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Analysis of Crossed Wires in a Plane-Wave Field (II)

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Abstract - The currents and charges induced in a pair of electrically thin crossed wires by a normally incident plane electromagnetic wave are derived by analytical methods. The boundary conditions at the ends and at the junction are explained. The solution of a new integro-differential equation for the currents is obtained in terms of trigonometric and integral-trigonometric functions. Depending on the electrical lengths of the crossed elements and the location of their junction a variety of quite different distributions of current and charge obtain. These determine the scattered near and far fields. Graphs of computed currents and charges per unit length on the four arms of several important cases are displayed. The accurate determination of the induced currents and charges on a mathematically tractable structure - the thin-wire cross - is an early step in a study that will proceed to electrically thick cylinders, wide strips, and their junctions in crossed configurations in an effort to gain a meaningful approximate understanding of the currents and charges induced on an aircraft by an electromagnetic pulse.

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1. Introduction

A knowledge of the distributions of current and charge induced on crossed conductors by a plane electromagnetic wave is needed to determine the scattered field. At distant points this is of interest in radar; near the metal surfaces it provides a means for estimating the penetration through holes and slots in a structure like an aircraft. Since the transverse dimensions are generally not electrically small, the determination of surface currents and charges is a formidable task. Attempts have been made to simulate an aircraft by crossed cylinders and to determine the currents and charges on these by thin-wire antenna theory which ignores transverse currents. Since such currents are significant on electrically thick conductors, the distributions calculated for crossed thin tubes are not representative of those on thick cylinders. Nevertheless, a determination of their properties is a useful step in the study of induced currents and charges on crossed conductors in general.

Studies of crossed thin wires excited by a plane wave have depended primarily on numerical methods [1]-[4] to solve coupled integral equations subject to boundary and junction conditions. Graphs of numerically computed distributions of the induced currents have been displayed [2]-[4] for crosses constructed of relatively short wires, but these are hardly adequate to provide insight into the behavior of currents and charges under conditions of resonance and antiresonance with their quite different standing-wave patterns [5]. This can perhaps be obtained best from an analytical solution which relates the distributions of current and charge to the lengths of the arms and the location of the junction.

2. Formulation of the Problem: Boundary and Junction Conditions

In the interest of simplicity the plane of the crossed wires is assumed to lie in a wave front of a normally incident plane wave with its electric vector parallel to one of the wires. For mutually perpendicular wires, the solution with the electric vector parallel to the other wire is obtained by a simple change in notation. A superposition of the solutions for the two polarizations gives the solution for an arbitrarily polarized, normally incident wave.

The crossed wires and the incident electric field are shown in Fig. 1. The wires extend from $x = -\ell_1$ to $x = \ell_2$ and from $z = -h_1$ to $z = h_2$ with the center of their junction at $x = y = z = 0$. The wires all have the same radius a and this is sufficiently small so that

$$ka = 2\pi a/\lambda \ll 1 \quad , \quad h_i/a \gg 1 \quad , \quad \ell_i/a \gg 1 \, , \tag{1}$$

where $i = 1$ or 2 and $k = \omega/c = 2\pi/\lambda$ is the wave number. The incident field is $E_z^{inc}(y) = E_z^{inc} e^{-jky}$ where E_z^{inc} is the value at $y = 0$.

Under the action of the incident field, standing-wave distributions of charge and current are induced on the vertical conductor and these, in turn, induce distributions on the horizontal arms. All of the currents and charges are distributed so that the total tangential electric field vanishes on the conducting surfaces. Subject to the condition $ka \ll 1$ all transverse currents are negligible. Since the excitation is not rotationally symmetric, the induced axial currents and associated charges also depart from rotational symmetry. However, when $ka \ll 1$, this can be disregarded and the components $K_x(x)$ and $K_z(z)$ of the surface density of current and the associated surface densities of charge $\eta(x)$ and $\eta(z)$ treated as functions of the relevant axial coordinate only. The total currents and charges per unit length and the equation of continuity they satisfy (with $y = x$ or z) are:

$$I_y(y) = 2\pi a K_y(y) \quad ; \quad q(y) = 2\pi a \eta(y) \quad ; \; [\partial I_y(y)/\partial y] + j\omega q(y) = 0 \, . \tag{2}$$

The four sets of currents and charges are $I_{1z}(z)$, $q_1(z)$ in the range $-h_1 \le z \le 0$, $I_{2z}(z)$, $q_2(z)$ in the range $0 \le z \le h_2$; $I_{3x}(x)$, $q_3(x)$ in the range $-\ell_1 \le x \le 0$; and $I_{4x}(x)$, $q_4(x)$ in the range $0 \le x \le \ell_2$.

At the open ends of tubular conductors, the total currents vanish so that

$$I_{1z}(-h_1) = I_{2z}(h_2) = I_{3x}(-\ell_1) = I_{4x}(\ell_2) = 0 \quad . \tag{3}$$

The specification of the currents and charges at the junction is difficult since the boundaries between the chargeable surfaces of the four arms and the

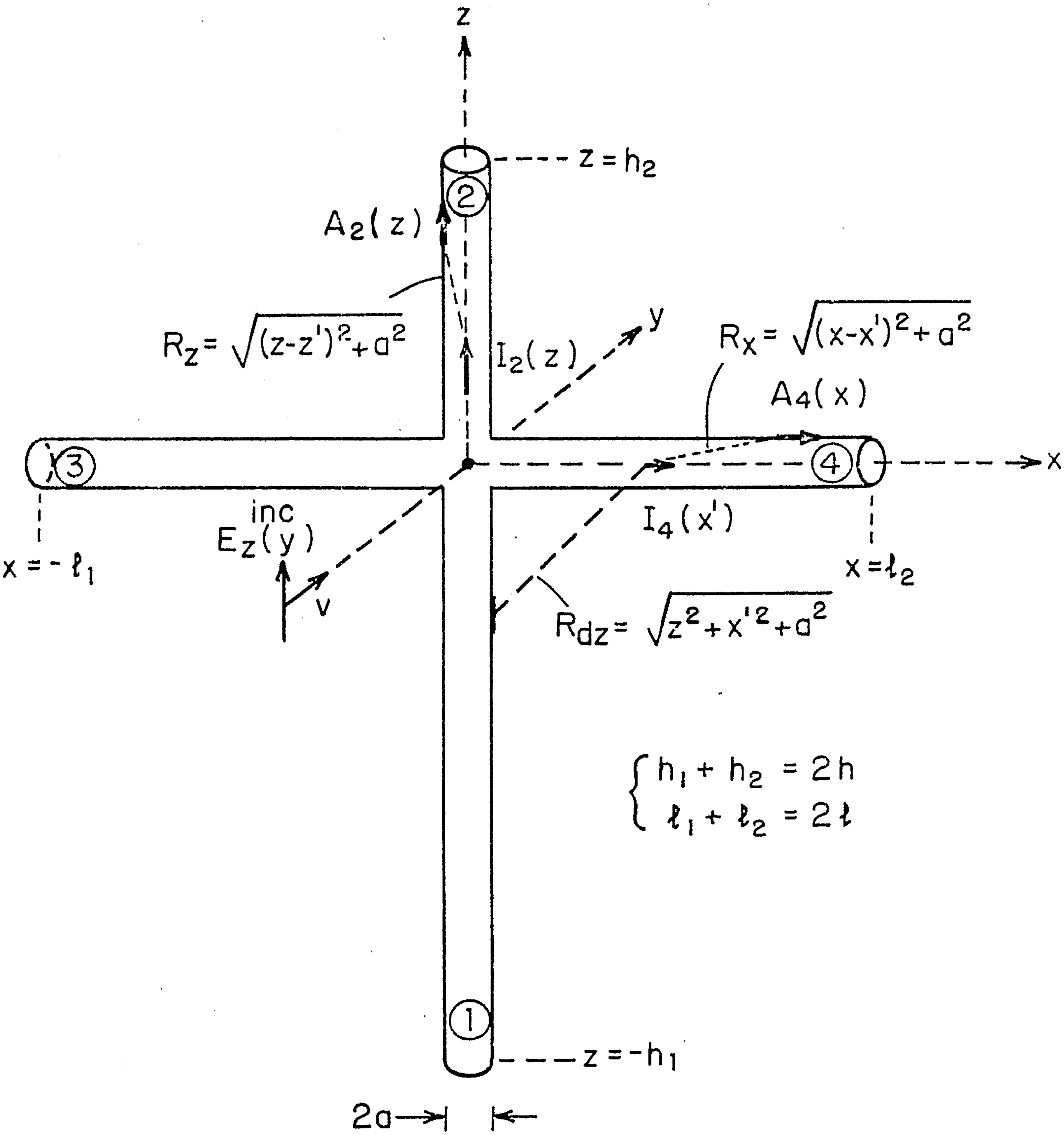


FIG. 1 CROSSED WIRES IN AN INCIDENT PLANE-WAVE FIELD

surface of the junction are ambiguous. However, since this latter is an area of the order a^2, it is negligible when $ka << 1$ and each of the arms effectively ends at $x = z = 0$. The small overlapping areas can be ignored. Alternatively, the surface currents and charges may be treated as if at average locations on the axes of the conductors. For them the junction is the point $x = z = 0$ with no chargeable surface. To complement the conditions (3) on the currents at the outer ends, four additional conditions are needed at the junction $x = z = 0$. Since $ka << 1$, all interactions associated with charges and currents near the junction are quasi-stationary and the conditions of low-frequency electric circuits obtain. With the continuity properties of the electric field it follows that,

$$I_{1z}(0) - I_{2z}(0) + I_{3x}(0) - I_{4x}(0) = 0 \tag{4}$$

$$q_1(0) = q_2(0) = q_3(0) = q_4(0). \tag{5}$$

With the equation of continuity an alternative form of (5) is

$$[\partial I_{1z}(z)/\partial z]_{z=0} = [\partial I_{2z}(z)/\partial z]_{z=0} = [\partial I_{3x}(x)/\partial x]_{x=0} = [\partial I_{4x}(x)/\partial x]_{x=0} .$$

The condition (4) is a consequence of the conservation of electric charge and the absence of significant chargeable surfaces on the junction. The condition (5) is usually not expressed in low-frequency circuit theory since there are no charges on the surfaces of the conductors (except the inner surfaces of condensers). In effect, all conductors and their junctions are at a maximum of current and a zero of charge per unit length in a standing-wave pattern. In transmission lines and antennas that are not electrically short, a junction may be located at an arbitrary point in a standing wave, so that large concentrations of charge may be present at and near a junction. The condition (5) assures that discontinuities in charge per unit length are ruled out in passing from one conductor to another across the junction. Such discontinuities cannot exist in the absence of delta-function generators. Note that the current in a driven antenna has a continuous slope everywhere except at the driving point and that it becomes continuous there when the driving voltage is reduced to zero. The condition (5) has been confirmed experimentally [6] for conductors with the same radius. Its extension to conductors with unequal radii is under study both theoretically [7] and experimentally but the results obtained transcend the present investigation.

3. Analytical Formulation

Since the conditions at the ends of the crossed wires and at their junction involve the currents and their derivatives, an integral equation in which the constants of integration appear in expressions for the potentials is not convenient. Therefore, new and somewhat different integral equations for the currents are derived from the boundary conditions on the conducting surfaces:

$$E_z(z) = E_z^{inc} - \frac{\partial\phi(z)}{\partial z} - j\omega A_z(z) = 0 \quad ; \quad -h_1 \leq z \leq h_2 \tag{6a}$$

$$E_x(x) = -\frac{\partial\phi(x)}{\partial x} - j\omega A_x(x) = 0 \quad ; \quad -\ell_1 \leq x \leq \ell_2 \tag{6b}$$

where

$$A_z(z) = \frac{\mu_0}{4\pi}\int_{-h_1}^{h_2} I_z(z')K(z,z')\,dz' \tag{7}$$

$$\phi(z) = \frac{1}{4\pi\epsilon_0}\left[\int_{-h_1}^{h_2} q(z')K(z,z')\,dz' + \int_{-\ell_1}^{\ell_2} q(x')K(z,x')\,dx'\right]. \tag{8}$$

$A_x(x)$ and $\phi(x)$ are obtained from (7) and (8) by interchanging z and x, h and ℓ. The average kernels are $K(z,z') = \exp(-jkR_z)/R_z$ with $R_z = [(z-z')^2 + a^2]^{1/2}$ and $K(z,x') = \exp(-jkR_{cz})/R_{cz}$ with $R_{cz} = [z^2 + x'^2 + a^2]^{1/2}$. Note that $K(z,z')$ $= K_R(z,z') + jK_I(z,z')$ where $K_R(z,z') = (\cos kR_z)/R_z$ and $K_I(z,z') = -(\sin kR_z)/R_z$.

In the analysis of single and parallel antennas it is customary to introduce the condition, $\nabla\cdot\vec{A} + j(k^2/\omega)\phi = 0$, in (6) to eliminate the scalar potential. This procedure is not followed here. Instead, the integrals in (7) and (8) are inserted directly in (6) to obtain the following equations:

$$\int_{-h_1}^{h_2} I(z')K(z,z')\,dz' - \frac{j\omega}{k^2}\frac{\partial}{\partial z}\left[\int_{-h_1}^{h_2} q(z')K(z,z')\,dz' + \int_{-\ell_1}^{\ell_2} q(x')K(z,x')\,dx'\right]$$
$$= -(j4\pi/\omega\mu)E_z^{inc} \tag{9 a}$$

$$\int_{-\ell_1}^{\ell_2} I(x')K(x,x')\,dx' - \frac{j\omega}{k^2}\frac{\partial}{\partial x}\left[\int_{-\ell_1}^{\ell_2} q(x')K(x,x')\,dx' + \int_{-h_1}^{h_2} q(z')K(x,z')\,dz'\right] = 0 \quad . \tag{9b}$$

These are to be solved for I and q in the two conductors subject to (3), (4) and (5). The equation of continuity interrelates the current and charge.

4. Formal Solution of the Integral Equations

Before obtaining a solution of (9a,b) it is convenient to apply the equation of continuity (2) to the middle integral in (9a). With $\partial K(z,z')/\partial z' = -\partial K(z,z')/\partial z$, the desired relations are:

$$J(z) \equiv j\omega \frac{\partial}{\partial z}\int_{-h_1}^{h_2} q(z')K(z,z')\,dz' = \int_{-h_1}^{h_2} \frac{\partial I(z')}{\partial z'}\frac{\partial}{\partial z'}K(z,z')\,dz' \quad . \tag{10}$$

Integration by parts now yields:

$$J(z) = -j\omega[q(h_2)K(z,h_2) - q(-h_1)K(z,-h_1)] - \int_{-h_1}^{h_2} \frac{\partial^2 I(z')}{\partial z'^2}K(z,z')\,dz' \quad . \tag{11}$$

With (11) and an expression like it with x substituted for z and ℓ for h, (9a,b) become:

$$\int_{-h_1}^{h_2}\left[\frac{\partial^2 I(z')}{\partial z'^2} + k^2 I(z')\right]K(z,z')\,dz' - F_2(z) - F_3(z) = \frac{-j4\pi k^2 E_z^{inc}}{\omega\mu} \tag{12a}$$

$$\int_{-\ell_1}^{\ell_2}\left[\frac{\partial^2 I(x')}{\partial x'^2} + k^2 I(x')\right]K(x,x')\,dx' - F_2(x) - F_3(x) = 0 \tag{12b}$$

where

$$F_2(z) = j\omega\frac{\partial}{\partial z}\int_{-\ell_1}^{\ell_2} q(x')K(z,x')\,dx' \tag{13a}$$

$$F_3(z) = -j\omega[q(h_2)K(z,h_2) - q(-h_1)K(z,-h_1)] \quad . \tag{13b}$$

The functions $F_2(x)$ and $F_3(x)$ are obtained from (13a,b) with the substitution of x for z and ℓ for h.

The equations (12a,b) can be simplified greatly if use is made of the peaking property of the real parts of the kernels of the integrals. These occur at z'

= z and x' = x where $K_R(z,z')$ and $K_R(x,x')$ become very large when ka << 1. As a consequence, the relation

$$\int_{-h_1}^{h_2} f(z')K_R(z,z')\,dz' = \int_{-h_1}^{h_2} f(z')\frac{\cos k\sqrt{(z-z')^2+a^2}}{\sqrt{(z-z')^2+a^2}}\,dz' \doteq \Psi f(z) \qquad (14a)$$

where Ψ is a constant is an excellent approximation for any function f(z) such as $(\partial^2/\partial z^2 + k^2)I(z)$. [Note that (14a) is not valid when ka is not small.] The constant parameter Ψ is defined by

$$\Psi = f^{-1}(z_m)\int_{-h_1}^{h_2} f(z')K_R(z_m,z')\,dz' \qquad (14b)$$

where z_m is a point near the maximum of f(z). This integral is readily evaluated for the problem at hand [8,Appendix A]. When the electrical length of the conductor is not smaller than $\pi/2$, $\Psi = 2[\ln(2/ka) - 0.5772]$. Evidently the same parameter Ψ applies to the transverse conductor if it has the same radius.

With (14) the coupled integral equations (12a,b) become:

$$\left[\frac{\partial^2}{\partial z^2} + k^2\right] I(z) = Ak^2 + \Psi^{-1}[F_1(z) + F_2(z) + F_3(z)] \qquad (15a)$$

$$\left[\frac{\partial^2}{\partial x^2} + k^2\right] I(x) = \Psi^{-1}[F_1(x) + F_2(x) + F_3(x)] \qquad (15b)$$

where

$$A = -(j4\pi E_z^{inc}/\omega\mu\Psi) = (-j/60\pi\Psi)(E_z^{inc}\lambda) \qquad (16)$$

and

$$F_1(z) = -j\int_{-h_1}^{h_2}\left[\frac{\partial^2 I(z')}{\partial z'^2} + k^2 I(z')\right] K_I(z,z')\,dz' \ . \qquad (17)$$

The function $F_1(x)$ is obtained from (17) with the substitution of x for z, ℓ for h.

The solutions of (15a,b) include the simple solutions of the homogeneous equations and sums of particular integrals due to the inhomogeneous terms. The formal solutions for the currents on the four arms of the crossed wires are:

$$I_1(z) = A[C_1' \cos kz + C_1'' \sin kz + 1] + H_h(z)/\Psi \quad ; \quad -h_1 \le z \le 0 \qquad (18a)$$

$$I_2(z) = A[C_2' \cos kz + C_2'' \sin kz + 1] + H_h(z)/\Psi \quad ; \quad 0 \le z \le h_2 \tag{18b}$$

$$I_3(x) = A[C_3' \cos kx + C_3'' \sin kx] + H_\ell(x)/\Psi \quad ; \quad -\ell_1 \le x \le 0 \tag{18c}$$

$$I_4(x) = A[C_4' \cos kx + C_4'' \sin kx] + H_\ell(x)/\Psi \quad ; \quad 0 \le x \le \ell_2 \tag{18d}$$

where the C's are arbitrary constants of integration and

$$H_h(z) = T_1(z) + T_2(z) + T_3(z) \quad , \quad H_\ell(x) = T_1(x) + T_2(x) + T_3(x) \tag{19a}$$

with

$$T_i(z) = k^{-1} \int_0^z F_i(s) \sin k(z - s)\, ds \quad , \quad i = 1, 2, 3 . \tag{19b}$$

The functions F_i are defined in (13a,b) and (17). The particular integral due to the first term on the right in (15a) is obtained from (19b) with k^2A substituted for $F_i(s)$. It contributes the term 1 in (18a,b) and to the arbitrary constants C_1' and C_2''. The other particular integrals are the functions $T_i(z)$ and $T_i(x)$.

The distributions of charge per unit length are obtained from the currents in (18a-d) with the equation of continuity. With $\partial H(z)/\partial z$ denoted by $H'(z)$,

$$q_1(z) = (jk/\omega)A[-C_1' \sin kz + C_1'' \cos kz] + (j/\omega\Psi)H_h'(z) \tag{20a}$$

$$q_2(z) = (jk/\omega)A[-C_2' \sin kz + C_2'' \cos kz] + (j/\omega\Psi)H_h'(z) \tag{20b}$$

$$q_3(x) = (jk/\omega)A[-C_3' \sin kx + C_3'' \cos kx] + (j/\omega\Psi)H_\ell'(x) \tag{20c}$$

$$q_4(x) = (jk/\omega)A[-C_4' \sin kx + C_4'' \cos kx] + (j/\omega\Psi)H_\ell'(x) . \tag{20d}$$

Since the currents and charges appear in the integrands of the particular integrals, (18a-d) and (20a-d) are not solutions but rearranged coupled integral equations. Approximate solutions can be obtained by iteration. Suitable zero-order solutions are given by the square brackets in (18a-d) and (20a-d). First-order solutions are obtained with the substitution of zero-order values into the integrands in $H_h(z)$ and $H_\ell(x)$ and their derivatives. Second-order solutions can be generated by the substitution of first-order values in $H_h(z)$ and $H_\ell(x)$. For present purposes first-order solutions are adequate.

5. First-Order Solutions

The substitution of zero-order currents and charges in (13a,b) and (17) yields zero-order values of the functions F_i. When these are used in (19) the following first-order integrals are obtained:

$$T_1(z) = -At_1(z) \quad ; \quad T_1(x) = 0 \tag{21}$$

$$T_2(z) = -A[C_3'G_s(z,\ell_1) - C_4'G_s(z,\ell_2) + C_3''G_c(z,\ell_1) + C_4''G_c(z,\ell_2)] \tag{22a}$$

$$T_2(x) = -A[C_1'G_s(x,h_1) - C_2'G_s(x,h_2) + C_1''G_c(x,h_1) + C_2''G_c(x,h_2)] \tag{22b}$$

$$T_3(z) = A[\mathcal{S}(z,h_2)(-C_2' \sin kh_2 + C_2'' \cos kh_2) - \mathcal{S}(z,-h_1)(C_1' \sin kh_1 + C_1'' \cos kh_1)] \tag{23a}$$

$$T_3(x) = A[\mathcal{S}(x,\ell_2)(-C_4' \sin k\ell_2 + C_4'' \cos k\ell_2) - \mathcal{S}(x,-\ell_1)(C_3' \sin k\ell_1 + C_3'' \cos k\ell_1)] . \tag{23b}$$

Formulas for the several functions in (21) - (23b) are given in the Glossary. Their evaluation is carried out in [8,Appendices B,C, and D].

When (21) - (23b) are substituted in (19), (18a-d) and (20a-d), first-order solutions for the currents and charges per unit length are obtained. It remains to evaluate the constants C_i' and C_i'' from the boundary and junction conditions.

6. Evaluation of Constants of Integration

Since $H_h(0) = H_\ell(0) = 0$ and $H_h'(0) = H_\ell'(0) = 0$, the junction conditions (4) and (5) are completely specified by the zero-order parts of (18a-d) and (20a-d). Thus (5) gives:

$$C_1'' = C_2'' = C_3'' = C_4'' \equiv C'' . \tag{24}$$

Similarly, from (4):

$$C_1' - C_2' + C_3' - C_4' = 0 . \tag{25}$$

The conditions (3) involve zero- and first-order terms. They yield the following four simultaneous equations for the C_j' , j = 1, 2, 3, 4:

$$\sum_{j=1}^{4} C_j' a_{ij} = R_i \quad ; \quad i = 1, 2, 3, 4 \tag{26a}$$

where

$$R_1 = -1 + \theta(-h_1) + C''(\sin kh_1 - b_1) \quad ; \quad R_3 = C''(\sin k\ell_1 - b_3) \tag{26b}$$

$$R_2 = -1 + \theta(h_2) - C''(\sin kh_2 + b_2) \quad ; \quad R_4 = -C''(\sin k\ell_2 + b_4) \ . \tag{26c}$$

The following coefficients are involved:

$$a_{11} = \cos kh_1 - \Psi^{-1}\mathcal{J}(-h_1,-h_1)\sin kh_1 \quad ; \quad a_{12} = -\Psi^{-1}\mathcal{J}(-h_1,h_2)\sin kh_2 \ ;$$
$$a_{13} = -\Psi^{-1}G_s(-h_1,\ell_1) \quad ; \quad a_{14} = \Psi^{-1}G_s(-h_1,\ell_2) \ ; \tag{27a}$$

$$a_{21} = -\Psi^{-1}\mathcal{J}(h_2,-h_1)\sin kh_1 \quad ; \quad a_{22} = \cos kh_2 - \Psi^{-1}\mathcal{J}(h_2,h_2)\sin kh_2 \quad ;$$
$$a_{23} = -\Psi^{-1}G_s(h_2,\ell_1) \quad ; \quad a_{24} = \Psi^{-1}G_s(h_2,\ell_2) \quad ; \tag{27b}$$

$$a_{31} = -\Psi^{-1}G_s(-\ell_1,h_1) \quad ; \quad a_{32} = \Psi^{-1}G_s(-\ell_1,h_2) \quad ; \quad a_{33} = \cos k\ell_1$$
$$- \Psi^{-1}\mathcal{J}(-\ell_1,-\ell_1)\sin k\ell_1 \quad ; \quad a_{34} = -\Psi^{-1}\mathcal{J}(-\ell_1,\ell_2)\sin k\ell_2 \ ; \tag{27c}$$

$$a_{41} = -\Psi^{-1}G_s(\ell_2,h_1) \quad ; \quad a_{42} = \Psi^{-1}G_s(\ell_2,h_2) \quad ; \quad a_{43} = -\Psi^{-1}\mathcal{J}(\ell_2,-\ell_1)\sin k\ell_1 \ ;$$
$$a_{44} = \cos k\ell_2 - \Psi^{-1}\mathcal{J}(\ell_2,\ell_2)\sin k\ell_2 \tag{27d}$$

$$b_1 = -\Psi^{-1}[\mathcal{J}(-h_1,-h_1)\cos kh_1 - \mathcal{J}(-h_1,h_2)\cos kh_2 + G_c(-h_1,\ell_1) + G_c(-h_1,\ell_2)] \tag{28a}$$

$$b_2 = -\Psi^{-1}[-\mathcal{J}(h_2,h_2)\cos kh_2 + \mathcal{J}(h_2,-h_1)\cos kh_1 + G_c(h_2,\ell_1) + G_c(h_2,\ell_2)] \tag{28b}$$

$$b_3 = -\Psi^{-1}[\mathcal{J}(-\ell_1,-\ell_1)\cos k\ell_1 - \mathcal{J}(-\ell_1,\ell_2)\cos k\ell_2 + G_c(-\ell_1,h_1) + G_c(-\ell_1,h_2)] \tag{28c}$$

$$b_4 = -\Psi^{-1}[-\mathcal{J}(\ell_2,\ell_2)\cos k\ell_2 + \mathcal{J}(\ell_2,-\ell_1)\cos k\ell_1 + G_c(\ell_2,h_1) + G_c(\ell_2,h_2)] \tag{28d}$$

$$\theta(z) = t_1(z)/\Psi \ . \tag{29}$$

The solutions of the simultaneous equations (26a) have the form:

$$C_j' = A_j/D \quad , \quad j = 1, 2, 3, 4 \tag{30}$$

where D is the determinant of the coefficients a_{ij}, and A_j the appropriate cofactor. General analytical formulas can be obtained quite simply when $\Psi >> 1$ - which is a necessary consequence of the condition $ka << 1$. When terms of the order Ψ^{-2} are neglected, the determinant of the coefficients a_{ij} reduces to the diagonal terms. Thus,

$$D \doteq a_{11}a_{22}a_{33}a_{44} \quad . \tag{31}$$

This contains terms of the order Ψ^{-1} which are important since the leading terms in a_{ii} can vanish when the cosine is zero. With (31)

$$A_1 \doteq R_1a_{22}a_{33}a_{44} - R_2a_{12}a_{33}a_{44} - R_3a_{13}a_{22}a_{44} - R_4a_{14}a_{22}a_{33} \tag{32a}$$

$$A_2 \doteq R_2a_{11}a_{33}a_{44} - R_1a_{21}a_{33}a_{44} - R_3a_{23}a_{11}a_{44} - R_4a_{24}a_{11}a_{33} \tag{32b}$$

$$A_3 \doteq R_3a_{11}a_{22}a_{44} - R_1a_{31}a_{22}a_{44} - R_2a_{32}a_{11}a_{44} - R_4a_{34}a_{11}a_{22} \tag{32c}$$

$$A_4 \doteq R_4a_{11}a_{22}a_{33} - R_1a_{41}a_{22}a_{33} - R_2a_{42}a_{11}a_{33} - R_3a_{43}a_{11}a_{22} \quad . \tag{32d}$$

It follows that

$$C_1' = A_1/D \doteq a_{11}^{-1}(R_1 - R_2a_{12}/a_{22} - R_3a_{13}/a_{33} - R_4a_{14}/a_{44}) \tag{33a}$$

$$C_2' = A_2/D \doteq a_{22}^{-1}(R_2 - R_1a_{21}/a_{11} - R_3a_{23}/a_{33} - R_4a_{24}/a_{44}) \tag{33b}$$

$$C_3' = A_3/D \doteq a_{33}^{-1}(R_3 - R_1a_{31}/a_{11} - R_2a_{32}/a_{22} - R_4a_{34}/a_{44}) \tag{33c}$$

$$C_4' = A_4/D \doteq a_{44}^{-1}(R_4 - R_1a_{41}/a_{11} - R_2a_{42}/a_{22} - R_3a_{43}/a_{33}) \quad . \tag{33d}$$

These expressions involve C" which occurs in the R's. It can be determined with (25) and the simplifying notation:

$$n_1 = \frac{a_{21}}{a_{22}} - \frac{a_{31}}{a_{33}} + \frac{a_{41}}{a_{44}} \quad ; \quad n_2 = \frac{a_{32}}{a_{33}} - \frac{a_{42}}{a_{44}} + \frac{a_{12}}{a_{11}} \tag{34a}$$

$$n_3 = \frac{a_{43}}{a_{44}} - \frac{a_{13}}{a_{11}} + \frac{a_{23}}{a_{22}} \quad ; \quad n_4 = \frac{a_{14}}{a_{11}} - \frac{a_{24}}{a_{22}} + \frac{a_{34}}{a_{33}} \quad . \tag{34b}$$

With (34a,b), (25) becomes:

$$(R_1/a_{11})(1 + n_1) - (R_2/a_{22})(1 + n_2) + (R_3/a_{33})(1 + n_3) - (R_4/a_{44})(1 + n_4) = 0 \quad . \tag{35}$$

The substitution for the R's from (26b,c) yields:

$$C'' = (T + M)^{-1}(a_{11}^{-1} - a_{22}^{-1} + N) \tag{36}$$

where

$$N = a_{11}^{-1}[n_1 - \theta(-h_1)] - a_{22}^{-1}[n_2 - \theta(h_2)] \tag{37}$$

$$M = a_{11}^{-1}[n_1 \sin kh_1 - b_1(1 + n_1)] + a_{22}^{-1}[n_2 \sin kh_2 + b_2(1 + n_2)]$$
$$+ a_{33}^{-1}[n_3 \sin k\ell_1 - b_3(1 + n_3)] + a_{44}^{-1}[n_4 \sin k\ell_2 + b_4(1 + n_4)] \tag{38}$$

$$T = a_{11}^{-1} \sin kh_1 + a_{22}^{-1} \sin kh_2 + F(\ell_1,\ell_2) \tag{39}$$

$$F(\ell_1,\ell_2) = a_{33}^{-1} \sin k\ell_1 + a_{44}^{-1} \sin k\ell_2 \;. \tag{40}$$

When (36) and (26a-d) are used in (33a-d), these yield explicit formulas for the four C's. It is convenient to separate the leading and higher-order terms as follows:

$$C_1' = a_{11}^{-1}[-1 + c_1' + C''(\sin kh_1 + c_1'')] \tag{41a}$$

$$C_2' = a_{22}^{-1}[-1 + c_2' - C''(\sin kh_2 + c_2'')] \tag{41b}$$

$$C_3' = a_{33}^{-1}[c_3' + C''(\sin k\ell_1 + c_3'')] \tag{41c}$$

$$C_4' = a_{44}^{-1}[c_4' - C''(\sin k\ell_2 + c_4'')] \tag{41d}$$

where

$$c_1' = \theta(-h_1) + a_{12}a_{22}^{-1}[1 - \theta(h_2)] \quad ; \quad c_2' = \theta(h_2) + a_{21}a_{11}^{-1}[1 - \theta(-h_1)] \tag{42a}$$

$$c_3' = a_{31}a_{11}^{-1}[1 - \theta(-h_1)] + a_{32}a_{22}^{-1}[1 - \theta(h_2)] \quad ;$$

$$c_4' = a_{41}a_{11}^{-1}[1 - \theta(-h_1)] + a_{42}a_{22}^{-1}[1 - \theta(h_2)] \quad ; \tag{42b}$$

$$c_1'' = -b_1 + a_{12}a_{22}^{-1}(\sin kh_2 + b_2) - a_{13}a_{33}^{-1}(\sin k\ell_1 - b_3) + a_{14}a_{44}^{-1}(\sin k\ell_2 + b_4) \tag{42c}$$

$$c_2'' = b_2 + a_{23}a_{33}^{-1}(\sin k\ell_1 - b_3) - a_{24}a_{44}^{-1}(\sin k\ell_2 + b_4) + a_{21}a_{11}^{-1}(\sin kh_1 - b_1) \tag{42d}$$

$$c_3'' = -b_3 + a_{34}a_{44}^{-1}(\sin k\ell_2 + b_4) - a_{31}a_{11}^{-1}(\sin kh_1 - b_1) + a_{32}a_{22}^{-1}(\sin kh_2 + b_2) \tag{42e}$$

$$c_4'' = b_4 + a_{41}a_{11}^{-1}(\sin kh_1 - b_1) - a_{42}a_{22}^{-1}(\sin kh_2 + b_2) + a_{43}a_{33}^{-1}(\sin k\ell_1 - b_3) \;. \tag{42f}$$

Since C'' is given explicitly in (36), the four constants C_1', C_2', C_3' and C_4' have been determined.

7. The Distribution of Current

The substitution of (41a-d) and (36) into (18a-d) gives the first-order currents. They are:

$$\begin{aligned} I_1(z) = {} & A[T + M]^{-1}\{a_{11}^{-1}(\sin kz + \sin kh_1) - a_{22}^{-1}(\sin kz - \sin kh_2) \\ & - a_{11}^{-1}a_{22}^{-1}(\sin kh_1 + \sin kh_2)\cos kz + [F(\ell_1,\ell_2) + M](1 - a_{11}^{-1}\cos kz) \\ & + N[\sin kz + a_{11}^{-1}\sin kh_1 \cos kz] + [c_1'(T + M) + c_1''(a_{11}^{-1} - a_{22}^{-1} + N)]a_{11}^{-1}\cos kz\} \\ & + H_h(z)\Psi^{-1} \end{aligned} \tag{43a}$$

$$I_2(z) = A[T + M]^{-1}\{a_{11}^{-1}(\sin kz + \sin kh_1) - a_{22}^{-1}(\sin kz - \sin kh_2)$$
$$- a_{11}^{-1}a_{22}^{-1}(\sin kh_1 + \sin kh_2)\cos kz + [F(\ell_1,\ell_2) + M](1 - a_{22}^{-1}\cos kz)$$
$$+ N[\sin kz - a_{22}^{-1}\sin kh_2 \cos kz] + [c_2'(T + M) - c_2''(a_{11}^{-1} - a_{22}^{-1} + N)]a_{22}^{-1}\cos kz\}$$
$$+ H_h(z)\Psi^{-1} \tag{43b}$$

$$I_3(x) = A[T + M]^{-1}\{(a_{11}^{-1} - a_{22}^{-1} + N)(\sin kx + a_{33}^{-1}\sin k\ell_1 \cos kx)$$
$$+ [c_3'(T + M) + c_3''(a_{11}^{-1} - a_{22}^{-1} + N)]a_{33}^{-1}\cos kx\} + H_\ell(x)\Psi^{-1} \tag{43c}$$

$$I_4(x) = A[T + M]^{-1}\{(a_{11}^{-1} - a_{22}^{-1} + N)(\sin kx - a_{44}^{-1}\sin k\ell_2 \cos kx)$$
$$+ [c_4'(T + M) - c_4''(a_{11}^{-1} - a_{22}^{-1} + N)]a_{44}^{-1}\cos kx\} + H_\ell(x)\Psi^{-1} \quad . \tag{43d}$$

With (19a), (21), (22a,b) and (23a,b), it follows that

$$H_h(z) = -A\{t_1(z) + C_1'\vartheta(z,-h_1)\sin kh_1 + C_2'\vartheta(z,h_2)\sin kh_2 + C_3'G_s(z,\ell_1) - C_4'G_s(z,\ell_2)$$
$$+ C''[\vartheta(z,-h_1)\cos kh_1 - \vartheta(z,h_2)\cos kh_2 + G_c(z,\ell_1) + G_c(z,\ell_2)]\} \tag{44a}$$

where the C's are given by (41a-d) and C" by (36). The functions t, G and ϑ are listed in the glossary. The corresponding formula for $H_\ell(x)$ is:

$$H_\ell(x) = -A\{C_3'\vartheta(x,-\ell_1)\sin k\ell_1 + C_4'\vartheta(x,\ell_2)\sin k\ell_2 + C_1'G_s(x,h_1) - C_2'G_s(x,h_2)$$
$$+ C''[\vartheta(x,-\ell_1)\cos k\ell_1 - \vartheta(x,\ell_2)\cos k\ell_2 + G_c(x,h_1) + G_c(x,h_2)]\} \quad . \tag{44b}$$

Note that $H_h(0) = H_\ell(0) = 0$, $H_h'(0) = H_\ell'(0) = 0$.

When the electrical lengths of the four arms differ from integral multiples of $\lambda/4$, simple zero-order formulas may be adequate when Ψ is sufficiently large. These are obtained by neglecting all terms with Ψ^{-1} as a factor. They are:

$$[I_1(z)]_0 = -AW[\sin k(h_1+ z) + \sin k(h_2- z) - \sin k(h_1+ h_2)$$
$$+ F(\ell_1,\ell_2)\cos kh_2(\cos kz - \cos kh_1)] \tag{45a}$$

$$[I_2(z)]_0 = -AW[\sin k(h_1+ z) + \sin k(h_2- z) - \sin k(h_1+ h_2)$$
$$+ F(\ell_1,\ell_2)\cos kh_1(\cos kz - \cos kh_2)] \tag{45b}$$

$$[I_3(x)]_0 = -AW\ (\cos kh_2 - \cos kh_1)\ \sec k\ell_1\ \sin k(\ell_1 + x) \tag{45c}$$

$$[I_4(x)]_0 = AW\ (\cos kh_2 - \cos kh_1)\ \sec k\ell_2\ \sin k(\ell_2 - x) \tag{45d}$$

where

$$W = [\sin k(h_1 + h_2) + F(\ell_1, \ell_2)\ \cos kh_1\ \cos kh_2]^{-1}\ . \tag{46}$$

A is defined in (16) and $F(\ell_1, \ell_2) = \tan k\ell_1 + \tan k\ell_2$.

8. The Distributions of Charge Per Unit Length

The first-order distributions of charge per unit length are obtained directly from the currents (43a-d) with the help of the equation of continuity, $q(y) = (j/\omega)[\partial I(y)/\partial y]$. The very simple zero-order formulas are:

$$[q_1(z)]_0 = (-jkAW/\omega)[\cos k(h_1 + z) - \cos k(h_2 - z) - F(\ell_1, \ell_2)\cos kh_2\ \sin kz] \tag{47a}$$

$$[q_2(z)]_0 = (-jkAW/\omega)[\cos k(h_1 + z) - \cos k(h_2 - z) - F(\ell_1, \ell_2)\cos kh_1\ \sin kz] \tag{47b}$$

$$[q_3(x)]_0 = (-jkAW/\omega)(\cos kh_2 - \cos kh_1)\ \sec k\ell_1\ \cos k(\ell_1 + x) \tag{47c}$$

$$[q_4(x)]_0 = (-jkAW/\omega)(\cos kh_2 - \cos kh_1)\ \sec k\ell_2\ \cos k(\ell_2 - x)\ . \tag{47d}$$

9. Special Cases

In order to gain insight into the numerous possible distributions of current and charge on crossed dipoles, it is important to study the special cases associated with conditions of resonance and antiresonance in the six possible circuits, each consisting of two arms. These are not adequately represented by the zero-order formulas. It is also of interest to examine the completely symmetrical case and the vertical section alone without side arms. This will be done first.

1) Junction at the center of the vertical element, $h_2 = h_1 = h$.

Under these conditions $I(x) = 0 = q(x)$ since the horizontal element is in the neutral plane. The vertical section behaves as if isolated. Specifically,

$$I(z) = A\{1 + C_1' \cos kz - \theta(z) - C_1' \Psi^{-1}[\mathcal{J}(z,-h) + \mathcal{J}(z,h)]\sin kh\} \quad ; \quad -h \le z \le 0 \tag{48}$$

$$q(z) = (-jkA/\omega)\left\{C_1'\sin kz + \frac{\theta'(z)}{k} + [C_1'/k\Psi][\mathcal{J}'(z,-h) + \mathcal{J}'(z,h)]\sin kh\right\} \; ; \quad -h \le z \le 0 \tag{49}$$

where the prime denotes differentiation with respect to z and where

$$C_1' = [\theta(-h) - 1]/[a_{11} + a_{12}] \; ; \quad \theta(z) = t_1(z)/\Psi \, . \tag{50}$$

Note that $I(-z) = I(z)$, $q(-z) = -q(z)$.

When $kh = \pi/2$, the antenna is near resonance with $a_{11} = -\mathcal{J}(h,h)/\Psi$, $a_{12} = -\mathcal{J}(-h,h)/\Psi$, $C_1' = -\Psi[\theta(-h) - 1]/[\mathcal{J}(h,h) + \mathcal{J}(-h,h)]$. The zero-order term is:

$$[I(z)]_0 = [A\Psi\cos kz]/[\mathcal{J}(h,h) + \mathcal{J}(-h,h)] \tag{51}$$

When $kh = \pi$, $a_{11} = a_{22} = -1$, $C_1' = 1 - t_1(-h)/\Psi$, so that

$$I_1(z) = A\left\{1 + \cos kz - \Psi^{-1}[t_1(z) + t_1(-h)\cos kz]\right\} \; ; \; -h \le z \le 0 \tag{52}$$

where $t_1(-h) = j2.95$ and

$$t_1(z) = -j\left\{\mathrm{Si}(\pi+kz) + \mathrm{Si}(\pi-kz) + (1/2)[\mathrm{Cin}\, 2(\pi+kz) - \mathrm{Cin}\, 2(\pi-kz)]\sin kz \right.$$
$$\left. + (1/2)[\mathrm{Si}\, 2(\pi+kz) + \mathrm{Si}\, 2(\pi-kz) - 4\,\mathrm{Si}\,\pi - 2\,\mathrm{Si}\, 2\pi]\cos kz\right\} . \tag{53}$$

Graphs of $I(z)/A$ for $kh = \pi$, $ka = 0.04$ are shown in Fig.2. They show the familiar distribution characteristic of forced currents.

2) Junction at minima of charges per unit length and maxima of currents along horizontal and vertical elements: $kh_1 = 5\pi/2$, $kh_2 = k\ell_1 = k\ell_2 = \pi/2$.

For the specified lengths, $a_{11} = -\mathcal{J}(5\lambda/4, 5\lambda/4)/\Psi$, $a_{22} = a_{33} = a_{44} = -\mathcal{J}(\lambda/4,\lambda/4)/\Psi$. With (36) and (41a-d), it is readily verified that C'' is of order 1, whereas the C's are of order $\Psi \gg 1$. The leading terms in I and q are:

$$I_i(z) \doteq AC_i' \cos kz \quad ; \quad q_i(z) \doteq (-jkA/\omega)\, C_i' \sin kz \quad ; \quad i = 1, 2 \tag{54a}$$
$$I_i(x) \doteq AC_i' \cos kx \quad ; \quad q_i(x) \doteq (-jkA/\omega)\, C_i' \sin kx \quad ; \quad i = 3, 4 . \tag{54b}$$

Graphs of the first-order currents calculated by machine with the complete determinant of the a_{ij} are shown in Fig. 3. It is seen that the leading-term representation is a good one for the three short arms, only approximate for the long arm which evidently carries a not insignificant component of forced current as well as a leading term of resonant current. This combination is typical of currents on resonant elements in a plane-wave field [5].

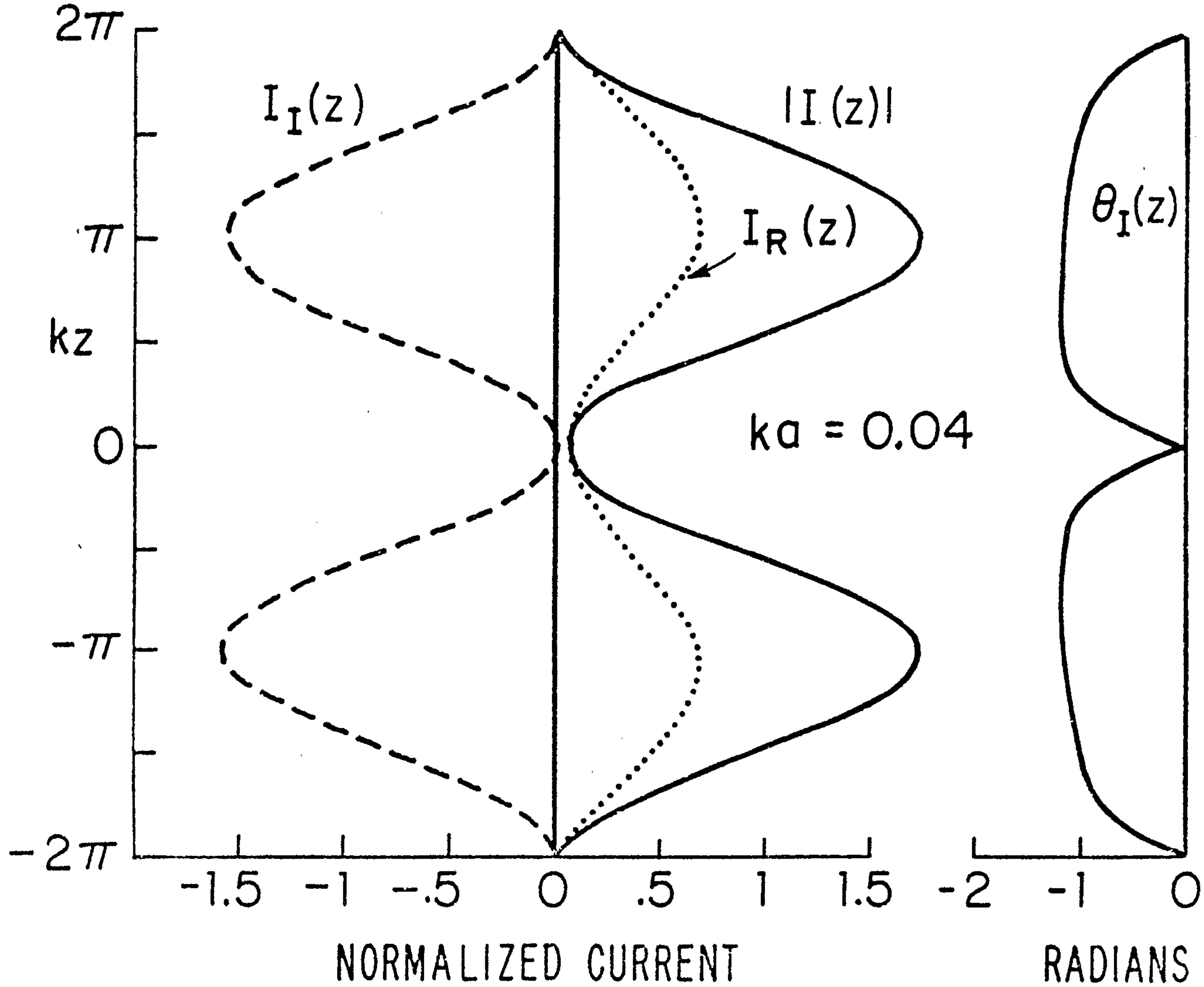
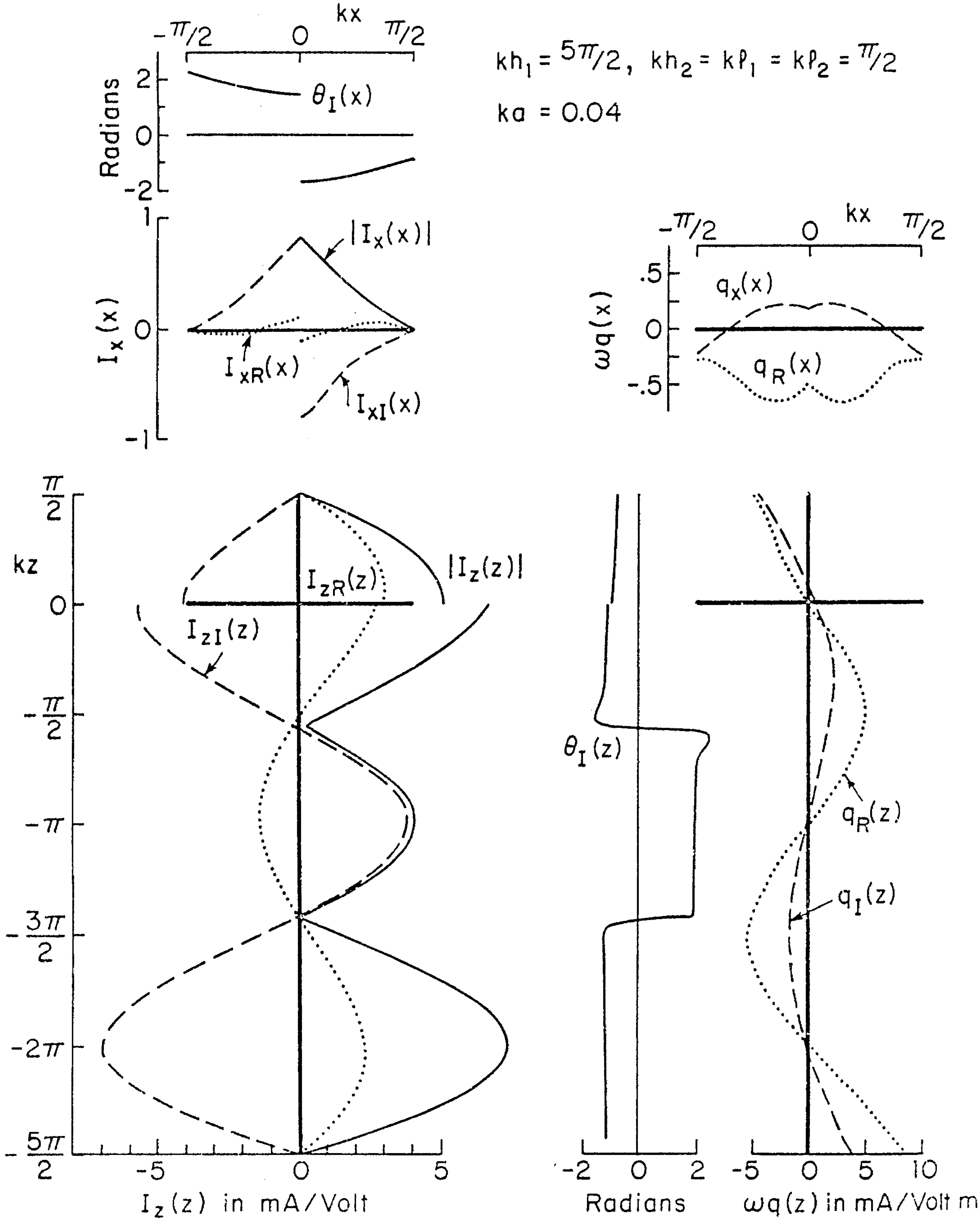


FIG. 2 CURRENT ON PARASITIC ANTENNA (KING-WU THEORY)



CURRENT AND CHARGE PER UNIT LENGTH ON CROSSED ANTENNA

FIG. 3

3) Junction at charge minimum, current maximum along vertical element; charge maximum and current minimum along horizontal element: $kh_1 = 3\pi$, $kh_2 = k\ell_1 = k\ell_2 = \pi$.

The following parameters apply: $a_{11} = a_{22} = a_{33} = a_{44} = -1$, $T = 0$, and $F(\ell_1, \ell_2) = 0$. The leading terms for the currents and charges are:

$$I_1(z) \doteq I_2(z) \doteq A[1 + \cos kz + (N/M)\sin kz] \;;\quad I_3(x) \doteq I_4(x) \doteq A(N/M)\sin kx \tag{55a}$$

$$q_1(z) \doteq q_2(z) \doteq (-jkA/\omega)[\sin kz - (N/M)\cos kz]$$

$$q_3(x) \doteq q_4(x) \doteq (jkA/\omega)(N/M)\cos kx \;. \tag{55b}$$

Note that at the junction $I_1(0)$ and $I_2(0)$ are maxima, $q_1(0)$ and $q_2(0)$ minima. On the other hand, $I_3(0)$ and $I_4(0)$ are minima, $q_3(0)$ and $q_4(0)$ maxima.

Complete first-order distributions of current and charge per unit length are shown in Fig. 4. It is seen that the leading terms in (55a,b) are reasonable approximations. In particular, the currents in the vertical arms are typical forced components since the lengths are antiresonant.

4) Junction at minima of charge and current along vertical element, maximum of charge and minimum of current along horizontal element: $kh_1 = 4\pi$, $kh_2 = 2\pi$, $k\ell_1 = k\ell_2 = \pi$.

For the specified lengths, $a_{11} = a_{22} = 1$, $a_{33} = a_{44} = -1$, $T = 0$, $F(\ell_1, \ell_2) = 0$. The leading terms for the current and charge are:

$$I_1(z) \doteq I_2(z) \doteq A[1 - \cos kz + (N/M)\sin kz] \;;\quad I_3(x) \doteq I_4(x) \doteq A(N/M)\sin kz \tag{56a}$$

$$q_1(z) \doteq q_2(z) \doteq (jkA/\omega)[\sin kz + (N/M)\cos kz]$$

$$q_3(x) \doteq q_4(x) \doteq (jkA/\omega)(N/M)\cos kx \;. \tag{56b}$$

At the junction $I_1(0)$, $I_2(0)$, $q_1(0)$, $q_2(0)$, $I_3(0)$ and $I_4(0)$ are all minima; $q_3(0)$ and $q_4(0)$ are maxima.

Graphs of first-order currents and charges are shown in Fig. 5 . The forced currents on the vertical arms are like those in Fig. 4 but with minima instead of maxima at the junction. Note that in this case $I_3(x)$ and $I_4(x)$ are very small.

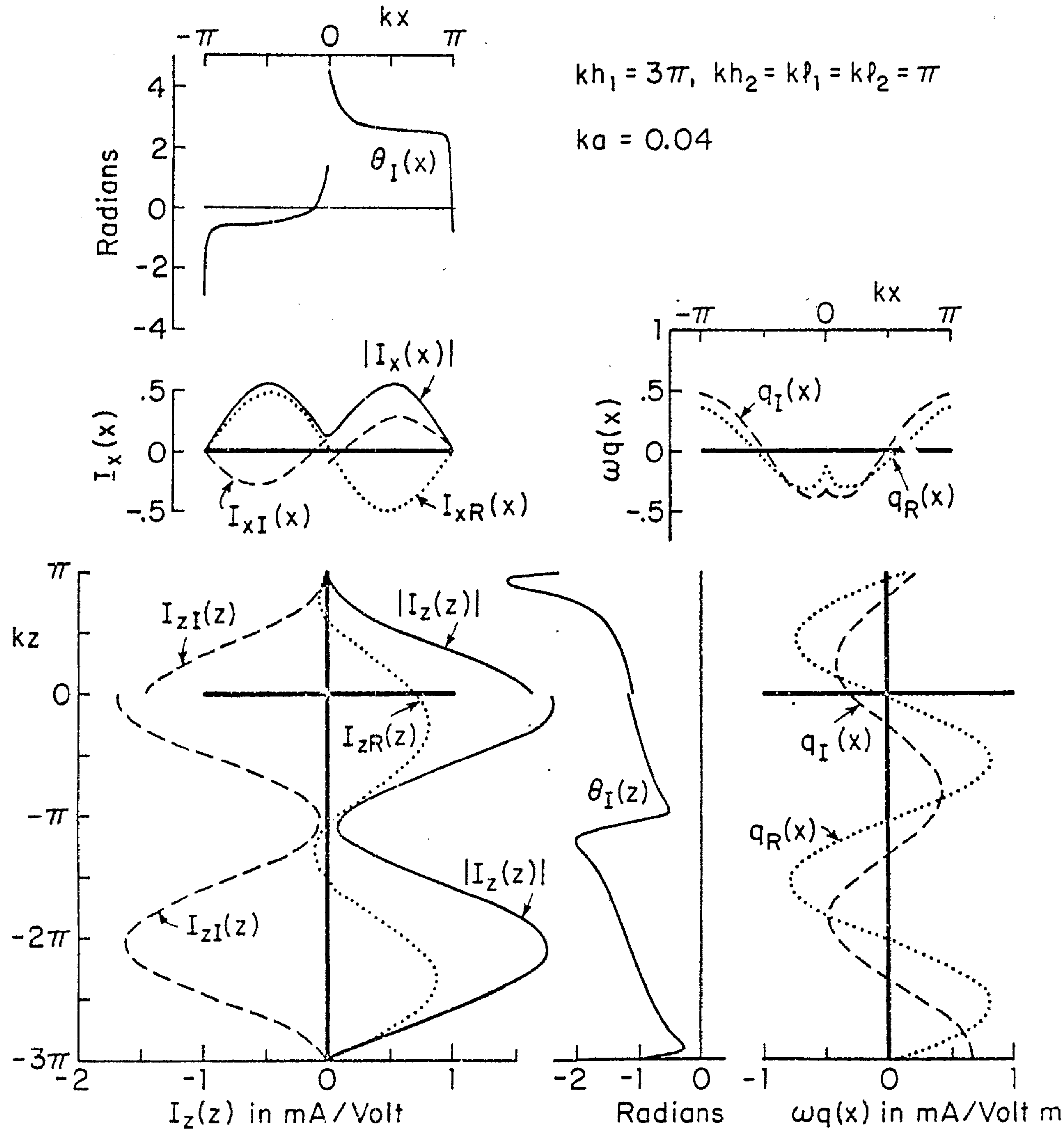


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FIG. 4

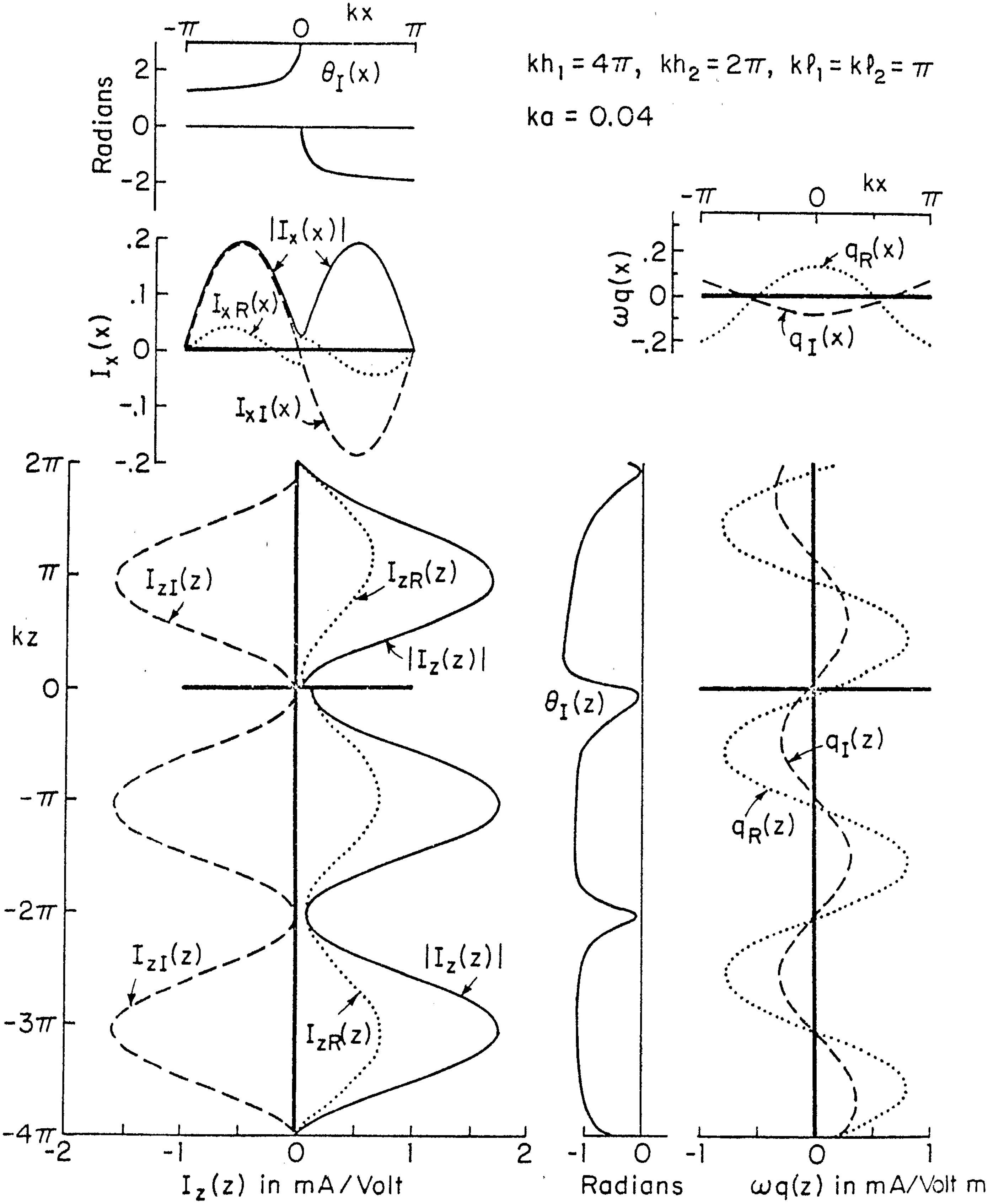

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FIG. 5

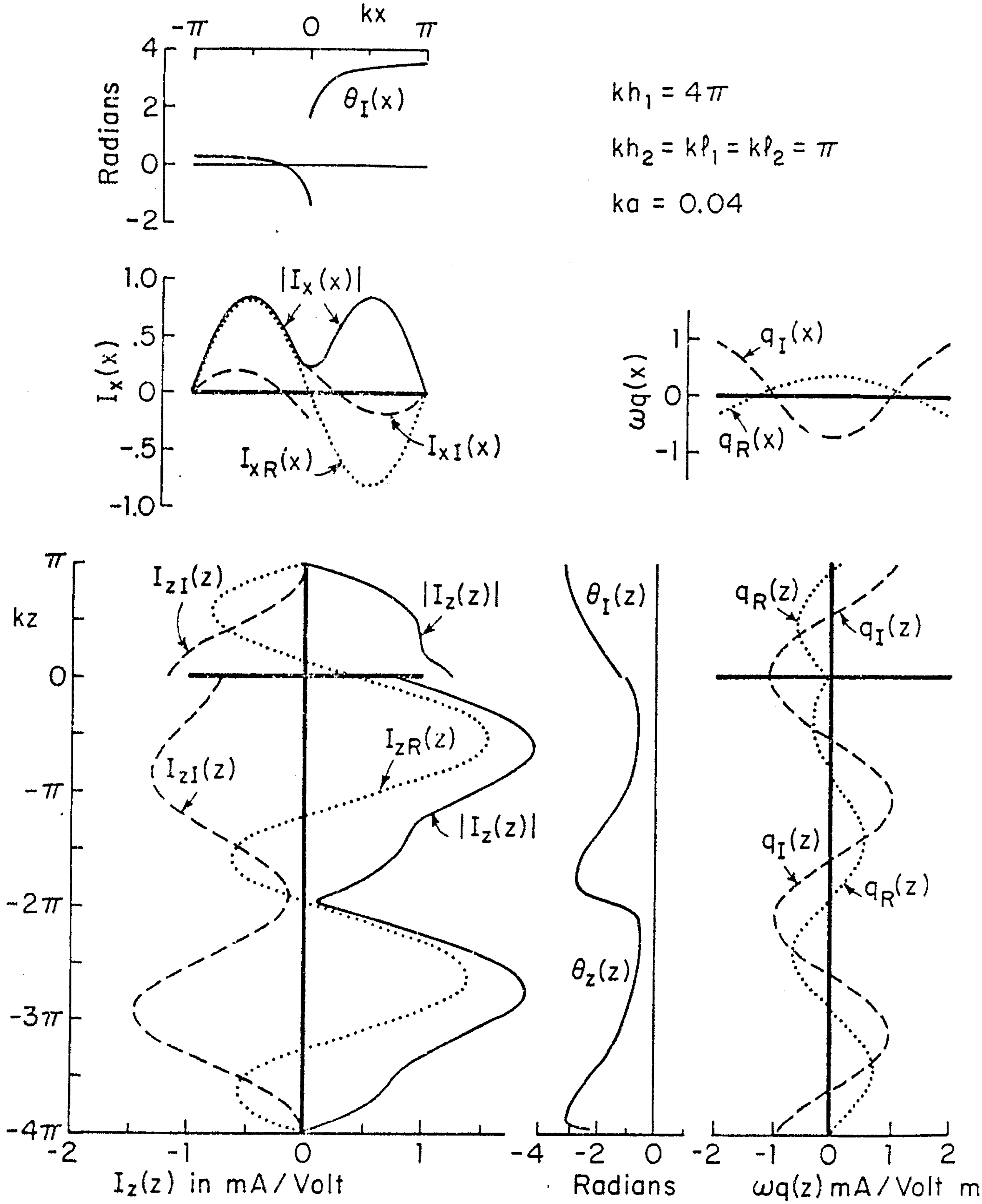

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FIG. 6

5) Large discontinuity at the junction in the z-directed currents: $kh_1 = 4\pi$, $kh_2 = k\ell_1 = k\ell_2 = \pi$.

With the specified lengths, $a_{11} = 1$, $a_{22} = a_{33} = a_{44} = -1$, $T = 0$, $F(\ell_1,\ell_2) = 0$. The largest components of current are $I_1(z) \doteq A(1 - \cos kz)$ and $I_2(z) \doteq A(1 + \cos kz)$. Evidently, $I_1(0) \doteq 0$, $I_2(0) \doteq 2A$. This large discontinuity in the leading part of the z-directed current indicates that the x-directed currents are necessarily relatively much greater than in the previous cases and that an approximation by leading terms alone is not adequate. The four currents are well approximated by the following expressions:

$$I_1(z) \doteq A\{1 - \cos kz - [(2 + N)/M](b_1 \cos kz - \sin kz)\} + H_h(z)/\Psi \qquad (57a)$$

$$I_2(z) \doteq A\{1 + \cos kz + [(2 + N)/M](b_2 \cos kz + \sin kz)\} + H_h(z)/\Psi \qquad (57b)$$

$$I_3(x) \doteq [A(2 + N)/M](b_3 \cos kx + \sin kx) + H_\ell(x)/\Psi \qquad (57c)$$

$$I_4(x) \doteq [A(2 + N)/M](b_4 \cos kx + \sin kx) + H_\ell(x)/\Psi \ . \qquad (57d)$$

At the junction, $I_1(0) \doteq -2Ab_1/M$; $I_2(0) \doteq A(2 + 2b_2/M)$; $I_3(0) \doteq 2Ab_3/M$; $I_4(0) \doteq 2Ab_4/M$. Also, $q_1(0) = q_2(0) = q_3(0) = q_4(0) \doteq (jkA/\omega M)(2 + N)$. It can be verified that the currents satisfy (4). The vanishing of the currents at the ends is accomplished with the terms $H_h(z)$ and $H_\ell(x)$.

Graphs of the first-order currents and charges are shown in Fig. 6. It is seen that the discontinuity in $I_z(z)$ at the junction is greatly reduced from the value of 2A obtained from the largest terms. It is, nevertheless, quite large and the currents in the x-directed arms are comparable in magnitude to those in the z-directed ones.

6) Horizontal element asymmetrical: $kh_1 = 5\pi/2$, $kh_2 = \pi$, $k\ell_1 = \pi/2$, $k\ell_2 = \pi$.

For the specified lengths the following parameters apply: $a_{11} = -\Psi^{-1}\mathcal{J}(-h_1,-h_1)$, $a_{22} = -1$, $a_{33} = -\Psi^{-1}\mathcal{J}(-\ell_1,-\ell_1)$, $a_{44} = -1$ where $|a_{11}^2| \ll 1$, $|a_{33}^2| \ll 1$. Also, $T = a_{11}^{-1} + a_{33}^{-1}$; $F(\ell_1,\ell_2) = a_{33}^{-1}$. The two largest terms are:

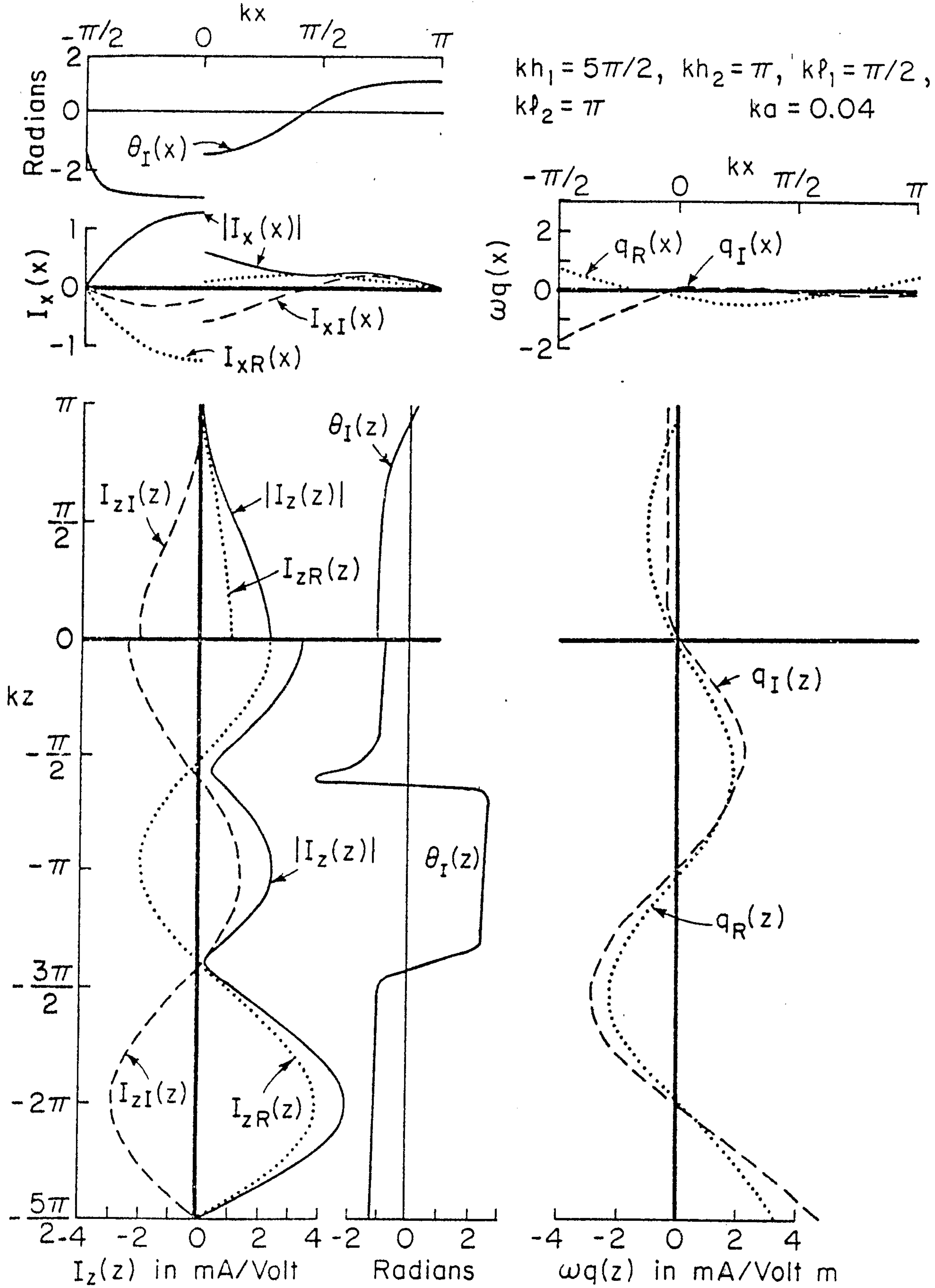
$I_1(z) \doteq -(A/Ta_{11}a_{33}) \cos kz$ and $I_3(x) \doteq (A/Ta_{11}a_{33}) \cos kx$. They indicate that a principal oscillation takes place along the vertical arm 1 and the horizontal arm 3. Owing to the wide range of amplitudes of the currents in the four arms all of the first-order terms are required. Computations from (43a-d) yield the graphs in Fig. 7. These confirm the large currents in arms 1 and 3. They also show a current of moderate amplitude on arm 2, a very small current on arm 4. Since $h_1 + \ell_1 = 3\lambda/2$, arms 1 and 3 together form a resonant circuit with a large current. In addition, there are forced currents on arms 1 and 2.

7) Crossed short dipoles: $kh_1 = 1.38$, $kh_2 = k\ell_1 = k\ell_2 = 0.69$.

The currents and charges per unit length on the arms of this cross are shown in Fig. 8. The currents are similar to those reported by Chao and Strait [4] for the same structure but their components do not have discontinuous slopes at the junction. Such discontinuities can also be observed in graphs shown by Logan [9]. Their occurrence is not surprising since numerical methods used in the solution enforced the junction condition (4) but not (5). Currents with discontinuous slopes are physically impossible in the absence of delta-function generators.

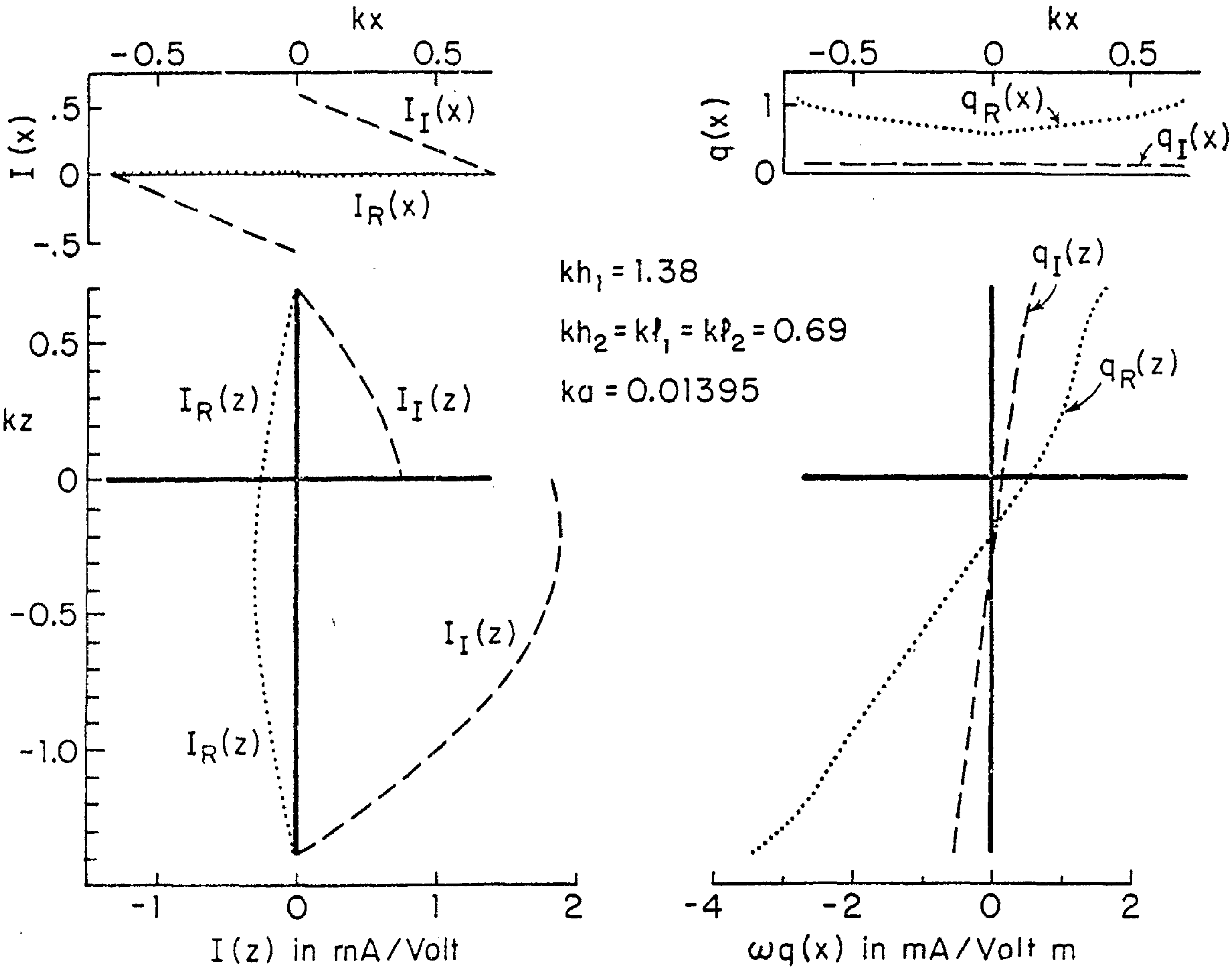
10. Conclusion

An analytical solution has been obtained for the currents and charges on the perfectly conducting, mutually perpendicular, and electrically thin arms of a crossed dipole antenna when excited by a normally incident, plane electromagnetic wave. The solution applies specifically when the electric field is parallel to one of the conductors but it is easily extended to an arbitrarily polarized, normally incident plane wave. The induced currents and charges determine the scattered field both far from and near the crossed conductors.



CURRENT AND CHARGE PER UNIT LENGTH ON CROSSED ANTENNA

FIG. 7



CURRENT AND CHARGE PER UNIT LENGTH ON CROSSED ANTENNA

FIG. 8

[1] C.D.Taylor, "Electromagnetic scattering from arbitrary configurations of wires," IEEE Trans. Antennas Propagat., Vol. AP-17, pp.662-663, Aept.1969.

[2] C.D.Taylor, S.-M. Lin and H.V.McAdams, "Scattering from crossed wires," IEEE Trans. Antennas Propagat., Vol. AP-18, pp.133-136, Jan. 1970.

[3] C.M.Butler, "Currents induced on a pair of skew crossed wires," IEEE Trans. Antennas Propagat., Vol. AP-20, pp. 731-736, Nov. 1972.

[4] H.H.Chao and B.J.Strait, "Radiation and scattering by a configuration of bent wires with junctions," IEEE Trans. Antennas Propagat., Vol. AP-19, pp. 701-702, Sept. 1971.

[5] R.W.Burton and R.W.P.King, "Induced currents and charges on thin cylinders in a time-varying electromagnetic field", Interaction Notes, Note 257, August 1975.

[6] R.W.Burton and R.W.P.King, "Measured currents and charges on thin crossed antennas in a plane-wave field," Interaction Notes, Note 258, September 1975.

[7] R.W.P.King and T.T.Wu, "Junction conditions for electrically thin conductors", 1975 USNC/URSI Meeting, URSI Session VI-6, Paper No. 4; University of Illinois, Champaign-Urbana, June 3-5.

[8] R.W.P.King and T.T.Wu,"Analysis of crossed wires in a plane-wave field*", Interaction Notes, Note 216, July 1974.

[9] J.C.Logan, "A comparison of techniques for treating radiation and scattering by wire configurations with junctions", Tech. Rept. TR-73-10, Dept. of Elec. and Computer Engineering, Syracuse University, Syracuse, N.Y., Aug. 1973.

* * *

*The following corrections to this report have been noted (numbers refer to equations): Insert - after = in (19b,c),(21b),(38a-d),(80a-d),(D-2),(D-3). Delete - after = in (30a,b), (33a,b). In (37a-d) change + to - before Ψ in $a_{11}, a_{22}, a_{33}, a_{44}$; insert - after = in $a_{12}, a_{21}, a_{34}, a_{43}$. In (48) insert $[1+n_i]$ between b_i and) with i = 1,2,3,4. Line after (50): change (42a) to (42a-d). In (54a,b) change second, third and fifth - to +; change third + to -. In (59) replace θ'(z) by θ'(z)/k. In (63) delete - before Ψ. In (64) add - before Ψ. In (67) insert - after first and last =. In (75) delete - after = in two places. Three lines before (85a): insert - after first and third =. In (88a) change first $1/a_{11}$ to $1/a_{33}$. In (89a) change $\sin kx/a_{11}$ to $\sin kx/a_{33}$. In (B-4), (B-5), and (B-6) change jkA to jk^2A. In (B-7) insert k before A.

Glossary

1. Expansion parameter when $k^2a^2 << 1$.

$$\Psi = \int_{-h_1}^{h_2} K_R(0,z')\,dz' = 2\,\ell n \frac{2\sqrt{h_1h_2}}{a} - \mathrm{Cin}\,kh_1 - \mathrm{Cin}\,kh_2$$

where $\mathrm{Cin}\,x = C + \ell n\,x - \mathrm{Ci}\,x$. When $kh_1 \gtrsim \pi/2$, $kh_2 \gtrsim \pi/2$,

$$\Psi = 2[\ell n(2/ka) - 0.5772]$$

2. Particular integral $T_1(z)$.

$$T_1(z) = jkA \int_0^z [\mathrm{Si}\,k(h_1 + s) + \mathrm{Si}\,k(h_2 - s)]\sin k(z - s)\,ds = -At_1(z)$$

$$\begin{aligned} t_1(z) = -j\{&\mathrm{Si}\,k(h_1 + z) - (1/2)\sin k(h_1 + z)[\mathrm{Cin}\,2k(h_1 + z) + 2\,\mathrm{Si}\,kh_1 \sin kh_1 \\ &- \mathrm{Cin}\,2kh_1] - (1/2)\cos k(h_1 + z)[\mathrm{Si}\,2k(h_1 + z) + 2\,\mathrm{Si}\,kh_1 \cos kh_1 \\ &- \mathrm{Si}\,2kh_1] + \mathrm{Si}\,k(h_2 - z) - (1/2)\sin k(h_2 - z)[\mathrm{Cin}\,2k(h_2 - z) \\ &+ 2\,\mathrm{Si}\,kh_2 \sin kh_2 - \mathrm{Cin}\,2kh_2] - (1/2)\cos k(h_2 - z)[\mathrm{Si}\,2k(h_2 - z) \\ &+ 2\,\mathrm{Si}\,kh_2 \cos kh_2 - \mathrm{Si}\,2kh_2]\} \end{aligned}$$

3. Particular integral $T_2(z)$.

$$T_2(z) = -A[C_3'G_s(Z,L_1) - C_4'G_s(Z,L_2) + C_3''G_c(Z,L_1) + C_4''G_c(Z,L_2)]$$

where

$$G_s(Z,L) = \int_0^L \sin X\,dX \int_0^Z \sin(Z - S)\,\frac{\partial}{\partial S}\,\frac{e^{-j\sqrt{S^2 + X^2 + A^2}}}{\sqrt{S^2 + X^2 + A^2}}\,dS$$

$$G_c(Z,L) = \int_0^L \cos X\,dX \int_0^Z \sin(Z - S)\,\frac{\partial}{\partial S}\,\frac{e^{-j\sqrt{S^2 + X^2 + A^2}}}{\sqrt{S^2 + X^2 + A^2}}\,dS$$

These integrals give

$$G_s(Z,L) = (j/2)\sin Z\,[\mathrm{Cin}\,2L + j\,\mathrm{Si}\,2L] + j\cos Z\,[\sin L\,\sinh^{-1}(Z/L) + \mathrm{Si}\,2Z]$$

$$+ j\sin Z\,[\mathrm{Cin}\,2Z + \mathrm{Cin}\,2L - \cos L\,\mathrm{Cin}\,L - \sin L\,\mathrm{Si}\,L] - j\,\mathrm{Si}\,Z$$

$$- (1/2)e^{jZ}\{\mathrm{Cin}(-L + Z + U) + j\,\mathrm{Si}(-L + Z + U) + \mathrm{Cin}(L + Z + U)$$

$$+ j\,\mathrm{Si}(L + Z + U) - [\mathrm{Cin}(Z + U) + j\,\mathrm{Si}(Z + U)]\cos L\}$$

$$+ (1/2)e^{-jZ}\{\mathrm{Cin}(-L - Z + U) + j\,\mathrm{Si}(-L - Z + U) + \mathrm{Cin}(L - Z + U)$$

$$+ j\,\mathrm{Si}(L - Z + U) - [\mathrm{Cin}(-Z + U) + j\,\mathrm{Si}(-Z + U)]\cos L\}$$

$$G_c(Z,L) = -\sin Z\,\sinh^{-1}(L/A) + (1/\ )\sin Z\,[\mathrm{Cin}\,2L + j\,\mathrm{Si}\,2L] - j\cos Z$$

$$\times (1 - \cos L)\sinh^{-1}(Z/L) + j(1 - \cos Z)\sinh^{-1}(L/Z) + j\sin Z$$

$$\times [\sin L\,\mathrm{Cin}\,L - \cos L\,\mathrm{Si}\,L + \mathrm{Si}\,2L] - (j/2)e^{jZ}\{\mathrm{Cin}(-L + Z + U)$$

$$+ j\,\mathrm{Si}(-L + Z + U) - \mathrm{Cin}(L + Z + U) - j\,\mathrm{Si}(L + Z + U)$$

$$- j[\mathrm{Cin}(Z + U) + j\,\mathrm{Si}(Z + U)]\sin L\} + (j/2)e^{-jZ}\{\mathrm{Cin}(-L - Z + U)$$

$$+ j\,\mathrm{Si}(-L - Z + U) - \mathrm{Cin}(L - Z + U) - j\,\mathrm{Si}(L - Z + U)$$

$$- j[\mathrm{Cin}(-Z + U) + j\,\mathrm{Si}(-Z + U)]\sin L\}$$

In these formulas $U = (Z^2 + L^2)^{1/2}$ and $Z = kz$, $L = k\ell$, $A = ka$; $\mathrm{Si}\,v = \int_0^v [(\sin v)/v]dv$, $\mathrm{Cin}\,v = \int_0^v [(1 - \cos v)/v]dv$. Note that $\mathrm{Si}(-v) = -\mathrm{Si}\,v$, $\mathrm{Cin}(-v) = \mathrm{Cin}\,v$. Also,

$$G_s(-Z,L) = -G_s(Z,L) \quad , \quad G_c(-Z,L) = -G_c(Z,L)$$

$$G_s(0,L) = 0 \quad , \quad G_c(0,L) = 0$$

4. Particular integral $T_3(z)$.

$$T_3(z) = -(j\omega/k)\{[q(h_2)]_0\vartheta(z,h_2) - [q(-h_1)]_0\vartheta(z,-h_1)\}$$

where

$$\vartheta(z,h_2) = \int_0^z \frac{e^{-jk\sqrt{(s - h_2)^2 + a^2}}}{\sqrt{(s - h_2)^2 + a^2}}\sin k(z - s)\,ds$$

$$\vartheta(z,-h_1) = \int_0^z \frac{e^{-jk\sqrt{(s+h_1)^2+a^2}}}{\sqrt{(s+h_1)^2+a^2}} \sin k(z-s)\, ds$$

These integrals can be expressed as follows:

$$\vartheta(z,h_2) = \{\sinh^{-1}[(h_2 - z)/a] - \sinh^{-1}(h_2/a)\}\sin k(h_2 - z) - (1/2)[\mathrm{Si}\ 2k(h_2 - z) - \mathrm{Si}\ 2kh_2 - j\ \mathrm{Cin}\ 2k(h_2 - z) + j\ \mathrm{Cin}\ 2kh_2]\exp[jk(h_2 - z)]$$

Similarly,

$$\vartheta(z,-h_1) = \{\sinh^{-1}[(h_1 + z)/a] - \sinh^{-1}(h_1/a)\}\sin k(h_1 + z) - (1/2)[\mathrm{Si}\ 2k(h_1 + z) - \mathrm{Si}\ 2kh_1 - j\ \mathrm{Cin}\ 2k(h_1 + z) + j\ \mathrm{Cin}\ 2kh_1]\exp[jk(h_1 + z)]$$

Note that

$$\vartheta(-z,h) = \vartheta(z,-h) \quad ; \quad \vartheta(-z,-h) = \vartheta(z,h) \quad ;$$

$$\vartheta(0,\pm h) = 0 \quad ; \quad \vartheta(-h,-h) = \vartheta(h,h) \quad ; \quad \vartheta(-h,h) = \vartheta(h,-h) \quad ;$$

$$\vartheta(h_2,h_2) = (1/2)[\mathrm{Si}\ 2kh_2 - j\ \mathrm{Cin}\ 2kh_2] \quad ;$$

$$\vartheta(-h_1,-h_1) = (1/2)[\mathrm{Si}\ 2kh_1 - j\ \mathrm{Cin}\ 2kh_1] \quad ;$$

$$\vartheta(-h_1,h_2) = \ln(1 + h_2/h_1)\sin k(h_1 + h_2) - (1/2)[\mathrm{Si}\ 2k(h_1 + h_2) - \mathrm{Si}\ 2kh_1 - j\ \mathrm{Cin}\ 2k(h_1 + h_2) + j\ \mathrm{Cin}\ 2kh_1]\exp[jk(h_1 + h_2)]$$

$$\vartheta(h_2,-h_1) = \ln(1 + h_1/h_2)\sin k(h_1 + h_2) - (1/2)[\mathrm{Si}\ 2k(h_1 + h_2) - \mathrm{Si}\ 2kh_2 - j\ \mathrm{Cin}\ 2k(h_1 + h_2) + j\ \mathrm{Cin}\ 2kh_2]\exp[jk(h_1 + h_2)]$$