{H}^i$

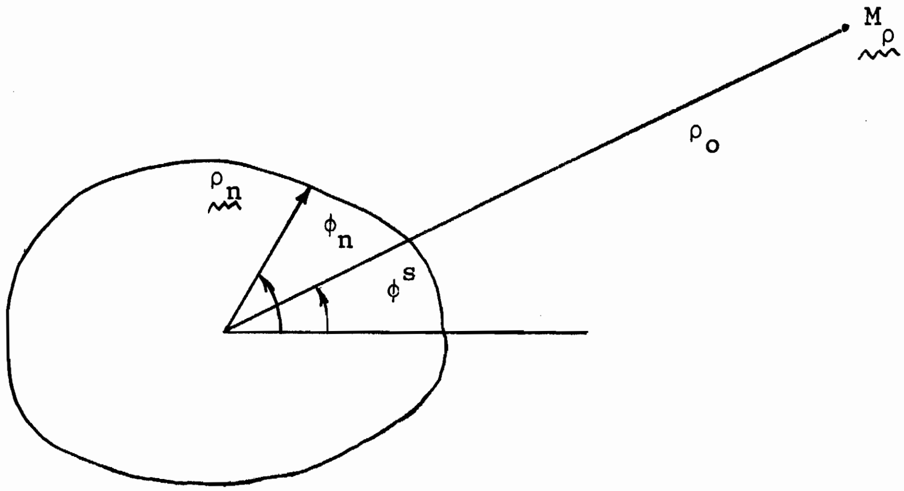


Fig. 2-10. A two-dimensional contour and z-directed magnetic current filament.

By reciprocity it is evident that

$$-H_\rho = \frac{1}{M} \int_C (\mathbf{E}^i \cdot \mathbf{J} - \mathbf{H}^i \cdot \mathbf{M})\, d\ell \tag{2-91}$$

or

$$-H_\rho = \frac{1}{M_\rho} \int_C \tilde{\mathbf{f}} \begin{bmatrix} \mathbf{E}^i \\ \\ j\mathbf{H}^i \end{bmatrix} d\ell \tag{2-92}$$

where

$$\mathbf{f} = \begin{bmatrix} \mathbf{J} \\ \\ j\mathbf{M} \end{bmatrix}$$

Equation (2-7) can be expressed in matrix form as

$$\mathbf{f} = \begin{bmatrix} [\tilde{I}_n] & [J_n] \\ \\ [\tilde{V}_n] & [M_n] \end{bmatrix} \tag{2-93}$$

With the help of equation (2-93), a new form for equation (2-92) is

$$- H_\rho = \frac{1}{M_\rho} [D] \begin{bmatrix} [I_n] \\ \\ [V_n] \end{bmatrix} \tag{2-94}$$

Note that $[I_n]$ and $[V_n]$ are column matrices, and the matrix [D] is

$$[D] = [[\int_C \underset{\sim}{E}^i \cdot [\underset{\sim}{\tilde{J}}_n] d\ell] \; [\int_C [\underset{\sim}{\tilde{M}}_n] \cdot j\underset{\sim}{H}^i d\ell]] \tag{2-95}$$

The values of I_n and V_n can be obtained from equation (2-14) by matrix inversion. The constant $1/M_\rho$ is that needed to produce a plane wave of unit amplitude at the origin, which is

$$\begin{aligned} \frac{1}{M_\rho} &= - \frac{k^2}{4\omega\mu} H_o^{(2)}(k\rho_o) \\ &= - \frac{\omega\varepsilon}{4} H_o^{(2)}(k\rho_o) \end{aligned} \tag{2-96}$$

Redefine equation (2-95) as

$$[D] = \{[D^e] \; [D^m]\} \tag{2-97}$$

where

$$D_n^e = \int_C \underset{\sim}{E}^i \cdot [\underset{\sim}{\tilde{J}}_n] d\ell \tag{2-98}$$

$$D_n^m = \int_C j\underset{\sim}{H}^i \cdot [\underset{\sim}{\tilde{M}}_n] d\ell \tag{2-99}$$

The evaluation of the integrals in equation (2-98) and (2-99) is straightforward, and the procedures involved are completely analogous to those used in the evaluation of V_m^i and I_m^i. Only the results will be given here

$$D_n^e = -\eta \sum_{p=1}^{4} T_p \, e^{jk(x_{np}\cos\phi^s + y_{np}\sin\phi^s)} [-\Delta x \sin\phi^s + \Delta y \cos\phi^s] \quad (2\text{-}100)$$

$$D_n^m = j \sum_{p=1}^{4} T_p \Delta t_p e^{jk(x_{np}\cos\phi^s + y_{np}\sin\phi^s)} \quad (2\text{-}101)$$

The scattered field can be expressed as

$$H_\rho = \frac{\omega\varepsilon}{4} H_o^{(2)}(k\rho_o)[[D_m^e][D_n^m]] \begin{bmatrix} [I_n] \\ \\ [V_n] \end{bmatrix} \quad (2\text{-}102)$$

or

$$H_\rho = \frac{\omega\varepsilon}{4} H_o^{(2)}(k\rho_o)[[D_n^e][D_n^m]] \begin{bmatrix} [Z] & [B] \\ \\ [C] & [Y] \end{bmatrix}^{-1} \begin{bmatrix} [V_n^i] \\ \\ [I_n^i] \end{bmatrix} \quad (2\text{-}103)$$

In the scattering problem, the bistatic scattering cross section σ is a parameter of interest. It is defined as the width for which the incident wave carries sufficient power to produce the field E_ρ, H_ρ by omnidirectional radiation. It may be expressed as

$$\sigma(\phi^s) = 2\pi \rho_o \lim_{\rho_o \to \infty} \left| \frac{E_\rho(\phi^s)}{\eta} \right|^2$$

or

$$= 2\pi \rho_o \lim_{\rho_o \to \infty} \left| H_\rho(\phi^s) \right|^2 \quad (2\text{-}104)$$

The large argument approximation for $H_o^{(2)}(kR)$ is

$$H_o^{(2)}(x) \xrightarrow[x\to\infty]{} \sqrt{\frac{2j}{\pi x}}\, e^{-jx} \tag{2-107}$$

The expression for the scattering cross section can be stated as

$$\sigma = \lim_{\rho_o \to \infty} 2\pi\rho_o \left| \frac{\omega\varepsilon}{4} \sqrt{\frac{2}{\pi k \rho_o}}\, h \right|^2$$

$$= \frac{k}{4\eta^2} |h|^2 \tag{2-106}$$

where

$$h = \left| [D] \begin{bmatrix} [Z] & [B] \\ [C] & [Y] \end{bmatrix}^{-1} \begin{bmatrix} [V_n^i] \\ [I_n^i] \end{bmatrix} \right|$$

CHAPTER 3

CHARACTERISTIC MODES - A SURFACE FORMULATION

3.1 Theoretical Development

The treatment of characteristic modes for perfectly conducting bodies, starting from the impedance operator for the conducting surface, has been given by Harrington and Mautz [2]. In terms of the polarization current and the magnetization current, a volume formulation of the characteristic mode theory for dielectric and magnetic bodies has also been treated [4]. In this chapter a theory of characteristic modes for material bodies (dielectric, magnetic, or both) based on a surface formulation is developed. The appropriate operator formulation of the problem is

$$\begin{bmatrix} Le & N \\ N & Lm \end{bmatrix} \begin{bmatrix} \underset{\sim}{J} \\ j\underset{\sim}{M} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{E}^{i} \\ j\underset{\sim}{H}^{i} \end{bmatrix} \tag{3-1}$$

To emphasize the symmetric nature of the matrix of operators, the off-diagonal operators are denoted by a single symbol, N. Define the following rise vectors

$$\underset{\sim}{f} = \begin{bmatrix} \underset{\sim}{J} \\ j\underset{\sim}{M} \end{bmatrix} , \qquad \underset{\sim}{g} = \begin{bmatrix} \underset{\sim}{E} \\ j\underset{\sim}{H} \end{bmatrix} \tag{3-2}$$

and the matrix of operators

$$T = \begin{bmatrix} Le & N \\ N & Lm \end{bmatrix} \tag{3-3}$$

where N = -jC.

Equation (3-1) can then be written as

$$\underset{\sim}{T}\underset{\sim}{f} = \underset{\sim}{g}^i$$

Define the symmetric product

$$\langle \underset{\sim}{f}, \underset{\sim}{g} \rangle = \iint_s \tilde{\underset{\sim}{f}}\underset{\sim}{g} \, ds$$

$$= \iint_s (\underset{\sim}{J} \cdot \underset{\sim}{E} - \underset{\sim}{M} \cdot \underset{\sim}{H}) ds \tag{3-4}$$

which, for f a source quantity and g a field quantity, is reaction. The product

$$\langle \underset{\sim}{f}^*, \underset{\sim}{g} \rangle = \iint_s \tilde{\underset{\sim}{f}}^* \underset{\sim}{g} \, ds$$

$$= \iint_s (\underset{\sim}{J}^* \cdot \underset{\sim}{E} + \underset{\sim}{M}^* \cdot \underset{\sim}{H}) ds \tag{3-5}$$

is a suitable inner product for the Hilbert space of functions $\underset{\sim}{f}$, $\underset{\sim}{g}$ in S. If $\underset{\sim}{f}$ is a source quantity and $\underset{\sim}{g}$ a field quantity, the real part of (3-5) is time average power, but the imaginary part of (3-5) differs from the usual imaginary power. It is easy to show that T is symmetric, that is, $\langle \underset{\sim}{f}_1, \underset{\sim}{T}\underset{\sim}{f}_2 \rangle = \langle \underset{\sim}{f}_2, \underset{\sim}{T}\underset{\sim}{f}_1 \rangle$ by reciprocity. The operator T can be expressed in terms of its Hermitian parts as $T = T_1 + jT_2$ where

$$T_1 = \frac{1}{2}(T + T^*) = \begin{bmatrix} R & N_1 \\ N_1 & G \end{bmatrix} \tag{3-6}$$

$$T_2 = \frac{1}{2j}(T - T^*) = \begin{bmatrix} X & N_2 \\ N_2 & B \end{bmatrix} \tag{3-7}$$

Here N_1 and N_2 are the Hermitian parts of N, R and X are the Hermitian parts of Z, G and B are the Hermitian parts of Y.

By equation (1-9), the fields due to J and M can be expressed as

$$L' \begin{bmatrix} J \\ \\ jM \end{bmatrix} = - \begin{bmatrix} E \\ \\ jH \end{bmatrix} \quad ; \quad L' = \begin{bmatrix} Le' & -jC' \\ \\ -jC' & Lm' \end{bmatrix} \tag{3-8}$$

As far as radiation is concerned, the contribution due to the doubly primed operators is zero. The power radiated by any $\underset{\sim}{J}$ and $\underset{\sim}{M}$ on S is given by

$$\begin{aligned} \text{Re}(P_s) &= - \text{Re} \iint (\underset{\sim}{E} \cdot \underset{\sim}{J}^* + \underset{\sim}{M} \cdot \underset{\sim}{H}^*) ds \\ &= - \text{Re} \iint (\underset{\sim}{E} \cdot \underset{\sim}{J}^* + \underset{\sim}{M}^* \cdot \underset{\sim}{H}) ds \\ &= \text{Re} \, \{\langle \underset{\sim}{f}^*, T\underset{\sim}{f} \rangle\} \end{aligned} \tag{3-9}$$

Hence the time average power delivered by a source f is

$$\text{Re}(P_s) = \text{Re} \, \langle \underset{\sim}{f}^*, T\underset{\sim}{f} \rangle \tag{3-10}$$

The imaginary part of $\langle \underset{\sim}{f}^*, T\underset{\sim}{f} \rangle$ is not simply related to reactive power.

Using six-vector notation, we formulate a theory of characteristic modes which parallels that of the volume formulation [4]. The eigenvalue equation defining the modes is

$$T_2(\underset{\sim}{f_n}) = \lambda_n T_1(\underset{\sim}{f_n}) \tag{3-11}$$

where T_1 and T_2 are real symmetric operators. Hence, all eigenvalues λ_n are real and all characteristic sources $\underset{\sim}{f_n}$ may be chosen real. In expanded form

$$\underset{\sim}{f}_n = \begin{bmatrix} \underset{\sim}{J}_n \\ j\underset{\sim}{M}_n \end{bmatrix} \tag{3-12}$$

which, for characteristic sources, implies that $\underset{\sim}{M}_n$ is imaginary and $\underset{\sim}{J}_n$ is real. The characteristic sources can be normalized to radiate unit power, and the usual orthogonality relationships expressed as

$$\langle \underset{\sim}{f}^*_m, T_1\underset{\sim}{f}_n \rangle = \langle \underset{\sim}{f}_m, T_1\underset{\sim}{f}_n \rangle = \delta_{mn}$$

$$\langle \underset{\sim}{f}^*_m, T_2\underset{\sim}{f}_n \rangle = \langle \underset{\sim}{f}_m, T_2\underset{\sim}{f}_n \rangle = \lambda_n\delta_{mn}$$

$$\langle \underset{\sim}{f}^*_m, T\underset{\sim}{f}_n \rangle = \langle \underset{\sim}{f}_m, T\underset{\sim}{f}_n \rangle = (1 + j\lambda_n)\delta_{mn} \tag{3-13}$$

where δ_{mn} is the Kronecker delta. The field

$$\underset{\sim}{g}_n = \begin{bmatrix} \underset{\sim}{E}_n \\ j\underset{\sim}{H}_n \end{bmatrix} \tag{3-14}$$

due to a source $\underset{\sim}{f}_n$ is called a characteristic field. In the radiation zone the characteristic field is of the form of an outward traveling wave, and it is completely characterized by either $\underset{\sim}{E}_n$ or $\underset{\sim}{H}_n$.

Let $\underset{\sim}{f}_n$ and $\underset{\sim}{f}_m$ be two characteristic sources. By equation (3-13), the following expression is true.

$$\langle \underset{\sim}{f}_m, T\underset{\sim}{f}_n \rangle = \left\langle [\underset{\sim}{J}_n \quad j\underset{\sim}{M}_n] \begin{bmatrix} Le & N \\ N & Lm \end{bmatrix} \begin{bmatrix} \underset{\sim}{J}_n \\ j\underset{\sim}{M}_n \end{bmatrix} \right\rangle$$

$$= 0 \quad \text{for } m \neq n \tag{3-15}$$

Equation (3-13) is essentially

$$\iint (\underset{\sim}{J}_m \cdot \underset{\sim}{E}_n - \underset{\sim}{H}_n \cdot \underset{\sim}{M}_m) ds = 0 \tag{3-]6}$$

where $\underset{\sim}{E}_n$ and $\underset{\sim}{H}_n$ are produced by $\underset{\sim}{f}_n$. Because $\underset{\sim}{J}_m$ is real and $\underset{\sim}{M}_m$ is imaginary, we have

$$\iint (\underset{\sim}{J}_m^* \cdot \underset{\sim}{E}_n + \underset{\sim}{H}_n \cdot \underset{\sim}{M}_m^*) ds = 0 \qquad (3\text{-}17)$$

It follows that

$$\text{Re} \iint (\underset{\sim}{E}_n \cdot \underset{\sim}{J}_m^* + \underset{\sim}{H}_n^* \cdot \underset{\sim}{M}_m) ds = 0 \qquad (3\text{-}18)$$

which means that the real part of the cross power is zero. In the radiation zone the characteristic waves are of the form of outward traveling wave, i.e.

$$\underset{\sim}{E}_n = \eta \, \underset{\sim}{H}_n \times \underset{\sim}{n} \qquad (3\text{-}19)$$

where $\underset{\sim}{n}$ is the unit radial vector on S_∞. The real part of the cross power can be expressed as

$$\text{Re} \iint \underset{\sim}{E}_m \times \underset{\sim}{H}_n^* \cdot ds = \text{Re} \iint \frac{\underset{\sim}{E}_m \cdot \underset{\sim}{E}_n^*}{\eta} ds = 0 \qquad (3\text{-}20)$$

The real part of the cross power between $j(\underset{\sim}{E}_n, \underset{\sim}{H}_n)$ and $(\underset{\sim}{E}_m, \underset{\sim}{H}_m)$ is also zero. Hence,

$$\frac{1}{\eta} \iint \underset{\sim}{E}_m \cdot \underset{\sim}{E}_n^* ds = \eta \iint \underset{\sim}{H}_m \cdot \underset{\sim}{H}_n^* ds = \delta_{mn} \qquad (3\text{-}21)$$

3.2 Characteristic Equation and Modal Representation

In the preceding section, the analytical development was based on the interpretation of operators. The reduction of operator equations to matrix equations can be effected in the usual manner by the method of moments.

Let

$$\underset{\sim}{f}_n = \sum_j (I_j \underset{\sim}{f}_j^e + V_j \underset{\sim}{f}_j^m) \tag{3-22}$$

where

$$\underset{\sim}{f}_j^e = \begin{bmatrix} \underset{\sim}{W}_j^e \\ 0 \end{bmatrix} \quad , \quad \underset{\sim}{f}_j^m = \begin{bmatrix} 0 \\ \underset{\sim}{W}_j^m \end{bmatrix} \tag{3-23}$$

After substituting equation (3-22) into equation (3-11), the following is obtained.

$$\{ \sum_j I_j T_2 \underset{\sim}{f}_j^e + \sum_j V_j T_2 \underset{\sim}{f}_j^m \} = \lambda_n \{ \sum_j I_j T_1 \underset{\sim}{f}_j^e + \sum_j V_j T_1 \underset{\sim}{f}_j^m \} \tag{3-24}$$

Perform inner product with electric testing function $\underset{\sim}{W}_i^e$,

$$\{ \sum_j I_j < \underset{\sim}{W}_i^e, T_2 \underset{\sim}{f}_j^e > + \sum_j V_j < \underset{\sim}{W}_i^e, T_2 \underset{\sim}{f}_j^m > \}$$

$$= \lambda_n \{ \sum_j I_j < \underset{\sim}{W}_i^e, T_1 \underset{\sim}{f}_j^e > + \sum_j V_j < \underset{\sim}{W}_i^e, T_1 \underset{\sim}{f}_j^m > \} \tag{3-25}$$

and with magnetic testing function $\underset{\sim}{W}_i^m$.

$$\{ \sum_j I_j < \underset{\sim}{W}_i^m, T_2 \underset{\sim}{f}_j^e > + \sum_j V_j < \underset{\sim}{W}_i^m, T_2 \underset{\sim}{f}_j^m > \}$$

$$= \lambda_n \{ \sum_j I_j < \underset{\sim}{W}_i^m, T_1 \underset{\sim}{f}_j^e > + \sum_j V_j < \underset{\sim}{W}_i^m, T_1 \underset{\sim}{f}_j^m > \} \tag{3-26}$$

Equation (3-25) and equation (3-26) can be put into one matrix equation.

$$\begin{bmatrix} [X] & [N_2] \\ [\widetilde{N_2}] & [B] \end{bmatrix} \begin{bmatrix} [I] \\ [V] \end{bmatrix}_n = \lambda_n \begin{bmatrix} [R] & [N_1] \\ [\widetilde{N_1}] & [G] \end{bmatrix} \begin{bmatrix} [I] \\ [V] \end{bmatrix}_n \tag{3-27}$$

The definitions of [X], $[N_2]$, [B], [R], $[N_1]$, [G], [I], and [V] are obvious by comparing equation (3-27) with equations (3-25) and (3-26).

Equation (3-27) is the eigenvalue equation which will be used in the actual computation of the modes. In abbreviated form, it becomes

$$[\ T_2\ ]\ [\ \underset{\sim}{f_n}\ ] = \lambda_n\ [\ T_1\ ]\ [\ \underset{\sim}{f_n}\ ] \tag{3-28}$$

Now, with the understanding that λ_n and $\underset{\sim}{f_n}$ can be found, the modal solution for $\underset{\sim}{f}$ can be expressed as

$$\underset{\sim}{f} = \sum_n \alpha_n \underset{\sim}{f_n} \tag{3-29}$$

Recall that

$$T\underset{\sim}{f} = \underset{\sim}{g^i} \tag{3-30}$$

After substituting equation (3-29) into equation (3-30) and performing the inner product with $\underset{\sim}{f_m}$, the following equation results.

$$\sum_n \alpha_n < \underset{\sim}{f_m},\ T\underset{\sim}{f_n} > = < \underset{\sim}{f_m},\ \underset{\sim}{g^i} > \tag{3-31}$$

Apply the orthogonality relationships given in equation (3-13). It follows that

$$\alpha_n = \frac{< \underset{\sim}{f_m},\ \underset{\sim}{g^i} >}{(1 + j\ \lambda_n) < \underset{\sim}{f_m},\ T\underset{\sim}{f_n} >} \tag{3-32}$$

Explicitly,

$$< \underset{\sim}{f_m},\ \underset{\sim}{g^i} > = \sum_i I_i < \underset{\sim}{f_i^e},\ \underset{\sim}{g^i} > + \sum_i V_i < \underset{\sim}{f_i^m},\ \underset{\sim}{g^i} > \tag{3-33}$$

The matrix equivalents of the orthogonality relationships for the characteristic currents, equation (3-13), are also of interest. For example, that for T_1 is

$$< \underset{\sim}{f}_m, T_1 \underset{\sim}{f}_n > = < \sum_i (I_i \underset{\sim}{f}_i^e + V_i \underset{\sim}{f}_i^m)_m, T_1 \sum_j (I_j \underset{\sim}{f}_j^e + V_j \underset{\sim}{f}_j^m)_n >$$

$$= \sum_i \sum_j \{ I_i I_j < \underset{\sim}{f}_i^e, T_1 \underset{\sim}{f}_j^e > + I_i V_j < \underset{\sim}{f}_i^e, T_1 \underset{\sim}{f}_j^m >$$

$$+ V_i I_j < \underset{\sim}{f}_i^m, T_1 \underset{\sim}{f}_j^e > + V_i V_j < \underset{\sim}{f}_i^m, T_1 \underset{\sim}{f}_j^m > \}$$

$$= [\widetilde{I}]_m [R] [I]_n + [\widetilde{I}]_m [N_1] [V]_n$$

$$+ [\widetilde{V}]_m [\widetilde{N_1}] [I]_n + [\widetilde{V}]_m [G] [V]_n$$

$$= [\widetilde{\underset{\sim}{f}_m}] [T_1] [\underset{\sim}{f}_n] = \delta_{mn} \qquad (3\text{-}34)$$

where denotes transpose. Similar derivations hold for T_2 and T.

3.3 Linear Measurement

Any scalar ρ linearly related to the generalized current, i.e. a linear functional of the equivalent electric and magnetic currents, will be called a linear measurement of the current.

Any linear functional of $\underset{\sim}{f}$ can be expressed as

$$\rho = < \underset{\sim}{g}^m, \underset{\sim}{f} > \qquad (3\text{-}35)$$

where g^m is a vector function which consists of an electric field and a magnetic field. By equations (3-32) and (3-33), the linear measurement of f can be stated as

$$\rho = \sum_n \frac{< \underset{\sim}{f}_n, \underset{\sim}{g}^i >}{(1 + j\lambda_n) < \underset{\sim}{f}_n, T_1 \underset{\sim}{f}_n >} < \underset{\sim}{g}^m, \underset{\sim}{f}_n > \qquad (3\text{-}36)$$

where

$$< \underset{\sim}{g}^m, \underset{\sim}{f}_n > = \sum I_{i_n} < \underset{\sim}{g}^m, \underset{\sim}{f}_i^e > = \sum_i V_{i_n} < \underset{\sim}{g}^m, \underset{\sim}{f}_i^m > \tag{3-37}$$

and define the following

$$K_n^m = <\underset{\sim}{g}^m, \underset{\sim}{f}_n> \;\; = \text{modal measurement coefficient} \tag{3-38}$$

$$<\underset{\sim}{g}^i, \underset{\sim}{f}_n> \;\; = \text{modal excitation coefficient} \tag{3-39}$$

Equation (3-36) is a symmetric bilinear functional of $\underset{\sim}{g}^i$ (the impressed field) and of $\underset{\sim}{g}^m$ (the measured field). The symmetry of (3-36) is a consquence of the symmetry of the original operator T. Equation (3-36) can be expressed as

$$\rho = \sum_n \frac{K_n^i K_n^m}{1 + j \lambda_n} \tag{3-40}$$

similarly, in terms of K_n^i equation (3-32) becomes

$$\alpha_n = \frac{K_n^i}{1 + j \lambda_n} \tag{3-41}$$

and equation (3-29) will take the form

$$\underset{\sim}{f} = \sum_n \frac{K_n^i}{1 + j \lambda_n} \underset{\sim}{f}_n \tag{3-42}$$

3.4 Characteristic Fields and Scattering Cross Section

The characteristic fields are linearly related to the characteristic currents, $\underset{\sim}{f}_n$, and hence can also be expressed in modal form.

$$\underset{\sim}{g} = \sum_n \frac{K_n^i}{1 + j \lambda_n} \underset{\sim}{g}_n \tag{3-43}$$

When K_n^i and $\mathbf{f}_n$ are known, the field pattern can be obtained by employing equation (3-43). A convenient way is to evaluate the modal measurement coefficient first. In the two-dimensional case, consider a magnetic current filament, $\mathbf{M} = M\,\mathbf{u}_m$ at (ρ, ϕ) on S_∞. See Fig. 3-1 below.

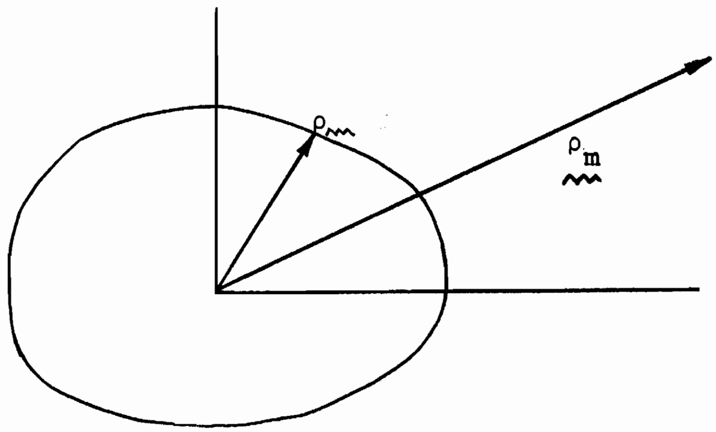


Fig. 3-1. A coordinate system for modal measurement coefficient

By reciprocity, it is readily seen that

$$-\mathbf{H}_n \cdot \mathbf{M} = \int (\mathbf{J}_n \cdot \mathbf{E}_m - \mathbf{M}_n \cdot \mathbf{H}_m)\, ds \tag{3-44}$$

where $\mathbf{H}_n$ is the characteristic field components produced by the mode current $\mathbf{f}_n$. $\mathbf{E}_m$ and $\mathbf{H}_m$ are the fields due to the magnetic current filament, $\mathbf{M}$. To simplify the analysis, the magnitude of the magnetic current, $\mathbf{M}$, is adjusted to produce a plane wave on the material body, i.e.

$$\mathbf{H}_m = \mathbf{u}_m\, e^{-j\mathbf{k}_m \cdot \mathbf{r}_m} \tag{3-45}$$

$$\mathbf{E}_m = \eta\, \mathbf{H}_m \times \mathbf{u}_{km} \tag{3-46}$$

where η is the wave impedance and $\underset{\sim}{u}_{km}$ is the unit vector in the direction of propagation. The right hand side of equation (3-44), in matrix form, is the modal measurement coefficient. Hence,

$$K_n^m = \int_c (\underset{\sim}{J}_n \cdot \underset{\sim}{E}_m - \underset{\sim}{M}_n \cdot \underset{\sim}{H}_m) \, d\ell \tag{3-47}$$

Explicitly, the electric field and the magnetic field can be extracted from equation (3-43) as

$$\underset{\sim}{E} = \sum_n \frac{K_n^i}{1 + j \lambda_n} \underset{\sim}{E}_n \tag{3-48}$$

$$\underset{\sim}{H} = \sum_n \frac{K_n^i}{1 + j \lambda_n} \underset{\sim}{H}_n \tag{3-49}$$

Since the magnetic field is currently under consideration, only equation (3-49) will be used. The component of the magnetic field on $\underset{\sim}{u}_m$ is

$$\underset{\sim}{H} \cdot \underset{\sim}{u}_m = \sum_n \frac{K_n^i}{1 + j \lambda_n} \underset{\sim}{H}_n \cdot \underset{\sim}{u}_m$$

$$= - \frac{1}{M} \sum_n \frac{K_n^i K_n^m}{1 + j \lambda_n}$$

$$= \frac{\omega \varepsilon}{4} H_o^{(2)} (k\rho) \sum_n \frac{K_n^i K_n^m}{1 + j \lambda_n} \tag{3-50}$$

Note that K_n^i is of the same functional form as K_n^m. Equation (2-98) has been used in deriving equation (3-50).

A commonly used parameter in plane wave scattering problems is the echo area. In two-dimensional problems the quantity " echo width " corresponds to the " echo area " of the three-dimensional problems.

The echo width is defined in equation (2-106).

$$\sigma = 2\pi \lim_{\rho_m \to \infty} \rho_m \left| \underset{\sim}{H} \cdot \underset{\sim}{u}_m \right|^2 \tag{3-51}$$

By equations (3-50) and (2-106), the following expression for the scattering cross section is obtained.

$$\sigma = \frac{k}{4\eta^2} \left| \sum_n \frac{K_n^i K_n^m}{1 + j\lambda_n} \right|^2 \tag{3-52}$$

3.5 <u>Computational Considerations</u>

The solution of the matrix eigenvalue problem, equation (3-28), will be discussed.

$$[\,T_2\,]\,[\,\underset{\sim}{f}\,] = \lambda\,[\,T_1\,]\,[\,\underset{\sim}{f}\,] \tag{3-53}$$

Note that the subscript n has been dropped for brevity. The conventional method for reducing (3-53) to a symmetric unweighted eigenvalue equation requires $[T_2]$ to be positive definite. In theory $[T_1]$ is positive semi-definite, but because of numerical inaccuracies it is actually indefinite, with some small negative eigenvalues. If the values of the matrix elements cover a very wide range, scaling will become desirable. The magnitude of the scale factor can be chosen as such that all scaled matrix elements will be brought, as close as possible, to the same order of magnitude. The conventional method will be modified as follows.

Let [D] be a diagonal matrix. After premultiplying by [D], equation (3-47) becomes

$$[\,D\,]\,[\,T_2\,]\,[\,\underset{\sim}{f}\,] = \lambda[\,D\,]\,[\,T_1\,]\,[\,\underset{\sim}{f}\,] \qquad (3\text{-}54)$$

Then observe that

$$[\,D\,][\,T_2\,][\,D\,]\,(\,[\,D\,]^{-1}[\,\underset{\sim}{f}\,]\,)$$
$$= \lambda\,[\,D\,][\,T_1\,][\,D\,]\,(\,[\,D\,]^{-1}[\,\underset{\sim}{f}\,]\,) \qquad (3\text{-}55)$$

The eigenvalue equation as given by (3-55) will have the same eigenvalues as the original unscaled equation, but the eigenvectors will be different. In other words the eigenvalues are not affected by the diagonal transformation. The original eigenvectors will be modified by [D] inverse. If the scale factor is s, [D] can be chosen as

$$[\,D\,] = \begin{bmatrix} 1/s & 0 \\ 0 & 1 \end{bmatrix} \qquad (3\text{-}56)$$

By equations (3-27), (3-55), and (3-56), the scaled eigenvalue equation is

$$[\,T_2^s\,][\,\underset{\sim}{f}^s\,] = \lambda[\,T_1^s\,][\,\underset{\sim}{f}^s\,] \qquad (3\text{-}57)$$

where

$$[\,T_2^s\,] = \begin{bmatrix} [\,X\,]/s^2 & [\,N_2\,]/s \\ [\,\widetilde{N_2}\,]/s & [\,B\,] \end{bmatrix} \qquad (3\text{-}58)$$

$$[\,T_1^s\,] = \begin{bmatrix} [\,R\,]/s^2 & [\,N_1\,]/s \\ [\,\widetilde{N_1}\,]/s & [\,G\,] \end{bmatrix} \qquad (3\text{-}59)$$

and

$$[\underset{\sim}{f}^s] = \begin{bmatrix} s[\ I\ ] \\ [\ V\ ] \end{bmatrix} \qquad (3\text{-}60)$$

Note that [I] and [V] ($\underset{\sim}{J}$ and $j\underset{\sim}{M}$) should first be recovered from the scaled eigenvectors before computing the surface currents and the scattered fields.

Rewrite equation (3-57) below

$$[\ T_2\ ][\ \underset{\sim}{f}\ ] = \lambda[\ T_1\ ][\ \underset{\sim}{f}\ ] \qquad (3\text{-}61)$$

Note that the superscripts have been dropped for brevity. An approximation will be used in finding the eigenvalues and eigenvectors. The eigenvalue equation

$$[\ T_1\ ]\ \underset{\sim}{r} = \mu\ \underset{\sim}{r} \qquad (3\text{-}62)$$

is used to find a set of basis functions for the T_1 vector space. An orthonormal set of vectors can be obtained by using the vectors $\{\ r_i\ \}$. Let $\{\ U_i\ \}$ be the set of orthonormal vectors, and let [U] be the orthogonal matrix which diagonalizes $[T_1]$ according to

$$[\ \widetilde{U}\ ][\ T_1\ ][\ U\ ] = \begin{bmatrix} \mu_1 & 0 & 0 & 0 & \cdots \\ 0 & \mu_2 & 0 & 0 & \cdots \\ 0 & 0 & \mu_3 & 0 & \cdots \\ 0 & 0 & 0 & \mu_4 & \cdots \\ \cdots & & & & \\ \cdots & & & & \\ \cdots & & & & \end{bmatrix} \qquad (3\text{-}63)$$

where the μ_i are the eigenvalues of $[T_1]$ ordered $\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 \geq \cdots\cdots$ Every column of [U] is in $\{\ U_i\}$. Only the larger μ_i can be considered accurate. All $\mu_i > M\mu_1$ are put in $[\ \mu_1\ ]$ where M is some small positive number set by the estimated accuracy of $[T_1]$. Usually M is anywhere between 10^{-3} and 10^{-6}. The diagonal matrix $[\mu]$ is then partitioned as

$$[\,\mu\,] = \begin{bmatrix} [\,\mu_1\,] & [\,0\,] \\ [\,0\,] & [\,\mu_2\,] \end{bmatrix} \tag{3-64}$$

where

$$[\,\mu_1\,] = \begin{bmatrix} \mu_1 & & & & 0 \\ & \mu_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \mu_m \end{bmatrix} \tag{3-65}$$

$$[\,\mu_2\,] = \begin{bmatrix} \mu_{m+1} & & & 0 \\ & \mu_{m+2} & & \\ & & \cdot & \\ 0 & & & \mu_n \end{bmatrix} \tag{3-66}$$

Now consider

$$\underset{\sim}{f} = \sum_{\ell=1}^{n} x_\ell \, \underset{\sim}{U}_\ell \tag{3-67}$$

where $\underset{\sim}{U}_\ell$ is a column vector of [U]. This is a valid expansion because the { $\underset{\sim}{U}_\ell$ } vectors form a basis for T_1 vector space. In matrix form equation (3-67) becomes

$$\underset{\sim}{f} = [\,U\,][\,\underset{\sim}{x}\,] \tag{3-68}$$

If [μ_2] is set to zero, it follows that all column vectors of [U] corresponding to all $\mu_k \in [\,\mu_2\,]$ are in the null space of T_1. This is illustrated in Fig. 3-2.

The expression for f as given in equation (3-67) can be written as

$$\underset{\sim}{f} = \sum_{i=1}^{m} x_i \underset{\sim}{U}_i + \sum_{k=m+1}^{n} x_k \underset{\sim}{U}_k \tag{3-69}$$

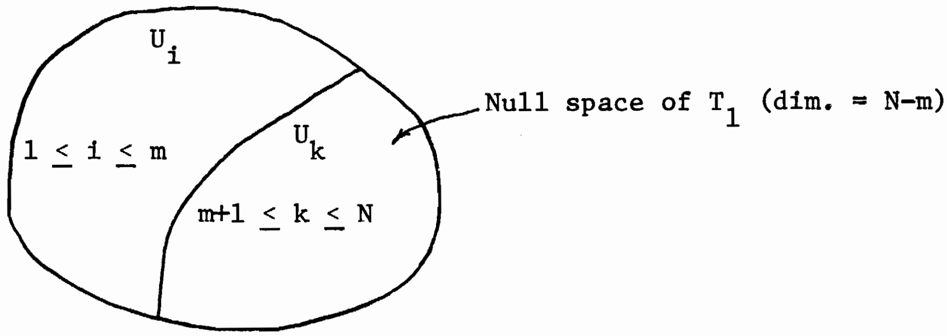


Fig. 3-2. T_1 vector space

The column vector $[\underset{\sim}{x}]$ in equation (3-68) can be partitioned as

$$[\ \underset{\sim}{x}\ ] = [\ \tilde{U}\ ][\ \underset{\sim}{f}\ ] = \begin{bmatrix} [\ \underset{\sim}{x}_1\ ] \\ [\ \underset{\sim}{x}_2\ ] \end{bmatrix} \tag{3-70}$$

where $[\underset{\sim}{x}_1]$ and $[\underset{\sim}{x}_2]$ are column vectors, and they are obtained by partitioning $[\underset{\sim}{x}]$ according to equation (3-69). Premultiply equation (3-61) by $[\tilde{U}]$ and use equations (3-63) and (3-68). The result is

$$[\ \tilde{U}\ ][\ T_2\ ][\ U\ ][\ \underset{\sim}{x}\ ] = \lambda[\ \mu\ ][\ \underset{\sim}{x}\ ] \tag{3-71}$$

Set $[\ \mu_2\ ]$ equal to zero and partition all other matrices conformably. The following two matrix equations are obtained.

$$[\ A_{11}\ ][\ \underset{\sim}{x}_1\ ] + [\ A_{12}\ ][\ \underset{\sim}{x}_2\ ] = \lambda[\ \mu_1\ ][\ \underset{\sim}{x}_1\ ] \tag{3-72}$$

$$[\ \tilde{A}_{12}\ ][\ \underset{\sim}{x}_1\ ] + [\ A_{22}\ ][\ \underset{\sim}{x}_2\ ] = 0 \tag{3-73}$$

Note that $[\ A\ ] = [\ \tilde{U}\ ][\ T_2\ ][\ U\ ]$. Equation (3-73) can be solved for $[\ x_2\ ]$ and the result substituted into equation (3-72) to get

$$[\ A_{11} - A_{12}A_{22}^{-1}\tilde{A}_{12}\ ][\ \underset{\sim}{x}_1\ ] = \lambda[\ \mu_1\ ][\ \underset{\sim}{x}_1\ ] \tag{3-74}$$

The brackets of submatrices have been dropped to conserve space.

Now $[\mu_1]$ has only positive diagonal elements as defined by equation (3-65). Observe that

$$[\ \mu_1\ ] = [\ \mu_1^{1/2}]\ [\ \mu_1^{1/2}] \qquad (3\text{-}75)$$

where

$$[\ \mu_1^{1/2}] = \begin{bmatrix} \mu_1^{1/2} & & & & & 0 \\ & \mu_2^{1/2} & & & & \\ & & \mu_3^{1/2} & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \mu_m^{1/2} \end{bmatrix} \qquad (3\text{-}76)$$

By equations (3-74) and (3-75) a new and unweighted eigenvalue equation is obtained.

$$[\ B\ ][\ \underset{\sim}{y}\ ] = \lambda[\ \underset{\sim}{y}\ ] \qquad (3\text{-}77)$$

where

$$[\ \underset{\sim}{y}\ ] = [\ \mu_1^{1/2}][\ \underset{\sim}{x_1}\ ] \qquad (3\text{-}78)$$

$$[\ B\ ] = [\ \mu_1^{-1/2}][\ A_{11} - A_{12}A_{22}^{-1}\tilde{A}_{12}\ ][\ \mu_1^{-1/2}] \qquad (3\text{-}79)$$

The eigenvalues of equation (3-77) are the smaller eigenvalues of the original equation (3-61), and the eigenvectors of (3-77) give the corresponding eigenvectors of equation (3-61) according to

$$\underset{\sim}{f} = [\ U\ ][\ \underset{\sim}{x}\ ] = [\ U\ ] \begin{bmatrix} [\ \delta\ ] \\ [-A_{22}^{-1}A_{12}] \end{bmatrix} [\ \mu_1^{-1/2}][\ \underset{\sim}{y}\ ] \qquad (3\text{-}80)$$

where $[\ \delta\ ]$ is the identity matrix.

Once the eigenvalues and the eigencurrents are known, the equivalent surface currents and scattered fields can be obtained by employing appropriate formulas for those quantities.

CHAPTER 4

RESULTS

The results of far field scattering calculations for some material cylinders are presented in this chapter. Equations used are those developed in Chapter 2 and Chapter 3.

The far field scattering patterns of circular material cylinders have been computed and the results are shown in Figures 4-1 through 4-16, for perpendicular polarization (TE). Figures 4-17 through 4-22 give the results for parallel polarization (TM). All results are compared with exact harmonic series solutions [7]. Figures 4-1 to 4-5 are obtained by using 15 triangle expansion functions. Twenty expansion functions have been used in obtaining Figures 4-6 to 4-22. In all figures the computed scattering cross section are normalized by πa, where "a" is the radius of the cylinder.

The normalized scattering cross sections of square cylinders are shown in Figures 4-23 through 4-27. All computed results are normalized by πb, where "b" is one-half the width of the square cylinder under consideration. Twenty expansions have been used in all computations for square cylinders of different material constants.

Figures 4-28 through 4-30 show the characteristic currents (or mode currents) for circular cylinders of different material constants. Fifteen expansion functions have been used for the computation of mode currents.

For representative computations, consider a circular cylinder with $ka = 0.7$ (where "a" is the radius of the cylinder, $\varepsilon_r = 9.5$, $\mu_r = 1.0$. The contour is approximated by 32 straight lines segments of equal length (the line segments can be of different length), and 15 expansion functions are used for both electric and magnetic surface currents. Figure 4-28 shows the characteristic currents plotted vs. the contour length variable in terms of a sequence of triangle functions. All the mode currents are composite currents. The first 15 points represent the electric mode current, and the second the magnetic current.

Figure 4-29 shows the characteristic currents for a circular cylinder with $\varepsilon_r = 50.0$, $\mu_r = 1.0$, and ka = 0.7. Figure 4-30 shows the characteristic currents for a circular cylinder of $\varepsilon_r = 2.56$. Note that every mode current is normalized by its maximum magnitude.

For perpendicular polarization (TE), the modal solution for the scattered field agrees extremely well with the scattered field computed directly from matrix inversion. The scattering cross sections using characteristic modes are almost identical to the matrix inversion solutions (the differences are less than 0.001 db).

To be specific, Fig. 4-1 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.5$, $\mu_r = 1.0$, and ka = 0.7 for perpendicular polarization (TE). The computed scattering cross section is in good agreement with harmonic solution [7]. The maximum deviation is 0.65 db. Figure 4-2 gives the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 20.0$, $\mu_r = 1.0$, and ka = 0.7, for perpendicular polarization (TE). The maximum deviation from exact harmonic series solution is 0.076 db. Figure 4-3 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 50.0$, $\mu_r = 1.0$, and ka = 0.7, for perpendicular polarization (TE). Maximum deviation from exact harmonic solution is 0.485 db. The scattering cross section shown in Fig. 4-4 is for a circular cylinder with $\varepsilon_r = 100.0$, $\mu_r = 0.01$, and ka = 0.7 for perpendicular polarization. The computed solution is in excellent agreement with the exact solution. Maximum deviation is 0.01 db. Figure 4-5 gives the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1000.0$, $\mu_r = 0.001$, and ka = 0.7, for perpendicular polarization. Note that the computed result is in excellent agreement with the calculations of a conducting cylinder. Maximum deviation is 0.01 db. The conducting cylinder problem can be viewed as a specialization of the more general material cylinder problem. This is expected to be true even for three-dimensional objects. Figure 4-6 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 1.0$, and ka = 2.0, for perpendicular polarization (TE). Maximum deviation from exact harmonic solution is 1.79 db. Better agreement can be reached, if more expansion functions are used. The scattering cross section given in Fig. 4-7 is for a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 1.0$, and ka = 1.0, for perpendicular polarization. The agreement with exact solution is excellent. Maximum deviation is 0.013 db. Figure 4-8 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 100.0$, and ka = 0.7, for perpendicular polarization. Agreement with exact solution is very good. Maximum deviation is 0.01 db. Figure 4-9 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 5.0$, and ka = 0.7, for perpendicular polarization. The computed result is in good agreement with exact solution. Maximum deviation is 0.3 db. The computed scattering cross section of a circular cylinder with $\varepsilon_r = 0.001$, $\mu_r = 1000.0$, and ka = 0.7, for perpendicular polarization is shown in Fig. 4-10. Note that the cylinder is highly magnetic. Maximum deviation from exact harmonic solution is 0.001 db. The agreement is excellent. Figure 4-11 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1.0$, $\mu_r = 1000.0$, and ka = 0.7, for perpendicular polarization. Maximum deviation from exact solution is 0.04 db. Figure 4-12 represents the computed scattering cross section of a circular cylinder with $\varepsilon_r = 1.0$, $\mu_r = 10.0$, and ka = 0.7, for perpendicular polarization. Maximum deviation from exact solution is 0.04 db. Figure 4-13 shows the computed scattering cross section of a circular cylinder with $\varepsilon_r = 1.0$, $\mu_r = 300$, and ka = 0.7, for perpendicular polarization. Maximum deviation from exact harmonic solution is 0.2 db. Figure 4-14 gives the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 2.56$, $\mu_r = 1.0$, and ka = 0.7, for perpendicular polarization. Maximum deviation is 0.6 db. Figure 4-15 shows the computed scattering cross section of a circular cylinder with $\varepsilon_r = 1000.0$, $\mu_r = 0.001$, and ka = 0.7, for perpendicular polarization. The computed solution is in excellent agreement with exact solution. Maximum deviation is 0.01 db. The computed scattering cross sections of a circular cylinder with ka = 0.7, are given in Fig. 4-16 for three different sets of material constants; i) $\varepsilon_r = 1000.0$, $\mu_r = 1.0$ i) $\varepsilon_r = 10000.0$, $\mu_r = 1.0$ iii) $\varepsilon_r = 5.0$, $\mu_r = 10^{-6}$. All are for perpendicular polarization. Figure 4-17 shows the normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1000.0$, $\mu_r = 0.001$, and ka = 0.7, for parallel polarization (TM). The solution agrees excellently with conducting cylinder solution. Maximum deviation is 0.023 db. The normalized scattering cross section of a circular cylinder with $\varepsilon_r = 2.56$, $\mu_r = 1.0$, and ka = 0.7, for parallel polarization is shown in Fig. 4-18. Maximum deviation from exact solution is 0.5 db. Figure 4-19 represents the computed scattering cross section of a circular cylinder with $\varepsilon_r = 20.0$, $\mu_r = 1.0$, and ka = 0.7, for parallel polarization. Maximum deviation from exact solution is 0.2 db. Figure 4-20 shows the computed scattering cross section of a circular cylinder with $\varepsilon_r = 50.0$, $\mu_r = 1.0$, and ka = 0.7, for parallel polarization. The computed solution is in excellent agreement with exact harmonic solution. Maximum deviation is 0.05 db. Figure 4-21 shows the computed scattering cross section of a circular cylinder with $\varepsilon_r = 4.0$, $\mu_r = 1.0$, and ka = 0.7, for parallel polarization. Maximum deviation from exact solution is 0.2 db. Figure 4-22 shows the computed scattering cross section of a circular cylinder with $\varepsilon_r = 9.5$, $\mu_r = 1.0$, and ka = 0.7, for parallel polarization. The computed solution is in excellent agreement with exact harmonic solution maximum deviatwon is 0.01 db. Figure 4-23 shows the computed scattering cross sections of a square cylinder with kb = 1.4, for two sets of material constants: i) $\varepsilon_r = 1000.0$, $\mu_r = 0.001$ ii) $\varepsilon_r = 1000.0$, $\mu_r = 1.0$, all for perpendicular polarization. For square cylinders, there are no exact solutions. Figure 4-24 shows the computed scattering cross section of a square cylinder with $\varepsilon_r = 10000.0$, $\mu_r = 0.0001$, and kb = 1.4, for perpendicular polarization. The computed solution has been compared with the solution of a conducting square cylinder by using E-field formulation [13]. Maximum deviation is 0.1 db. Figure 4-25 shows the computed scattering cross section of a square cylinder with $\varepsilon_r = 9.0$, $\mu_r = 1.0$, and kb = 1.4, for perpendicular polarization. Figure 4-26 shows the computed scattering cross section of a square cylinder with $\varepsilon_r = 100.0$, $\mu_r = 1.0$, and kb = 1.4, for parallel polarization. Figure 4-27 shows the scattering cross section of a square cylinder with $\varepsilon_r = 10000.0$, $\mu_r = 0.0001$, and kb = 1.4, for parallel polarization. The computed result is in excellent agreement with conducting square cylinder solution. Figure 4-28 shows the lowest order characteristic currents, plotted as a function of the contour variable. The currents are normalized by choosing their maximum amplitude to be unity. The characteristic currents are for a circular cylinder with $\varepsilon_r = 9.5$, $\mu_r = 1.0$, and ka = 0.7, for perpendicular polarization. The electric part of each characteristic current is circumferentially directed and the magnetic part is axially directed. The scattering cross section computed from modal solution is almost identical to that from matrix inversion. Figure 4-29 shows the normalized characteristic currents for a circular cylinder with $\varepsilon_r = 50.0$, $\mu_r = 1.0$, and ka = 0.7, for perpendicular polarization. Figure 4-30 gives the normalized characteristic currents for a circular cylinder with $\varepsilon_r = 2.56$, $\mu_r = 1.0$, and ka = 0.7, for perpendicular polarization.

The purpose of this work is to show the feasibility that a surface formulation for the theory characteristic modes can be applied to the solution of scattering from material objects. For large cylinders, more expansion functions are needed. No attempt has been made to treat large objects. It is expected that this is one of the important areas for future research. Many questions are still left unanswered in the interpretation and application of characteristic modes to material objects. It is hoped that this work will be of some value to future researchers in their effort to gain a complete understanding of the theory of characteristic modes.

The eigenvalue equation (3-61) is

$$[T_2][f] = \lambda\ [T_1][f] \tag{4-1}$$

and the expression for the Rayleigh quotient associated with equation (4-1) is

$$\lambda_i = \frac{[\widetilde{f_i}][T_2][f_i]}{[f_i][T_1][f_i]} \tag{4-2}$$

The computed eigenvalues and their corresponding eigenvectors should satisfy equation (4-2). The Rayleigh quotient check is important because it gives some verification to the approximations used in numerical computation.

The quadratic term $[f_i][T_1][f_i]$ deserves some elaboration since it qppears frequently in equations. Note that

$$\begin{aligned} [\widetilde{f}][T_1][f] &= [\widetilde{x}][\widetilde{U}][T_1][U][x] \\ &= [\widetilde{x}][\mu][x] \\ &= [\widetilde{x_1}][\mu_1][x_1] + [\widetilde{x_2}][\mu_2][x_2] \end{aligned} \tag{4-3}$$

It has already been pointed out in Chapter 3 that $[x_2]$ is the component of an eigencurrent f that lies within the null space of T_1, in other words, $[x_2]$ does not radiate. Since approximations are made in the computational procedures, the eigencurrents will not be absolutely exact. Consequently, the second quadratic term on the right hand side of equation (4-3) will differ from zero, but it should be much smaller than the first quadratic term. To a certain degree, this will give some indication of the accuracy of the computed eigencurrents. The first quadratic term at the right hand side of equation (4-3) can be further expressed as

$$\widetilde{[x_1]}[\mu_1][x_1]$$

$$= \widetilde{[\mu_1^{-1/2}][y]}[\mu_1][\mu_1^{-1/2}][y]$$

$$= \widetilde{[y]}\widetilde{[\mu_1^{-1/2}]}[\mu_1][\mu_1^{-1/2}][y]$$

$$= \widetilde{[y]}[y]$$

$$= 1 \text{ (if } \{y_i\} \text{ are orthonormal)} \qquad (4\text{-}4)$$

In numerical computation, approximations are inevitable. Some special analytical manipulations such as those discussed above can often provide added insight to the correctness of the numerical results.

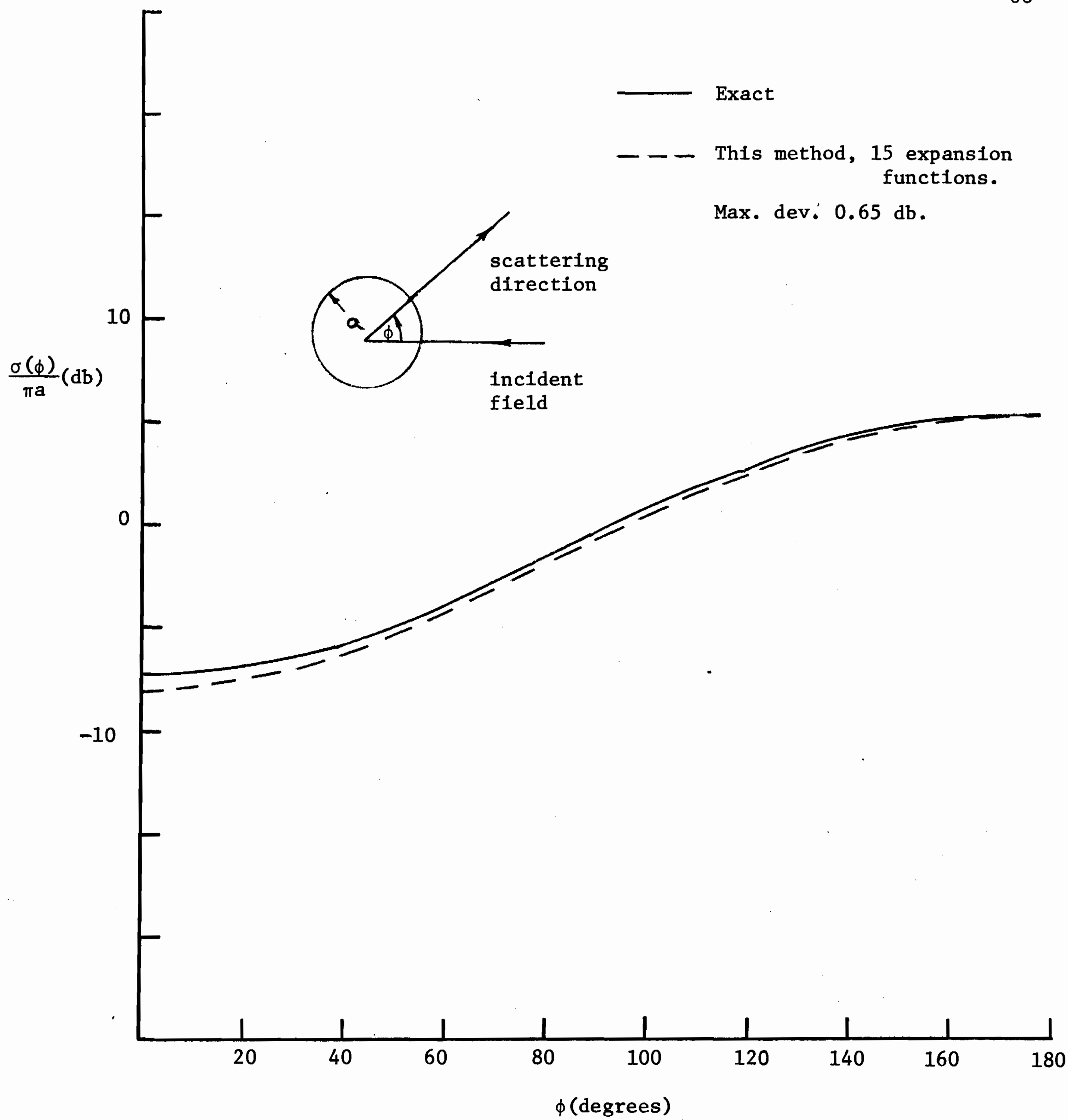


Fig. 4-1. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.5$, $\mu_r = 1.0$, $ka = 0.7$, perpendicular polarization (TE).

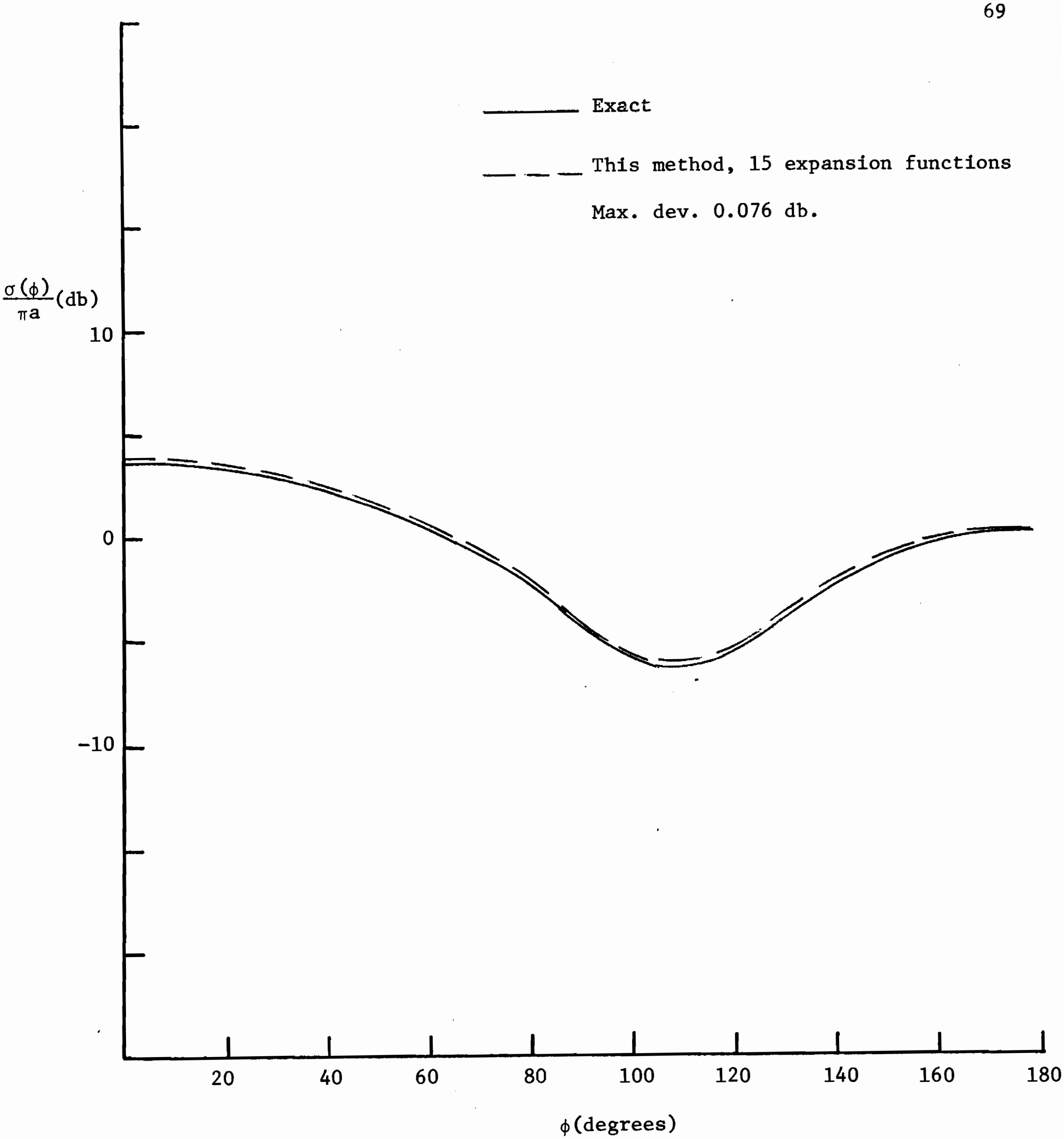


Fig. 4-2. Normalized scattering cross section of a circular cylinder with ε_r = 20.0, μ_r = 1.0, ka = 0.7, perpendicular polarization (TE).

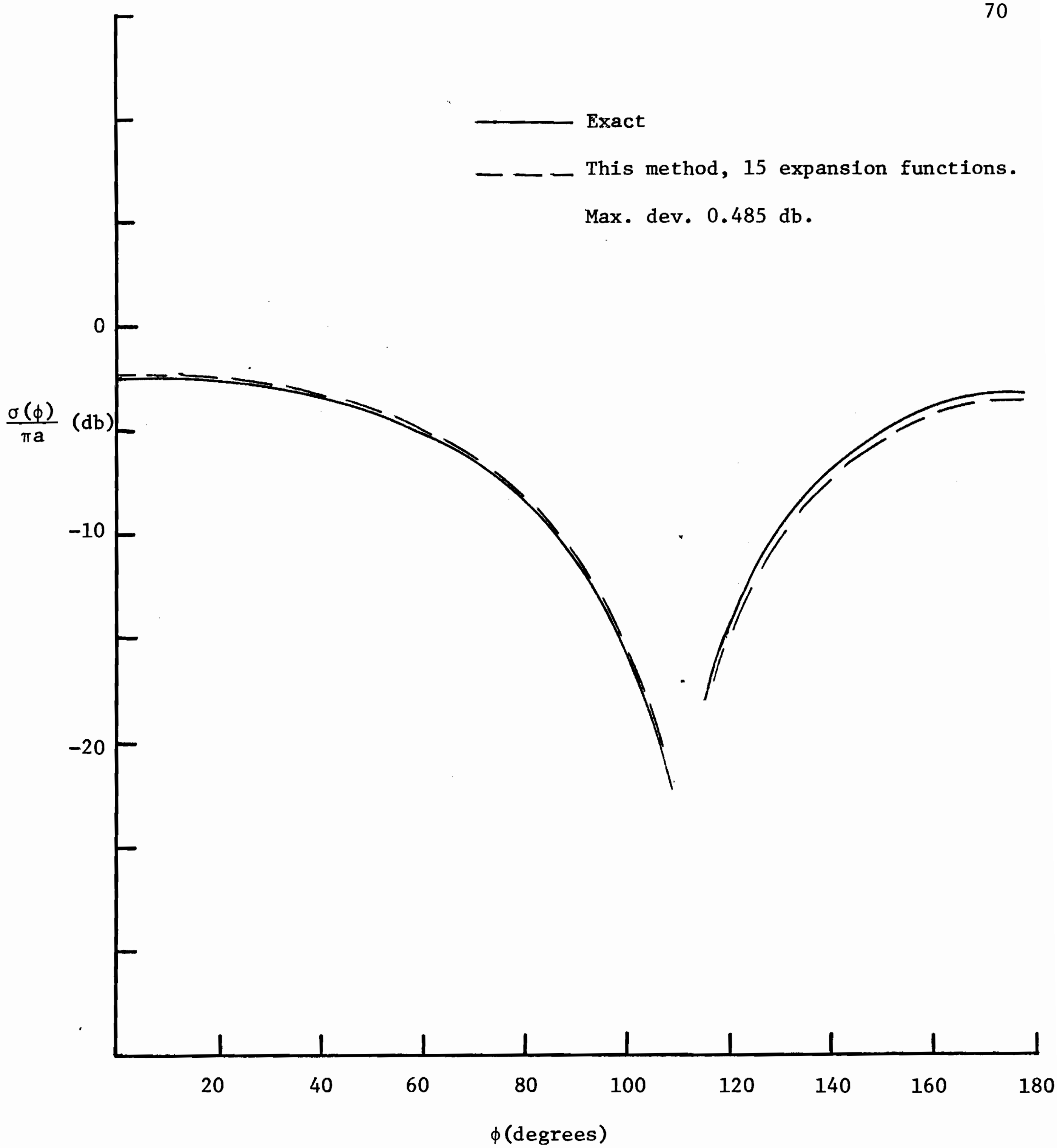


Fig. 4-3. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 50.0$, $\mu_r = 1.0$, ka = 0.7, perpendicular polarization (TE).

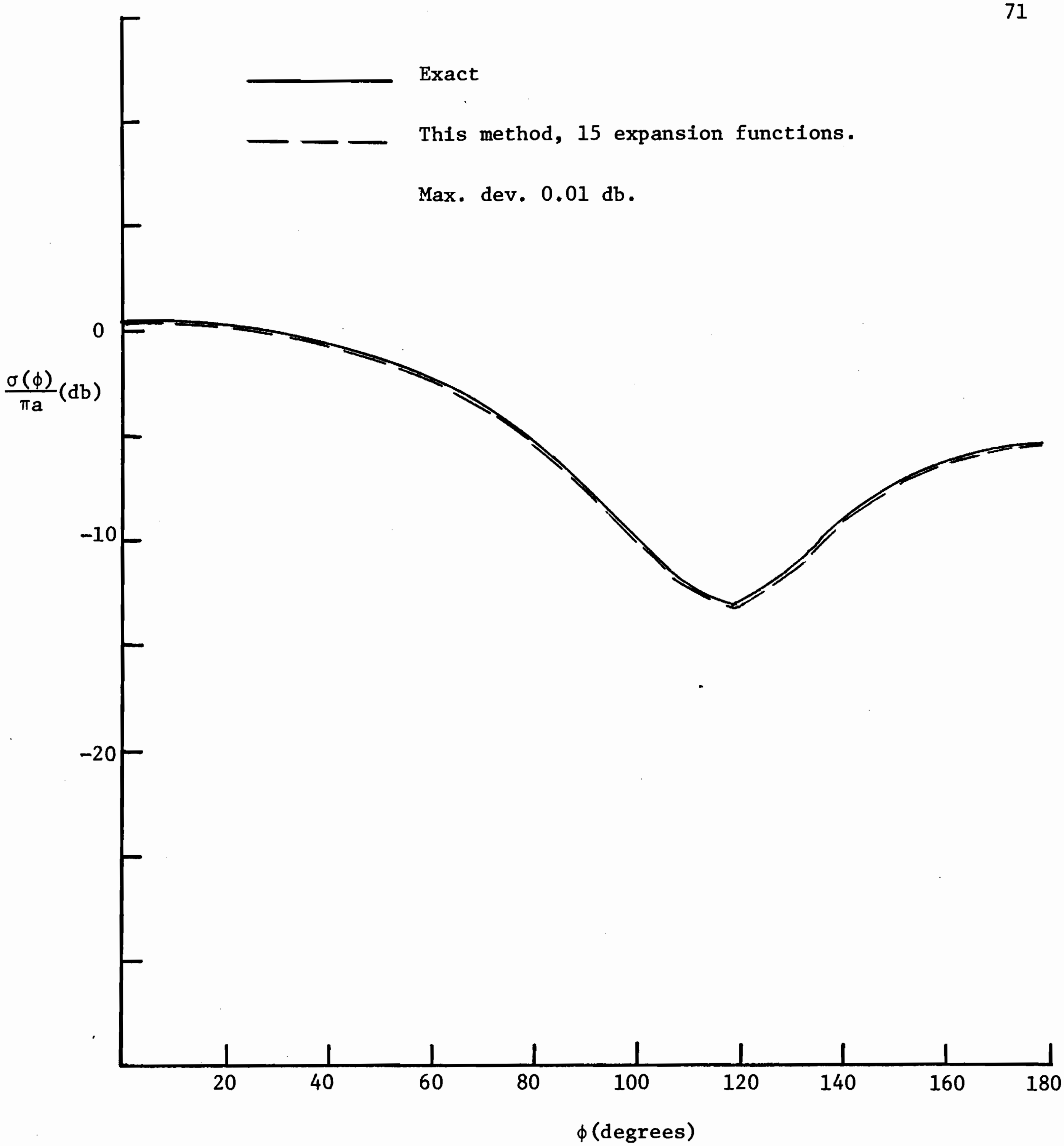


Fig. 4-4. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 100.0$, $\mu_r = 0.01$, $ka = 0.7$, perpendicular polarization (TE).

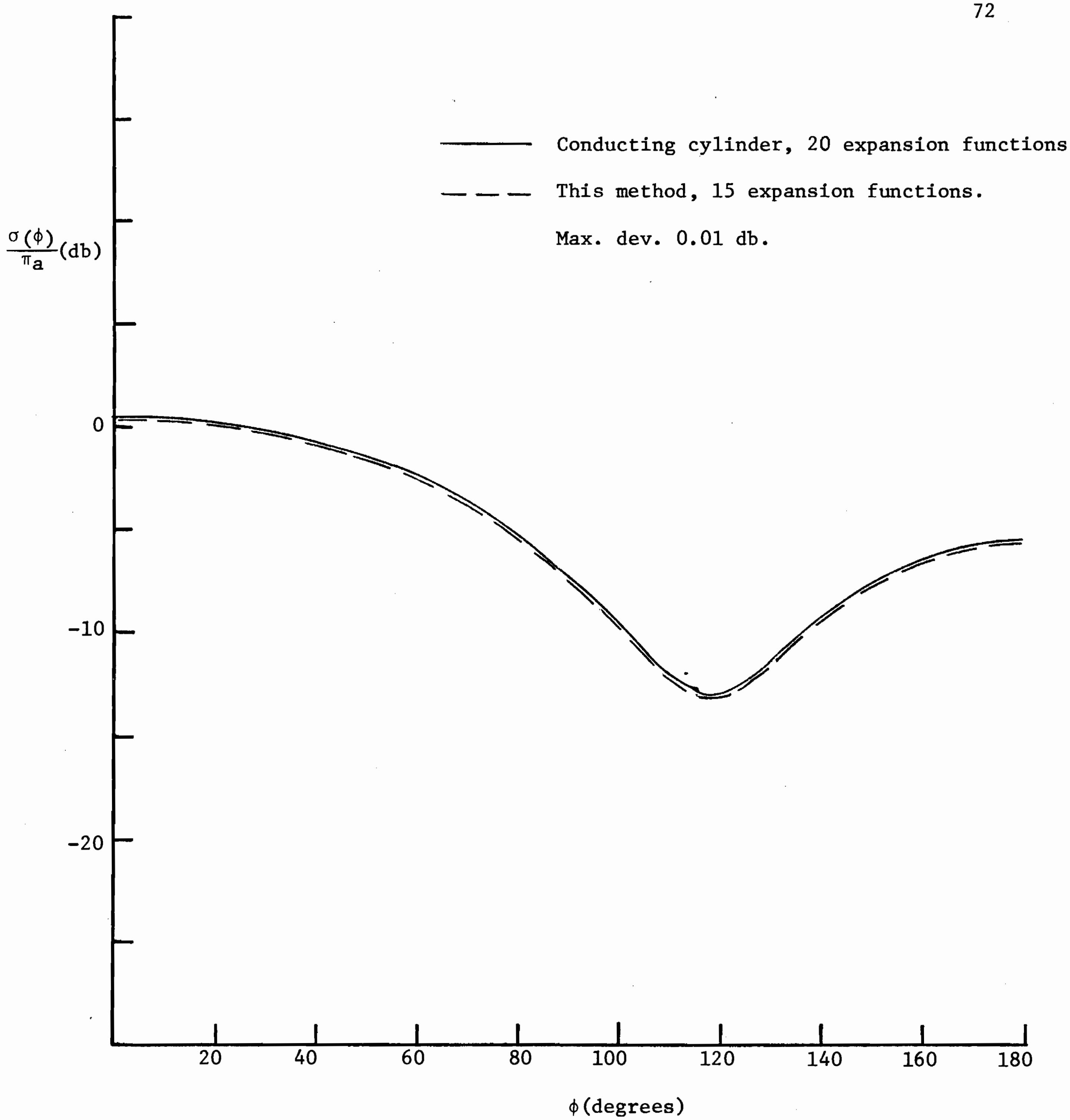


Fig. 4-5. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1000.0$, $\mu_r = 0.001$, ka = 0.7, perpendicular polarization (TE).

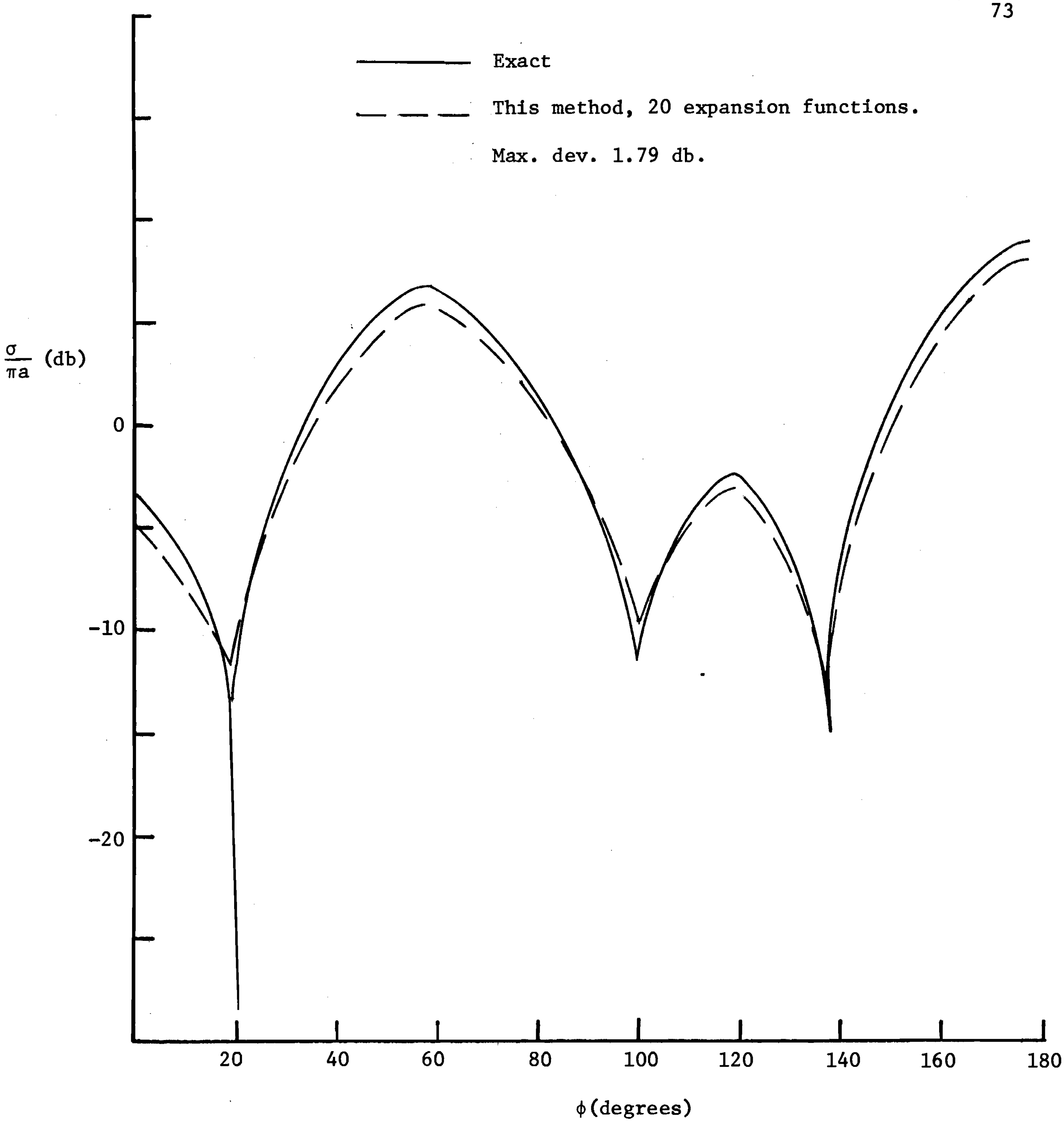


Fig. 4-6. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 1.0$, ka = 2.0, perpendicular polarization (TE).

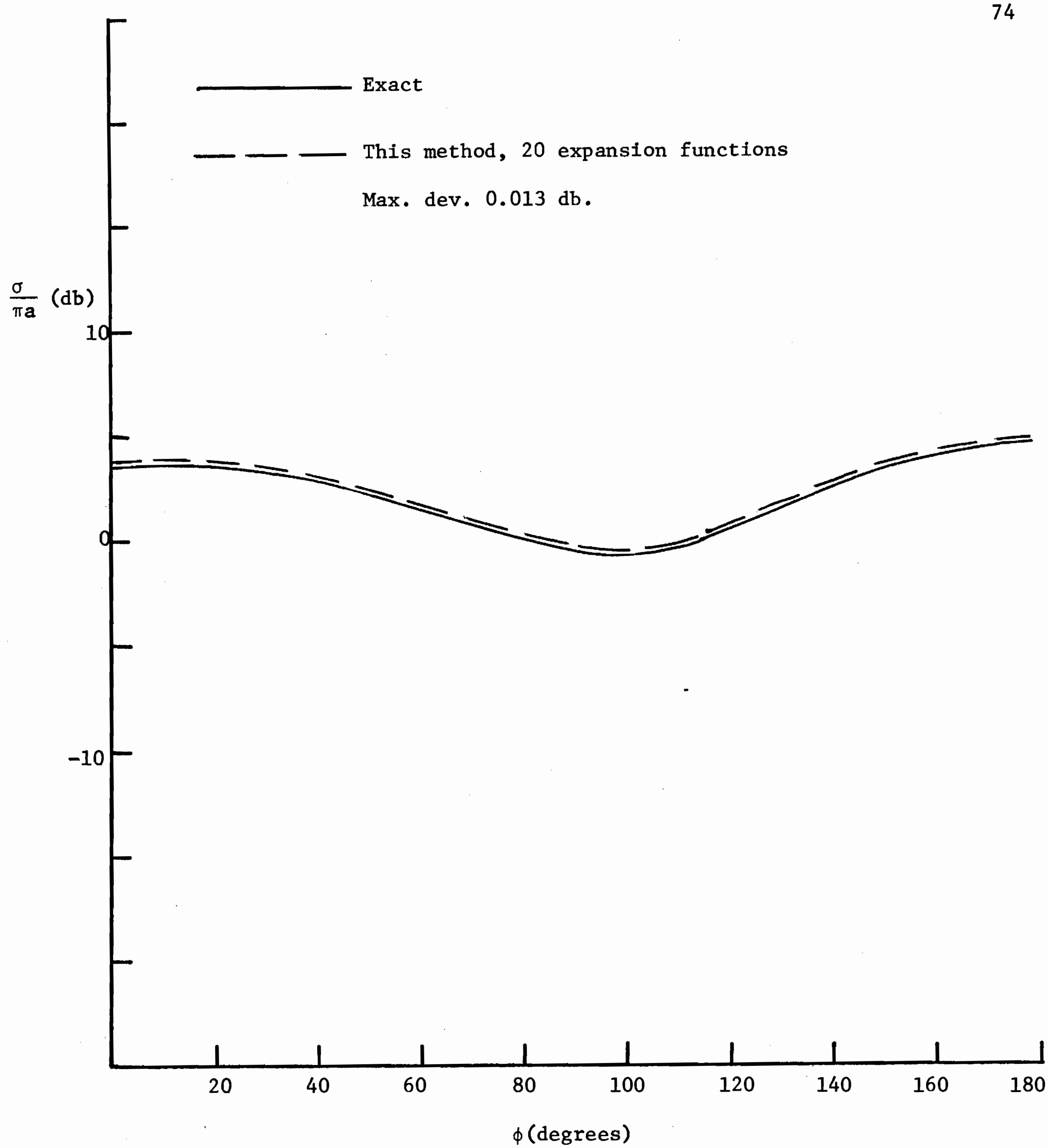


Fig. 4-7. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 1.0$, ka = 1.0, perpendicular polarization (TE).

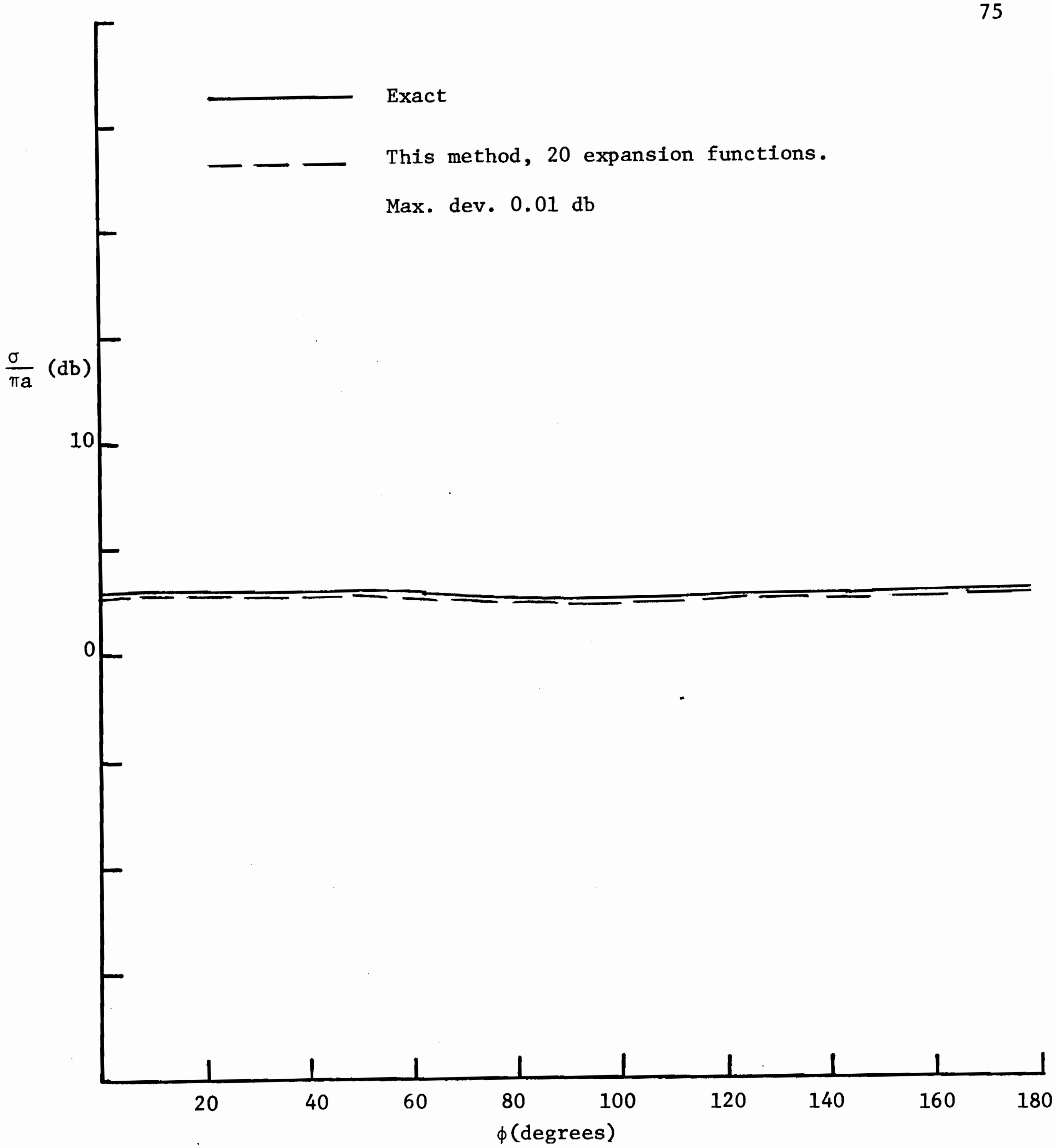


Fig. 4-8. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 100.0$, ka = 0.7, perpendicular polarization (TE).

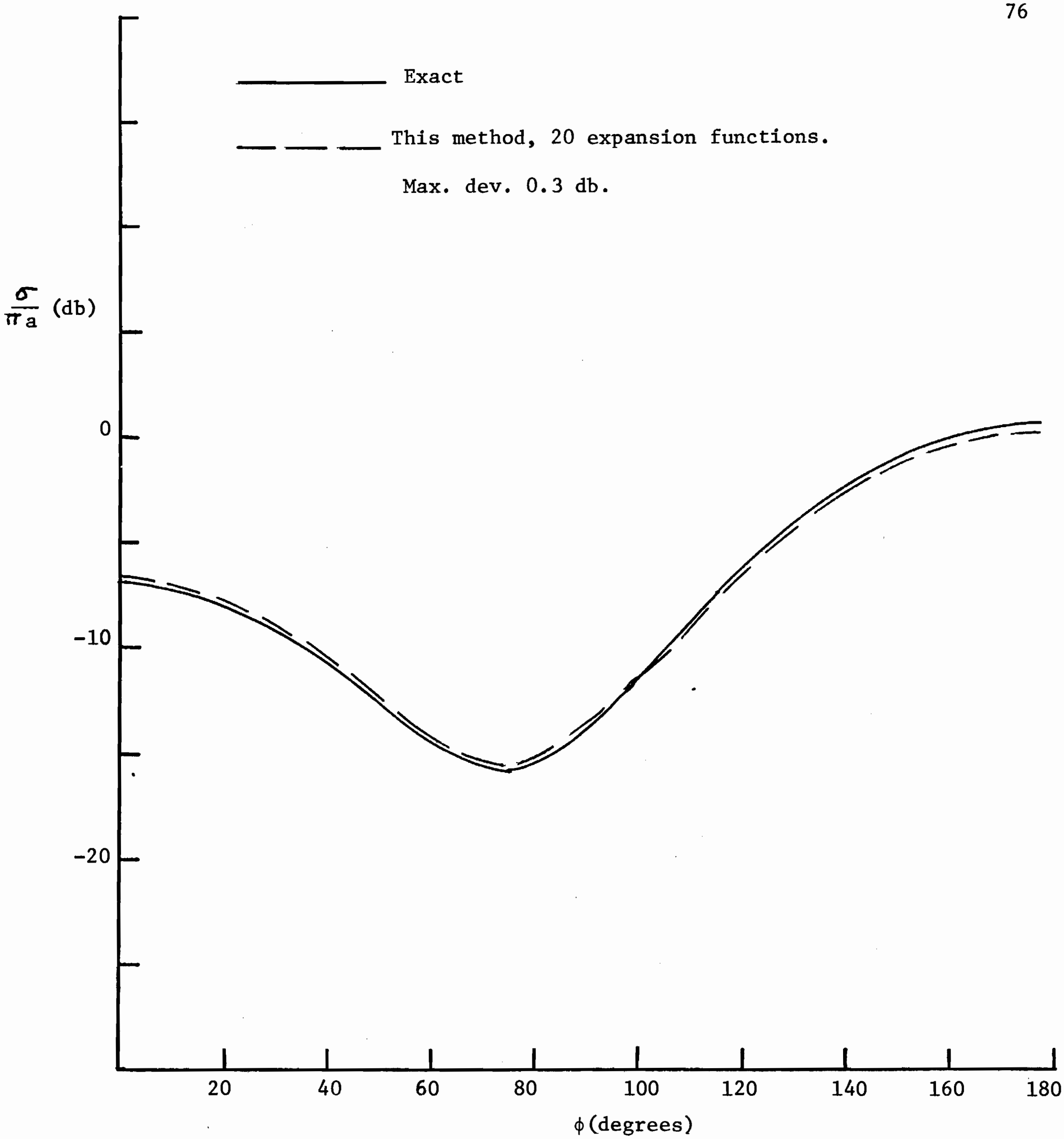


Fig. 4-9. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.0$, $\mu_r = 5.0$, ka = 0.7, perpendicular polarization (TE).

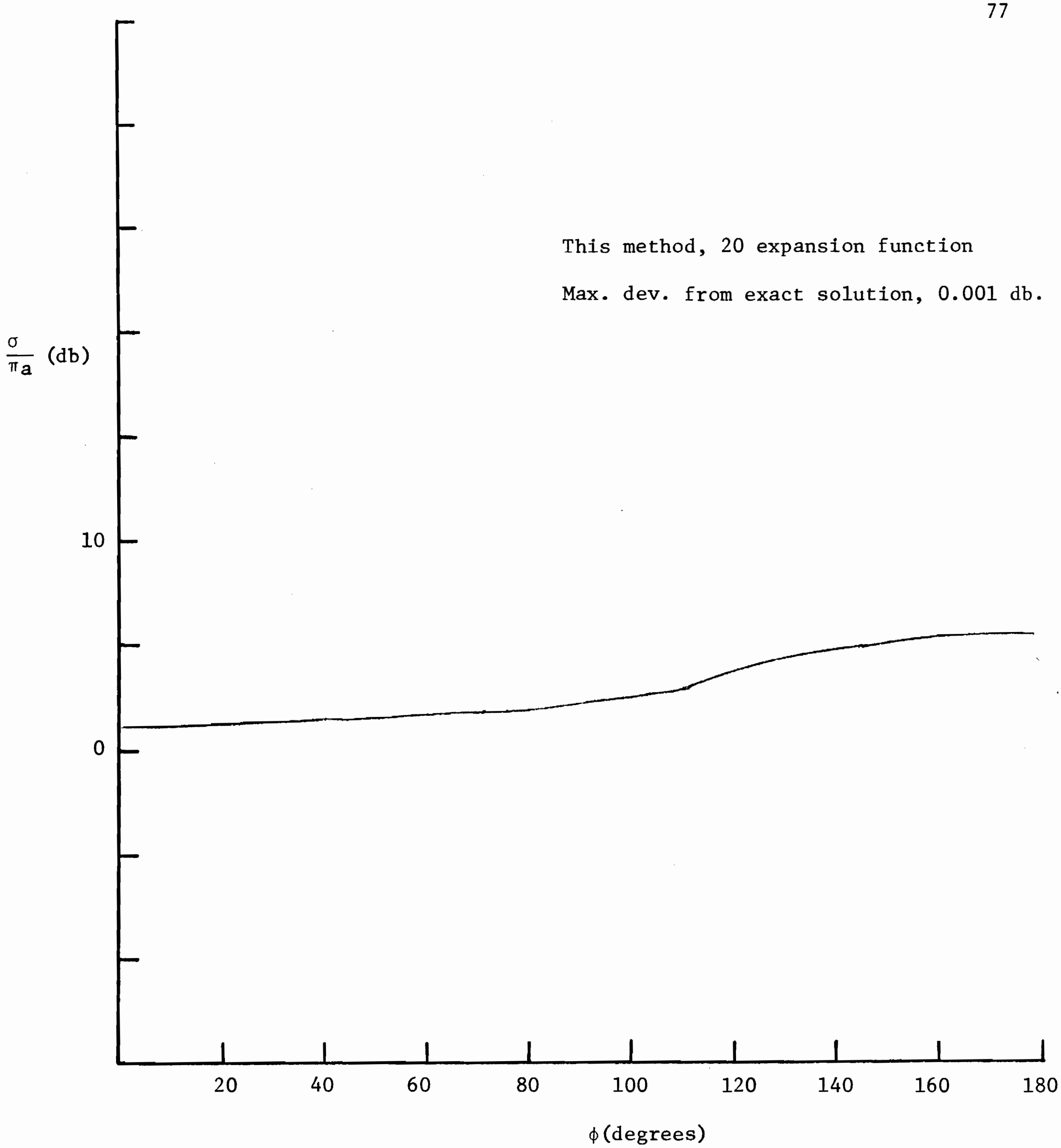


Fig. 4-10. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 0.001$, $\mu_r = 1000.0$, ka = 0.7, perpendicular polarization (TE).

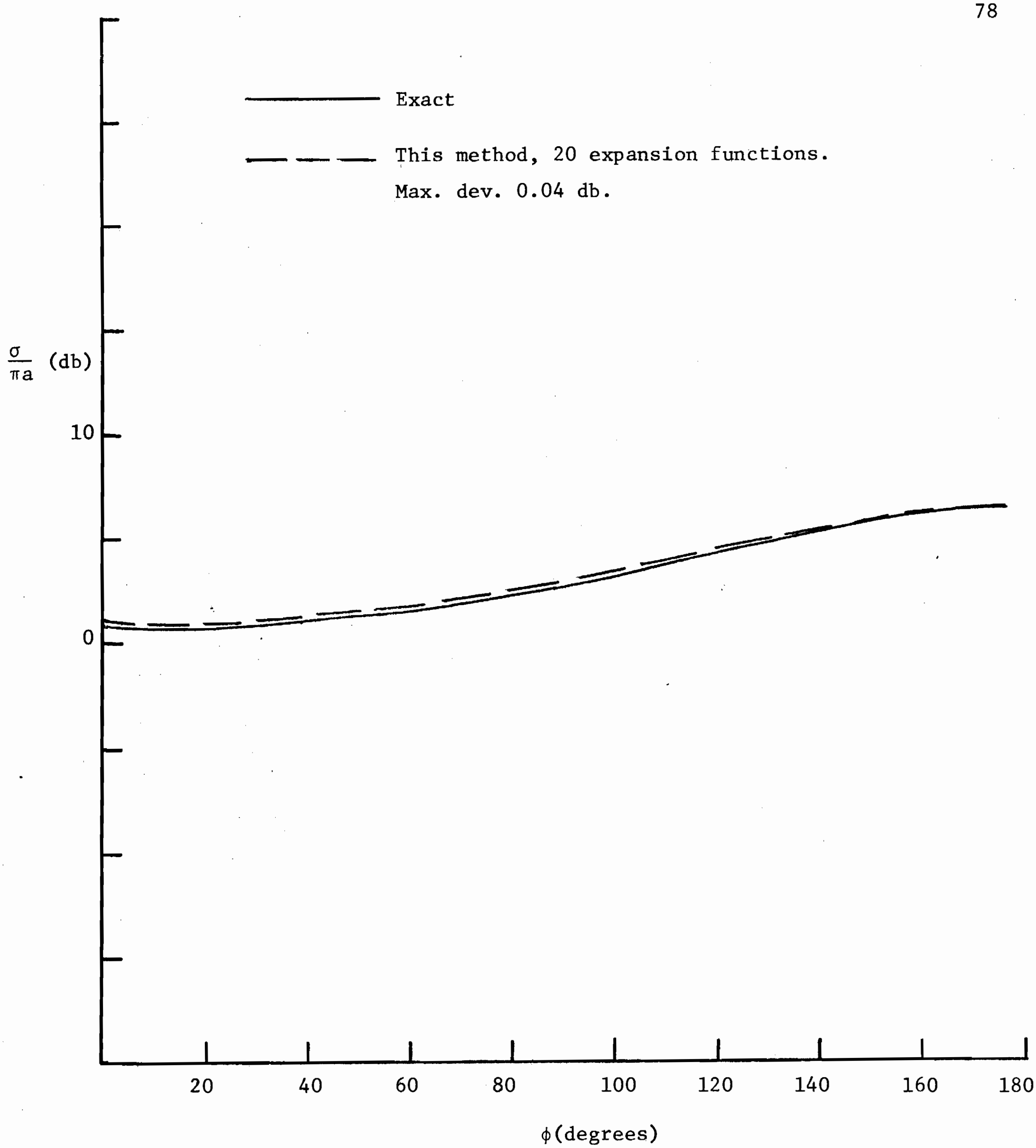


Fig. 4-11. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1$, $\mu_r = 1000$, ka = 0.7, perpendicular polarization (TE).

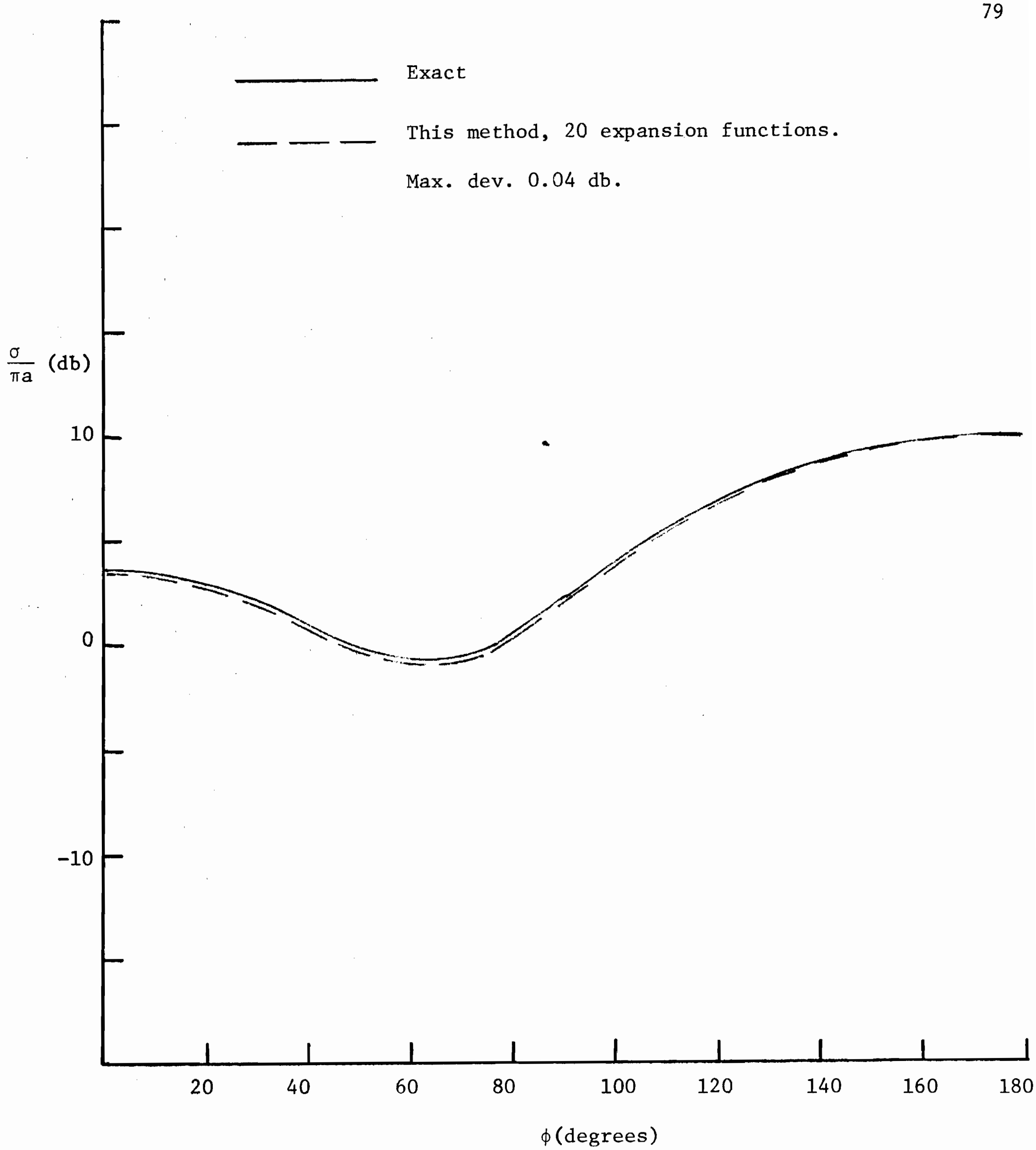


Fig. 4-12. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1.0$, $\mu_r = 10.0$, ka = 0.7, perpendicular polarization (TE).

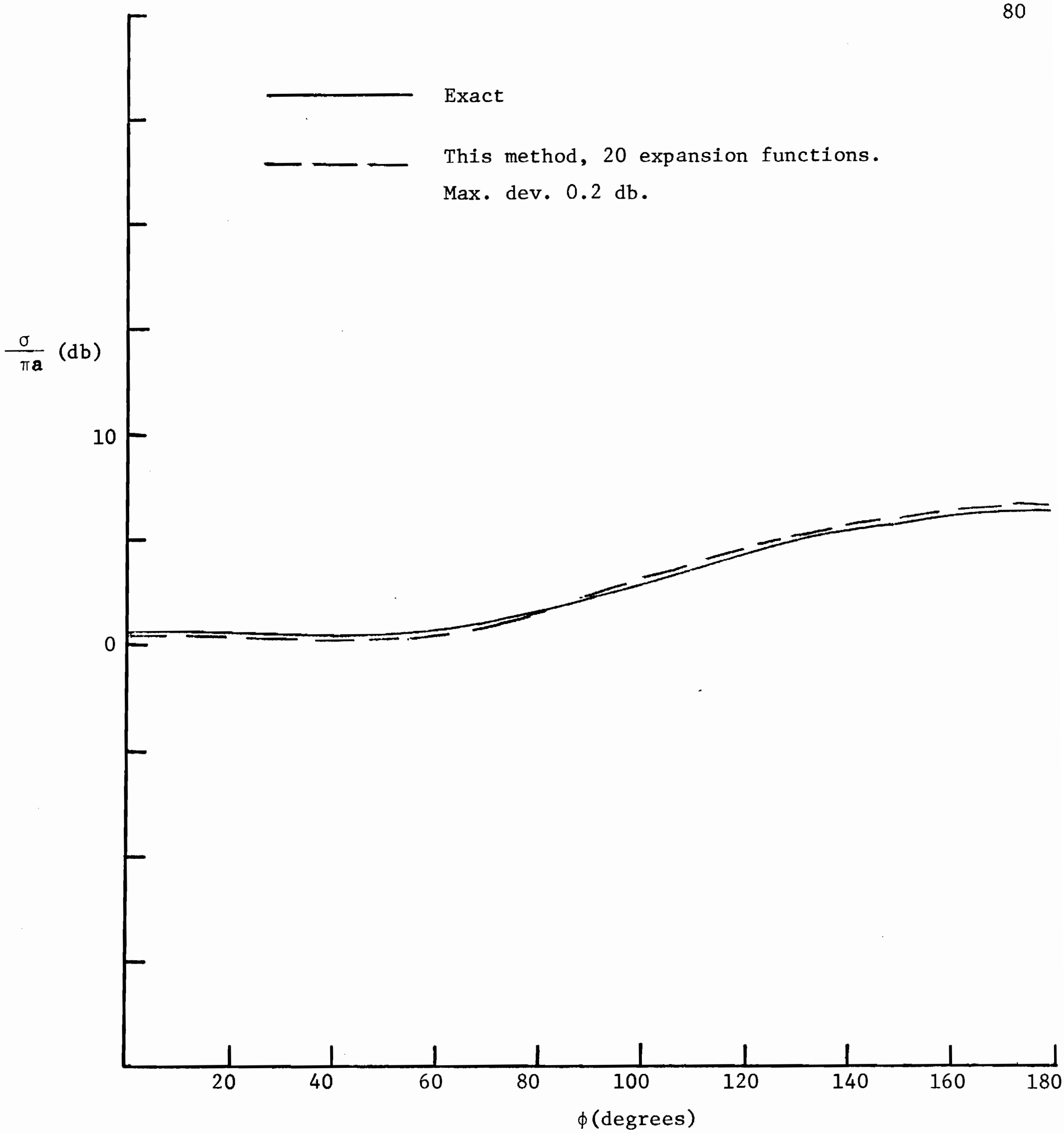


Fig. 4-13. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 1.0$, $\mu_r = 300$, ka = 0.7, perpendicular polarization (TE).

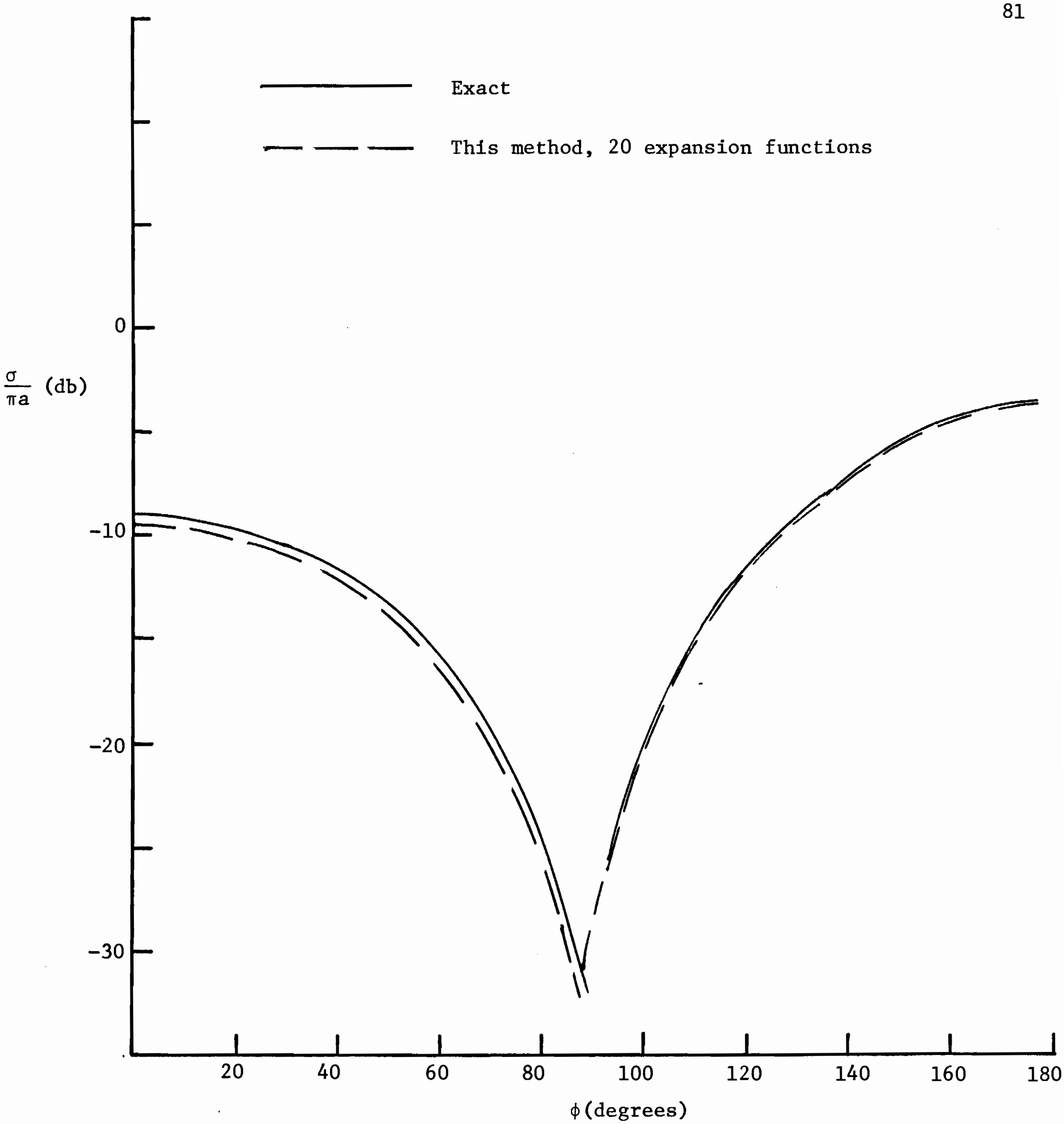


Fig. 4-14. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 2.56$, $\mu_r = 1.0$, ka = 0.7, perpendicular polarization (TE).

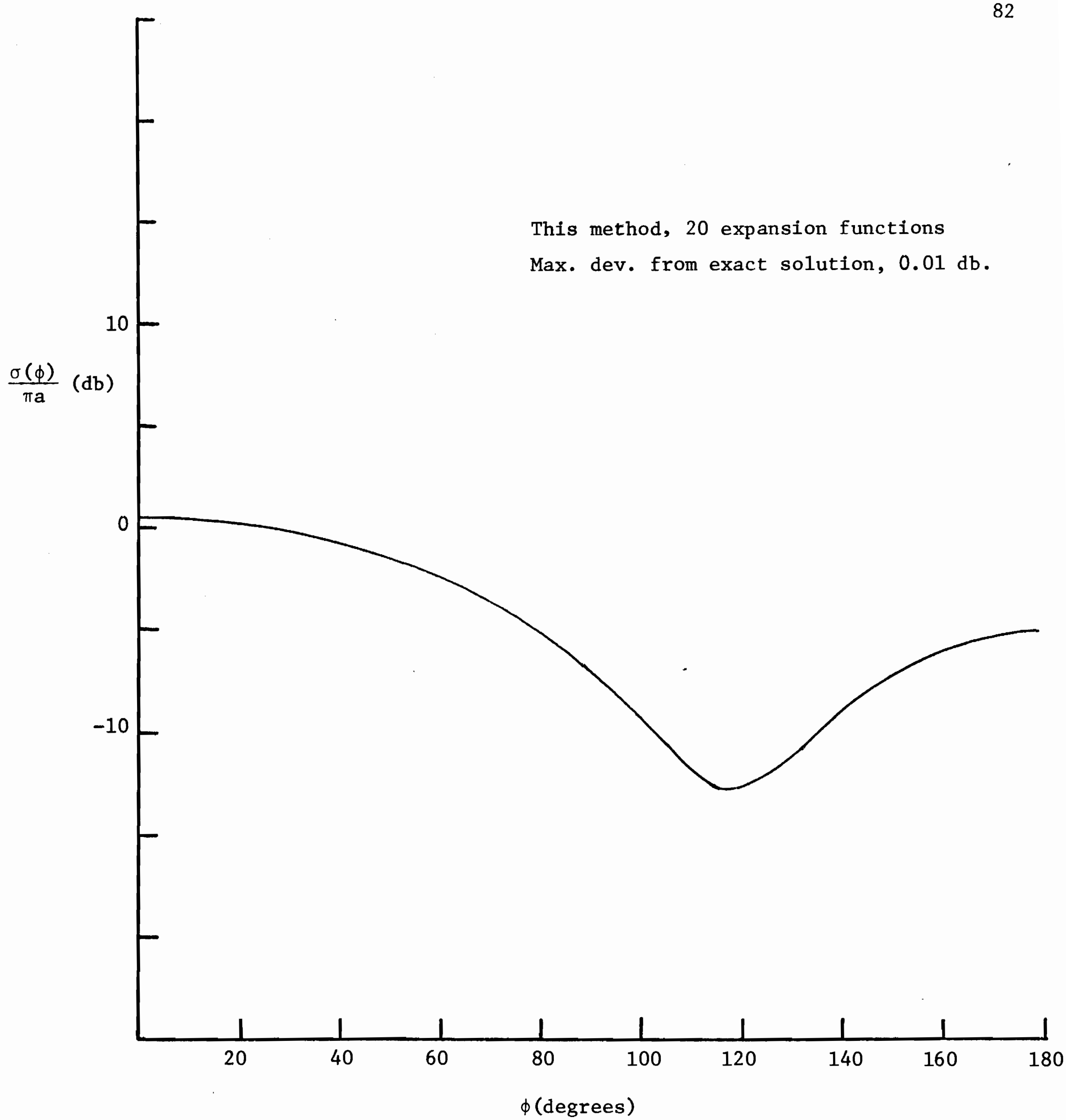


Fig. 4-15. Normalized scattering cross section of a circular cylinder with ε_r = 1000.0, μ_r = 0.001, ka = 0.7, perpendicular polarization (TE).

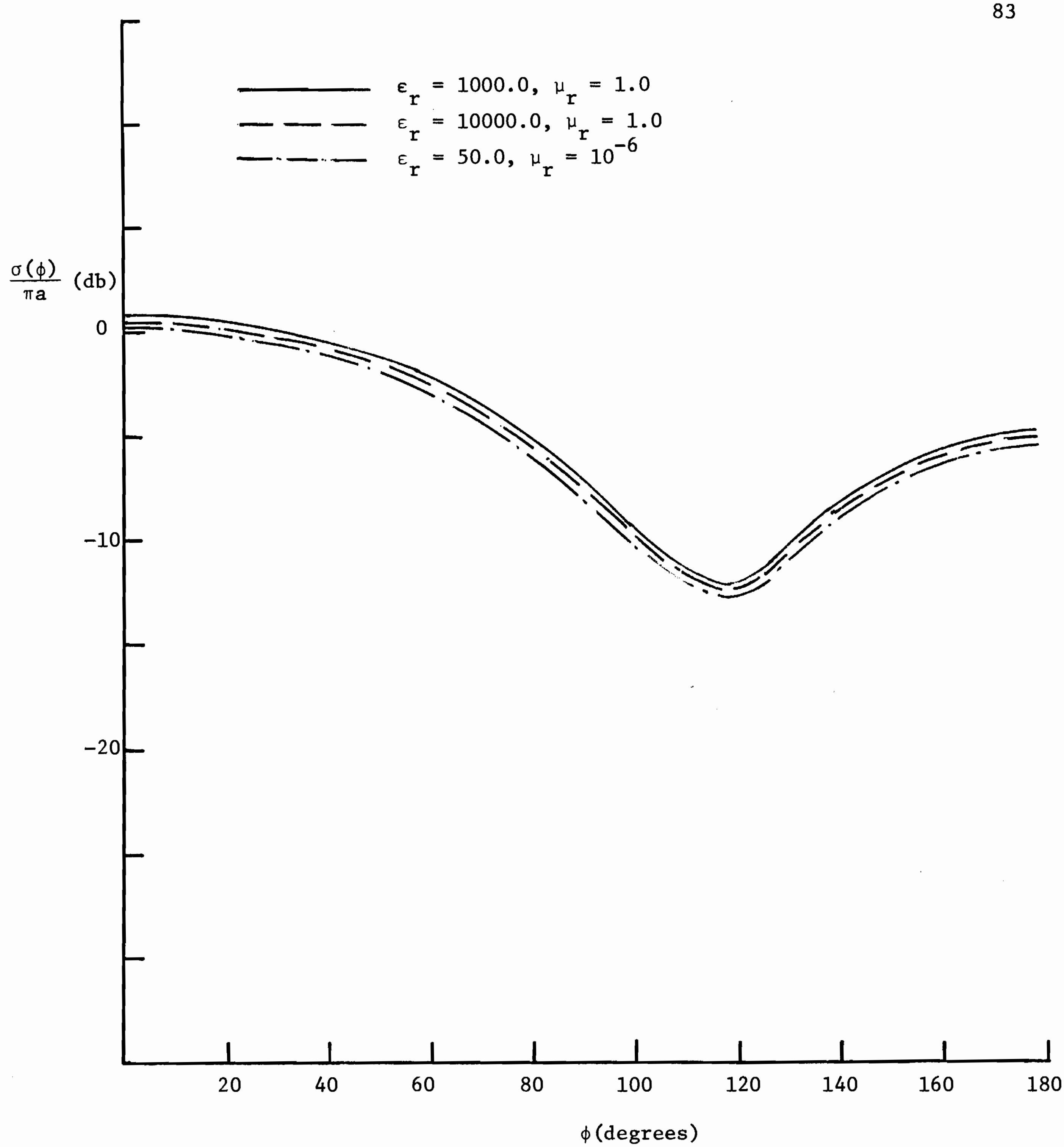


Fig. 4-16. Normalized scattering cross section of a circular cylinder with ka = 0.7, perpendicular polarization (TE).

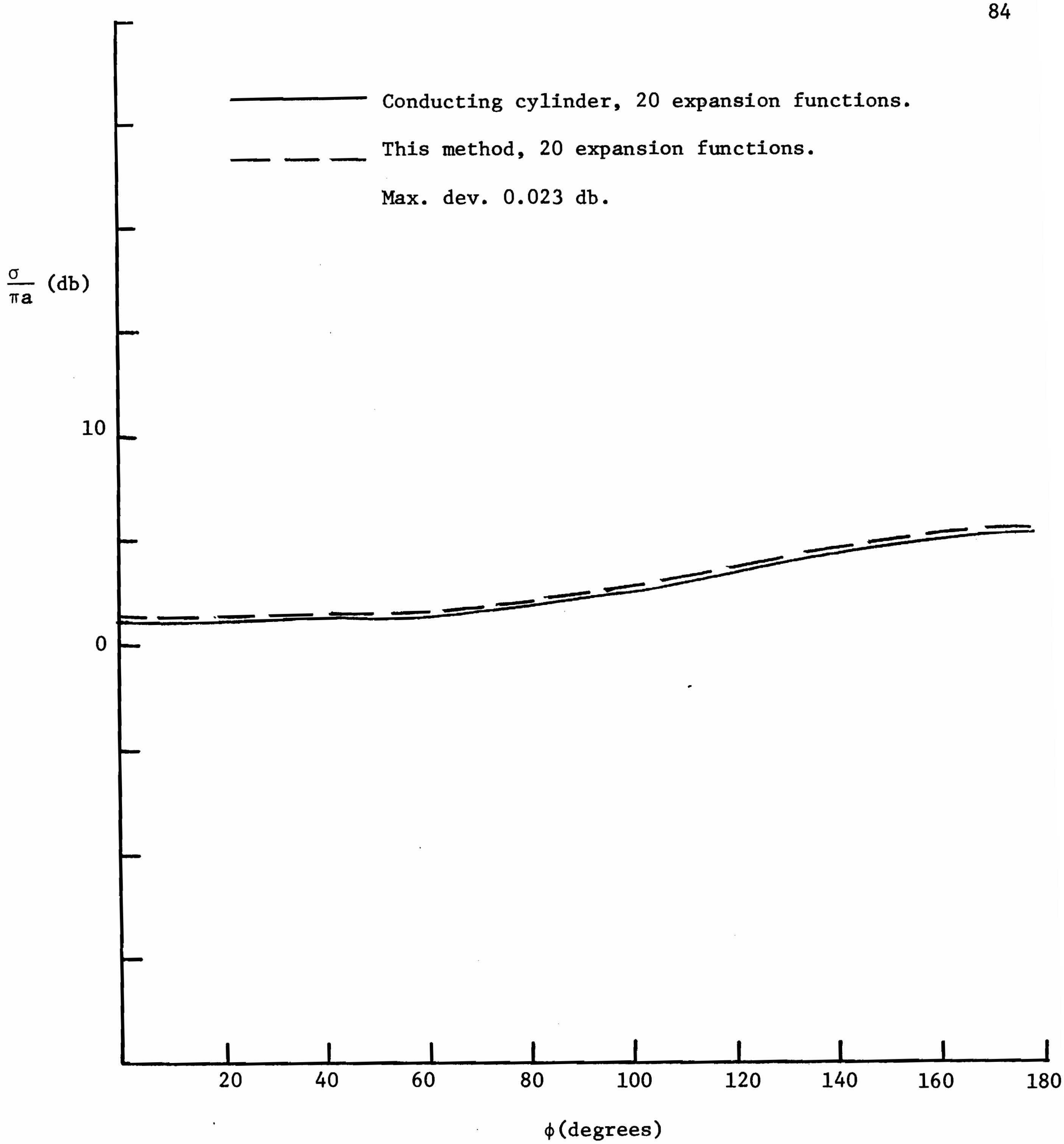


Fig. 4-17. Normalized scattering cross section of a circular cylinder with ε_r = 1000.0, μ_r = 0.001, ka = 0.7, parallel polarization (TM).

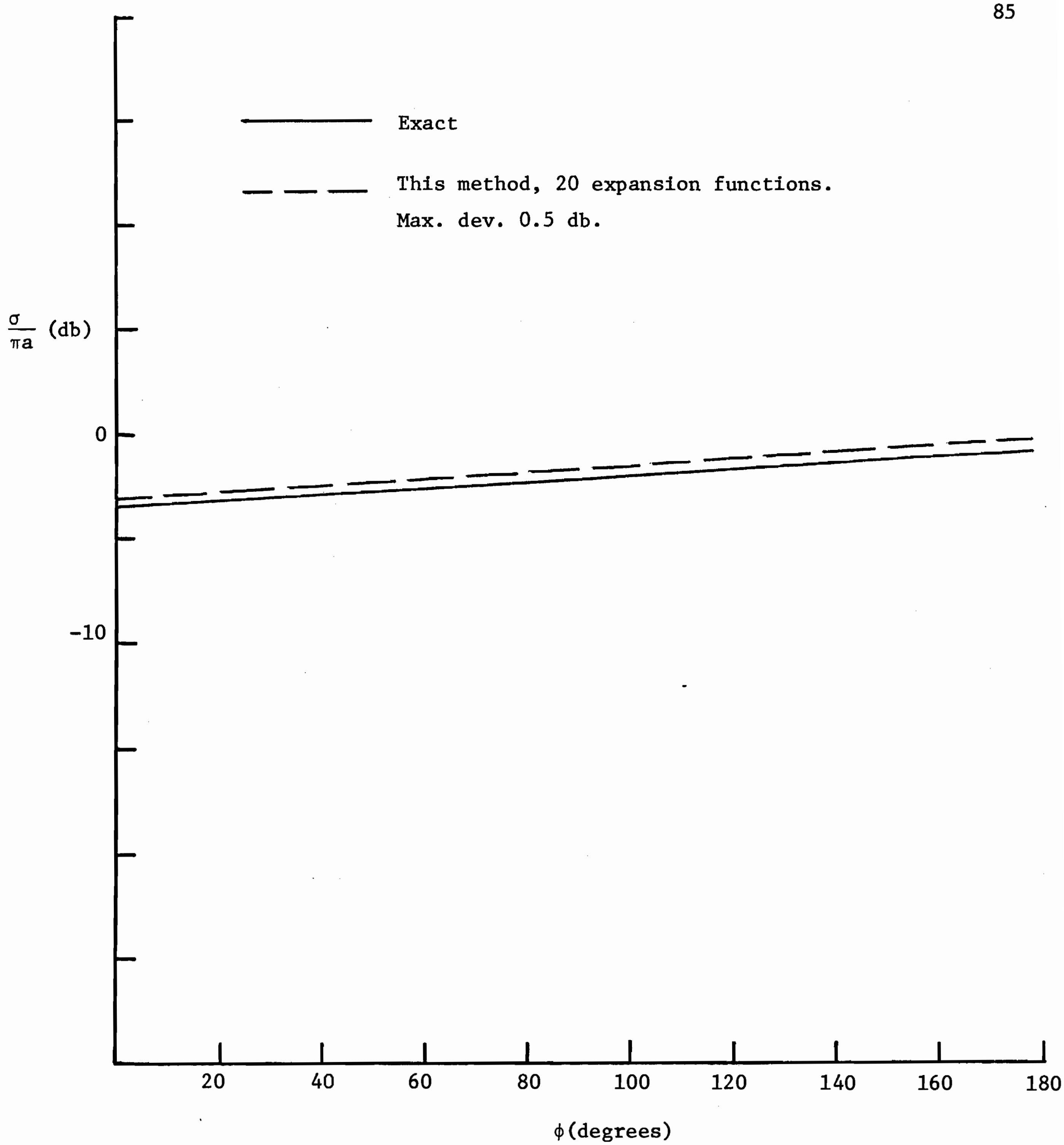


Fig. 4-18. Normalized scattering cross section of a circular cylinder with $\epsilon_r = 2.56$, $\mu_r = 1.0$, ka = 0.7, parallel polarization (TM).

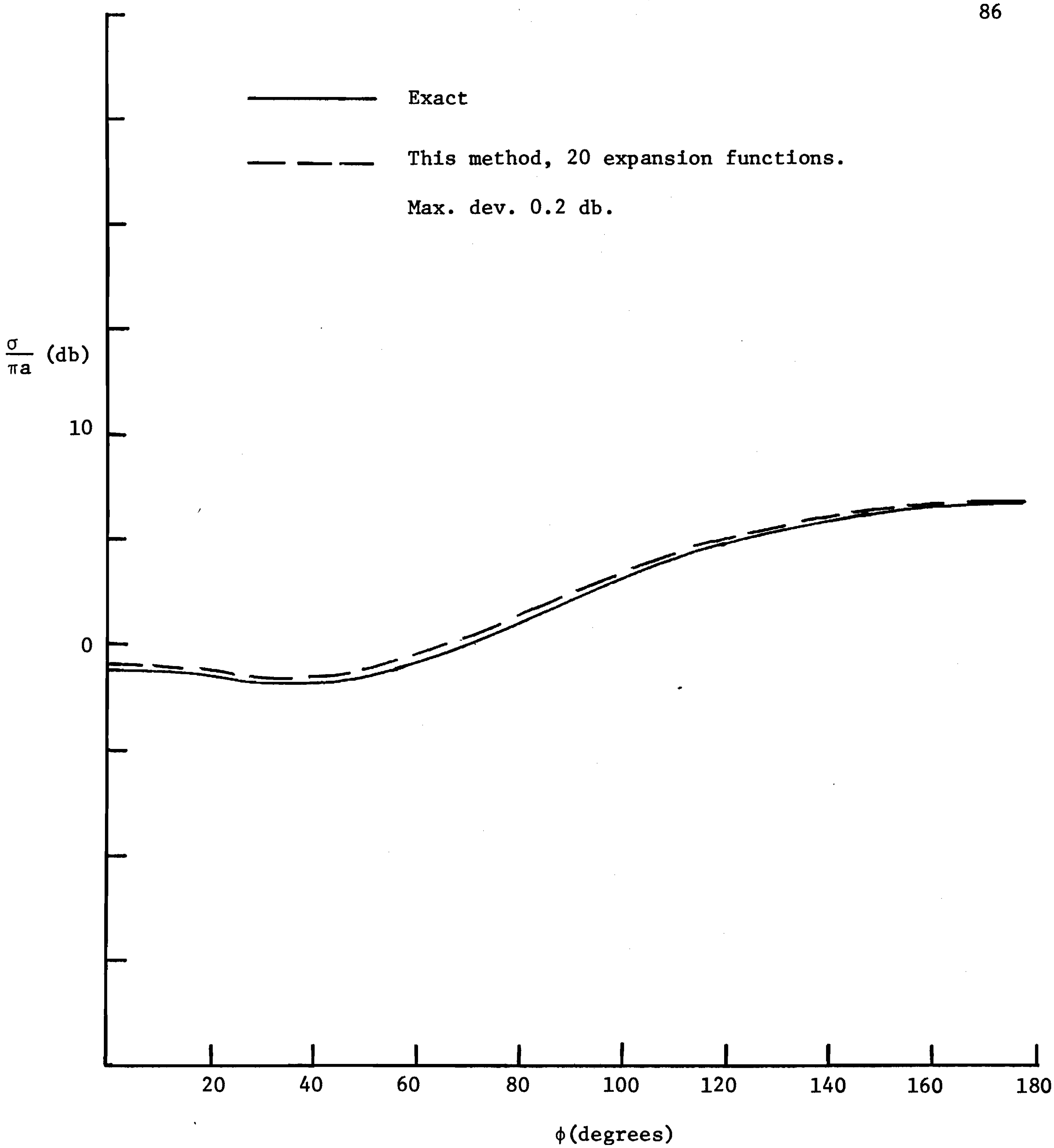


Fig. 4-19. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 20.0$, $\mu_r = 1.0$, ka = 0.7, parallel polarization (TM).

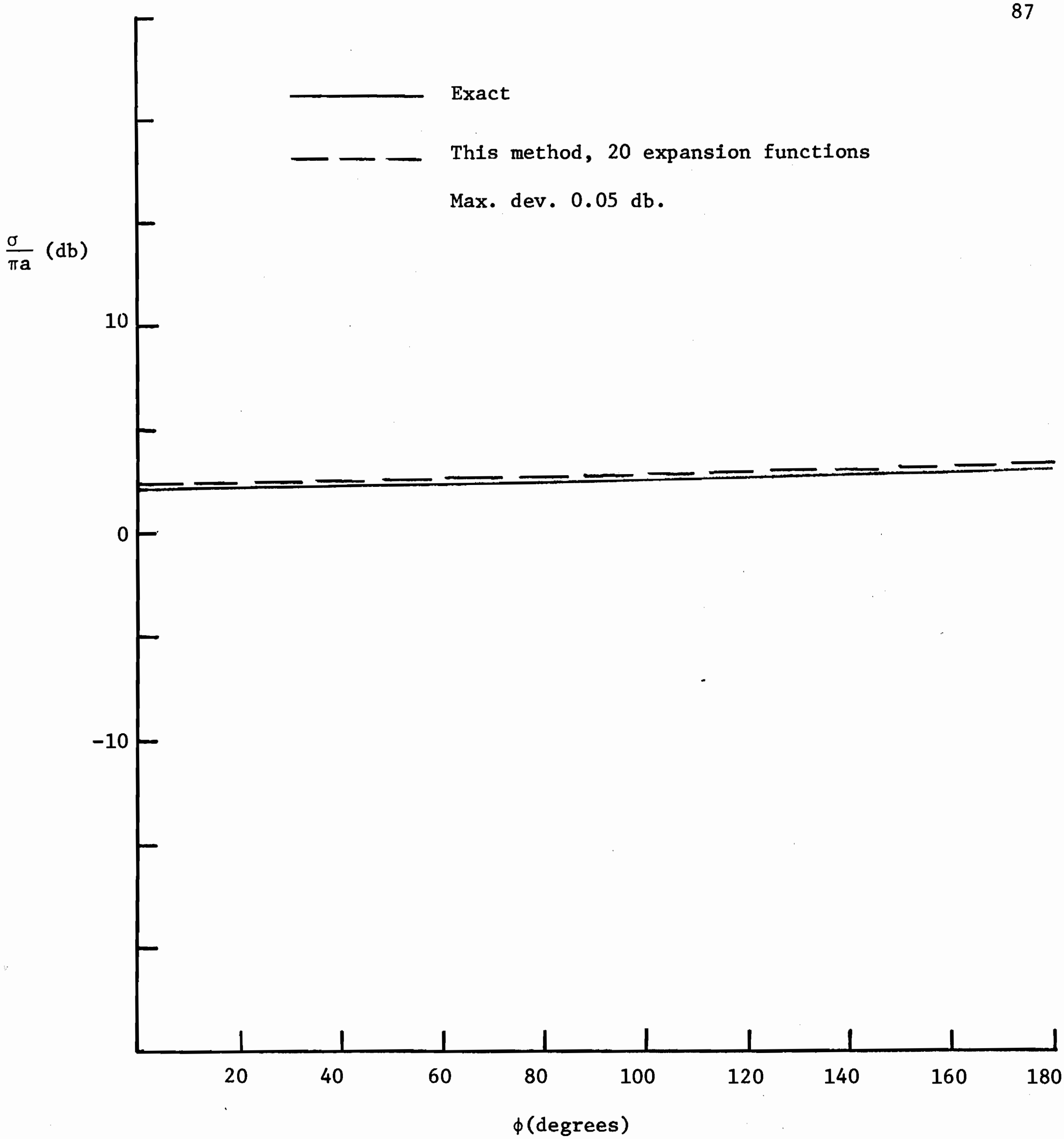


Fig. 4-20. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 50.0$, $\mu_r = 1.0$, ka = 0.7, parallel polarization (TM).

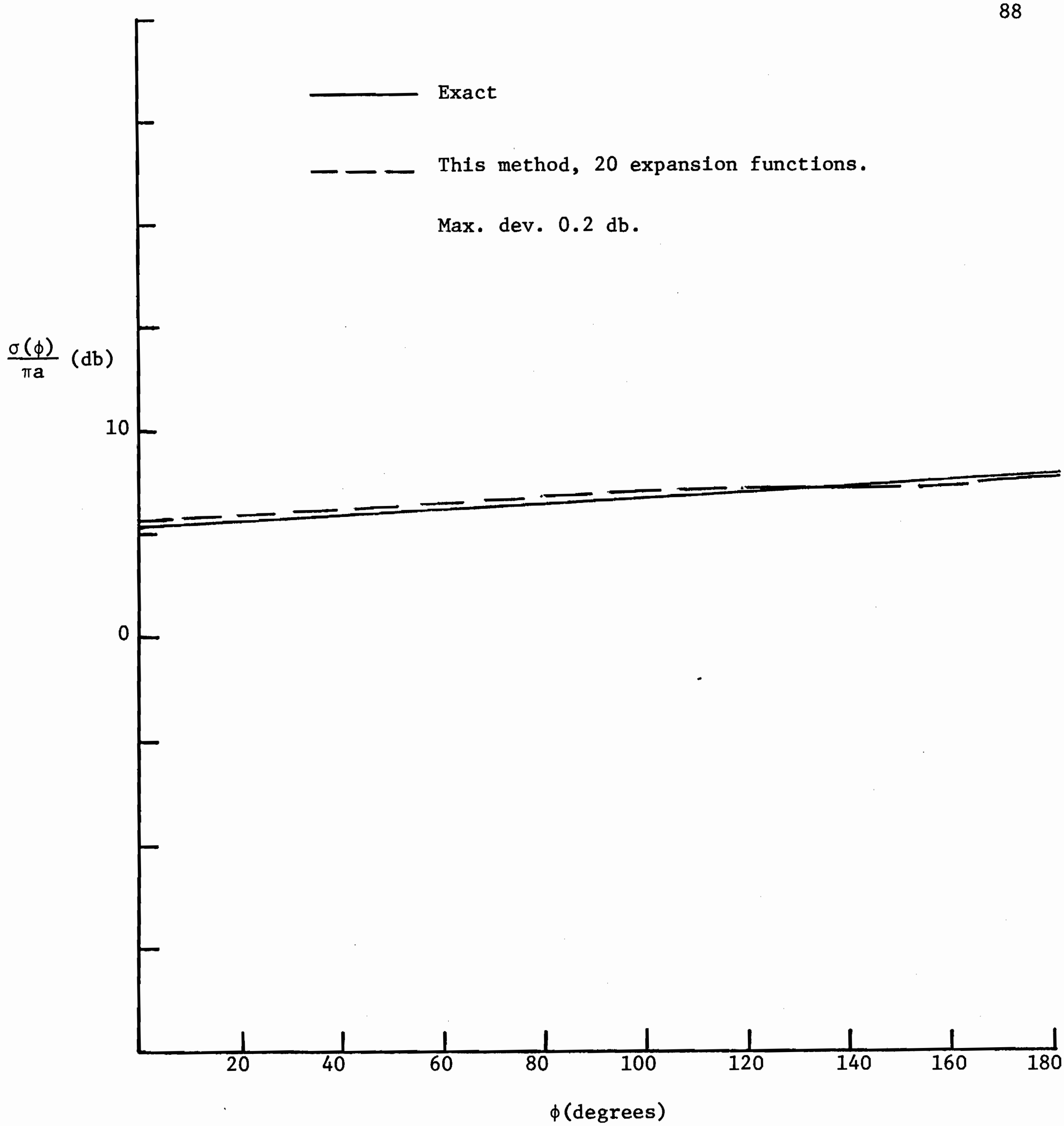


Fig. 4-21. Normalized scattering cross section of a circular material cylinder, with $\varepsilon_r = 4.0$, $\mu_r = 1.0$, ka = 0.7 parallel polarization (TM).

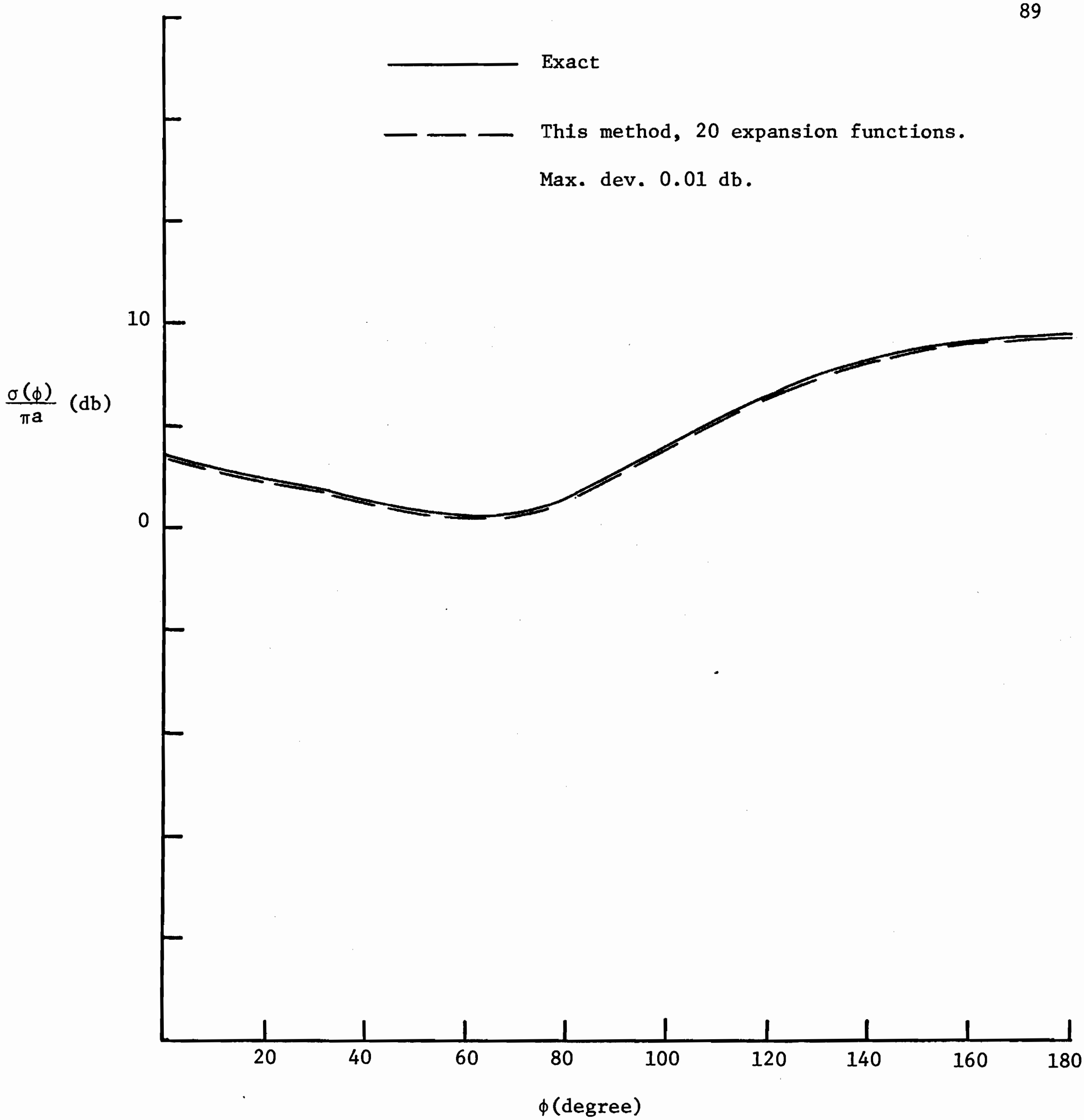


Fig. 22. Normalized scattering cross section of a circular cylinder with $\varepsilon_r = 9.5$, $\mu_r = 1.0$.

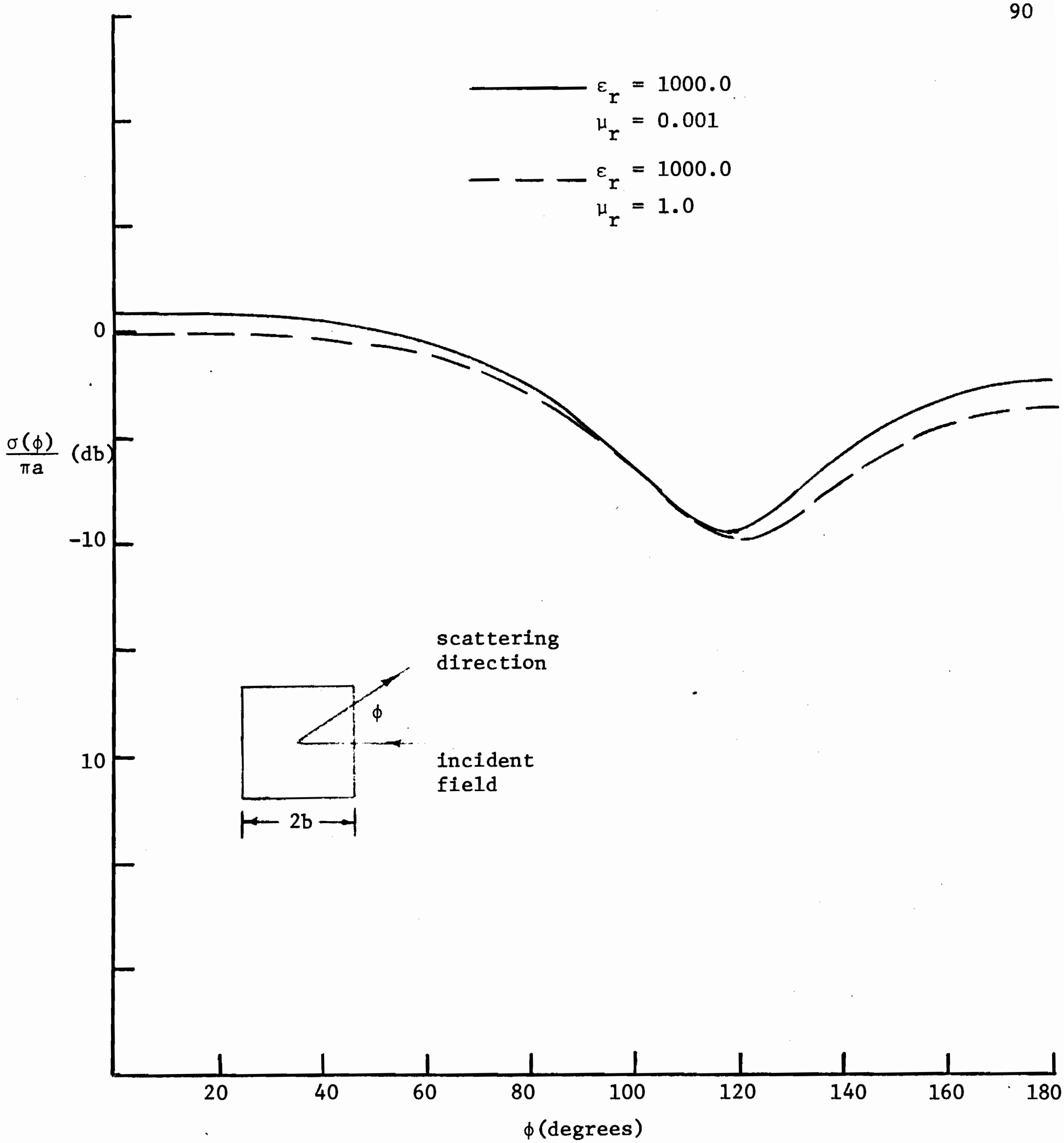


Fig. 4-23. Normalized scattering cross section of a square cylinder with kb = 1.4, perpendicular polarization (TE).

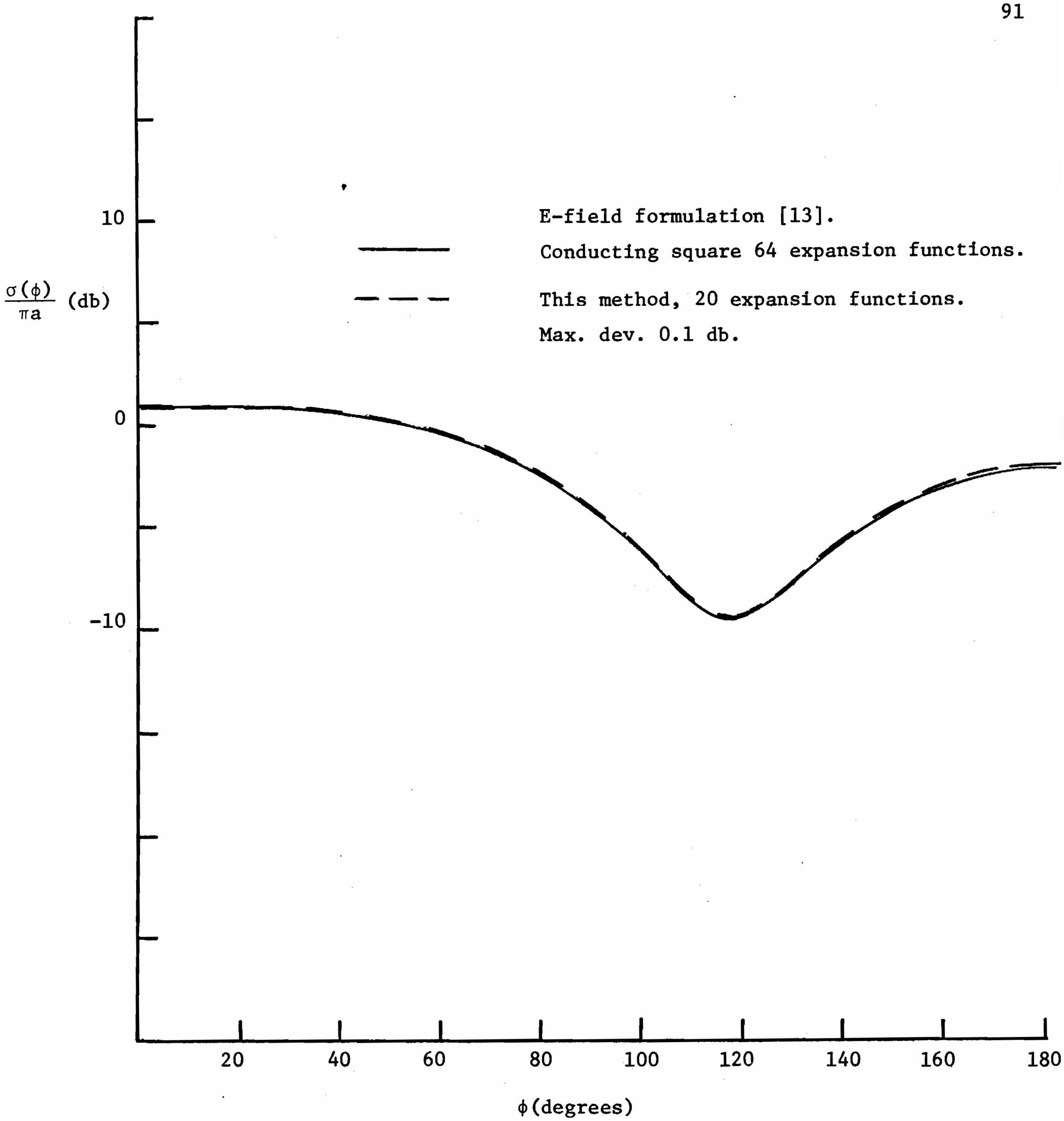


Fig. 4-24. Normalized scattering cross section of a square cylinder with ε_r = 10000.0, μ_r = 0.0001, kb = 1.4, perpendicular polarization (TE).

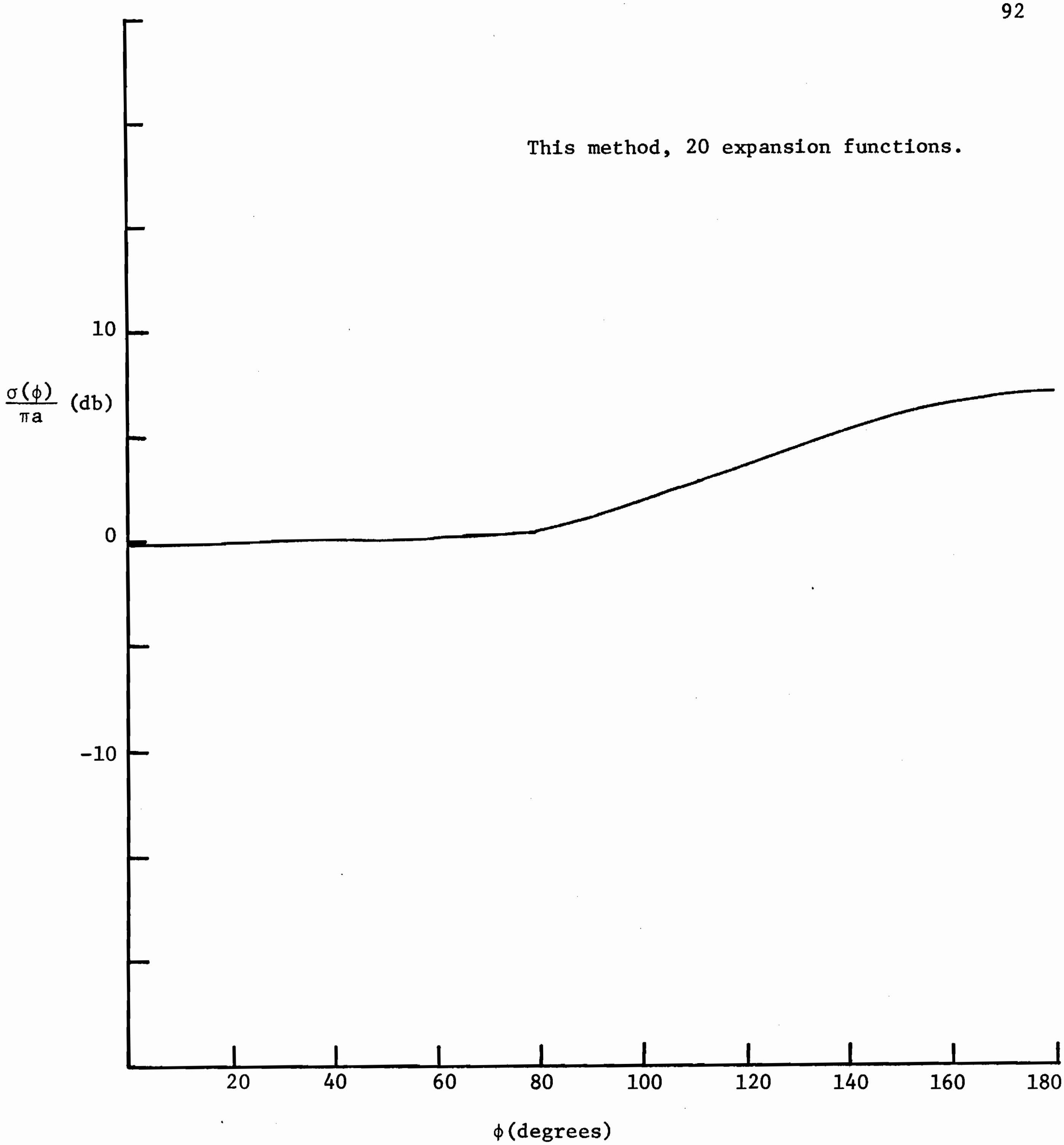


Fig. 4-25. Normalized scattering cross section of a square cylinder with $\varepsilon_r = 9.0$, $\mu_r = 1.0$, kb = 1.4, perpendicular polarization (TE).

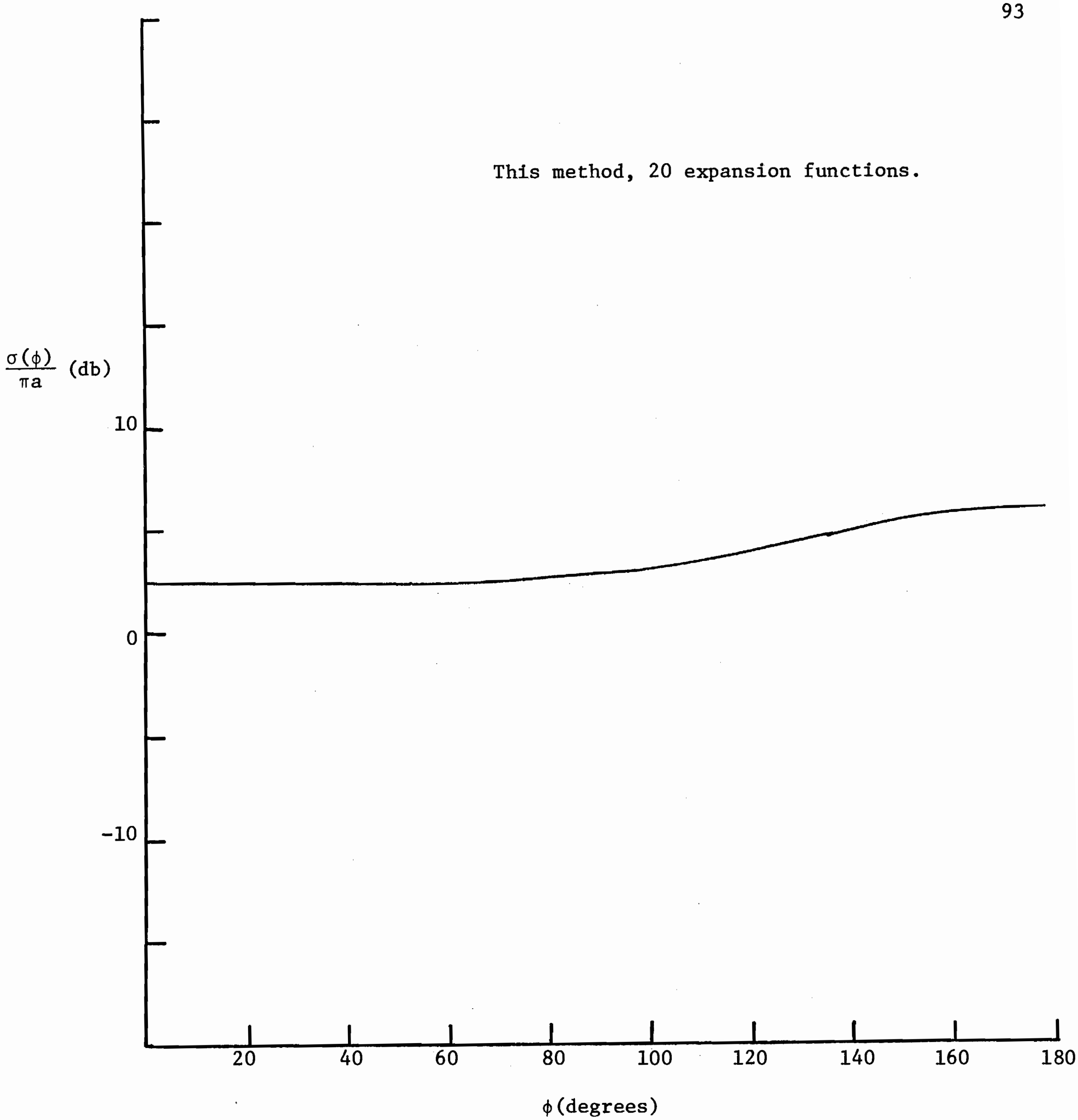


Fig. 4-26. Normalized scattering cross section of a square cylinder with ε_r = 100.0, μ_r = 1.0, kb = 1.4, parallel polarization (TM).

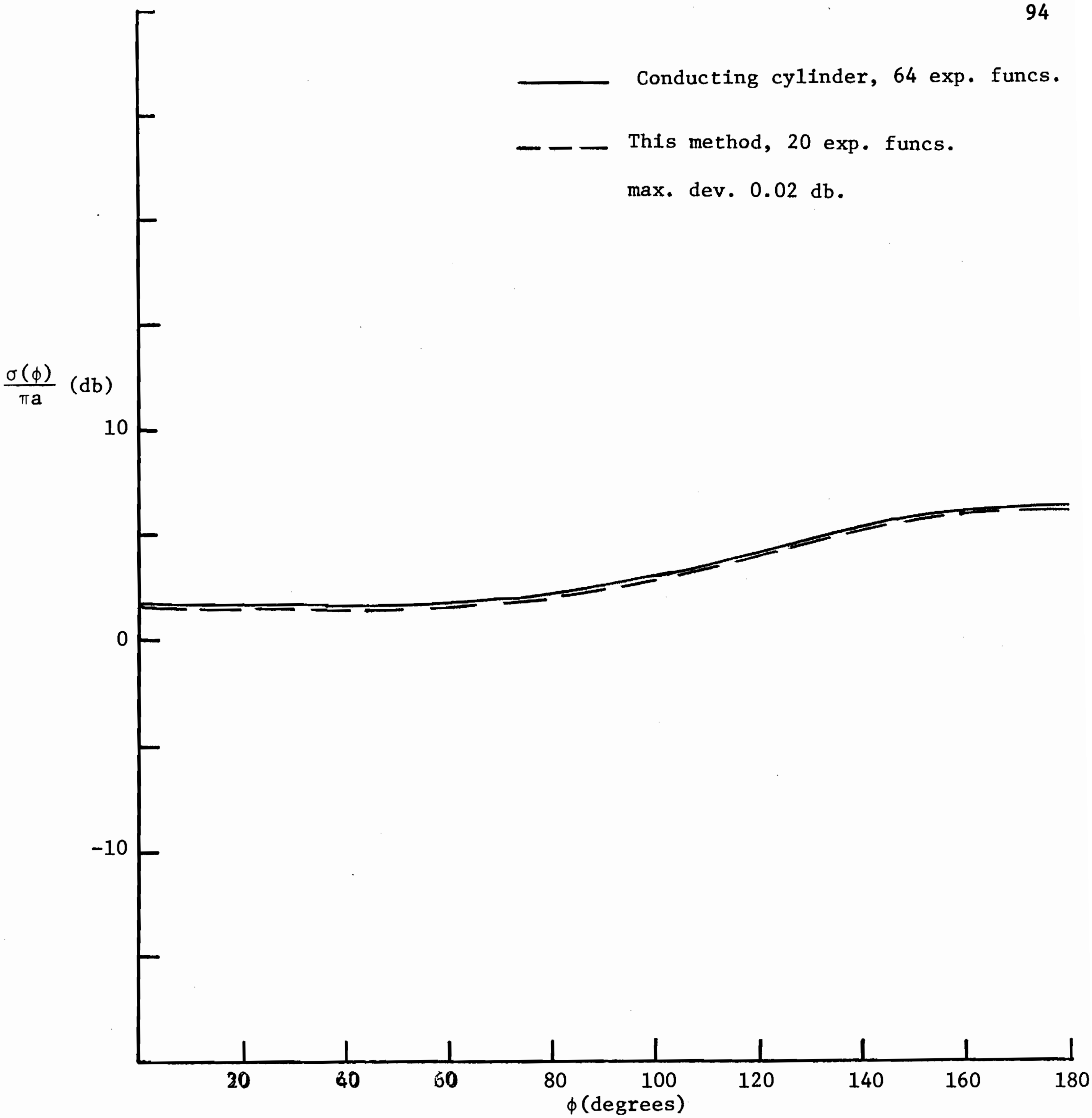


Fig. 4-27. Normalized scattering cross section of a square cylinder with ε_r = 10000.0, μ_r = 0.0001, kb = 1.4, parallel polarization (TM).

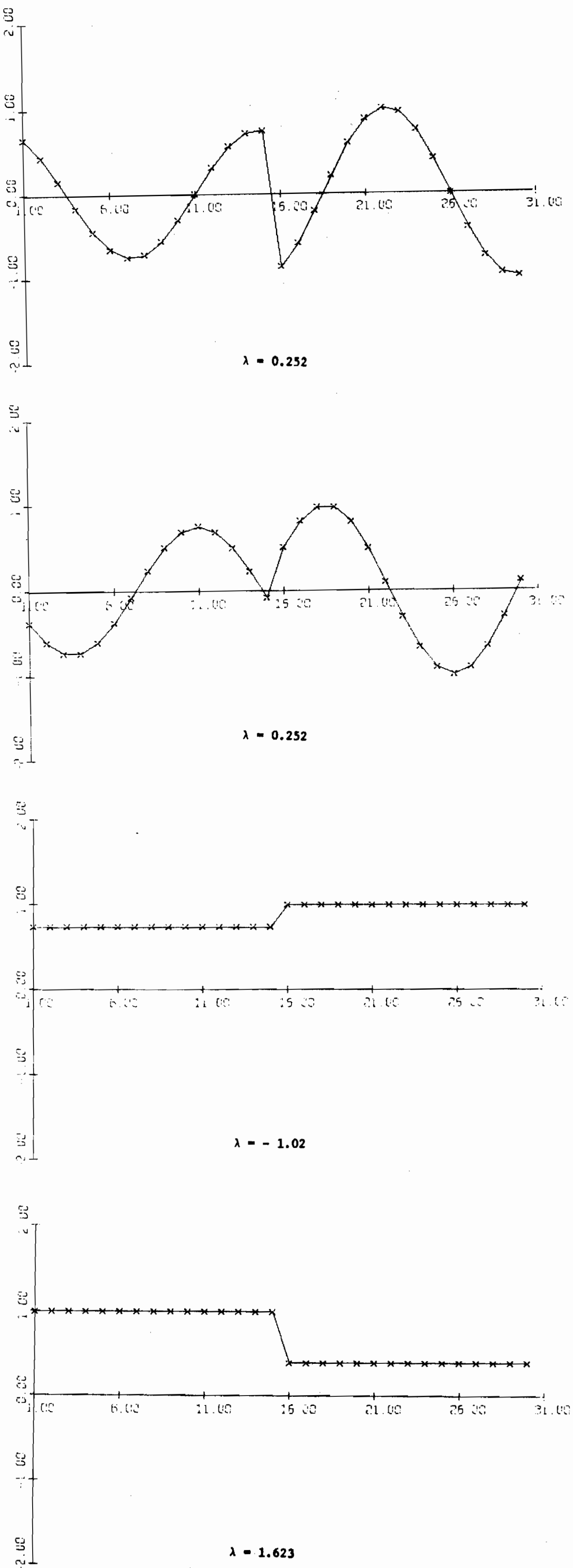
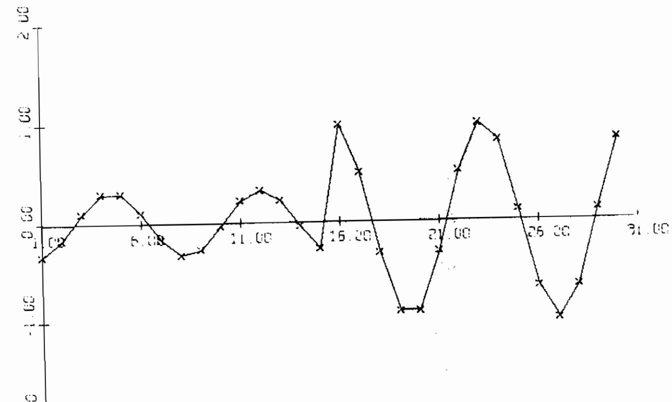
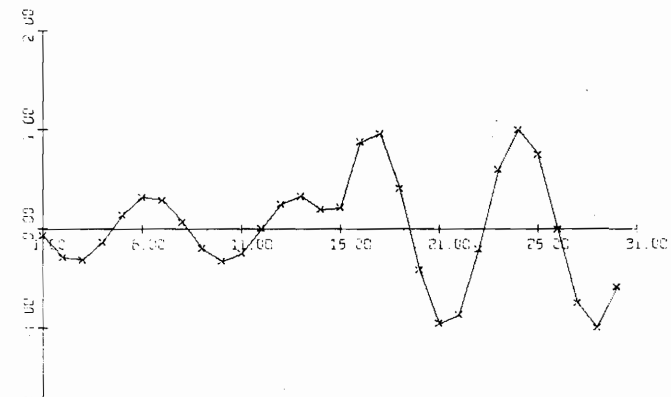


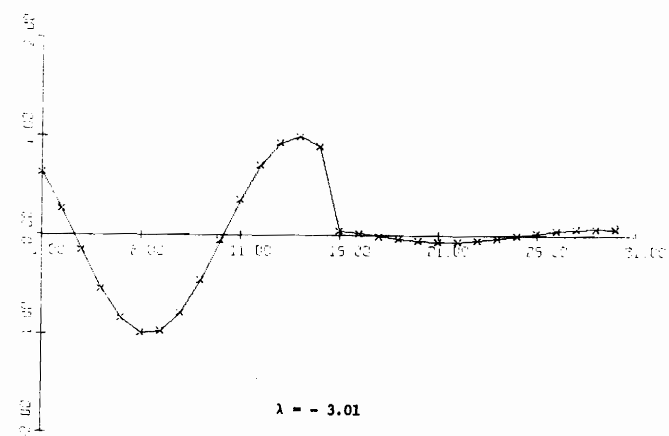
Fig. 4-28. Characteristic currents for a circular cylinder, $\epsilon_r = 9.5$, $ka = 0.7$, perpendicular polarization.



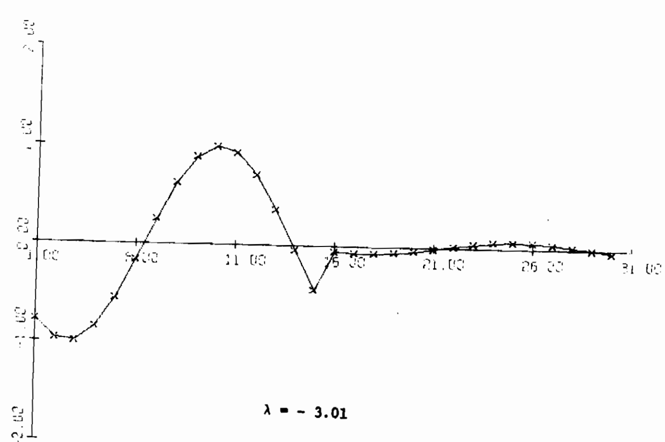
λ = 1.627



λ = 1.627

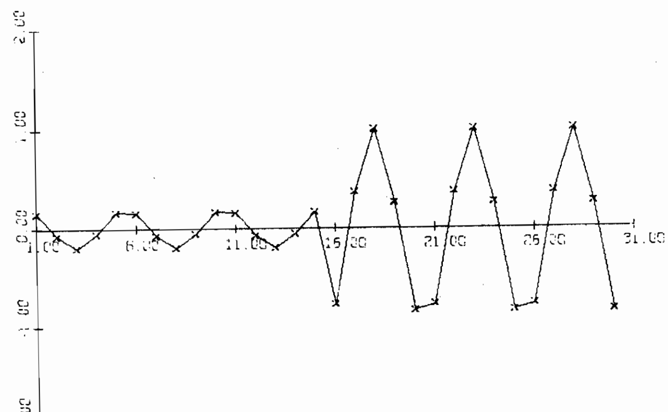


λ = − 3.01

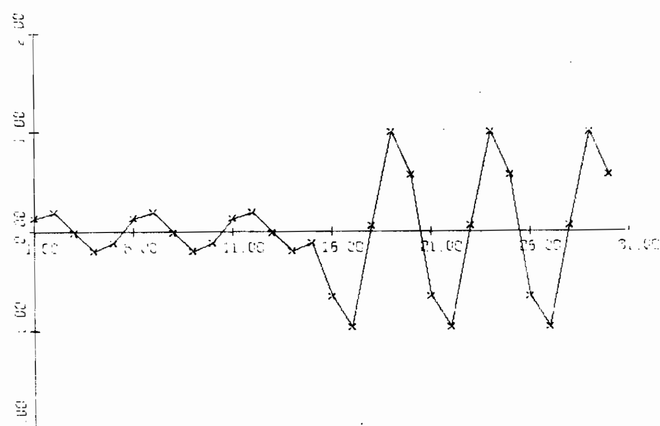


λ = − 3.01

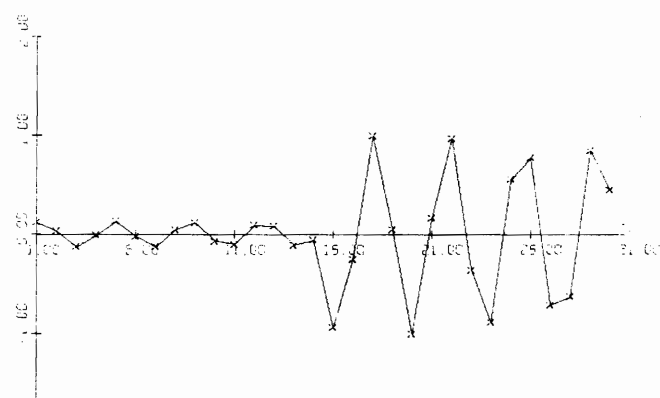
Fig. 4-28 continued



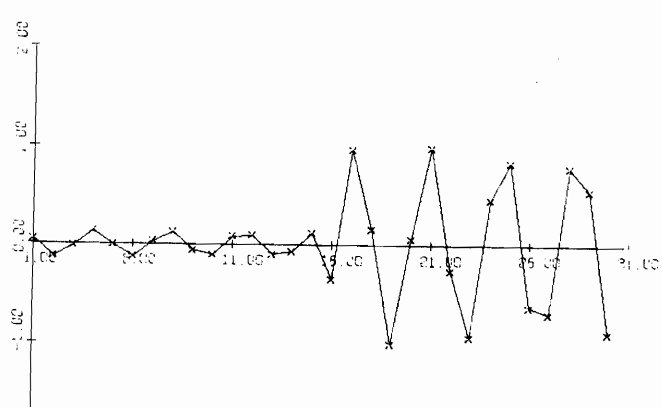
λ = 7.251



λ = 7.251

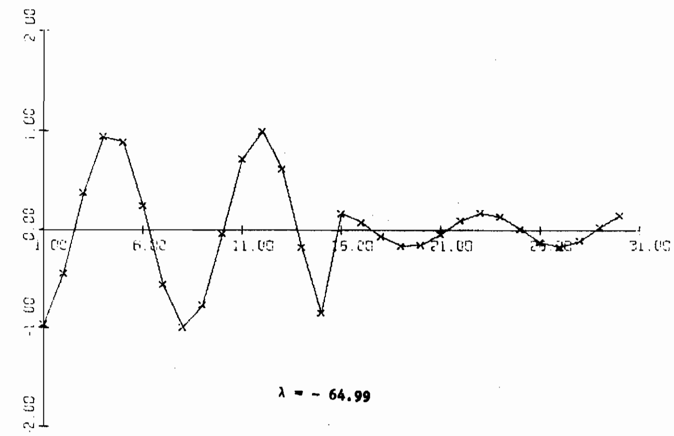


λ = 56.93

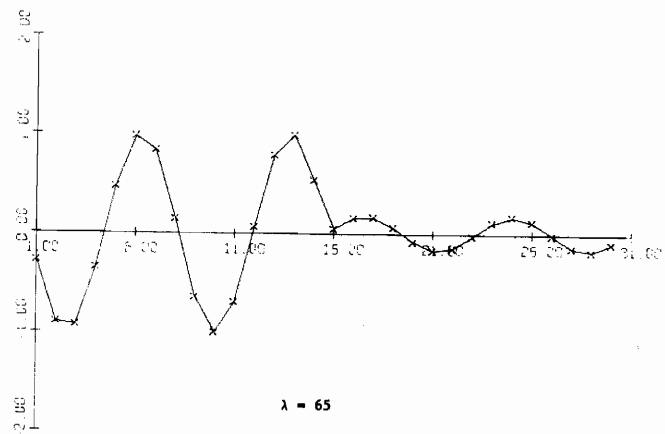


λ = 56.94

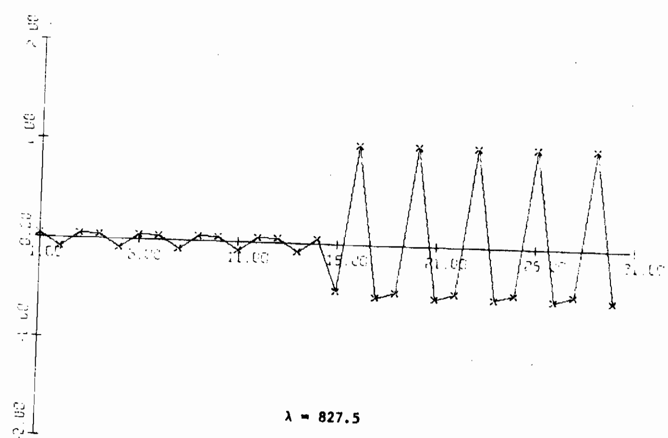
Fig. 4-28 continued



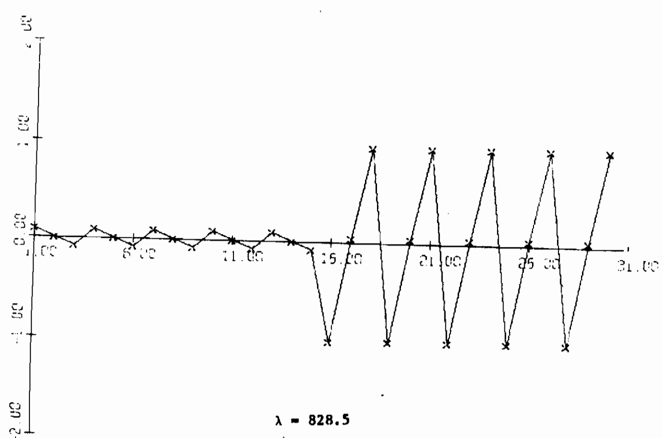
λ = - 64.99



λ = 65



λ = 827.5



λ = 828.5

Fig. 4-28 continued.

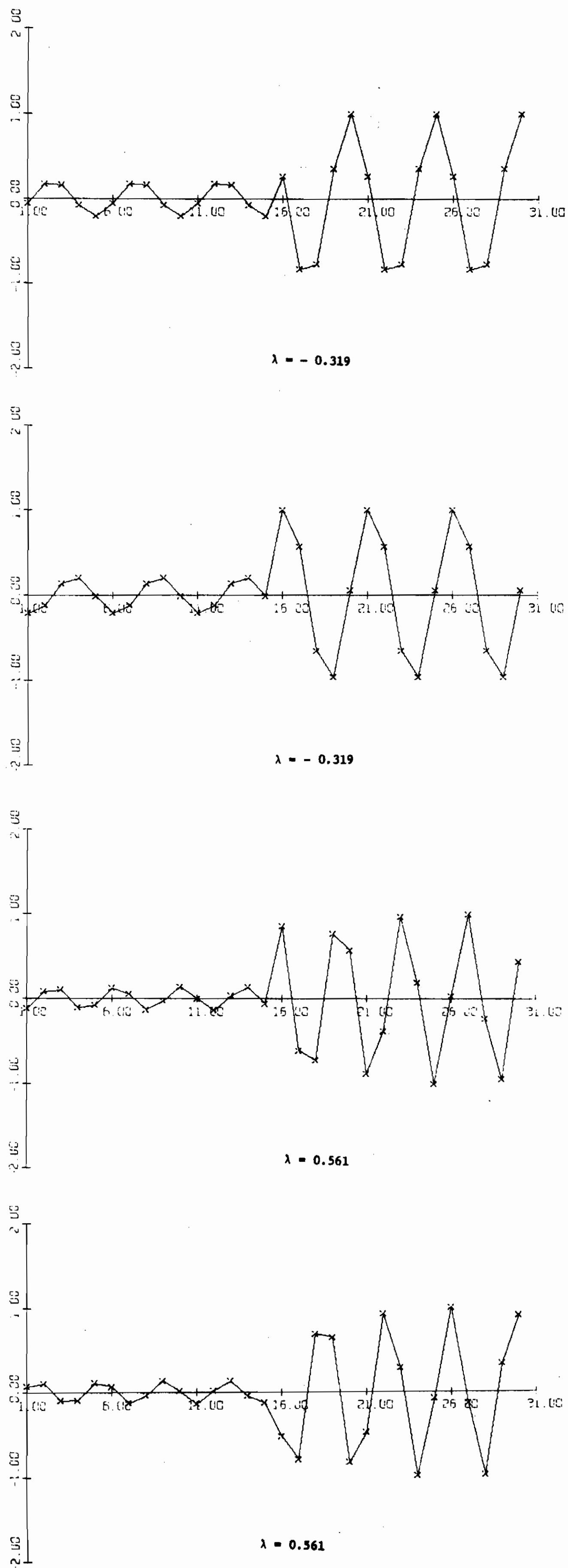


Fig. 4-29. Characteristic currents for a circular cylinder, $\epsilon_r = 50.0$, ka = 0.7, perpendicular polarization.

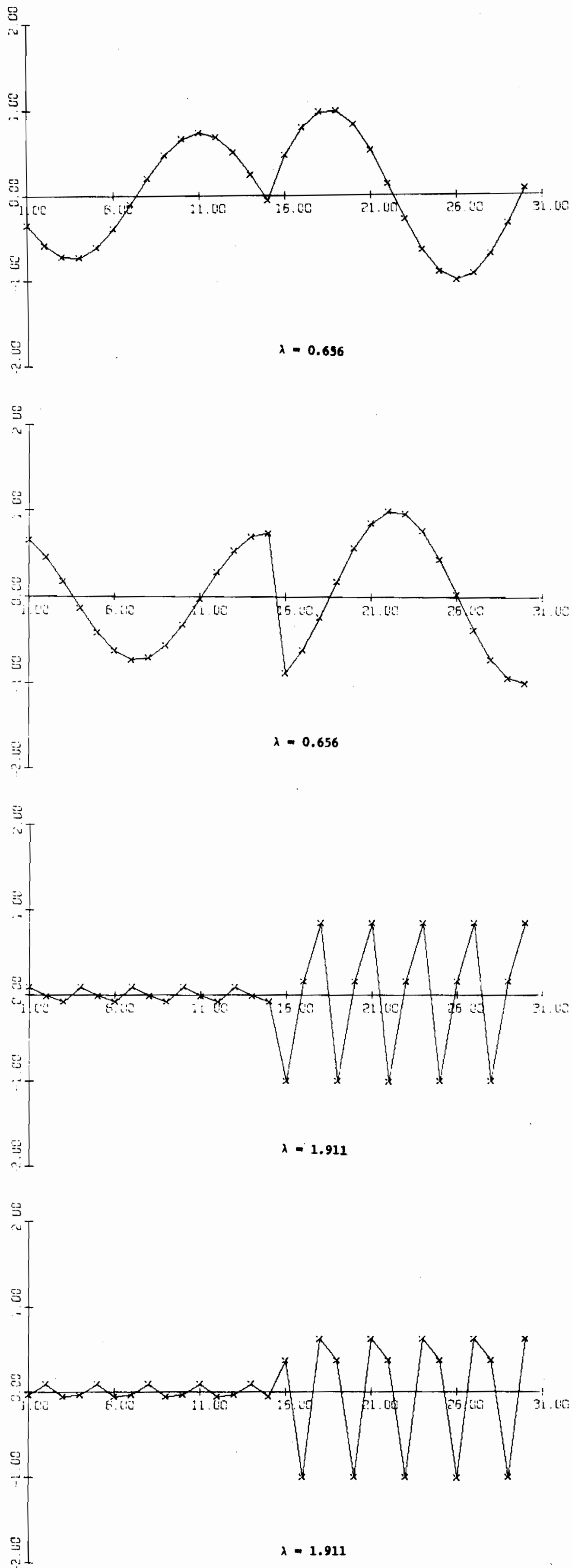


Fig. 4-29. continued.

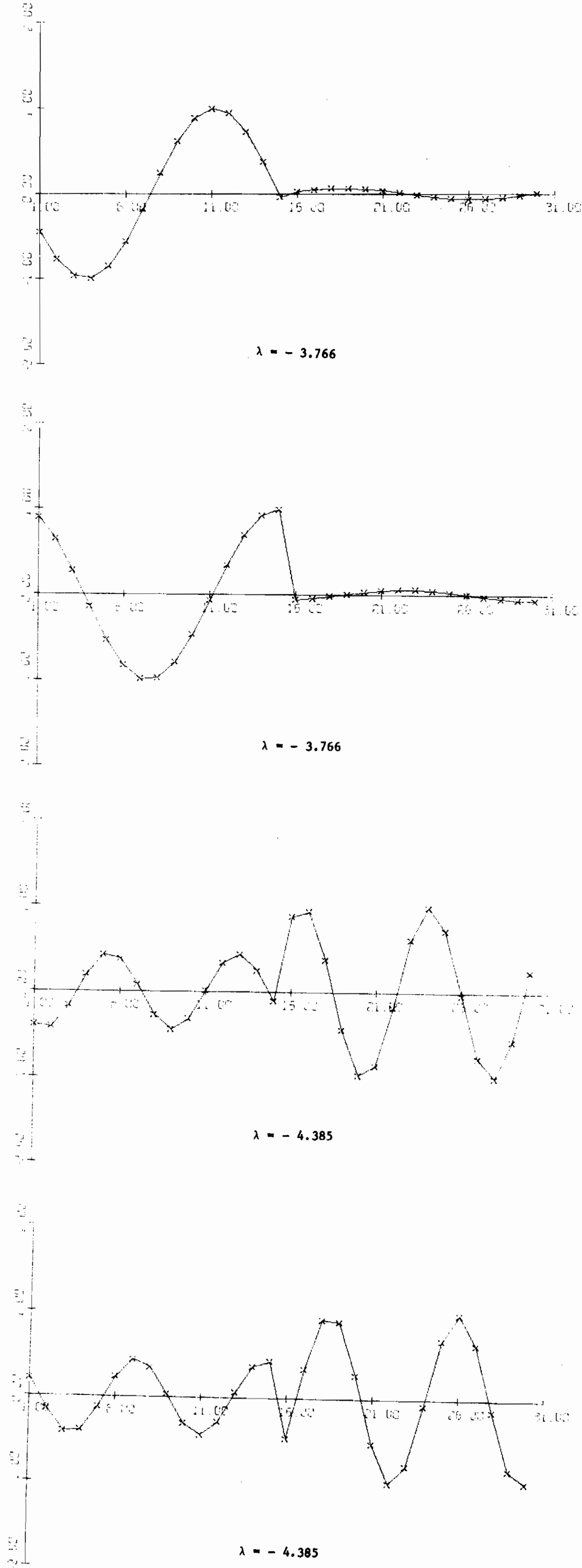


Fig. 4-29. continued.

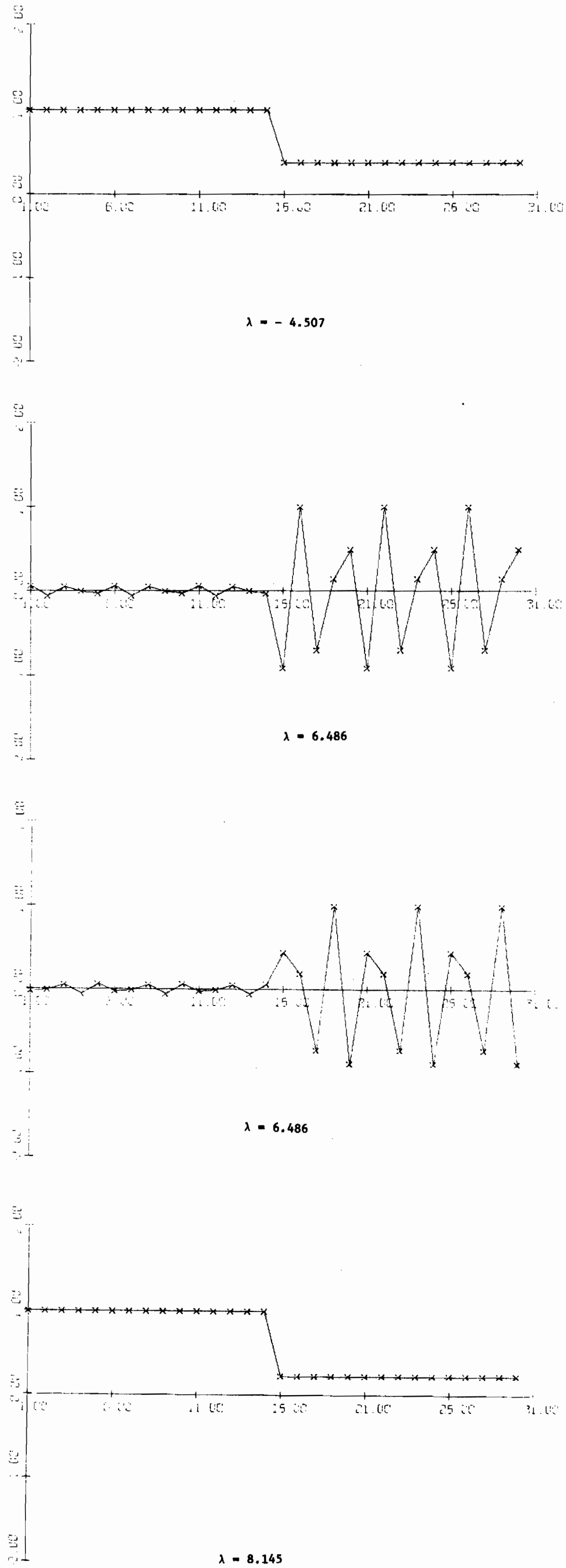


Fig. 4-29. Continued.

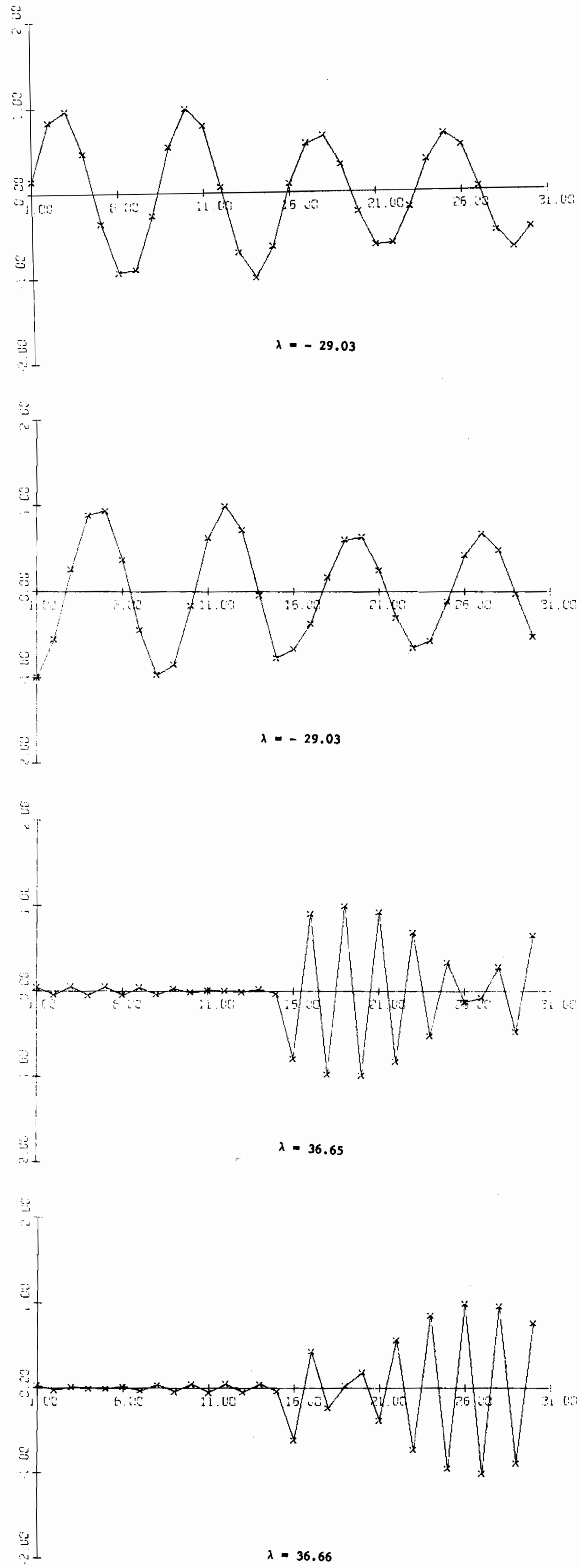


Fig. 4-29. continued.

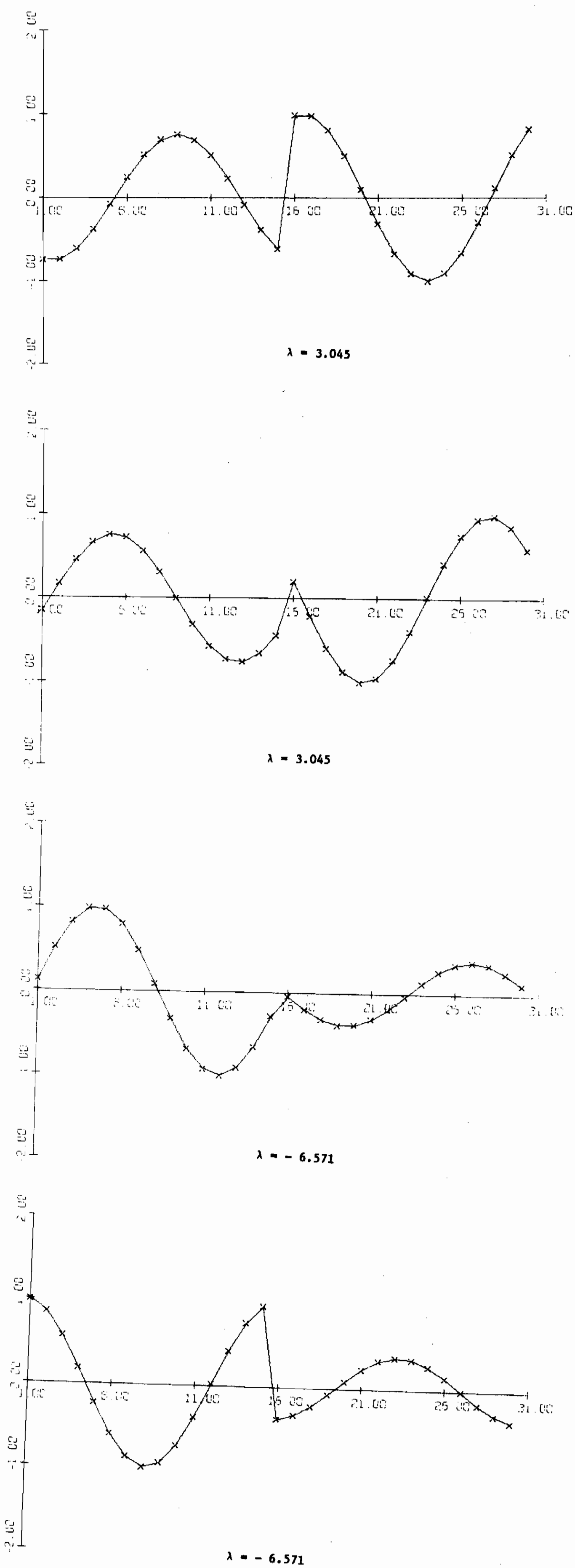


Fig. 4-30. Characteristic currents for a circular cylinder, $\epsilon_r = 2.56$, ka = 0.7, perpendicular polarization.

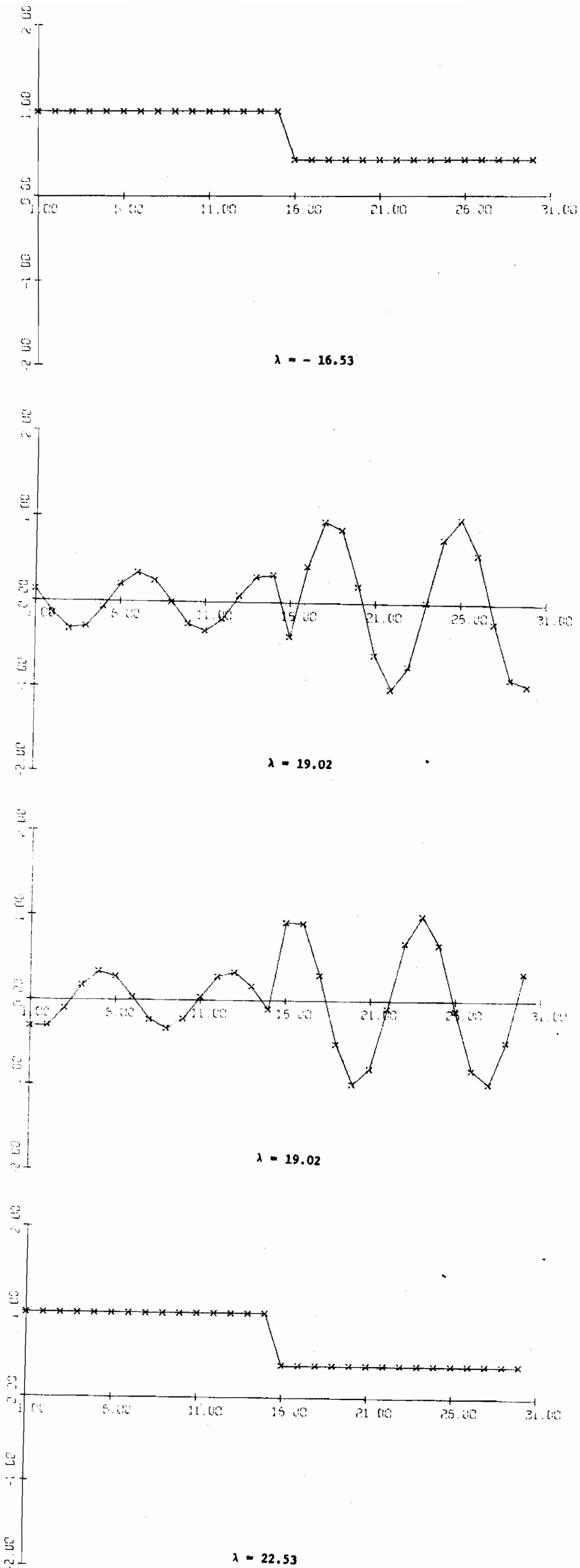
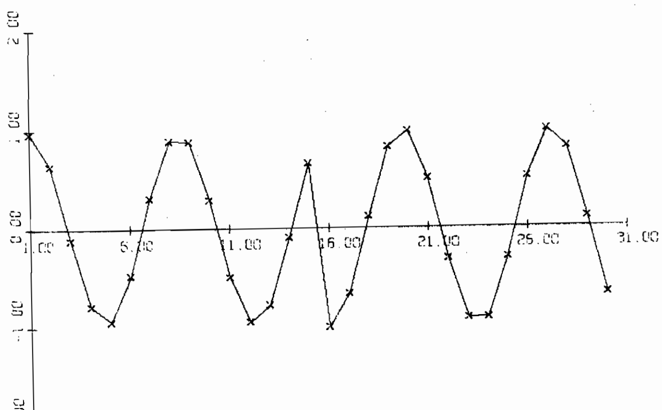
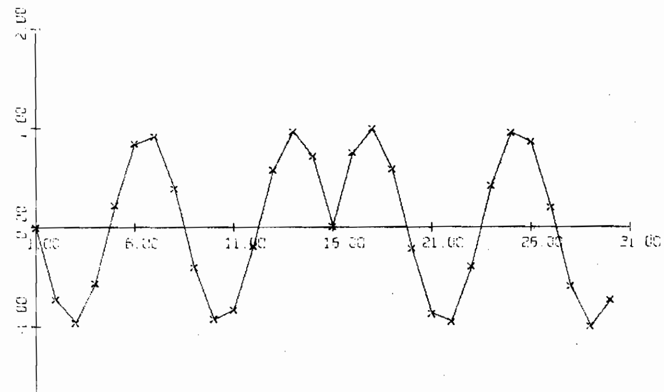


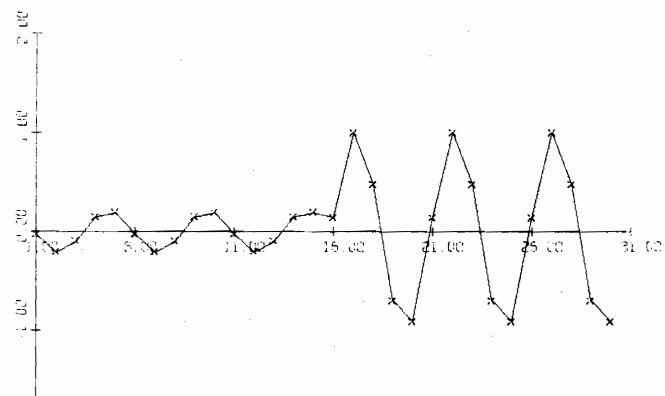
Fig. 4-30. continued.



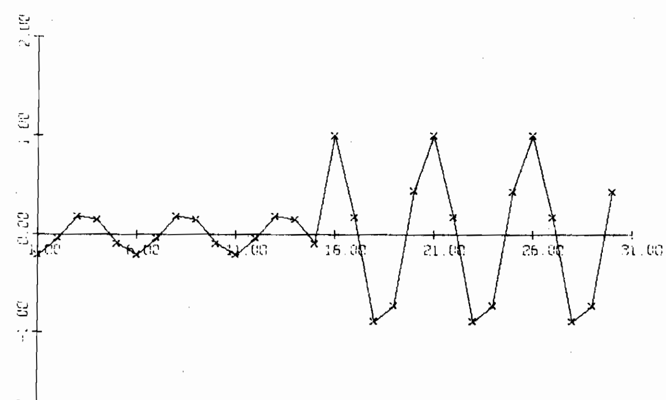
λ = − 141.2



λ = − 141.3



λ = 313.4



λ = 313.3

Fig. 4-30. continued.

CHAPTER 5

DISCUSSION

A surface formulation is developed for solving two-dimensional electromagnetic scattering problems. A basic theory for characteristic modes of dielectric and magnetic bodies based on the surface formulation is derived. The method of computing characteristic modes can be used for homogeneous material bodies of arbitrary shape provided the body is not electrically large. The characteristic modes of material bodies have most of the properties of those for perfectly conducting bodies, and should find similar uses. The theory presented here is in contrast to that for the volume formulation [4]. The basic difference is that the current in the material body has been treated as equivalent surface currents instead of a volume distribution. The characteristic currents are real and their corresponding eigenvalues are also real. The eigenvectors given by equation (3-73) are those corresponding to the lowest eigenvalues, and they are usually very efficient radiators. Characteristic currents associated with large eigenvalues generally indicate higher order modes which do not radiate very much.

Two ways for computing the scattered fields are given here. The simple material cylinders. The matrix inversion method is easier to use and gives very good results. The characteristic mode method may require slightly longer computing time, but it does provide more insight into the problem. As in the conducting body case, the characteristic mode method should prove to be of value, both theoretically and computationally for scattering and radiation problems. The versatility of characteristic modes has been adequately demonstrated in analysis and synthesis problems dealing with conducting bodies. The two approaches are based on a surface formulation, and they require the material body to be homogeneous since the unknowns are surface currents. For inhomogeneous bodies the surface formulation is not appropriate, and a volume current distribution must be used which requires sample points inside the scattering body.

APPENDIX A

MATRIX ELEMENTS FOR PARALLEL POLARIZATION

For the incident field

$$\underset{\sim}{E} = \underset{\sim}{u}_z \, e^{-j\underset{\sim}{k}\cdot\underset{\sim}{\rho}} \tag{A-1}$$

the following formulas are obtained (u_z is the axially directed unit vector). The procedures involved are identical to that given in Chapter 2, except that the directions of the surface currents are different.

A.1 Formulas for [Z] Matrix Elements

$$Z_{mn} = \frac{\omega\mu}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} T_p \, T_q \, \dot{Z} \tag{A-2}$$

where

$$Z = \Delta t_p \, \Delta t_q \, H_o^{(2)}(kR_{pq}) \qquad \text{(non-coincident intervals)}$$

$$= t_p \left[1 - j\frac{2}{\pi} \, \ell og \, \frac{\gamma k \Delta t_p}{4e}\right] \qquad \text{(coincident intervals)}$$

A.2 Formulas for [B] Matrix Elements

$$B_{mn} = -\frac{1}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} \Delta t_q \, T_p \, T_q \, B \tag{A-3}$$

$$B = -\frac{kH_1^{(2)}(kR_{pq})}{R_{pq}} \left[-(x_p - x_q)\Delta y_p + (y_p - y_q)\Delta x_p \right]$$

(non-coincident intervals)

$$= -j2 \qquad \text{(coincident intervals)}$$

A.3 Formulas for [C] Matrix Elements

$$C_{mn} = - \frac{1}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} \Delta t_p \; T_p \; T_q \; C \tag{A-4}$$

$$C = - \frac{kH_1^{(2)}(kR_{pq})}{R_{pq}} \left[-(y_p - y_q)\Delta x_q + (x_p - x_q)\Delta y_q \right]$$

(non-coincident intervals)

$$= -j2$$

(coincident intervals)

A.4 Formulas for [Y] Matrix Elements

$$Y_{mn} = \frac{1}{4} \sum_{q=1}^{4} \sum_{p=1}^{4} \Delta t_p \; \Delta t_q \left[\omega\varepsilon T_p \; T_q \; (u_p . u_q) - \frac{1}{\omega\mu} T_p' \; T_q' \right] Y \tag{A-5}$$

where

$$Y = H_o^{(2)} (k \, R_{pq})$$

(non-coincident intervals)

$$= 1 - \frac{2}{\pi} \ell og \frac{\gamma k \Delta t_p}{4e}$$

(coincident intervals)

A.5 Excitation Matrix Elements

$$V_m^i = \sum_{p=1}^{4} T_p \; \Delta t_p \; e^{jk(x_{mp} \cos\phi^i + y_{mp} \sin\phi^i)} \tag{A-6}$$

$$I_m^i = - \frac{j}{\eta} \sum_{p=1}^{4} T_p \left[-\Delta x \sin\phi^i + \Delta y \cos\phi^i \right] e^{jk(x_{mp}\cos\phi^i + y_{mp}\sin\phi^i)} \tag{A-7}$$

APPENDIX B

COMPUTER PROGRAMS

B.1 Listing of the program to compute scattering cross sections, perpendicular polarization(TE).

```
//YUCHANG JOB (0639,EE,2M59S),'CHANG.Y',REGION=250K,CLASS=A
// EXEC WATFIV
//GO.SYSIN  DD  *
$JOB          CHANG,TIME=3,PAGES=30
      SUBROUTINE FPO(Z,FO,PO)
      Y=3.0/Z
      FO=0.79788456+Y*(-0.00000077+Y*(-0.0055274+Y*(-0.00009512+Y*(0.001
     137237+Y*(-0.00072805+Y*0.00014476)))))
      PO=0.78539816+Y*(0.04166397+Y*(0.00003954+Y*(-0.00262573+Y*(0.0005
     14125+Y*(0.00029333-Y*0.0001355)))))
      RETURN
      END
      SUBROUTINE FP1(Z,F1,P1)
      Y=3.0/Z
      F1=0.79788456+Y*(0.00000156+Y*(0.01659667+Y*(0.00017105+Y*(-0.0024
     19511+Y*(0.00113653-Y*0.00020033)))))
      P1=0.78539816+Y*(-0.12499612+Y*(-0.00005650+Y*(0.00637879+Y*(-0.00
     107434R+Y*(-0.00079824+Y*0.00029166)))))
      RETURN
      END
      FUNCTION BSJO(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSJO=1.0
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSJO=1.0+Y*(-2.2499997+Y*(1.2656208+Y*(-0.3163866+Y*(0.0444479+Y*(
     1-0.0039444+Y*0.00021)))))
      RETURN
    1 CALL FPO(Z,FO,PO)
      BSJO=FO*COS(Z-PO)/SQRT(Z)
      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTE-
     1RED IN CALCULATING A BESSEL FUNCTION OF ORDER ZERO'/)
      END
      FUNCTION BSYO(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSYO=-1.0E75
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSYO=0.63661977*ALOG(0.5*Z)*BSJO(Z)+0.36746691+Y*(0.60559366+Y*(-0
     1.74350384+Y*(0.25300117+Y*(-0.04261214+Y*(0.00427916-Y*0.00024846)
     2))))
      RETURN
    1 CALL FPO(Z,FO,PO)
      BSYO=FO*SIN(Z-PO)/SQRT(Z)
      RETURN


   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A NEUMANN FUNCTION OF ORDER ZERO ABS(X) USED'/)
      END
      FUNCTION BSJ1(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSJ1=0.0
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0) GO TO 1
      Y=Z*Z/9.0
      BSJ1=X*(0.5+Y*(-0.56249985+Y*(0.21093573+Y*(-0.03954289+Y*(0.00443
     1319+Y*(-0.00031761+Y*0.00001109))))))
      RETURN
    1 CALL FP1(Z,F1,P1)
      BSJ1=F1*SIN(Z-P1)/SQRT(Z)
      IF(X.LT.0.0)BSJ1=-BSJ1
      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A BESSEL FUNCTION OF ORDER ONE'/)
      END
      FUNCTION BSY1(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSY1=-1.0E75
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSY1=(-0.63661977+Y*(0.2212091+Y*(2.1682709+Y*(-1.3164827+Y*(0.312
     13951+Y*(-0.0400976+Y*0.0027873))))))/Z+0.63661977*ALOG(0.5*Z)*BSJ1
     2(Z)
      RETURN
    1 CALL FP1(Z,F1,P1)
      BSY1=-F1*COS(Z-P1)/SQRT(Z)
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A NEUMANN FUNCTION OF ORDER ONE ABS(X) USED'/)
      RETURN
      END
      SUBROUTINE LINEQ(LL,C)
      COMPLEX C(3600),STOR,STO,ST,S
      DIMENSION LR(60)
      DO 20 I=1,LL
      LR(I)=I
   20 CONTINUE
      M1=0
      DO 18 M=1,LL
      K=M
      K2=M1+K
      S1=ABS(REAL(C(K2)))+ABS(AIMAG(C(K2)))
      DO 2 I=M,LL
      K1=M1+I
      S2=ABS(REAL(C(K1)))+ABS(AIMAG(C(K1)))
      IF(S2-S1) 2,2,6
    6 K=I
      S1=S2
    2 CONTINUE
      LS=LR(M)
      LR(M)=LR(K)
      LR(K)=LS
      K2=M1+K
      STOR=C(K2)
      J1=0
      DO 7 J=1,LL
      K1=J1+K
      K2=J1+M
      STO=C(K1)
      C(K1)=C(K2)
      C(K2)=STO/STOR
      J1=J1+LL
    7 CONTINUE
      K1=M1+M
      C(K1)=1./STOR
      DO 11 I=1,LL
      IF(I-M) 12,11,12
   12 K1=M1+I
      ST=C(K1)
      C(K1)=0.
      J1=0
      DO 10 J=1,LL
      K1=J1+I
      K2=J1+M
      C(K1)=C(K1)-C(K2)*ST
      J1=J1+LL
   10 CONTINUE
   11 CONTINUE
      M1=M1+LL
   18 CONTINUE
      J1=0
      DO 9 J=1,LL
      IF(J-LR(J)) 14,8,14
   14 LRJ=LR(J)
      J2=(LRJ-1)*LL
   21 DO 13 I=1,LL
      K2=J2+I
      K1=J1+I
      S=C(K2)
      C(K2)=C(K1)
      C(K1)=S
   13 CONTINUE
      LR(J)=LR(LRJ)
      LR(LRJ)=LRJ
      IF(J-LR(J)) 14,8,14
    8 J1=J1+LL
    9 CONTINUE
      RETURN
      END
      SUBROUTINE HANK(GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      PI=3.141593
      P=2.0/3.141593
      N1=N2+1
      FE=4.*2.71828
      EL=1.781072
      I1=0
      DO 30 J=1,N1
      DO 40 I=1,J
      I1=I1+1


      IF(I .EQ. J)GO TO 25
      IF(I .EQ. 1 .AND. J .EQ. N2)GO TO 25
      IF(I .EQ. 2 .AND. J .EQ. N1)GO TO 25
      RK=RR(I1)*GK
      H(I1)=BSJO(RK)-CJ*BSYO(RK)
      HF(I1)=BSJ1(RK)-CJ*BSY1(RK)
      GO TO 40
   25 CONTINUE
      AA=(EL*GK*DL(I))/FE
      H(I1)=1.0-CJ*P*ALOG(AA)
   40 CONTINUE
   30 CONTINUE
      RETURN
      END
      SUBROUTINE CALZY(WE,WU)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMPLEX ZZ
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),C
      COMMON XP(63),YP(63),N2
      PI=3.141593
      II=0
      L=0
      DO 50 J3=3,N2,2
      J1=J3-2
      J2=J3-1
      J4=J3+1
      K=0
      DO 60 I3=3,N2,2
      II=II+1
      I1=I3-2
      I2=I1+1
      I4=I3+1
      Z(II)=0.0
      Y(II)=0.0
      DO 70 M=J1,J4
      LL=M+L
      DO 80 N=I1,I4
      KK=N+K
      IF(M-N)100,120,120
  100 JJ=(N*(N-1))/2+M
      GO TO 130
  120 JJ=(M*(M-1))/2+N
  130 CONTINUE
      DC=(DX(M)*DX(N)+DY(M)*DY(N))/(DL(M)*DL(N))
      ZT1=WU*T(KK)*T(LL)*DC
      ZT2=-TD(KK)*TD(LL)/WE
      ZT3=ZT1+ZT2
      ZZ=H(JJ)
      Z(II)=0.25*DL(N)*DL(M)*ZT3*ZZ+Z(II)
      Y(II)=-0.25*WE*T(KK)*T(LL)*DL(M)*DL(N)*ZZ+Y(II)
   80 CONTINUE
   70 CONTINUE
      Y(II)=-Y(II)
      Z(II)=Z(II)/(377.0*377.0)
      K=K+2
   60 CONTINUE
      L=L+2
   50 CONTINUE
```

```
      RETURN
      END
      SUBROUTINE CALC(GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMPLEX HERR
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      N1=N2+1
      II=0
      L=0
      DO 50 J3=3,N2,2
      J1=J3-2
      J2=J3-1
      J4=J3+1
      K=0
      DO 60 I3=3,N2,2
      II=II+1
      I1=I3-2
      I2=I1+1
      I4=I3+1
      Z(II)=0.0
      DO 70 M=J1,J4
      LL=M+L
      DO 80 N=I1,I4
      KK=N+K
      IF(M-N)100,150,120
  100 JJ=(N*(N-1))/2+M
      IF(M .EQ. 1 .AND. N .EQ. N2) GO TO 150
      IF(M .EQ. 2 .AND. N .EQ. N1) GO TO 150
      GO TO 130
  120 JJ=(M*(M-1))/2+N
      IF(N .EQ. 1 .AND. M .EQ. N2) GO TO 150
      IF(N .EQ. 2 .AND. M .EQ. N1) GO TO 150
  130 CONTINUE
      HERR=HF(JJ)/RR(JJ)
      DTT=DL(M)*T(LL)*T(KK)
      CZ=-(XM(N)-XM(M))*DY(N)+(YM(N)-YM(M))*DX(N)
      Z(II)=0.25*CJ*DTT*HERR*GK*CZ+Z(II)
      GO TO 80
  150 CONTINUE
   80 CONTINUE
   70 CONTINUE
      Z(II)=-CJ*Z(II)
      Z(II)=Z(II)/377.0
      K=K+2
   60 CONTINUE
      L=L+2
   50 CONTINUE
      RETURN
      END
      SUBROUTINE EXMX(PHI,GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMPLEX PP
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      ETA=376.7301
      N=(N2-1)/2
      CP=COS(PHI)
      SP=SIN(PHI)
      II=0
      L=0
      DO 200 I3=3,N2,2
      II=II+1
      JJ=II+N
      I1=I3-2
      I2=I3-1
      I4=I3+1
      EV(II)=0.
      EV(JJ)=0.0
      DO 210 M=I1,I4
      LL=M+L
      EP=(XM(M)*CP+YM(M)*SP)*GK
      DP=-DX(M)*SP+DY(M)*CP
      CEP=COS(EP)
      SEP=SIN(EP)
      PP=(CEP+CJ*SEP)*T(LL)
      EV(II)=PP*DP+EV(II)
      EV(JJ)=-PP*DL(M)+EV(JJ)
  210 CONTINUE
      EV(II)=-ETA*EV(II)
      EV(JJ)=-CJ*EV(JJ)
      L=L+2
  200 CONTINUE
      RETURN
      END
      COMPLEX HED
      COMPLEX AE(60)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      A9=9.0
      A10=10.0
      PI=3.141593
      P4=180.0/3.141593
      NP=43
      N1=NP-1
      N2=NP-2
      N=(N1-2)/2
      NT=4*N
      WL=1.0
      ETA=376.7301
      P1=3.141593/180.0
      ANS=0.0
      GK=2.0*3.141593/WL
      WE=GK/ETA
      A1=2.405
      RD=A1/GK
      WU=GK*ETA
      NH=2*N
      UR=0.000I
      ER=10000.0
      GKM=SQRT(UR*ER)*GK
      WEM=WE*ER
      WUM=WU*UR
      CJ=(0.,1.)
      PA=RD*PI
      P2=A9*PI
      THETA=0.0
      DO 5 I=1,NP
      XP(I)=RD*COS(THETA)*WL
      YP(I)=RD*SIN(THETA)*WL
      THETA=THETA+P2
    5 CONTINUE
      WRITE(3,4) (XP(I),YP(I),I=1,NP)
      DO 10 J=1,N1
      J1=J+1
      DX(J)=XP(J1)-XP(J)
      DY(J)=YP(J1)-YP(J)
      DL(J)=SQRT(DX(J)**2+DY(J)**2)
      XM(J)=0.5*(XP(J1)+XP(J))
      YM(J)=0.5*(YP(J1)+YP(J))
   10 CONTINUE
      WRITE(3,45)
   45 FORMAT(/////)
      WRITE(3,4) (DL(I),I=1,N1)
      WRITE(3,45)
      WRITE(3,4) (DX(I),I=1,N1)
      WRITE(3,45)
      WRITE(3,4) (DY(I),I=1,N1)
      I1=0
      DO 30 J=1,N1
      DO 40 I=1,J
      I1=I1+1
      IF(I .EQ. J)GO TO 25
      IF(I .EQ. 1 .AND. J .EQ. N2)GO TO 25
      IF(I .EQ. 2 .AND. J .EQ. N1)GO TO 25
      XPQ=XM(I)-XM(J)
      YPQ=YM(I)-YM(J)
      RR(I1)=SQRT(XPQ*XPQ+YPQ*YPQ)
   25 CONTINUE
   40 CONTINUE
   30 CONTINUE
      L=0
      DO 20 M3=3,N2,2
      M1=M3-2
      M4=M3+1
      M2=M3-1
      L1=M1+L
      L2=L1+1
      L3=L2+1
      L4=L3+1
      T(L1)=.5*DL(M1)/(DL(M1)+DL(M2))
      T(L2)=(DL(M1)+.5*DL(M2))/(DL(M1)+DL(M2))
      T(L3)=(0.5*DL(M3)+DL(M4))/(DL(M3)+DL(M4))
      T(L4)=0.5*DL(M4)/(DL(M3)+DL(M4))
      TD(L1)=1./(DL(M1)+DL(M2))
      TD(L2)=TD(L1)
      TD(L3)=-1.0/(DL(M3)+DL(M4))
      TD(L4)=TD(L3)
      L=L+2
   20 CONTINUE
      WRITE(3,45)
      WRITE(3,4) (T(I),I=1,NT)
      NS=N**2
      CALL HANK(GK)
      CALL CALZY(WE,WU)
      K=0
      L=0
      NI=2*NS+N
      DO 500 J=1,N
      DO 510 I=1,N
      L=L+1
      KK=K+I
      II=NI+L
      A(L)=Z(KK)
      A(II)=Y(KK)
  510 CONTINUE
      L=L+N
      K=K+N
  500 CONTINUE
      CALL CALC(GK)
      K=0
      II=N
      DO 540 J=1,N
      DO 550 I=1,N
      KK=K+I
      II=II+1
      A(II)=Z(KK)
  550 CONTINUE
      K=K+N
      II=II+N
  540 CONTINUE
      CALL HANK(GKM)
      CALL CALZY(WEM,WUM)
      K=0
      II=0
      DO 520 J=1,N
      DO 530 I=1,N
      KK=K+I
      II=II+1
      III=NI+II
      A(II)=Z(KK)+A(II)
      A(III)=Y(KK)+A(III)
  530 CONTINUE
      K=K+N
      II=II+N
  520 CONTINUE
      CALL CALC(GKM)
      K=0
      II=N
      DO 560 J=1,N
      DO 570 I=1,N
      KK=K+I
      II=II+1
      A(II)=Z(KK)+A(II)
  570 CONTINUE
      K=K+N
      II=II+N
  560 CONTINUE
      NK=2*N
      NJ=2*NS
      L=0
      DO 580 J=1,N
      K=N
```

```
      DO 590 I=1,N
      KK=J+K
      L=L+1
      II=L+NJ
      A(II)=A(KK)
      K=K+NK
  590 CONTINUE
      L=L+N
  580 CONTINUE
    4 FORMAT(//,1H ,10E11.4)
      CALL LINEQ(NH,A)
      CALL EXMX(ANS,GK)
      WRITE(3,45)
      WRITE(3,4) (EV(I),I=1,NH)
      WRITE(3,304)
  304 FORMAT(1H ,'CURRENT'/'  I',6X,'REAL',10X,'IMAG',10X,'MAGNITUDE',7X
     1,'PHASE')
      DO 50 I=1,N
      EV(I)=EV(I)/377.0
   50 CONTINUE
      DO 270 I1=1,NH
      AE(I1)=0.0
      DO 300 I=1,NH
      J=(I-1)*NH+I1
      AE(I1)=A(J)*EV(I)+AE(I1)
  300 CONTINUE
  270 CONTINUE
      DO 70 I=1,N
      AE(I)=AE(I)/377.0
   70 CONTINUE
      DO 80 I1=1,NH
      CA=CABS(AE(I1))
      CPH=ATAN2(AIMAG(AE(I1)),REAL(AE(I1)))*57.2858
      WRITE(3,305)I1,AE(I1),CA,CPH
  305 FORMAT(1H ,1X,I3,3E14.6,F10.3)
   80 CONTINUE
      WRITE(3,275)
  275 FORMAT(///,' SCATTERING ANGLE - PHI',10X,'ECHO LENGTH/WAVELENGTH')
      AN=0.0
      P3=A10*P1
      DO 320 I=1,36
      CALL EXMX(AN,GK)
      HED=0.0
      DO 330 J=1,NH
      HED=EV(J)*AE(J)+HED
  330 CONTINUE
      CH=CABS(HED)
      ECHO=GK*CH**2/4.0
      ECL=ECHO/(WL*ETA**2)
      PRD=ECL/PA
      ECL=10.0*ALOG10(PRD)
      WRITE(3,335)AN,ECL
  335 FORMAT(1H ,10X,F7.2,20X,F14.7)
      AN=AN+P3
  320 CONTINUE
      STOP
      END
$DATA
$STOP
/*
```

B.2 <u>Listing of the program to compute characteristic currents.</u>

```
//YUCHANG JOB (0639,EE,2,2,900),'CHANGY',REGION=190K,CLASS=A
// EXEC FORTGCLG
//FORT.SYSIN DD *
C
C
C   ***WEIGHTED EIGENVALUE EQ. - INPUT COMPLEX Z***
C
      COMPLEX Z(900),U1,ZL(30)
      COMPLEX CJ,EC(30)
      DIMENSION U(900),R(900),T2(900),A22(900),B(900),X(900),A(900)
      DIMENSION Y(900),T3(900),EI(900),EU(30),RU(30),AMD(30),LB(30)
      DIMENSION RR(900),RE(900),ERE(30)
      DIMENSION AB(900),B22(900)
      DIMENSION XX(900)
      DIMENSION QXQ(30),RQ(30)
      DIMENSION DU(30)
      DIMENSION EL(30)
      DIMENSION RL(30)
      EQUIVALENCE (R(1),T2(1),A22(1),B(1)),(X(1),A(1),Y(1))
      EQUIVALENCE (T3(1),EI(1)),(EU(1),AMD(1))
      CJ=(0.0,1.0)
      READ(1,26) EPS
   26 FORMAT(E11.4)
      READ(1,7) N
    7 FORMAT(I3)
      WRITE(3,3) N,EPS
    3 FORMAT('0  N     EPS'/1X,I3,F11.4)
      NZ=N*N
      N2=N*N
      N2S=(N+N2)/2
      READ(1,200)(Z(I),I=1,N2S)
  200 FORMAT(5E15.7)
   22 NN=N
      DO 50 I=1,N2S
      R(I)=REAL(Z(I))
      X(I)=AIMAG(Z(I))
   50 CONTINUE
      DO 51 I=1,N2S
      RR(I)=R(I)
   51 CONTINUE
      II=N2S+1
      DO 40 K=1,N
      J=N+1-K
      DO 30 L=1,J
      I=N*(J-1)+J-L+1
      II=II-1
      RR(I)=RR(II)
      X(I)=X(II)
   30 CONTINUE
   40 CONTINUE
      KK=0
      NM=N-1
      DO 45 K=1,NM
      J=K+1
      DO 35 L=J,N
      I=L+KK
      JJ=(L-1)*N+K
      RR(I)=RR(JJ)
      X(I)=X(JJ)
   35 CONTINUE
      KK=KK+N
   45 CONTINUE
      DO 46 I=1,NZ
      XX(I)=X(I)
   46 CONTINUE
      WRITE(3,201) N
  201 FORMAT(/1X,'R MATRIX (STORED IN SYM. MODE) OF ORDER',I3)
      WRITE(3,202) (R(I),I=1,N2S)
  202 FORMAT(8(1X,E14.7))
      WRITE(3,203) N
  203 FORMAT(/1X,'X MATRIX (STORED COLUMNWISE) OF ORDER',I3)
      WRITE(3,202) (X(I),I=1,N2)
  130 CALL EIGEN(R,U,N,0)
      J1=0
      DO 104 J=1,N
      J1=J1+J
      EU(J)=R(J1)
      EL(J)=EU(J)
      RU(J)=1./SQRT(ABS(EU(J)))
  104 CONTINUE
      WRITE(3,141)(EU(J),J=1,N)
  141 FORMAT('0EIGENVALUES OF THE MATRIX R'/(1X,7E11.4))
      DO 75 J=1,N
      J1=(J-1)*N
      DO 76 I=1,N
      J2=J1+I
      T2(J2)=0.
      J3=(I-1)*N
      DO 77 K=1,N
      K1=K+J3
      K2=K+J1
      T2(J2)=T2(J2)+X(K1)*U(K2)
   77 CONTINUE
   76 CONTINUE
   75 CONTINUE
      DO 78 J=1,N
      J1=(J-1)*N
      DO 79 I=1,J
      J2=J1+I
      A(J2)=0.
      J3=(I-1)*N
      DO 80 K=1,N
      K1=K+J3
      K2=K+J1
      A(J2)=A(J2)+U(K1)*T2(K2)
   80 CONTINUE
      J4=J3+J
      A(J4)=A(J2)
   79 CONTINUE
   78 CONTINUE


      X2=EU(1)*EPS
      DO 70 J=1,N
      IF(EU(J)-X2) 72,144,144
  144 JM=J
   70 CONTINUE
   72 JN=N-JM
      JM1=JM+1
      IF(JN) 145,146,145
C
C  IF NO  EIGENVALUES OF R ARE SET TO ZERO (JN=0) THE INSTRUCTIONS
C  BETWEEN 146 AND 151 ARE CARRIED OUT.
C
  146 J2=0
      DO 148 J=1,N
      J3=(J-1)*N
      DO 149 I=1,J
      J2=J2+1
      J4=J3+I
      B(J2)=A(J4)*RU(J)*RU(I)
  149 CONTINUE
  148 CONTINUE
      CALL EIGEN(B,Y,JM,0)
      J1=0
      DO 150 J=1,N
      J1=J1+J
      AMD(J)=B(J1)
  150 CONTINUE
      WRITE(3,58)(AMD(J),J=1,N)
      DO 151 J=1,N
      J1=(J-1)*N
      DO 152 I=1,N
      J2=I+J1
      T2(J2)=Y(J2)*RU(I)
  152 CONTINUE
  151 CONTINUE
      GO TO 147
C
C  IF SOME EIGENVALUES OF R ARE NOT SET TO ZERO (JN NOT EQ. TO 0)
C  THEN INSTRUCTIONS BETWEEN 145 AND 147 ARE CARRIED OUT.
C
  145 J1=0
      DO 73 J=JM1,N
      J2=(J-1)*N
      DO 74 I=JM1,N
      J1=J1+1
      J3=J2+I
      A22(J1)=A(J3)
      B22(J1)=A22(J1)
   74 CONTINUE
   73 CONTINUE
      JNN=JN*JN
      WRITE(3,600) (A22(I),I=1,JNN)
  600 FORMAT('0A22 MATRIX'/(10(1X,F11.4)))
  128 CALL LINER(JN,A22,LB)
      WRITE(3,601) (A22(I),I=1,JNN)
  601 FORMAT('0A22 INVERSE'/(10(1X,F11.4)))
      II=0
      DO 700 I=1,JN
      K1=(I-1)*JN
      DO 710 J=1,JN
```

```
      II=II+1
      AB(II)=0.0
      DO 720 K=1,JN
      KK=K1+K
      JJ=(K-1)*JN+J
      AB(II)=A22(JJ)*B22(KK)+AB(II)
  720 CONTINUE
  710 CONTINUE
  700 CONTINUE
      WRITE(3,730) (AB(I),I=1,JNN)
  730 FORMAT('0A22*A22 INVERSE'/(10(1X,E11.4)))
      J1=0
      DO 81 J=1,JM
      J3=(J-1)*N+JM
      DO 82 I=1,JN
      J2=(I-1)*JN
      J1=J1+1
      T3(J1)=0.
      DO 83 K=1,JN
      K1=J2+K
      K2=J3+K
      T3(J1)=T3(J1)+A22(K1)*A(K2)
   83 CONTINUE
   82 CONTINUE
   81 CONTINUE
      J2=0
      DO 84 J=1,JM
      J3=(J-1)*N
      J5=(J-1)*JN
      DO 85 I=1,J
      J2=J2+1
      J4=J3+I
      B(J2)=A(J4)
      J6=(I-1)*N+JM
      DO 86 K=1,JN
      K1=K+J6
      K2=K+J5
      B(J2)=B(J2)-A(K1)*T3(K2)
   86 CONTINUE
      B(J2)=B(J2)*RU(J)*RU(I)
   85 CONTINUE
   84 CONTINUE
  129 CALL EIGEN(B,Y,JM,0)
      J1=0
      DO 107 J=1,JM
      J1=J1+J
      AMD(J)=B(J1)
  107 CONTINUE
      WRITE(3,58)(AMD(J),J=1,JM)
   58 FORMAT('0EIGENVALUES OF THE MATRIX B'/(1X,5E14.7))
      DO 91 J=1,JM
      J1=(J-1)*JM
      J4=(J-1)*N
      DO 92 I=1,JM
      J3=I+J4
      J2=I+J1
      T2(J3)=Y(J2)*RU(I)
   92 CONTINUE
   91 CONTINUE
      L=0


      JJ=0
      DO 240  I=1,JM
      JJ=JJ+1
      DU(JJ)=0.0
      DO 250 J=1,JM
      LL=L+J
      DU(JJ)=EL(J)*T2(LL)*T2(LL)+DU(JJ)
  250 CONTINUE
      L=L+N
  240 CONTINUE
      WRITE(3,251) (DU(JJ),JJ=1,JM)
  251 FORMAT('0QUADRATIC FORM X1U11X1'/(8(1X,E14.7)))
      S1=0.
      DO 93 J=1,JM
      J1=(J-1)*N
      DO 94 I=1,JN
      J2=J1+I+JM
      T2(J2)=0.
      DO 95 K=1,JM
      K1=(K-1)*JN+I
      K2=K+J1
      T2(J2)=T2(J2)-T3(K1)*T2(K2)
   95 CONTINUE
   94 CONTINUE
   93 CONTINUE
      K=JM+1
      L=0
      DO 260 I=1,JM
      DU(I)=0.0
      DO 270 J=K,N
      LL=L+J
      DU(I)=EL(J)*T2(LL)*T2(LL)+DU(I)
  270 CONTINUE
      L=L+N
  260 CONTINUE
      WRITE(3,261) (DU(I),I=1,JM)
  261 FORMAT('0QUADRATIC FORM X2U22X2'/(8(1X,E14.7)))
      JMM=JM*N
      WRITE(3,401) (T2(I),I=1,JMM)
  401 FORMAT('0MATRIX X'/(8(1X,E14.7)))
  147 DO 96 J=1,JM
      S1=0.
      J1=(J-1)*N
      J6=(J-1)*NN
      J7=0
      DO 97 I=1,NN
      J2=J6+I
      FI(J2)=0.
   34 J7=J7+1
      DO 98 K=1,N
      K2=K+J1
      K1=(K-1)*N+J7
      FI(J2)=FI(J2)+U(K1)*T2(K2)
   98 CONTINUE
   97 CONTINUE
      J2=J6+I
      J3=J6+NN
      WRITE(3,138) AMD(J)
  138 FORMAT('0EIGENCURRENT FOR WHICH LAMBDA = ',E11.4)
      WRITE(3,60)(FI(I),I=J2,J3)
   60 FORMAT(1X,8(1X,E14.7))
   96 CONTINUE
      DO 310 J=1,JM
      J1=(J-1)*N
      FRF(J)=0.0
      QXQ(J)=0.0
      DO 320 I=1,N
      S1=0.0
      X1=0.0
      J4=(I-1)*N
      DO 330 K=1,N
      J3=J1+K
      J2=J4+K
      S1=S1+RR(J2)*FI(J3)
      X1=X1+XX(J2)*FI(J3)
  330 CONTINUE
      J2=J1+I
      FRF(J)=FRF(J)+S1*FI(J2)
      QXQ(J)=QXQ(J)+X1*FI(J2)
  320 CONTINUE
      RQ(J)=QXQ(J)/FRF(J)
      EC(J)=1.0/(FRF(J)+CJ*QXQ(J))
  310 CONTINUE
      WRITE(3,332) (FRF(J),J=1,JM)
  332 FORMAT('0FRF'/(8(1X,F14.7)))
      WRITE(3,333) (QXQ(J),J=1,JM)
  333 FORMAT('0QXQ'/(8(1X,E14.7)))
      WRITE(3,331) (RQ(I),I=1,JM)
  331 FORMAT('0RAYLEIGH QUOTIENTS'/(8(1X,F14.7)))
      WRITE(3,311) (EC(I),I=1,JM)
  311 FORMAT('0NORMALIZATION COEFFS'/(8(1X,E14.7)))
      WRITE(3,312) (AMD(I),I=1,JM)
  312 FORMAT('0LAMDA'/(8(1X,F14.7)))
      WRITE(3,100) (FRF(I),I=1,JM)
  100 FORMAT('0NORMALIZATION FACTOR (J,RJ)'/(8(1X,E14.7)))
      NJ=JM*N
      PUNCH 101,(AMD(I),I=1,JM)
      PUNCH 101,(EC(I),I=1,JM)
      PUNCH 101,(FI(I),I=1,NJ)
  101 FORMAT(5E15.7)
  500 STOP
      END
      SUBROUTINE EIGEN(A,R,N,MV)
      DIMENSION A(465),R(900)
      IF(MV-1) 10,25,10
   10 IQ=-N
      DO 20 J=1,N
      IQ=IQ+N
      DO 20 I=1,N
      IJ=IQ+I
      R(IJ)=0.0
      IF(I-J) 20,15,20
   15 R(IJ)=1.0
   20 CONTINUE
   25 ANORM=0.0
      DO 35 I=1,N
      DO 35 J=I,N
      IF(I-J) 30,35,30
   30 IA=I+(J*J-J)/2
      ANORM=ANORM+A(IA)*A(IA)


   35 CONTINUE
      IF(ANORM) 165,165,40
   40 ANORM=1.414*SQRT(ANORM)
      ANRMX=ANORM*1.0E-6/FLOAT(N)
      IND=0
      THR=ANORM
   45 THR=THR/FLOAT(N)
   50 L=1
   55 M=L+1
   60 MQ=(M*M-M)/2
      LQ=(L*L-L)/2
      LM=L+MQ
   62 IF(ABS(A(LM))-THR) 130,65,65
   65 IND=1
      LL=L+LQ
      MM=M+MQ
      X=0.5*(A(LL)-A(MM))
   68 Y=-A(LM)/SQRT(A(LM)*A(LM)+X*X)
      IF(X) 70,75,75
   70 Y=-Y
   75 SINX=Y/SQRT(2.0*(1.0+(SQRT(1.0-Y*Y))))
      SINX2=SINX*SINX
   78 COSX=SQRT(1.0-SINX2)
      COSX2=COSX*COSX
      SINCS=SINX*COSX
      ILQ=N*(L-1)
      IMQ=N*(M-1)
      DO 125 I=1,N
      IQ=(I*I-I)/2
      IF(I-L) 80,115,80
   80 IF(I-M) 85,115,90
   85 IM=I+MQ
      GO TO 95
   90 IM=M+IQ
   95 IF(I-L) 100,105,105
  100 IL=I+LQ
      GO TO 110
  105 IL=L+IQ
  110 X=A(IL)*COSX-A(IM)*SINX
      A(IM)=A(IL)*SINX+A(IM)*COSX
      A(IL)=X
  115 IF(MV-1) 120,125,120
  120 ILR=ILQ+I
      IMR=IMQ+I
      X=R(ILR)*COSX-R(IMR)*SINX
      R(IMR)=R(ILR)*SINX+R(IMR)*COSX
      R(ILR)=X
  125 CONTINUE
      X=2.0*A(LM)*SINCS
      Y=A(LL)*COSX2+A(MM)*SINX2-X
      X=A(LL)*SINX2+A(MM)*COSX2+X
      A(LM)=(A(LL)-A(MM))*SINCS+A(LM)*(COSX2-SINX2)
      A(LL)=Y
      A(MM)=X
  130 IF(M-N) 135,140,135
  135 M=M+1
      GO TO 60
  140 IF(L-(N-1)) 145,150,145
  145 L=L+1
      GO TO 55
```

```
  150 IF(IND-1) 160,155,160
  155 IND=0
      GO TO 50
  160 IF(THR-ANRMX) 165,165,45
  165 IQ=-N
      DO 185 I=1,N
      IQ=IQ+N
      LL=I+(I*I-I)/2
      JQ=N*(I-2)
      DO 185 J=I,N
      JQ=JQ+N
      MM=J+(J*J-J)/2
      IF(A(LL)-A(MM)) 170,185,185
  170 X=A(LL)
      A(LL)=A(MM)
      A(MM)=X
      IF(MV-1) 175,185,175
  175 DO 180 K=1,N
      ILR=IQ+K
      IMR=JQ+K
      X=R(ILR)
      R(ILR)=R(IMR)
  180 R(IMR)=X
  185 CONTINUE
      RETURN
      END
      SUBROUTINE LINER(LL,C,LR)
      DIMENSION LR(30),C(900)
      DO 20 I=1,LL
      LR(I)=I
   20 CONTINUE
      M1=0
      DO 18 M=1,LL
      K=M
      DO 2 I=M,LL
      K1=M1+I
      K2=M1+K
      IF(ABS(C(K1))-ABS(C(K2))) 2,2,6
    6 K=I
    2 CONTINUE
      LS=LR(M)
      LR(M)=LR(K)
      LR(K)=LS
      K2=M1+K
      STOR=C(K2)
      J1=0
      DO 7 J=1,LL
      K1=J1+K
      K2=J1+M
      STO=C(K1)
      C(K1)=C(K2)
      C(K2)=STO/STOR
      J1=J1+LL
    7 CONTINUE
      K1=M1+M
      C(K1)=1./STOR
      DO 11 I=1,LL
      IF(I-M) 12,11,12
   12 K1=M1+I
      ST=C(K1)


      C(K1)=0.
      J1=0
      DO 10 J=1,LL
      K1=J1+I
      K2=J1+M
      C(K1)=C(K1)-C(K2)*ST
      J1=J1+LL
   10 CONTINUE
   11 CONTINUE
      M1=M1+LL
   18 CONTINUE
      J1=0
      DO 9 J=1,LL
      IF(J-LR(J)) 14,8,14
   14 LRJ=LR(J)
      J2=(LRJ-1)*LL
   21 DO 13 I=1,LL
      K2=J2+I
      K1=J1+I
      S=C(K2)
      C(K2)=C(K1)
      C(K1)=S
   13 CONTINUE
      LR(J)=LR(LRJ)
      LR(LRJ)=LRJ
      IF(J-LR(J)) 14,8,14
    8 J1=J1+LL
    9 CONTINUE
      RETURN
      END
/*
//GO.SYSIN DD *
 0.1000E-05
 30
/*
//
```

B.3 <u>Listing of the program to compute generalized impedance elements, perpendicular polarization(TE).</u>

```
//YUCHANG JOB (0639,EE,2,2,900),'CHANG.Y',REGION=250K,CLASS=A
//  EXEC WATFIV
//GO.FT02F001  DD  SYSOUT=B
//GO.SYSIN  DD  *
$JOB            CHANG
      SUBROUTINE FPO(Z,FO,PO)
      Y=3.0/Z
      FO=0.79788456+Y*(-0.00000077+Y*(-0.0055274+Y*(-0.00009512+Y*(0.001
     137237+Y*(-0.00072805+Y*0.00014476)))))
      PO=0.78539816+Y*(0.04166397+Y*(0.00003954+Y*(-0.00262573+Y*(0.0005
     14125+Y*(0.00029333-Y*0.0001355)))))
      RETURN
      END
      SUBROUTINE FP1(Z,F1,P1)
      Y=3.0/Z
      F1=0.79788456+Y*(0.00000156+Y*(0.01659667+Y*(0.00017105+Y*(-0.0024
     19511+Y*(0.00113653-Y*0.00020033)))))
      P1=0.78539816+Y*(-0.12499612+Y*(-0.00005650+Y*(0.00637879+Y*(-0.00
     1074348+Y*(-0.00079824+Y*0.00029166)))))
      RETURN
      END
      FUNCTION BSJO(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSJO=1.0
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSJO=1.0+Y*(-2.2499997+Y*(1.2656208+Y*(-0.3163866+Y*(0.0444479+Y*(
     1-0.0039444+Y*0.00021)))))
      RETURN
    1 CALL FPO(Z,FO,PO)
      BSJO=FO*COS(Z-PO)/SQRT(Z)
      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTE-
     1RED IN CALCULATING A BESSEL FUNCTION OF ORDER ZERO'/)
      END
      FUNCTION BSYO(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSYO=-1.0E75
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSYO=0.63661977*ALOG(0.5*Z)*BSJO(Z)+0.36746691+Y*(0.60559366+Y*(-0
     1.74350384+Y*(0.25300117+Y*(-0.04261214+Y*(0.00427916-Y*0.00024846)
     2))))
      RETURN
    1 CALL FPO(Z,FO,PO)
      BSYO=FO*SIN(Z-PO)/SQRT(Z)


      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A NEUMANN FUNCTION OF ORDER ZERO ABS(X) USED'/)
      END
      FUNCTION BSJ1(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSJ1=0.0
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0) GO TO 1
      Y=Z*Z/9.0
      BSJ1=X*(0.5+Y*(-0.56249985+Y*(0.21093573+Y*(-0.03954289+Y*(0.0044
     1319+Y*(-0.00031761+Y*0.00001109))))))
      RETURN
    1 CALL FP1(Z,F1,P1)
      BSJ1=F1*SIN(Z-P1)/SQRT(Z)
      IF(X.LT.0.0)BSJ1=-BSJ1
      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',F15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A BESSEL FUNCTION OF ORDER ONE'/)
      END
      FUNCTION BSY1(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSY1=-1.0E75
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSY1=(-0.63661977+Y*(0.2212091+Y*(2.1682709+Y*(-1.3164827+Y*(0.312
     13951+Y*(-0.0400976+Y*0.0027873))))))/Z+0.63661977*ALOG(0.5*Z)*BSJ1
     2(Z)
      RETURN
    1 CALL FP1(Z,F1,P1)
      BSY1=-F1*COS(Z-P1)/SQRT(Z)
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',F15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A NEUMANN FUNCTION OF ORDER ONE ABS(X) USED'/)
      RETURN
      END
      SUBROUTINE HANK(GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),EV(60),CJ
      COMMON T(120),TD(120),RR(1953)
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON XP(63),YP(63),N2
      PI=3.141593
      P=2.0/3.141593
      N1=N2+1
      FF=4.*2.71828
      EL=1.781072
      I1=0
      DO 30 J=1,N1
      DO 40 I=1,J
      I1=I1+1
      IF(I .EQ. J)GO TO 25
      IF(I .EQ. 1 .AND. J .EQ. N2)GO TO 25
      IF(I .EQ. 2 .AND. J .EQ. N1)GO TO 25
      RK=RR(I1)*GK
      H(I1)=BSJO(RK)-CJ*BSYO(RK)
      HF(I1)=BSJ1(RK)-CJ*BSY1(RK)
      GO TO 40
```

```
 25 CONTINUE
    AA=(EL*GK*DL(I))/FF
    H(I1)=1.0-CJ*P*ALOG(AA)
 40 CONTINUE
 30 CONTINUE
    RETURN
    END
    SUBROUTINE CALC(GK)
    COMPLEX CJ,H,HF,A,Z,Y,EV
    COMPLEX HFRR
    COMPLEX CK
    COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),EV(60),CJ
    COMMON T(120),TD(120),RR(1953)
    COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
    COMMON XP(63),YP(63),N2
    N1=N2+1
    CK=0.25*CJ*GK
    II=0
    L=0
    DO 50 J3=3,N2,2
    J1=J3-2
    J2=J3-1
    J4=J3+1
    K=0
    DO 60 I3=3,N2,2
    II=II+1
    I1=I3-2
    I2=I1+1
    I4=I3+1
    Z(II)=0.0
    DO 70 M=J1,J4
    LL=M+L
    DO 80 N=I1,I4
    KK=N+K
    IF(M-N)100,150,120
100 JJ=(N*(N-1))/2+M
    IF(M .EQ. 1 .AND. N .EQ. N2) GO TO 150
    IF(M .EQ. 2 .AND. N .EQ. N1) GO TO 150
    GO TO 130
120 JJ=(M*(M-1))/2+N
    IF(N .EQ. 1 .AND. M .EQ. N2) GO TO 150
    IF(N .EQ. 2 .AND. M .EQ. N1) GO TO 150
130 CONTINUE
    HFRR=HF(JJ)/RR(JJ)
    DTT=DL(M)*T(LL)*T(KK)
    CZ=-(XM(N)-XM(M))*DY(N)+(YM(N)-YM(M))*DX(N)
    Z(II)=DTT*HFRR*CZ+Z(II)
    GO TO 80
150 CONTINUE
 80 CONTINUE
 70 CONTINUE
    Z(II)=CK*Z(II)
    Z(II)=-CJ*Z(II)
    Z(II)=Z(II)/377.0
    K=K+2
 60 CONTINUE
    L=L+2
 50 CONTINUE
    RETURN
    END


    SUBROUTINE CALZY(WE,WU)
    COMPLEX CJ,H,HF,A,Z,Y,EV
    COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),EV(60),CJ
    COMMON T(120),TD(120),RR(1953)
    COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
    COMMON XP(63),YP(63),N2
    PI=3.141593
    WEE=-0.25*WE
    II=0
    L=0
    DO 50 J3=3,N2,2
    J1=J3-2
    J2=J3-1
    J4=J3+1
    K=0
    DO 60 I3=3,J3,2
    II=II+1
    I1=I3-2
    I2=I1+1
    I4=I3+1
    Z(II)=0.0
    Y(II)=0.0
    DO 70 M=J1,J4
    LL=M+L
    DO 80 N=I1,I4
    KK=N+K
    DLL=DL(N)*DL(M)
    TT=T(KK)*T(LL)
    IF(M-N)100,120,120
100 JJ=(N*(N-1))/2+M
    GO TO 130
120 JJ=(M*(M-1))/2+N
130 CONTINUE
    DC=(DX(M)*DX(N)+DY(M)*DY(N))/(DL(M)*DL(N))
    ZT1=WU*TT*DC
    ZT2=-TD(KK)*TD(LL)/WE
    ZT3=ZT1+ZT2
    Z(II)=DLL*ZT3*H(JJ)+Z(II)
    Y(II)=TT*DLL*H(JJ)+Y(II)
 80 CONTINUE
 70 CONTINUE
    Z(II)=0.25*Z(II)
    Y(II)=WEE*Y(II)
    Y(II)=-Y(II)
    Z(II)=Z(II)/(377.0*377.0)
    K=K+2
 60 CONTINUE
    L=L+2
 50 CONTINUE
    N=(N2-1)/2
    II=(N*(N-1))/2+1+N
    DO 40 K=1,N
    J=N+1-K
    DO 30 L=1,J
    I=N*(J-1)+J-L+1
    II=II-1
    Z(I)=Z(II)
    Y(I)=Y(II)
 30 CONTINUE
 40 CONTINUE
    KK=0
    NM=N-1
    DO 45 K=1,NM
    J=K+1
    DO 35 L=J,N
    I=L+KK
    JJ=(L-1)*N+K
    Z(I)=Z(JJ)
    Y(I)=Y(JJ)
 35 CONTINUE
    KK=KK+N
 45 CONTINUE
    RETURN
    END
    COMPLEX HED
    COMPLEX AE(60)
    COMPLEX CJ,H,HF,A,Z,Y,EV
    COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),EV(60),CJ
    COMMON T(120),TD(120),RR(1953)
    COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
    COMMON XP(63),YP(63),N2
    A9=12.0
    A10=10.0
    PI=3.141593
    P4=180.0/3.141593
    NP=33
    N1=NP-1
    N2=NP-2
    N=(N1-2)/2
    NT=4*N
    WL=1.0
    ETA=376.7301
    P1=3.141593/180.0
    ANS=0.0
    GK=2.0*3.141593/WL
    WE=GK/ETA
    A1=0.7
    RD=A1/GK
    WU=GK*ETA
    NH=2*N
    ER=9.5
    UR=1.0
    GKM=SQRT(UR*ER)*GK
    WEM=WE*ER
    WUM=WU*UR
    CJ=(0.,1.)
    PA=RD*PI
    P2=A9*P1
    THETA=0.0
    DO 5 I=1,NP
    XP(I)=RD*COS(THETA)*WL
    YP(I)=RD*SIN(THETA)*WL
    THETA=THETA+P2
  5 CONTINUE
    WRITE(3,4) (XP(I),YP(I),I=1,NP)
    DO 10 J=1,N1
    J1=J+1
    DX(J)=XP(J1)-XP(J)
    DY(J)=YP(J1)-YP(J)
    DL(J)=SQRT(DX(J)**2+DY(J)**2)


    XM(J)=0.5*(XP(J1)+XP(J))
    YM(J)=0.5*(YP(J1)+YP(J))
 10 CONTINUE
    WRITE(3,45)
 45 FORMAT(/////)
    WRITE(3,4) (DL(I),I=1,N1)
    WRITE(3,45)
    WRITE(3,4) (DX(I),I=1,N1)
    WRITE(3,45)
    WRITE(3,4) (DY(I),I=1,N1)
    I1=0
    DO 30 J=1,N1
    DO 40 I=1,J
    I1=I1+1
    IF(I .EQ. J)GO TO 25
    IF(I .EQ. 1 .AND. J .EQ. N2)GO TO 25
    IF(I .EQ. 2 .AND. J .EQ. N1)GO TO 25
    XPQ=XM(I)-XM(J)
    YPQ=YM(I)-YM(J)
    RR(I1)=SQRT(XPQ*XPQ+YPQ*YPQ)
 25 CONTINUE
 40 CONTINUE
 30 CONTINUE
    L=0
    DO 20 M3=3,N2,2
    M1=M3-2
    M4=M3+1
    M2=M3-1
    L1=M1+L
    L2=L1+1
    L3=L2+1
    L4=L3+1
    T(L1)=.5*DL(M1)/(DL(M1)+DL(M2))
    T(L2)=(DL(M1)+.5*DL(M2))/(DL(M1)+DL(M2))
    T(L3)=(0.5*DL(M3)+DL(M4))/(DL(M3)+DL(M4))
    T(L4)=0.5*DL(M4)/(DL(M3)+DL(M4))
    TD(L1)=1./(DL(M1)+DL(M2))
    TD(L2)=TD(L1)
    TD(L3)=-1.0/(DL(M3)+DL(M4))
    TD(L4)=TD(L3)
    L=L+2
 20 CONTINUE
    WRITE(3,45)
    WRITE(3,4) (T(I),I=1,NT)
    NS=N**2
    CALL HANK(GK)
    CALL CALZY(WE,WU)
    K=0
    L=0
    N1=2*NS+N
    DO 500 J=1,N
    DO 510 I=1,N
    L=L+1
    KK=K+I
    II=N1+L
    A(L)=Z(KK)
    A(II)=Y(KK)
510 CONTINUE
    L=L+N
    K=K+N
```

```
  500 CONTINUE
      CALL CALC(GK)
      K=0
      II=N
      DO 540 J=1,N
      DO 550 I=1,N
      KK=K+I
      II=II+1
      A(II)=Z(KK)
  550 CONTINUE
      K=K+N
      II=II+N
  540 CONTINUE
      CALL HANK(GKM)
      CALL CALZY(WEM,WUM)
      K=0
      II=0
      DO 520 J=1,N
      DO 530 I=1,N
      KK=K+I
      II=II+1
      III=NI+II
      A(II)=Z(KK)+A(II)
      A(III)=Y(KK)+A(III)
  530 CONTINUE
      K=K+N
      II=II+N
  520 CONTINUE
      CALL CALC(GKM)
      K=0
      II=N
      DO 560 J=1,N
      DO 570 I=1,N
      KK=K+I
      II=II+1
      A(II)=Z(KK)+A(II)
  570 CONTINUE
      K=K+N
      II=II+N
  560 CONTINUE
      NK=2*N
      NJ=2*NS
      L=0
      DO 580 J=1,N
      K=N
      DO 590 I=1,N
      KK=J+K
      L=L+1
      II=L+NJ
      A(II)=A(KK)
      K=K+NK
  590 CONTINUE
      L=L+N
  580 CONTINUE
    4 FORMAT(//,1H ,10F11.4)
      K=0
      II=0
      DO 350 J=1,NH
      DO 250 I=1,J
      II=II+1


      JJ=K+I
      A(II)=A(JJ)
  250 CONTINUE
      K=K+NH
  350 CONTINUE
      NSS=(NH*(NH+1))/2
      WRITE(3,202) (A(I),I=1,NSS)
  202 FORMAT(8(1X,E14.7))
      PUNCH 200,(A(I),I=1,NSS)
  200 FORMAT(5E15.7)
      STOP
      END
$DATA
$STOP
/*
//
```

B.4 <u>Listing of the program to compute scattering cross sections, parallel polarization(TM).</u>

```
//YUCHANG JOB (0639,FF,2M59S),'CHANG.Y',REGION=250K,CLASS=A
//   EXEC WATFIV
//GO.SYSIN  DD  *
$JOB              CHANG,TIME=3,PAGES=30
      SUBROUTINE FPO(Z,FO,PO)
      Y=3.0/Z
      FO=0.79788456+Y*(-0.00000077+Y*(-0.0055274+Y*(-0.00009512+Y*(0.001
     137237+Y*(-0.00072805+Y*0.00014476)))))
      PO=0.78539816+Y*(0.04166397+Y*(0.00003954+Y*(-0.00262573+Y*(0.0005
     14125+Y*(0.00029333-Y*0.0001355)))))
      RETURN
      END
      SUBROUTINE FP1(Z,F1,P1)
      Y=3.0/Z
      F1=0.79788456+Y*(0.00000156+Y*(0.01659667+Y*(0.00017105+Y*(-0.0024
     19511+Y*(0.00113653-Y*0.00020033)))))
      P1=0.78539816+Y*(-0.12499612+Y*(-0.00005650+Y*(0.00637879+Y*(-0.00
     1074348+Y*(-0.00079824+Y*0.00029166)))))
      RETURN
      END
      FUNCTION BSJO(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSJO=1.0
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSJO=1.0+Y*(-2.2499997+Y*(1.2656208+Y*(-0.3163866+Y*(0.0444479+Y*(
     1-0.0039444+Y*0.00021)))))
      RETURN
    1 CALL FPO(Z,FO,PO)
      BSJO=FO*COS(Z-PO)/SQRT(Z)
      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTE-
     1RED IN CALCULATING A BESSEL FUNCTION OF ORDER ZERO'/)
      END
      FUNCTION BSYO(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSYO=-1.0E75
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSYO=0.63661977*ALOG(0.5*Z)*BSJO(Z)+0.36746691+Y*(0.60559366+Y*(-0
     1.74350384+Y*(0.25300117+Y*(-0.04261214+Y*(0.00427916-Y*0.00024846)
     2))))
      RETURN
    1 CALL FPO(Z,FO,PO)
      BSYO=FO*SIN(Z-PO)/SQRT(Z)
      RETURN


   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A NEUMANN FUNCTION OF ORDER ZERO ABS(X) USED'/)
      END
      FUNCTION BSJ1(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSJ1=0.0
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0) GO TO 1
      Y=Z*Z/9.0
      BSJ1=X*(0.5+Y*(-0.56249985+Y*(0.21093573+Y*(-0.03954289+Y*(0.00443
     1319+Y*(-0.00031761+Y*0.00001109))))))
      RETURN
    1 CALL FP1(Z,F1,P1)
      BSJ1=F1*SIN(Z-P1)/SQRT(Z)
      IF(X.LT.0.0)BSJ1=-BSJ1
      RETURN
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A BESSEL FUNCTION OF ORDER ONE'/)
      END
      FUNCTION BSY1(X)
      IF(X.LE.0.0)WRITE(3,10)X
      BSY1=-1.0E75
      IF(X.EQ.0.0)RETURN
      Z=ABS(X)
      IF(Z.GT.3.0)GO TO 1
      Y=Z*Z/9.0
      BSY1=(-0.63661977+Y*(0.2212091+Y*(2.1682709+Y*(-1.3164827+Y*(0.312
     13951+Y*(-0.0400976+Y*0.0027873))))))/Z+0.63661977*ALOG(0.5*Z)*BSJ1
     2(Z)
      RETURN
    1 CALL FP1(Z,F1,P1)
      BSY1=-F1*COS(Z-P1)/SQRT(Z)
   10 FORMAT(1H ,'WARNING - AN ARGUMENT OF',E15.4,3X,'HAS BEEN ENCOUNTER
     1ED IN CALCULATING A NEUMANN FUNCTION OF ORDER ONE ABS(X) USED'/)
      RETURN
      END
      SUBROUTINE LINEQ(LL,C)
      COMPLEX C(3600),STOR,STO,ST,S
      DIMENSION LR(60)
      DO 20 I=1,LL
      LR(I)=I
   20 CONTINUE
      M1=0
      DO 18 M=1,LL
      K=M
      K2=M1+K
      S1=ABS(REAL(C(K2)))+ABS(AIMAG(C(K2)))
      DO 2 I=M,LL
      K1=M1+I
      S2=ABS(REAL(C(K1)))+ABS(AIMAG(C(K1)))
      IF(S2-S1) 2,2,6
    6 K=I
      S1=S2
    2 CONTINUE
      LS=LR(M)
      LR(M)=LR(K)
      LR(K)=LS
      K2=M1+K
      STOR=C(K2)
```

```
      J1=0
      DO 7 J=1,LL
      K1=J1+K
      K2=J1+M
      STO=C(K1)
      C(K1)=C(K2)
      C(K2)=STO/STOR
      J1=J1+LL
    7 CONTINUE
      K1=M1+M
      C(K1)=1./STOR
      DO 11 I=1,LL
      IF(I-M) 12,11,12
   12 K1=M1+I
      ST=C(K1)
      C(K1)=0.
      J1=0
      DO 10 J=1,LL
      K1=J1+I
      K2=J1+M
      C(K1)=C(K1)-C(K2)*ST
      J1=J1+LL
   10 CONTINUE
   11 CONTINUE
      M1=M1+LL
   18 CONTINUE
      J1=0
      DO 9 J=1,LL
      IF(J-LR(J)) 14,8,14
   14 LRJ=LR(J)
      J2=(LRJ-1)*LL
   21 DO 13 I=1,LL
      K2=J2+I
      K1=J1+I
      S=C(K2)
      C(K2)=C(K1)
      C(K1)=S
   13 CONTINUE
      LR(J)=LR(LRJ)
      LR(LRJ)=LRJ
      IF(J-LR(J)) 14,8,14
    8 J1=J1+LL
    9 CONTINUE
      RETURN
      END
      SUBROUTINE HANK(GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      PI=3.141593
      P=2.0/3.141593
      N1=N2+1
      FE=4.*2.71828
      EL=1.781072
      I1=0
      DO 30 J=1,N1
      DO 40 I=1,J
      I1=I1+1
      IF(I .EQ. J)GO TO 25
      IF(I .EQ. 1 .AND. J .EQ. N2)GO TO 25
      IF(I .EQ. 2 .AND. J .EQ. N1)GO TO 25
      RK=RR(I1)*GK
      H(I1)=BSJO(RK)-CJ*BSYO(RK)
      HF(I1)=BSJ1(RK)-CJ*BSY1(RK)
      GO TO 40
   25 CONTINUE
      AA=(EL*GK*DL(I))/FE
      H(I1)=1.0-CJ*P*ALOG(AA)
   40 CONTINUE
   30 CONTINUE
      RETURN
      END
      SUBROUTINE CALZY(WE,WU)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMPLEX ZZ
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      PI=3.141593
      II=0
      L=0
      DO 50 J3=3,N2,2
      J1=J3-2
      J2=J3-1
      J4=J3+1
      K=0
      DO 60 I3=3,N2,2
      II=II+1
      I1=I3-2
      I2=I1+1
      I4=I3+1
      Z(II)=0.0
      Y(II)=0.0
      DO 70 M=J1,J4
      LL=M+L
      DO 80 N=I1,I4
      KK=N+K
      IF(M-N)100,120,120
  100 JJ=(N*(N-1))/2+M
      GO TO 130
  120 JJ=(M*(M-1))/2+N
  130 CONTINUE
      DC=(DX(M)*DX(N)+DY(M)*DY(N))/(DL(M)*DL(N))
      YT1=WE*T(KK)*T(LL)*DC
      YT2=-TD(KK)*TD(LL)/WU
      YT3=YT1+YT2
      ZZ=H(JJ)
      Y(II)=0.25*DL(N)*DL(M)*YT3*ZZ+Y(II)
      Z(II)=-0.25*WU*T(KK)*T(LL)*DL(M)*DL(N)*ZZ+Z(II)
   80 CONTINUE
   70 CONTINUE
      Z(II)=-Z(II)
      K=K+2
   60 CONTINUE
      L=L+2
   50 CONTINUE
      RETURN
      END
      SUBROUTINE CALC(GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMPLEX HFRR
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      N1=N2+1
      II=0
      L=0
      DO 50 J3=3,N2,2
      J1=J3-2
      J2=J3-1
      J4=J3+1
      K=0
      DO 60 I3=3,N2,2
      II=II+1
      I1=I3-2
      I2=I1+1
      I4=I3+1
      Z(II)=0.0
      DO 70 M=J1,J4
      LL=M+L
      DO 80 N=I1,I4
      KK=N+K
      IF(M-N)100,150,120
  100 JJ=(N*(N-1))/2+M
      IF(M .EQ. 1 .AND. N .EQ. N2) GO TO 150
      IF(M .EQ. 2 .AND. N .EQ. N1) GO TO 150
      GO TO 130
  120 JJ=(M*(M-1))/2+N
      IF(N .EQ. 1 .AND. M .EQ. N2) GO TO 150
      IF(N .EQ. 2 .AND. M .EQ. N1) GO TO 150
  130 CONTINUE
      HFRR=HF(JJ)/RR(JJ)
      DTT=DL(M)*T(LL)*T(KK)
      CZ=-(XM(N)-XM(M))*DY(N)+(YM(N)-YM(M))*DX(N)
      Z(II)=0.25*CJ*DTT*HFRR*GK*CZ+Z(II)
      GO TO 80
  150 CONTINUE
   80 CONTINUE
   70 CONTINUE
      Z(II)=-CJ*Z(II)
      K=K+2
   60 CONTINUE
      L=L+2
   50 CONTINUE
      RETURN
      END
      SUBROUTINE EXMX(PHI,GK)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMPLEX PP
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      ETA=376.7301
      N=(N2-1)/2
      CP=COS(PHI)
      SP=SIN(PHI)
      II=0
      L=0
      DO 200 I3=3,N2,2
      II=II+1
      JJ=II+N
      I1=I3-2
      I2=I3-1
      I4=I3+1
      EV(II)=0.
      EV(JJ)=0.0
      DO 210 M=I1,I4
      LL=M+L
      EP=(XM(M)*CP+YM(M)*SP)*GK
      DP=-DX(M)*SP+DY(M)*CP
      CEP=COS(EP)
      SEP=SIN(EP)
      PP=(CEP+CJ*SEP)*T(LL)
      EV(II)=PP*DL(M)+EV(II)
      EV(JJ)=PP*DP+EV(JJ)
  210 CONTINUE
      EV(JJ)=CJ*EV(JJ)/ETA
      L=L+2
  200 CONTINUE
      RETURN
      END
      COMPLEX HED
      COMPLEX AE(60)
      COMPLEX CJ,H,HF,A,Z,Y,EV
      COMMON DX(62),DY(62),DL(62),XM(62),YM(62)
      COMMON T(120),TD(120),EV(60),RR(1953)
      COMMON H(1953),HF(1953),A(3600),Z(900),Y(900),CJ
      COMMON XP(63),YP(63),N2
      A9=9.0
      A10=10.0
      PI=3.141593
      P4=180.0/3.141593
      NP=43
      N1=NP-1
      N2=NP-2
      N=(N1-2)/2
      NT=4*N
      WL=1.0
      ETA=376.7301
      P1=3.141593/180.0
      ANS=0.0
      GK=2.0*3.141593/WL
      WE=GK/ETA
      A1=0.7
      RD=A1/GK
      WU=GK*ETA
      NH=2*N
      UR=1.0
      ER=100.0
      GKM=SQRT(UR*ER)*GK
      WEM=WE*ER
      WUM=WU*UR
      CJ=(0.,1.)
      PA=RD*PI
      P2=A9*P1
```

```
      AS=RD/5.0
      SG=-AS
      DO 811 I=1,6
      SG=SG+AS
      XP(I)=RD
      YP(I)=SG
  811 CONTINUE
      SG=RD
      DO 822 I=7,11
      SG=SG-AS
      XP(I)=SG
      YP(I)=RD
  822 CONTINUE
      J=11
      DO 833 I=12,16
      J=J-1
      XP(I)=-XP(J)
      YP(I)=YP(J)
  833 CONTINUE
      J=6
      DO 844 I=17,21
      J=J-1
      XP(I)=-XP(J)
      YP(I)=YP(J)
  844 CONTINUE
      J=21
      DO 855 I=22,26
      J=J-1
      XP(I)=XP(J)
      YP(I)=-YP(J)
  855 CONTINUE
      J=16
      DO 866 I=27,36
      J=J-1
      XP(I)=XP(J)
      YP(I)=-YP(J)
  866 CONTINUE
      J=26
      DO 877 I=37,43
      J=J-1
      XP(I)=-XP(J)
      YP(I)=YP(J)
  877 CONTINUE
      WRITE(3,4) (XP(I),YP(I),I=1,NP)
      DO 10 J=1,N1
      J1=J+1
      DX(J)=XP(J1)-XP(J)
      DY(J)=YP(J1)-YP(J)
      DL(J)=SQRT(DX(J)**2+DY(J)**2)
      XM(J)=0.5*(XP(J1)+XP(J))
      YM(J)=0.5*(YP(J1)+YP(J))
   10 CONTINUE
      WRITE(3,45)
   45 FORMAT(/////)
      WRITE(3,4) (DL(I),I=1,N1)
      WRITE(3,45)
      WRITE(3,4) (DX(I),I=1,N1)
      WRITE(3,45)
      WRITE(3,4) (DY(I),I=1,N1)
      I1=0


      DO 30 J=1,N1
      DO 40 I=1,J
      I1=I1+1
      IF(I .EQ. J)GO TO 25
      IF(I .EQ. 1 .AND. J .EQ. N2)GO TO 25
      IF(I .EQ. 2 .AND. J .EQ. N1)GO TO 25
      XPQ=XM(I)-XM(J)
      YPQ=YM(I)-YM(J)
      RR(I1)=SQRT(XPQ*XPQ+YPQ*YPQ)
   25 CONTINUE
   40 CONTINUE
   30 CONTINUE
      L=0
      DO 20 M3=3,N2,2
      M1=M3-2
      M4=M3+1
      M2=M3-1
      L1=M1+L
      L2=L1+1
      L3=L2+1
      L4=L3+1
      T(L1)=.5*DL(M1)/(DL(M1)+DL(M2))
      T(L2)=(DL(M1)+.5*DL(M2))/(DL(M1)+DL(M2))
      T(L3)=(0.5*DL(M3)+DL(M4))/(DL(M3)+DL(M4))
      T(L4)=0.5*DL(M4)/(DL(M3)+DL(M4))
      TD(L1)=1./(DL(M1)+DL(M2))
      TD(L2)=TD(L1)
      TD(L3)=-1.0/(DL(M3)+DL(M4))
      TD(L4)=TD(L3)
      L=L+2
   20 CONTINUE
      WRITE(3,45)
      WRITE(3,4) (T(I),I=1,NT)
      NS=N**2
      CALL HANK(GK)
      CALL CALZY(WE,WI)
      K=0
      L=0
      NI=2*NS+N
      DO 500 J=1,N
      DO 510 I=1,N
      L=L+1
      KK=K+I
      II=NI+L
      A(L)=Z(KK)
      A(II)=Y(KK)
  510 CONTINUE
      L=L+N
      K=K+N
  500 CONTINUE
      CALL CALC(GK)
      K=0
      II=N
      DO 540 J=1,N
      DO 550 I=1,N
      KK=K+I
      II=II+1
      A(II)=Z(KK)
  550 CONTINUE
      K=K+N
      II=II+N
  540 CONTINUE
      CALL HANK(GKM)
      CALL CALZY(WEM,WIM)
      K=0
      II=0
      DO 520 J=1,N
      DO 530 I=1,N
      KK=K+I
      II=II+1
      III=NI+II
      A(II)=Z(KK)+A(II)
      A(III)=Y(KK)+A(III)
  530 CONTINUE
      K=K+N
      II=II+N
  520 CONTINUE
      CALL CALC(GKM)
      K=0
      II=N
      DO 560 J=1,N
      DO 570 I=1,N
      KK=K+I
      II=II+1
      A(II)=Z(KK)+A(II)
  570 CONTINUE
      K=K+N
      II=II+N
  560 CONTINUE
      NK=2*N
      NJ=2*NS
      L=0
      DO 580 J=1,N
      K=N
      DO 590 I=1,N
      KK=J+K
      L=L+1
      II=L+NJ
      A(II)=A(KK)
      K=K+NK
  590 CONTINUE
      L=L+N
  580 CONTINUE
    4 FORMAT(//,1H ,10F11.4)
      CALL LINEQ(NH,A)
      CALL EXMX(ANS,GK)
      WRITE(3,45)
      WRITE(3,4) (EV(I),I=1,NH)
      WRITE(3,304)
  304 FORMAT(1H ,'CURRENT'/'  I',6X,'REAL',10X,'IMAG',10X,'MAGNITUDE',7X
     1,'PHASE')
      DO 270 I1=1,NH
      AE(I1)=0.0
      DO 300 I=1,NH
      J=(I-1)*NH+I1
      AE(I1)=A(J)*EV(I)+AE(I1)
  300 CONTINUE
  270 CONTINUE
      DO 80 I1=1,NH
      CA=CABS(AE(I1))


      CPH=ATAN2(AIMAG(AE(I1)),REAL(AE(I1)))*57.2958
      WRITE(3,305)I1,AE(I1),CA,CPH
  305 FORMAT(1H ,1X,I3,3F14.6,F10.3)
   80 CONTINUE
      WRITE(3,275)
  275 FORMAT(///,' SCATTERING ANGLE - PHI',10X,'ECHO LENGTH/WAVELENGTH')
      AN=0.0
      P3=A10*PI
      DO 320 I=1,36
      CALL EXMX(AN,GK)
      HED=0.0
      DO 330 J=1,NH
      HED=EV(J)*AE(J)+HED
  330 CONTINUE
      CH=CABS(HED)
      ECHO=GK*CH**2/4.0
      ECL=ECHO*ETA*ETA
      PRD=ECL/PA
      ECL=10.0*ALOG10(PRD)
      WRITE(3,335)AN,ECL
  335 FORMAT(1H ,10X,F7.2,20X,F14.7)
      AN=AN+P3
  320 CONTINUE
      STOP
      END
$DATA
$STOP
/*
//
```

B.5 <u>Listing of the program to compute scattering cross sections from characteristic currents.</u>

```
//YUCHANG JOB (0639,FF,2,2,900),'CHANGY',REGION=180K,CLASS=A
//  EXEC WATFIV
//GO.FT02F001  DD  SYSOUT=B
//GO.SYSIN  DD  *
$JOB              CHANG
      SUBROUTINE EXMX(PHI,GK)
      COMPLEX CJ
      COMPLEX PP
      COMPLEX VNI
      COMPLEX AE,AJ,EV
      COMMON AJ(900),AE(30),VNI(30),EV(30),CJ
      COMMON FI(900),T(60),DX(32),DY(32),XM(32),YM(32),DL(32)
      COMMON N2,N,NH,JM
      ETA=376.7301
      N=(N2-1)/2
      CP=COS(PHI)
      SP=SIN(PHI)
      II=0
      L=0
      DO 200 I3=3,N2,2
      II=II+1
      JJ=II+N
      I1=I3-2
      I2=I3-1
      I4=I3+1
      EV(II)=0.
      EV(JJ)=0.0
      DO 210 M=I1,I4
      LL=M+L
      EP=(XM(M)*CP+YM(M)*SP)*GK
      DP=-DX(M)*SP+DY(M)*CP
      CEP=COS(EP)
      SEP=SIN(EP)
      PP=(CEP+CJ*SEP)*T(LL)
      EV(II)=PP*DP+EV(II)
      EV(JJ)=-PP*DL(M)+EV(JJ)
  210 CONTINUE
      EV(II)=-ETA*EV(II)
      EV(JJ)=-CJ*EV(JJ)
      L=L+2
  200 CONTINUE
      RETURN
      END
      COMPLEX EC(30)
      COMPLEX CAMD(30),VA(30)
      COMPLEX VNI,CJ,HED
      COMPLEX AE,AJ,EV
      COMMON AJ(900),AE(30),VNI(30),EV(30),CJ
      COMMON FI(900),T(60),DX(32),DY(32),XM(32),YM(32),DL(32)
      COMMON N2,N,NH,JM


      DIMENSION TD(60),AMD(30)
      DIMENSION XP(33),YP(33)
      A9=12.0
      A10=10.0
      PI=3.141593
      P4=180.0/3.141593
      NP=33
      N1=NP-1
      N2=NP-2
      N=(N1-2)/2
      NT=4*N
      WL=1.0
      ETA=376.7301
      P1=3.141593/180.0
      ANS=0.0
      GK=2.0*3.141593/WL
      WE=GK/ETA
      A1=0.7
      RD=A1/GK
      WU=GK*ETA
      NH=2*N
      ER=2.56
      UR=1.0
      GKM=SQRT(UR*ER)*GK
      WEM=WE*ER
      WUM=WU*UR
      CJ=(0.,1.)
      PA=RD*PI
      P2=A9*P1
      THETA=0.0
      DO 5 I=1,NP
      XP(I)=RD*COS(THETA)*WL
      YP(I)=RD*SIN(THETA)*WL
      THETA=THETA+P2
    5 CONTINUE
      DO 10 J=1,N1
      J1=J+1
      DX(J)=XP(J1)-XP(J)
      DY(J)=YP(J1)-YP(J)
      DL(J)=SQRT(DX(J)**2+DY(J)**2)
      XM(J)=0.5*(XP(J1)+XP(J))
      YM(J)=0.5*(YP(J1)+YP(J))
   10 CONTINUE
      L=0
      DO 20 M3=3,N2,2
      M1=M3-2
      M4=M3+1
      M2=M3-1
      L1=M1+L
      L2=L1+1
      L3=L2+1
      L4=L3+1
      T(L1)=.5*DL(M1)/(DL(M1)+DL(M2))
      T(L2)=(DL(M1)+.5*DL(M2))/(DL(M1)+DL(M2))
      T(L3)=(0.5*DL(M3)+DL(M4))/(DL(M3)+DL(M4))
      T(L4)=0.5*DL(M4)/(DL(M3)+DL(M4))
      TD(L1)=1./(DL(M1)+DL(M2))
      TD(L2)=TD(L1)
      TD(L3)=-1.0/(DL(M3)+DL(M4))
      TD(L4)=TD(L3)
      L=L+2
   20 CONTINUE
  101 FORMAT(5E15.7)
      READ(1,201) JM
  201 FORMAT(I3)
      WRITE(3,201) JM
      NJ=JM*NH
      READ(1,101) (AMD(I),I=1,JM)
      READ(1,101) (EC(I),I=1,JM)
      READ(1,101) (FI(I),I=1,NJ)
      WRITE(3,202) (AMD(I),I=1,JM)
      WRITE(3,101) (EC(I),I=1,JM)
      WRITE(3,203) (FI(I),I=1,NJ)
  202 FORMAT('OLAMDA'/(8(1X,E14.7)))
  203 FORMAT('OEIGENCURRENT'/(8(1X,E14.7)))
      DO 50 J=1,JM
      JJ=(J-1)*NH
      DO 60 I=1,N
      II=JJ+I
      FI(II)=FI(II)/377.0
   60 CONTINUE
   50 CONTINUE
      CALL EXMX(ANS,GK)
      LL=0
      DO 100 I=1,JM
      VNI(I)=0.0
      DO 110 J=1,NH
      L=J+LL
      VNI(I)=EV(J)*FI(L)+VNI(I)
  110 CONTINUE
      LL=LL+NH
  100 CONTINUE
      DO 210 I=1,JM
      VA(I)=VNI(I)*EC(I)
  210 CONTINUE
      LL=-NH
      DO 220 I=1,JM
      LL=LL+NH
      DO 225 J=1,NH
      L=LL+J
      AJ(L)=VA(I)*FI(L)
  225 CONTINUE
  220 CONTINUE
      DO 230 I=1,NH
      AE(I)=0.0
      DO 235 J=1,JM
      K=(J-1)*NH+I
      AE(I)=AE(I)+AJ(K)
  235 CONTINUE
  230 CONTINUE
      WRITE(3,275)
  275 FORMAT(///,' SCATTERING ANGLE - PHI',10X,'ECHO LENGTH/WAVELENGTH')
      AN=0.0
      P3=A10*PI
      DO 320 I=1,36
      CALL EXMX(AN,GK)
      HED=0.0
      DO 330 J=1,NH
      HED=EV(J)*AE(J)+HED
  330 CONTINUE


      CH=CABS(HED)
      ECHO=GK*CH**2/4.0
      ECL=ECHO/(WL*ETA**2)
      PRD=ECL/PA
      ECL=10.0*ALOG10(PRD)
      WRITE(3,335)AN,ECL
  335 FORMAT(1H ,10X,F7.2,20X,E14.7)
      AN=AN+P3
  320 CONTINUE
      STOP
      END
$DATA
 18
$STOP
/*
//
```

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