Interaction Notes

Note 253

A Simple Way of Solving Transient Thin-Wire Problems

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Abstract

A simple way of calculating the transient response of a thin wire is presented. The method is based on (i) the asymptotic antenna theory for thin wires and (ii) an expansion of the induced current in terms of the so-called natural modes. An explicit series representation of the transient induced current in terms of the resonances of the wire is obtained and this representation is used to numerically evaluate the wire current using a desk calculator. A comparison is made between the results obtained using the asymptotic theory presented here and those obtained from a numerical solution of a space-time integral equation. Good agreement between the results is found except for early times. The method points to a fast way of estimating the transient induced currents on more complicated structures such as a stick model of an aircraft.

I. Introduction

So many papers have been written about scattering and radiation from thin wires since Pocklington [1] presented his in 1897 that one more paper on this topic is hard to justify. Almost all work done in recent years involving transient scattering and radiation from thin wires has been based on numerical calculations on a computer [2]-[5]. These numerical calculations tend to become very complicated and there are many cases where a simplified analysis yields sufficiently accurate results. A simple and physically appealing way of calculating the radiation field from a pulsed antenna is presented by Franceschetti and Papas [6]. The simple results so obtained are valid only in a time scale which is large compared to the transit time across the antenna, and thus fail to give the correct early-time response. However, the early-time response of a linear antenna has already been evaluated by Latham and Lee [7] in their study of the transient properties of an infinitely long cylindrical antenna.

In this paper, a technique based on the natural-mode method [8] is used to obtain approximate analytical expressions for the time history of the current induced on a thin wire when it is excited either by an incident step-function plane wave or by a slice generator whose output voltage is a step-function in time. The advantage of the method lies in the fact that all expressions are of simple analytical form and can be evaluated using a desk calculator. The method is currently being used to obtain the currents induced on a thin wire when a charged particle moves near the wire. It is also used to study the external resonances of crossed wires (a simplified model of an aircraft) and two parallel wires (staggered and nonstaggered).

II. Mathematical Formulation

The induced surface current density $\underline{j}$ on a conducting body satisfies the integro-differential equation

$$\mathcal{L}\cdot\underline{j} = - s\epsilon_o \underline{n}\times\underline{n}\times\underline{E}^{inc}, \quad \mathcal{L}\cdot\underline{j} \equiv \underline{n}\times\underline{n}\times(s^2/c^2 - \nabla\nabla\cdot)\int_S G\underline{j}dS' \tag{1}$$

where

$$G(\underline{r},\underline{r}') = (4\pi|\underline{r}-\underline{r}'|)^{-1} \exp(-s|\underline{r}-\underline{r}'|/c)$$

is the free space Green's function, $\underline{E}^{inc}$ is the incident electric field, $\underline{n}$ is the outward unit normal to the surface S of the scattering body and the time dependence $\exp(st)$ is understood. It is known that the singularities in the complex frequency (s) plane of the surface currents induced on a finite-size, perfectly conducting body can be identified either as the singularities of the incident field or as poles [8]. The locations of the poles are uniquely determined by the shape of the perfectly conducting scattering body. These properties of the induced current have been shown using the magnetic field integral equation and they imply that the inverse operator $\mathcal{L}^{-1}(s)$ is a meromorphic function of s and that the locations of its poles (the natural frequencies) are given by all those s_n for which the homogeneous equation

$$\mathcal{L}(s_n)\cdot\underline{j} = 0 \tag{2}$$

has a nontrivial solution $\underline{j}_n$.

The operator $\mathcal{L}(s)$ is a symmetric operator when operating on functions that are tangential to S, i.e.,

$$\mathcal{L}^T(s) = \mathcal{L}(s) \tag{3}$$

where the "transposed" operator $\mathcal{L}^T(s)$ is defined from the identity

$$<\mathcal{L}(s)\cdot\underline{f},\underline{g}> \equiv <\underline{f},\mathcal{L}^T(s)\cdot\underline{g}> \tag{4}$$

and the scalar product is defined by

$$<\underline{f},\underline{g}> \equiv \int_S \underline{f}(\underline{r})\cdot\underline{g}(\underline{r})dS.$$

To show that $\mathcal{L}$ is a symmetric operator one merely uses the definition (1) of $\mathcal{L}$ together with (4) and some simple vector-algebraic manipulations. The proof is given in Appendix A. Using the fact that $\mathcal{L}^{-1}$ is symmetric (an immediate consequence of the fact that $\mathcal{L}$ is symmetric) and assuming that $\mathcal{L}^{-1}$ has only simple poles (which has been substantiated in all cases investigated numerically), we obtain the following forced solution of (1) by using the Mittag-Leffler theorem, cf. [8]:

$$\underline{j} = -\varepsilon_o s\sum_n \left[(s-s_n)^{-1}<\mathcal{L}'(s_n)\underline{j}_n,\underline{j}_n>^{-1}<\underline{E}^{inc},\underline{j}_n>\underline{j}_n + \mathfrak{R}_n(s)\cdot\underline{E}^{inc}\right]$$
$$-\varepsilon_o s\mathcal{E}(s)\cdot\underline{E}^{inc} \tag{5}$$

where $\mathcal{L}' = d\mathcal{L}/ds$, $\mathfrak{R}_n(s)$ is an operator-valued polynomial of s and $\mathcal{E}(s)$ is an entire operator-valued function of s. The quantity $\underline{j}_n$ is the current distribution of a natural mode and it has the characteristics of a standing wave.

Some comments are in order here concerning (5). First, the expression (5) is valid provided there is no degeneracy, i.e., for each s_n there is only one linearly independent current distribution $\underline{j}_n$ that satisfies (2). The case of degeneracy can easily be incorporated into the series representation (5) of the induced current in the same way as degeneracy was incorporated into the corresponding expression in ref. [8]. It is however left out here since degeneracy does not seem to occur in the case of a thin wire. Second, the solution of (2) gives both the exterior and interior (cavity) resonances. On physical grounds it is also clear that s_n is purely imaginary for interior resonances and that $Re\{s_n\} < 0$ for exterior resonances. The fact that $Re\{s_n\} \neq 0$ for exterior resonances is an immediate consequence of the following two observations (cf. [9]): (i) the eigenvalues of the scattering operator have poles at the exterior resonances, and (ii) the scattering operator of a lossless object is a unitary operator for real frequencies (s purely imaginary). Also, the interior (exterior) resonances do not contribute to the sum (5) if the sources of the incident electromagnetic field are inside (outside) the scattering object. In fact, when the sources are outside the object it is shown in Appendix B that the scalar product $<\underline{E}^{inc}, \underline{j}_n>$ vanishes at the interior resonances showing that the residues of the interior modes vanish. This in turn implies that the sum (5) can be limited to include only the external resonances.

III. Resonances of a Thin Wire

The vector equation (1) can be reduced to a scalar equation for the total axial current $I(z)$ when the scattering object is a wire of length ℓ and radius a,

$$\left(\frac{d^2}{dz^2} - \frac{s^2}{c^2}\right) \int_0^{\ell} K(z-z')\, I(z')dz' = - s\varepsilon_o E_z^{inc}, \quad 0 \le z \le \ell \tag{6}$$

where

$$K(z-z') = \int_0^{2\pi} (8\pi^2 R)^{-1} \exp(-sR/c)d\phi$$

and

$$R^2 = 4a^2 \sin^2(\phi/2) + (z-z')^2.$$

Equation (6) can be solved asymptotically in terms of the "antenna parameter" $\Omega = 2\,\ell n(\ell/a)$ when $\ell >> a$. From this asymptotic solution the natural frequencies of a thin wire are found to be [10]

$$s_n = (c/\ell)\{in\pi - \Omega^{-1}[\ell n(2|n|\pi\Gamma) - Ci(2n\pi) + i\,Si(2n\pi)] + O(\Omega^{-2})\},\ n = \pm 1, \pm 2, \ldots \tag{7}$$

where $\Gamma = 1.781\ldots$ is the exponential of Euler's constant, and $Ci(x)$ and $Si(x)$ are the cosine and sine integrals, respectively. Also, to the first approximation, the current distributions of the natural modes are given by

$$I_n(z) = 2\pi a\, j_n(z) = \sin(n\pi z/\ell) + O(\Omega^{-1}), \quad 0 \le z \le \ell. \tag{8}$$

Expressions for the natural frequencies which are asymptotically correct up to and including order Ω^{-2} can be found in ref. [11] where expressions for $I_n(z)$ up to order Ω^{-1} also are presented.

To get some quantitative information about the accuracy of the asymptotic expansion (7) we have in Fig. 1 graphed three different representations of the fundamental natural frequency of a straight thin wire, namely (i) the asymptotic form (7) correct up to order Ω^{-1} (labeled 1^{st} order appr.), (ii) an asymptotic form correct up to order Ω^{-2} (labeled 2^{nd} order appr.), and (iii) the numerical results obtained in ref. [3]. We note that for $a/\ell = 0.01$ ($\Omega = 9.2$), the natural frequencies calculated from these different methods all differ about 20% from each other. We also note that the second order approximation gives too large a value of the damping constant $|Re\{s_n\}|$ whereas the first order approximation yields a somewhat too large value of $Im\{s_n\}$. Since the convergence of the asymptotic series is doubtful [11] and judging from the results presented in Fig. 1, it is questionable if the accuracy of the approximate solution would be improved by including the Ω^{-2}-order term when $a/\ell = 0.01$. For that reason and also for the reason that the calculation of the Ω^{-2}-order term is rather involved for more complicated structures we choose here to include only terms of order Ω^{-1} in s_n and terms of order $1(=\Omega^0)$ in all other quantities.

Some comments are in order here concerning the expansion parameter $\Omega = 2\,\ell n(\ell/a)$ that is used in this paper. This parameter is not the only expansion parameter that has been employed in the asymptotic solution of cylindrical antenna problems. The expansion parameters used in refs [12,13] depend on the frequency and they may lead to more accurate expansions of the induced current in cèrtain cases. Since this approach is intended to be a simple way of solving transient thin-wire problems (but not necessarily the most accurate one) we find Ω to be the most convenient expansion parameter. It should also be mentioned that expressions similar to (7) for the complex resonance frequencies of a thin wire have been calculated by Weinstein [14] who used the Wiener-Hopf technique to obtain approximate expressions for the induced current on a finite-length wire.

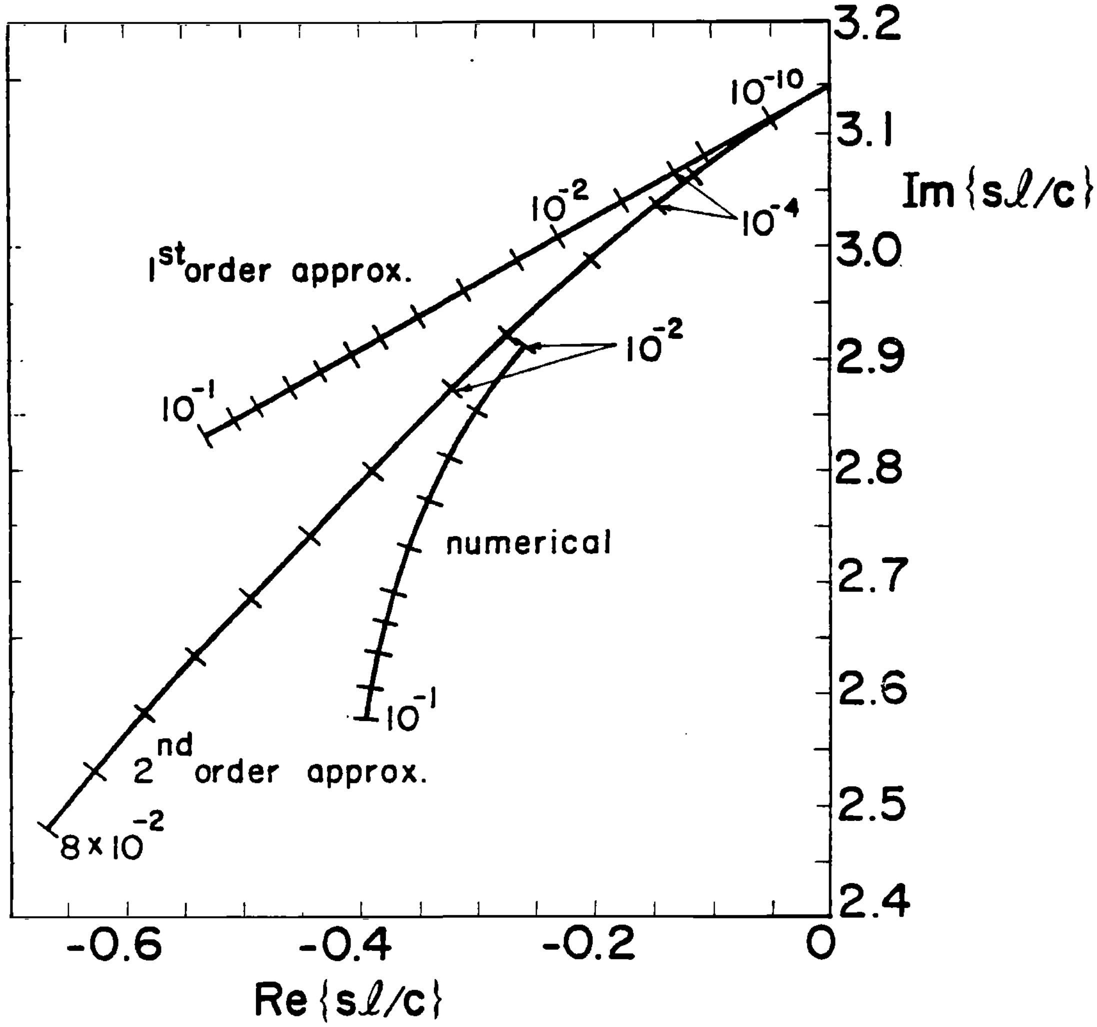

Fig. 1. The fundamental natural frequency s_1 of a thin wire. The natural frequencies for $a/\ell = 10^{-10}$, 10^{-5}, 10^{-4}, 10^{-3}, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09, 0.1 are indicated in the figure.

IV. Current Response of a Thin Wire

Using the approximate expressions (7) and (8) one can evaluate certain scalar products in (5) to get the following expression for the induced current on a thin wire:

$$I(z,s) = \frac{4\pi c}{Z_o \Omega \ell} \sum_n \frac{s}{s_n(s-s_n)} \left[\int_0^\ell \sin(n\pi z'/\ell) E_z^{inc}(z',s)dz'\right] \sin(n\pi z/\ell) + O(\Omega^{-2}) \tag{9}$$

where the quantity $E_z^{inc}(z,s)$ is the z-component of the incident electric field. In this expression we have deleted the polynomial operator $\mathfrak{R}_n$ and the entire operator $\mathcal{E}$ for two reasons: (i) the expression (9) directly leads to the quasi-static solution for low frequencies and (ii) in the limit of a very thin wire so that $s_n = in\pi c/\ell$, (9) is a solution of the differential equation

$$(d^2/dz^2 - s^2/c^2)I = -4\pi\Omega^{-1} s\varepsilon_o E_z^{inc}, \quad 0 \le z \le \ell \tag{10}$$

with the boundary conditions $I(0,s) = I(\ell,s) = 0$. Proofs of these two properties are given in Appendix C.

A. Gap Excitation

When a wire is center-driven by a slice generator whose output voltage is a step-function in time and has strength V_o, one obtains the following expression:

$$I(z,t) = \frac{8V_o}{Z_o \Omega} U(ct-|z-\ell/2|) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sin\frac{(2n+1)\pi z}{\ell} \sin(\omega_{2n+1}t)\exp(-\sigma_{2n+1}t) \tag{11}$$

where $-\sigma_n + i\omega_n = s_n$ and $U(t)$ is the Heaviside unit step function. In Fig. 2 we graph the time history of the current at $z = \ell/4$ and $z = \ell/2 + \varepsilon$, $a \ll \varepsilon \ll \ell$. A comparison is also made with the results obtained by numerically solving a space-time domain integral equation [2]. The approximate solution obtained here agrees with that in ref. [2] within 25% for $ct/\ell > 1/4$. It should be pointed out that the early-time response where the asymptotic theory yields poor accuracy can be obtained from the results for an infinitely long cylindrical

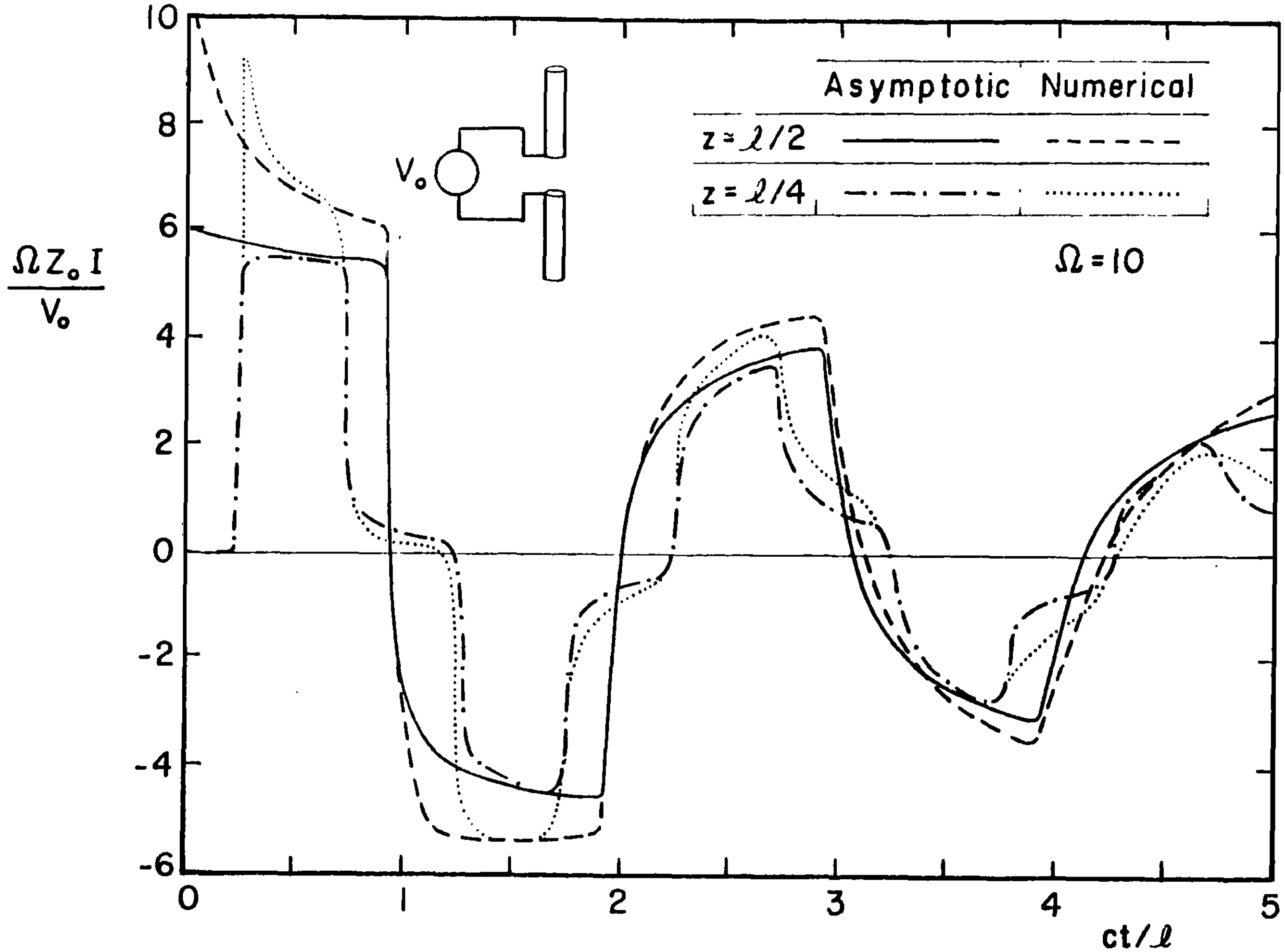

Fig. 2. Step-function response of the current at $z \approx \ell/2$ and $z = \ell/4$ for a wire center-driven by a slice generator with output voltage V_o. Also included for comparison are the corresponding results obtained by numerically solving a space-time domain integral equation.

antenna excited at a δ-gap [7].

B. Plane Wave Excitation

When the wire is excited by a step-function plane wave whose direction of propagation makes the angle θ with the positive z-axis and is so polarized that the electric field vector (strength E_o) makes the angle $\pi/2 - \theta$, $\theta < \pi/2$ with the positive z-axis, one gets the following asymptotic expression for the induced current:

$$I(z,t) = \frac{8E_o \ell}{\pi \Omega Z_o \sin\theta} U(ct - z\cos\theta)$$

$$\cdot \sum_{n=1}^{\infty} \left\{ \frac{1}{n^2} \sin\frac{n\pi z}{\ell} \left[\sin(\omega_n t) - (-1)^n \sin(\omega_n t - n\pi\cos\theta) \right] \exp(-\sigma_n t) \right\}. \quad (12)$$

We note that the time origin is so chosen that the wavefront hits the wire end point $z = 0$ at $t = 0$. The asymptotic expression (12) was used to numerically calculate on a desk calculator the time history of the induced current at different positions on the wire and at different angles of incidence of the plane wave. A comparison between these results and those obtained from a numerical solution of a space-time domain integral equation [2] is shown in Fig. 3 for two angles of incidence, $\theta = 30^o$ and 90^o. It is observed in Fig. 3 that the asymptotic theory results exhibit faster oscillations than those of the numerical solution. The oscillations are due mainly to the fundamental resonance mode. An inspection of Fig. 1 reveals that indeed, the fundamental natural frequency of the asymptotic theory has a larger imaginary part, implying faster oscillations than those obtained by numerically solving the integral equation.

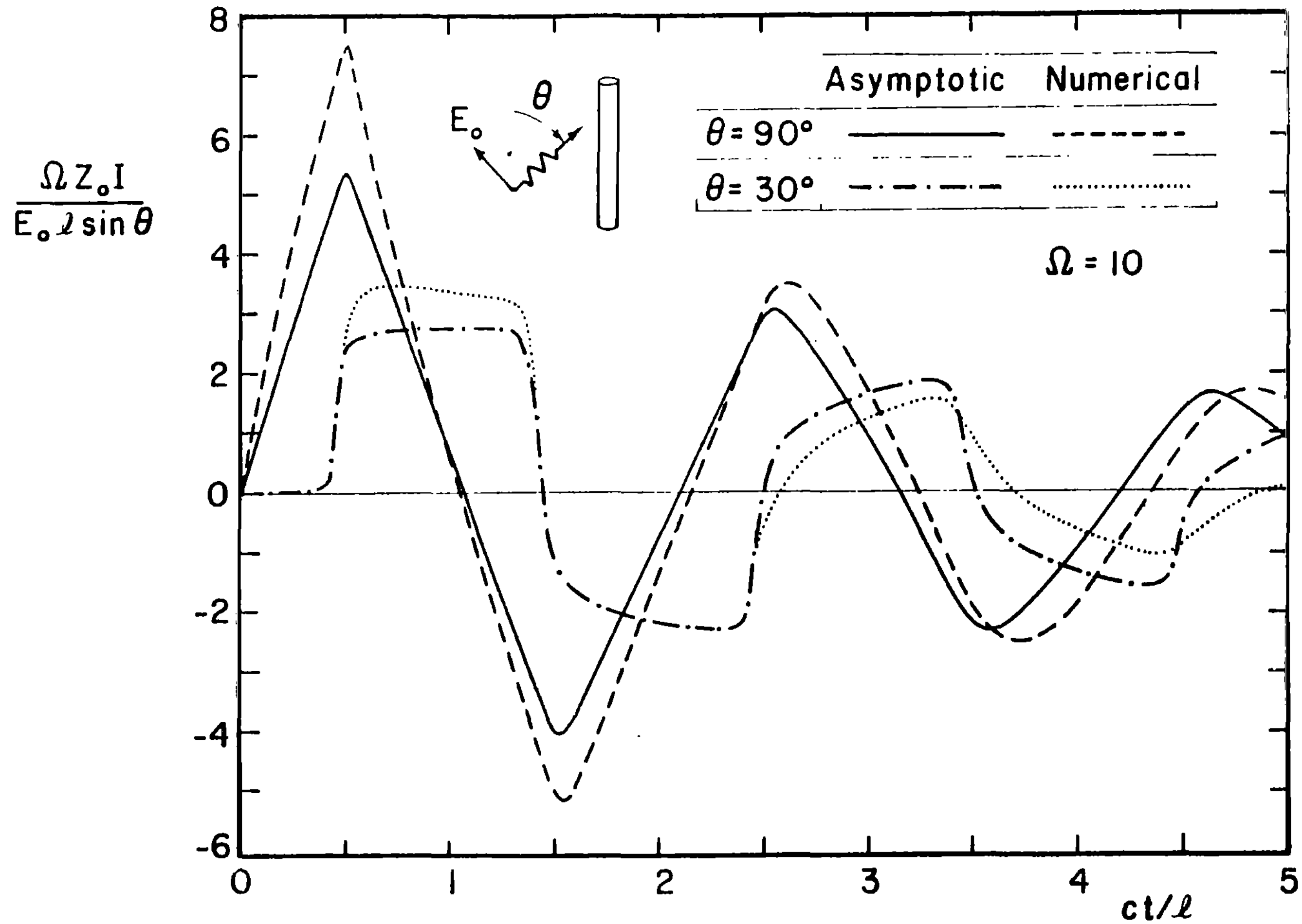

Fig. 3. Step-function response of the mid-point current for a wire illuminated by a plane wave with electric field strength E_o. The cases $\theta = 30^o$ and 90^o are shown. Also included for comparison are the corresponding results obtained by numerically solving a space-time domain integral equation.

Appendix A

In this appendix we show that the operator $\mathcal{L}$ defined by (1) in the text is a symmetric operator when operating on vector functions that are tangential to the closed surface S. Let $\underline{f}$ and $\underline{g}$ be two differentiable vector functions on the surface S such that $\underline{n}\cdot\underline{f} = \underline{n}\cdot\underline{g} = 0$ on S where $\underline{n}$ is the outward unit normal to S. We then have

$$\langle\mathcal{L}\cdot\underline{f},\underline{g}\rangle = \int_S \left\{\underline{n}(\underline{r})\times\left[\underline{n}(\underline{r})\times(s^2/c^2-\nabla\nabla\cdot)\int_S G(\underline{r},\underline{r}')\underline{f}(\underline{r}')dS'\right]\right\}\cdot\underline{g}(\underline{r})dS$$

$$= -\int_S\left[\nabla \,⨍_S \nabla' G(\underline{r},\underline{r}')\cdot\underline{f}(\underline{r}')dS' + s^2/c^2\int_S G(\underline{r},\underline{r}')\underline{f}(\underline{r}')dS'\right]\cdot\underline{g}(\underline{r})dS$$

$$= -\int_{S\times S}\left[G(\underline{r},\underline{r}')\nabla\cdot\underline{f}(\underline{r}')\nabla\cdot\underline{g}(\underline{r}') + s^2/c^2\, G(\underline{r},\underline{r}')\underline{f}(\underline{r}')\cdot\underline{g}(\underline{r})\right]dSdS'. \quad \text{(A1)}$$

where $⨍$ denotes the principal-value integral. In view of (A1) it is clear that

$$\langle\mathcal{L}\cdot\underline{f},\underline{g}\rangle = \langle\underline{f},\mathcal{L}\cdot\underline{g}\rangle \quad \text{(A2)}$$

for arbitrary differentiable functions $\underline{f}$ and $\underline{g}$ that are tangential to the closed surface S. Equation (A2) shows that $\mathcal{L}$ is a symmetric operator, i.e., $\mathcal{L} = \mathcal{L}^T$.

Appendix B

In this appendix we show that the scalar product $\langle \underline{E}^{inc}, \underline{j}_n \rangle$ vanishes at the interior resonance frequencies when all the sources of the incident field are located outside the scattering body.

Let $\underline{j}_n$ denote the surface current density on S of an interior mode, and $\underline{E}_n$, $\underline{H}_n$ the corresponding electromagnetic field of this cavity mode in V, the region inside S. The incident field evaluated at $s = s_n$ is denoted by $\underline{E}_n^{inc}$, $\underline{H}_n^{inc}$. Some vector algebraic manipulations combined with Maxwell's equations then give

$$\langle \underline{E}^{inc}, \underline{j}_n \rangle \Big|_{s_n} = \int_S \underline{E}_n^{inc} \cdot (\underline{n} \times \underline{H}_n) dS$$

$$= - \int_S \underline{H}_n^{inc} \cdot (\underline{n} \times \underline{E}_n) dS - \int_V \underline{i}_n^{inc} \cdot \underline{E}_n dV. \tag{B1}$$

where $\underline{i}^{inc}$ denotes the sources of the incident field. The surface integral of the last expression in (B1) vanishes since the tangential electric field of the modes is zero on S. When all the sources of the incident field are outside S we therefore have

$$\langle \underline{E}^{inc}, \underline{j}_n \rangle \Big|_{s_n} = 0. \tag{B2}$$

Equation (B2) shows that the solution of the integro-differential equation (1) can be viewed as a superposition of an "interior solution" and an "exterior solution". The interior (exterior) solution is due to sources inside (outside) the surface S. Another way of expressing the result (B2) is to say that all interior modes are orthogonal to any incident field whose sources are outside the scattering object. Finally, we also point out that the left-hand-side of (1) is the same for both the exterior and the interior problem. It is therefore to be expected that the inverse operator $\mathcal{L}^{-1}(s)$ is singular at both the interior and exterior resonances of the body. The result (B2) shows however that there always exists a solution of (1) even at the interior resonances when the sources of the field are outside the object. The solution of (1) is of course not unique at these resonances although on physical grounds we define (5) to be the forced solution of (1) even at the interior resonances.

Appendix C

In this appendix we give two reasons for deleting the polynomial operators $\mathfrak{N}_n$ and the entire operator $\mathcal{E}$ in (9) for the case of a thin wire.

To see that the expression (9) reduces to the quasi-static solution when $s \to 0$ one lets $s_n = in\pi c/\ell$, $n = \pm 1, \pm 2, \ldots$ in (9) and obtains

$$I(z,s) \simeq \frac{8\pi cs}{Z_o \Omega \ell} \int_0^{\ell} E_z^{inc}(z',s) \sum_{n>0} \frac{\sin(n\pi z'/\ell)\ \sin(n\pi z/\ell)}{s^2 + (n\pi c/\ell)^2}\, dz'. \tag{C1}$$

The series in (C1), after dropping the s^2 term, can be summed as [15]

$$\sum_{n>0} (n\pi c/\ell)^{-2} \sin(n\pi z'/\ell)\ \sin(n\pi z/\ell) = \begin{cases} (\ell - z)z'/2c^2, & z>z' \\ (\ell - z')z/2c^2, & z<z'. \end{cases} \tag{C2}$$

Thus, (C1) becomes

$$I(z,s) \simeq 4\pi\varepsilon_o s\Omega^{-1} \left[\int_0^z z'(1-z/\ell)E_z^{inc}(z',s)dz' + \int_z^{\ell} z(1-z'/\ell)E_z^{inc}(z',s)dz' \right] \tag{C3}$$

and in the time domain we have

$$I(z,t) \simeq 4\pi\varepsilon_o \Omega^{-1} \frac{\partial}{\partial t} \left[\int_0^z z'(1-z/\ell)E_z^{inc}(z',t)dz' + \int_z^{\ell} z(1-z'/\ell)E_z^{inc}(z',t)dz' \right]. \tag{C4}$$

To show that (C4) is the correct quasi-static expression for the induced current we note that the linear charge density $\tau(z,t)$ induced on a thin wire by a quasi-electrostatic field with potential $\Phi(x,y,z,t)$ is given by [16]

$$\tau(z,t) = 4\pi\varepsilon_o \Omega^{-1} [\Phi_o(t) - \Phi(0,0,z,t)], \quad 0 \le z \le \ell \tag{C5}$$

where $\Phi_o(t)$ is a function so that the wire has no net charge, i.e.,

$$\int_0^{\ell} \tau(z,t)dz = 0. \tag{C6}$$

Using the continuity equation we obtain the following expression for the induced current:

$$I(z,t) = -\frac{\partial}{\partial t}\int_0^z \tau(z',t)dz' \tag{C7}$$

which upon using the fact that $E_z^{inc}(z,t) = -(\partial/\partial z)\,\Phi(0,0,z,t)$ reduces to (C4) after integration by parts.

To see that (9) with $s_n = in\pi c/\ell$ satisfies the differential equation (10) we simply note that the Green's function

$$G(z,z') = \frac{2}{\ell}\sum_{n>0}\frac{\sin(n\pi z/\ell)\,\sin(n\pi z'/\ell)}{s^2 + (n\pi c/\ell)^2} \tag{C8}$$

satisfies the differential equation

$$\left(\frac{d^2}{dz^2} - \frac{s^2}{c^2}\right)G(z,z') = -\,\delta(z-z') \tag{C9}$$

and the boundary conditions $G(0,z') = G(\ell,z') = 0$. In view of (C8) and (C9) it is now clear that the expression (9) with $s_n = in\pi c/\ell$ satisfies (10). It has to be emphasized, however, that (10) together with the zero boundary conditions gives only undamped oscillations which are exactly the same as those given by (9) provided that radiation damping is neglected.

Finally, we mention that the polynomial operators $\mathfrak{R}_n$ and the entire operator $\mathcal{E}$ are not present in the series representation of the induced surface current density in the following cases: (i) all interior (cavity) scattering cases, (ii) scattering from a sphere and (iii) scattering from a thin wire (in the asymptotic sense). Without a mathematical proof, we expect this to be the case for any finite-size perfectly conducting scattering object.

Acknowledgment

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