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HYBRID EQUATIONS FOR THIN PERFECTLY-CONDUCTING SCATTERERS

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ABSTRACT

A new set of integral equations for electromagnetic scattering problems, the "hybrid" equations, are presented. The advantages of these equations for thin conductors are discussed in comparison to the magnetic and electric field integral equations. Specific comparisons are made with the solution of the electric field integral equation for a finite hollow cylinder. The hybrid equations are also obtained for flat plates with solutions available elsewhere in the literature.

INTRODUCTION

Since the advent of the digital computer and associated numerical techniques, integral equations have gained a significant place in the solution of electromagnetic scattering problems. The primary sets of equations used for solving perfect conductor scattering problems have been the magnetic and electric field integral equations (MFIE and EFIE). The difficulties associated with solving these equations for thin surface structures will be presented in addition to a new set of equations developed by the author [1] which alleviate the difficulties of the previous equations.

INTEGRAL EQUATIONS

The total fields due to an induced electric current $\bar{J}$ on a surface S may be written

$$\bar{H}(\bar{r}) = \bar{H}^{i}(\bar{r}) + \nabla \times \int_S \bar{J}(\bar{r}')\ \Phi(\bar{r}-\bar{r}')\ ds' \tag{1}$$

and

$$\bar{E}(\bar{r}) = \bar{E}^{i}(\bar{r}) + \frac{1}{j\omega\epsilon} \int_S \left[k^2\ \bar{J}(\bar{r}')\ \Phi(\bar{r}-\bar{r}') + (\bar{J}(\bar{r}')\cdot\nabla)\ \nabla\Phi(\bar{r}-\bar{r}')\right] ds' \tag{2}$$

where the superscript i indicates the incident field and Φ is the free space Green's function

$$\Phi(\bar{r}-\bar{r}') = \frac{e^{-jkR}}{4\pi R}\ ,\ R = |\bar{r}-\bar{r}'|. \tag{3}$$

If the surface is thin, then $\bar{J}$ represents the sum of the currents on both sides of the surface, $\bar{J} = \bar{J}_+ + \bar{J}_-$. Approaching the (+) side of the surface, (1) becomes

$$\frac{\bar{J}_+ - \bar{J}_-}{2} = \hat{n}_+ X\bar{H}^i - \hat{n}_+ X \fint_S [\bar{J} X \nabla \Phi] \, ds' \tag{4}$$

where $\bar{J}_+ = \hat{n}_+ X\bar{H}(\bar{r}_+)$ by equivalence and the integral is written as a Cauchy principle value with the residue $\bar{J}/2$. It should be clear that (4) cannot be used to solve for the required current $\bar{J}$ on a thin scatterer, but is simply an expression for the difference current in terms of the sum current.

The EFIE to be obtained from (2) does not suffer from the same difficulty as the MFIE in (4). To show this, we again approach the (+) side of the surface to obtain

$$-\hat{n}_+ X\bar{E}^i = \frac{\hat{n}_+ X}{j\omega\epsilon} \fint_S \left[k^2 \bar{J}\Phi + (\bar{J}\cdot\nabla) \nabla\Phi\right] ds' \tag{5}$$

where the integral is interpreted as a Hadamard principal value[1]. One may also exchange the differention and integration or integrate by parts to obtain more conventional integral definitions. Indeed, (5) is an integral equation for the current $\bar{J}$ over the entire scatterer except at corners and edges for which only the component equation of E parallel to an edge exists. To complete the description at edges, one can include the zero edge behavior of the current component perpendicular to the edge.

The difficulty in using the EFIE is directly related to the stability of the solution. The difficulty arises from the strong coupling between the two component equations of (5). Such coupling may cause one of the current

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1. For simplicity, the Hadamard principal value may be obtained by taking the finite or convergent part of the Cauchy principal value [2].

components to dominate both equations in certain regions of the scatterer and thus cause instability in the solution for the other current component in these regions. To be more explicit, let us consider a finite, perfectly conducting, hollow cylinder of radius "a" extending from z = -L/2 to L/2. Separating (5) into its Fourier components in terms of the spatial angle ϕ, we obtain the following equations for the n th harmonic with $z \in (-L/2, L/2)$:

$$-j\omega\epsilon E_{\phi n}^{i} = \int_{-L/2}^{L/2} \left[J_{\phi n} \left(k^2 G_{1n} - \frac{n^2}{a^2} G_{0n}\right) + \frac{jn}{a} J_{zn} \frac{\partial G_{0n}}{\partial z}\right] dz' \qquad (6a)$$

$$-j\omega\epsilon E_{zn}^{i} = \int_{-L/2}^{L/2} \left[J_{zn} \left(k^2 + \frac{\partial^2}{\partial z^2}\right) G_{0n} + \frac{jn}{a} J_{\phi n} \frac{\partial G_{0n}}{\partial z}\right] dz' \qquad (6b)$$

where

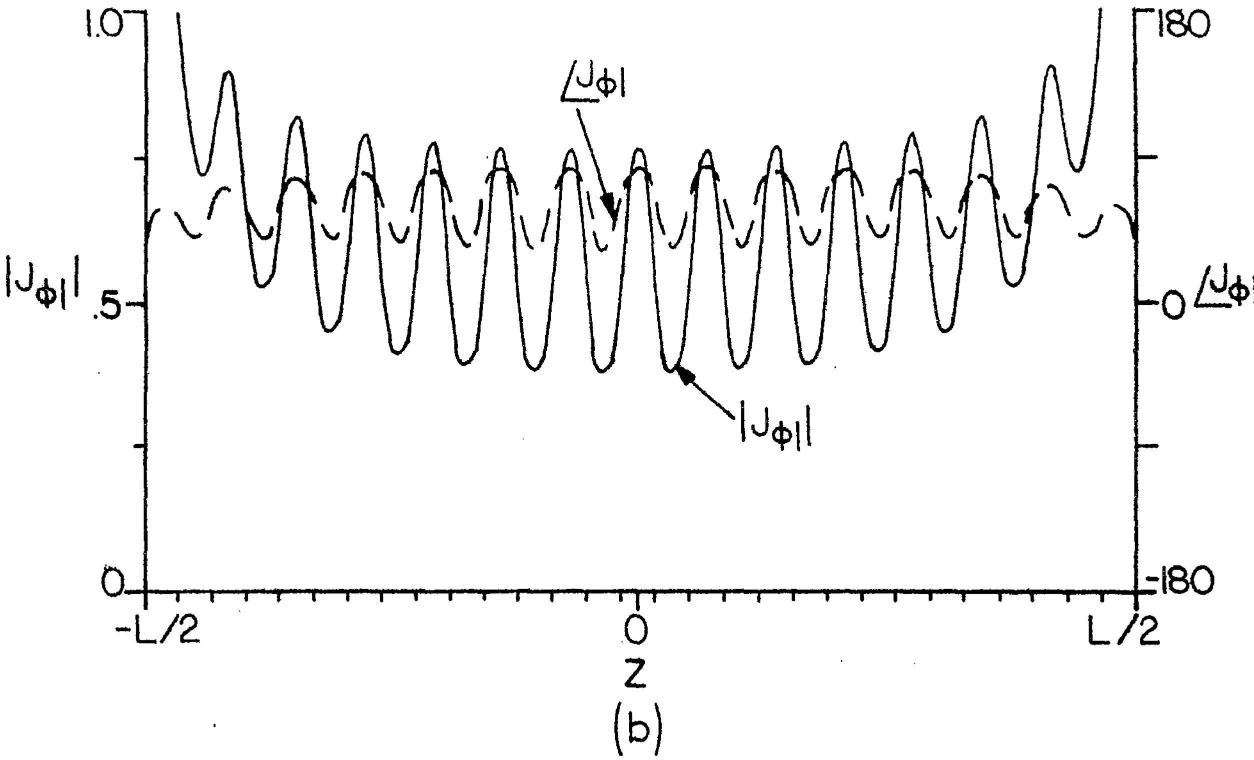
$$G_{mn}(z-z') = \int_0^{2\pi} \Phi \cos(m\alpha) \cos(n\alpha)\, a d\alpha, \qquad \alpha = \phi' - \phi$$

$$= -\frac{1}{2\pi} \ln |z-z'| + \text{Residue} \qquad (7)$$

and

$$\bar{E}^i = \sum_{n=-\infty}^{\infty} \bar{E}_n^{\,i} e^{jn\phi}$$

If $k = \pm n/a$, then the kernel on $J_{\phi n}$ in (6a) is bounded and J_{zn} becomes the dominate behavior in both equations, though $J_{\phi n}$ may still dominate (6b) in the regions about the ends due to edge behavior. The oscillatory behavior which characterizes the instability of $J_{\phi n}$ is clearly observed in Figure 1, while Figure 2 shows the instability in a more pronounced state. It is reasonable to assume that these problems may also occur for other structures, such as plates, with the appearance of either oscillatory behavior or incorrect edge behavior.



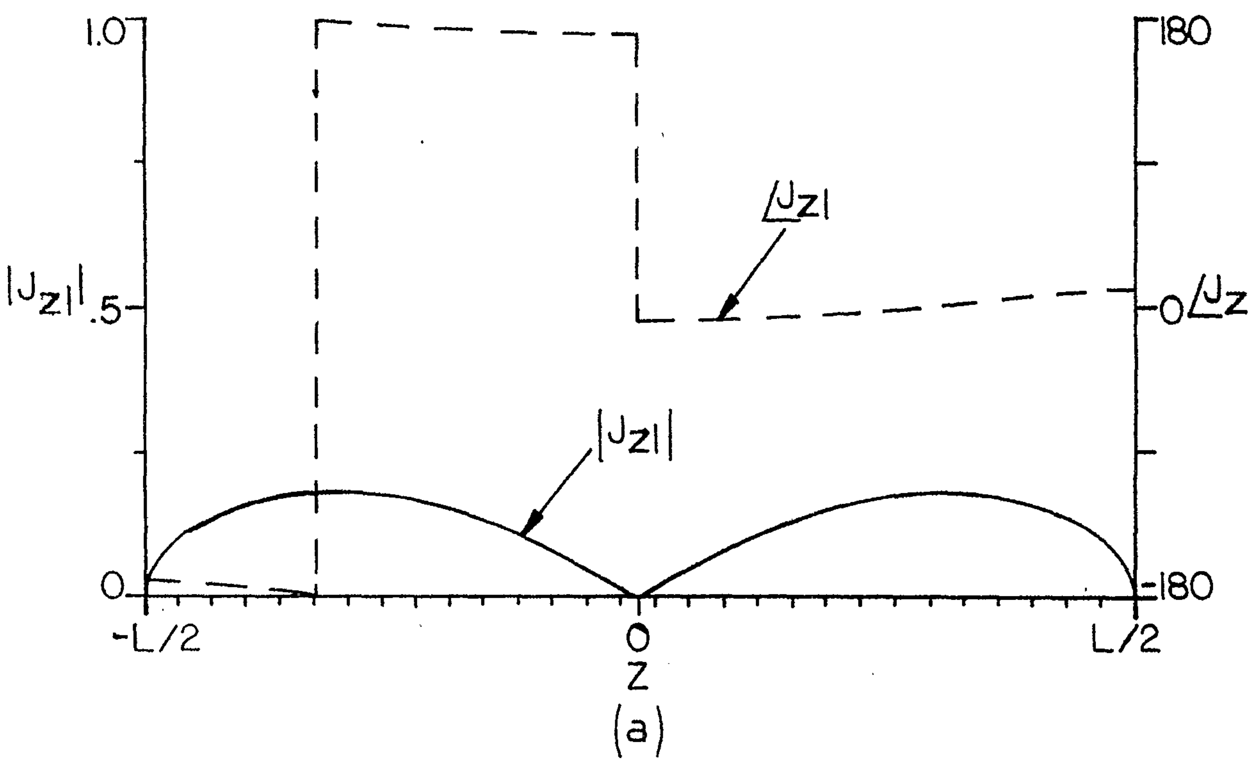


Figure 1. First-harmonic currents on a one-wavelength cylinder using the EFIE's for ka=1. The incident field is H-polarized and normally incident with $E_{\phi 1} = \eta/4$.

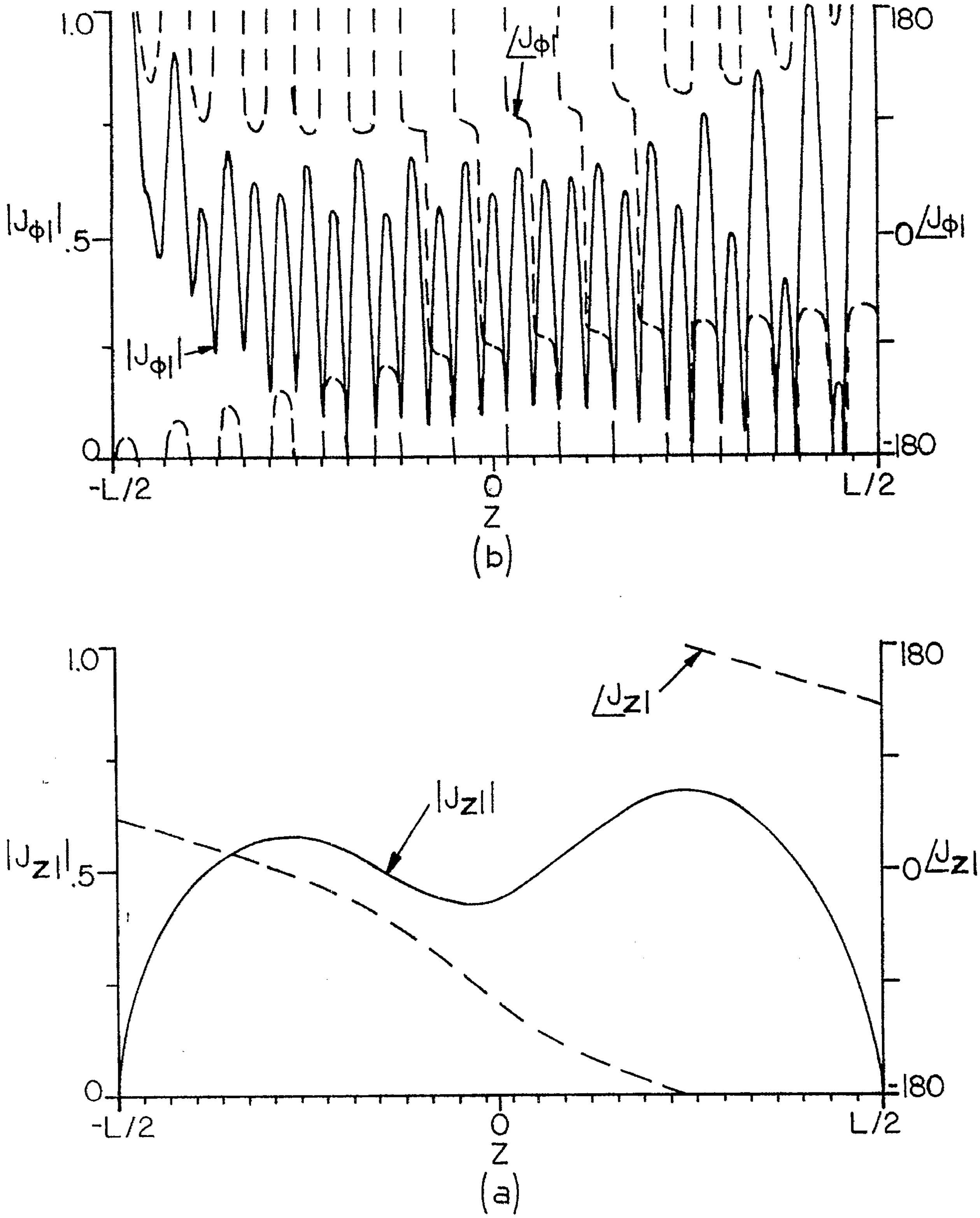


Figure 2. First-harmonic currents on a one-wavelength cylinder using the EFIE's for ka=1. The incident field is axial incident with $E_{\rho 1} = \eta/4$ at z=0.

We can minimize the coupling and resultant problems of the EFIE by combining the EFIE with the tangential components of the curl of the normal magnetic field. This combination results in the "hybrid" equations given by

$$\hat{n}_+ X \left[\nabla X \hat{n}_+ (\hat{n}_+ \cdot \bar{H}^i) - j\omega\epsilon\bar{E}^i\right]$$
$$= \hat{n}_+ X \oint_S \left[k^2 \bar{J}\Phi + \nabla(\nabla\cdot\bar{J}\Phi) + \nabla X \hat{n}_+(\hat{n}_+\cdot\bar{J}X\nabla\Phi)\right] ds'. \qquad (8)$$

From Maxwell's equations, the left-hand side of (8) may be written in terms of the tangential magnetic field as $\hat{n}_+X[-\nabla X\bar{H}_t{}^i]$. Hence, the hybrid equation is nothing other than the normal derivative of the MFIE in the sense of the curl. To point out the dominate behaviors in (8) we rewrite it as

$$-\hat{n}_+X[\nabla X\bar{H}_t{}^i] = \hat{n}_+X \oint_S \left[\bar{J}(k^2 + \nabla_t{}^2)\Phi - (\hat{n}_+\cdot\bar{J})(\nabla X(\hat{n}_+X\nabla\Phi))\right.$$
$$\left. - ((\hat{n}_+\cdot\nabla)\hat{n}_+)X(\bar{J}X\nabla\Phi) + (\hat{n}_+\cdot\nabla\Phi)(\bar{J}\cdot\nabla)\hat{n}_+\right] ds' \qquad (9)$$

Where $\nabla_t{}^2$ is the 2-dimensional Laplacian form given by Van Bladel.[2] We observe that the last three terms only require a Cauchy principal value, whereas the first term requires the Hadamard principal value. Moreover, the last three terms drop out for a flat surface and result in no coupling. Hence the first term is the dominant term of the equation.

To demonstrate the stabilization that may be obtained, consider (8) for the finite, hollow cylinder solved previously using the EFIE. The hybrid equations for this case are

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2. Van Bladel [4] defines $\nabla_t{}^2$ in curvilinear coordinates with $\hat{u}_3$ as the normal by

$$\nabla_t{}^2 f = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial v_1}\left(\frac{h_2}{h_1}\frac{\partial f}{\partial v_1}\right) + \frac{\partial}{\partial v_2}\left(\frac{h_1}{h_2}\frac{\partial f}{\partial v_2}\right)\right].$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H^i_{\phi n}) = \int_{-L/2}^{L/2} \Big[J_{zn} \Big(k^2 - \frac{n^2}{a^2} - \frac{d^2}{dz^2}\Big) G_{0n}$$

$$+ \frac{jn}{a} J_{\phi n} \frac{d}{dz} (G_{0n} - G_{1n}) \Big] dz' \qquad (10a)$$

$$\frac{\partial}{\partial\rho} H^i_{zn} = \int_{-L/2}^{L/2} \Big\{ J_{\phi n} \Big[\Big(k^2 + \frac{d^2}{dz^2}\Big) G_{1n} - \frac{n^2}{a^2} G_{0n}\Big]\Big\} dz', \qquad (10b)$$

where G_{mn} is defined in (7). The decoupling is apparent with J_{zn} eliminated from (10b) and $J_{\phi n}$ occurring in (10a) with a bounded kernel. Hence J_{zn} and $J_{\phi n}$ dominate (10a) and (10b) respectively. The implication of this decoupling is to stabilize the current solutions as shown in Figures 3 and 4 for the same cases previously shown in Figures 1 and 2 respectively.

For the EFIE,the behavior of the perpendicular component of current at the edge is specified to complete the problem since one of the EFIEs is not defined at the edge. This behavior must also be specified for the hybrid equations in addition to a second constraint, since neither of the hybrid equations are defined at the edge. This latter requirement is consistent with enforcement of the boundary conditions on tangential $\bar{E}$ and normal $\bar{H}$. In fact, without the constraint the normal magnetic field may be a homogeneous solution to

$$[k^2 + \nabla_t^2 + \hat{n}_+\cdot(\nabla X \hat{n}_+) X \nabla - \hat{n}_+\cdot(\nabla X \nabla X \hat{n}_+)] H_{n+} = 0 \qquad (11)$$

The constraint needed is to force the solution to (11) to be zero, thus implying zero tangential electric fields. For the hollow cylinder, (11) becomes

$$\Big(k^2 - \frac{n^2}{a^2} + \frac{\partial^2}{\partial z^2}\Big) H_{\rho n} = 0 \qquad (12)$$

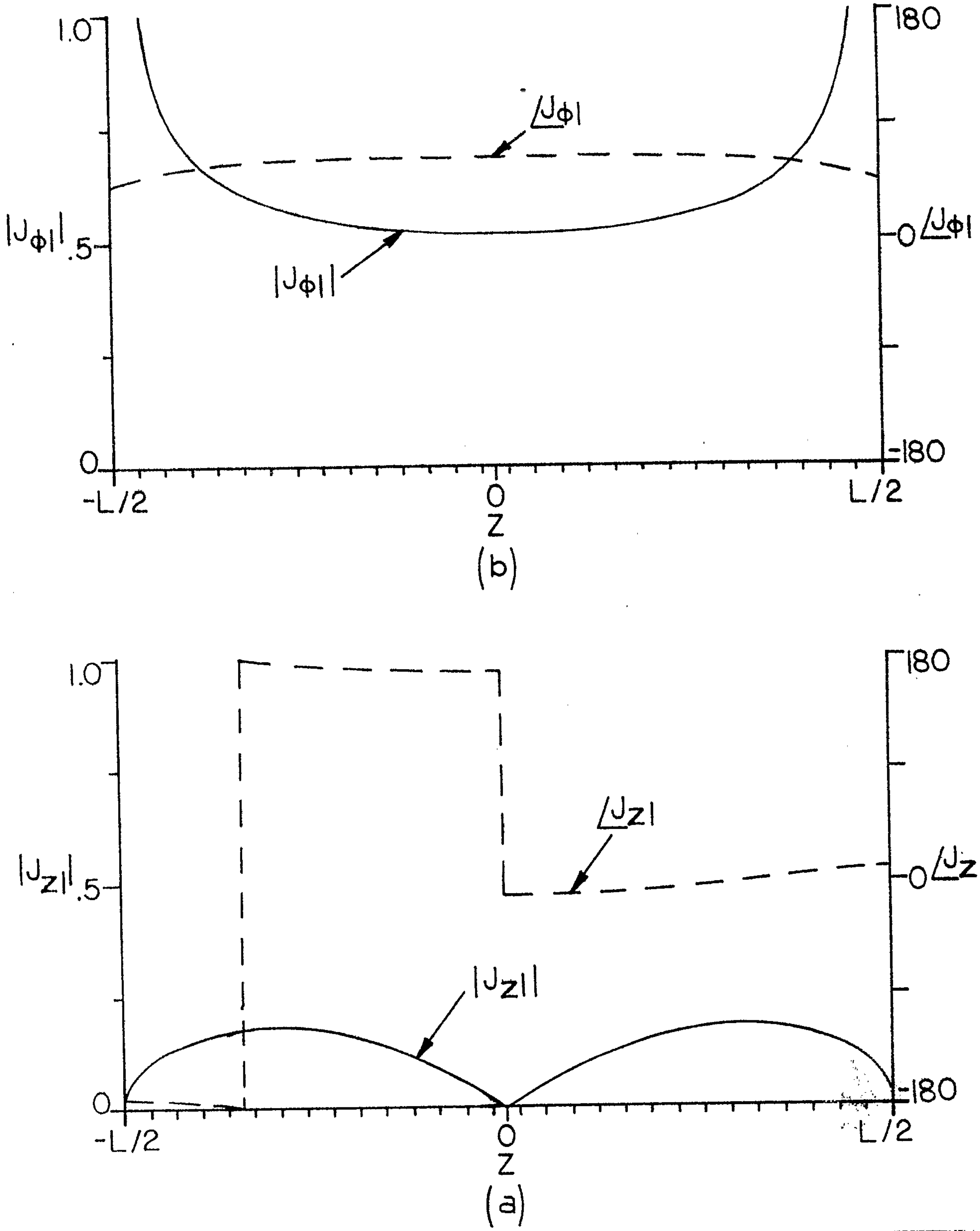

Figure 3. First-harmonic currents on a one-wavelength cylinder using the hybrid equations for ka=1. The incident field is H-polarized and normally incident with $E_{\phi 1} = \eta/4$.

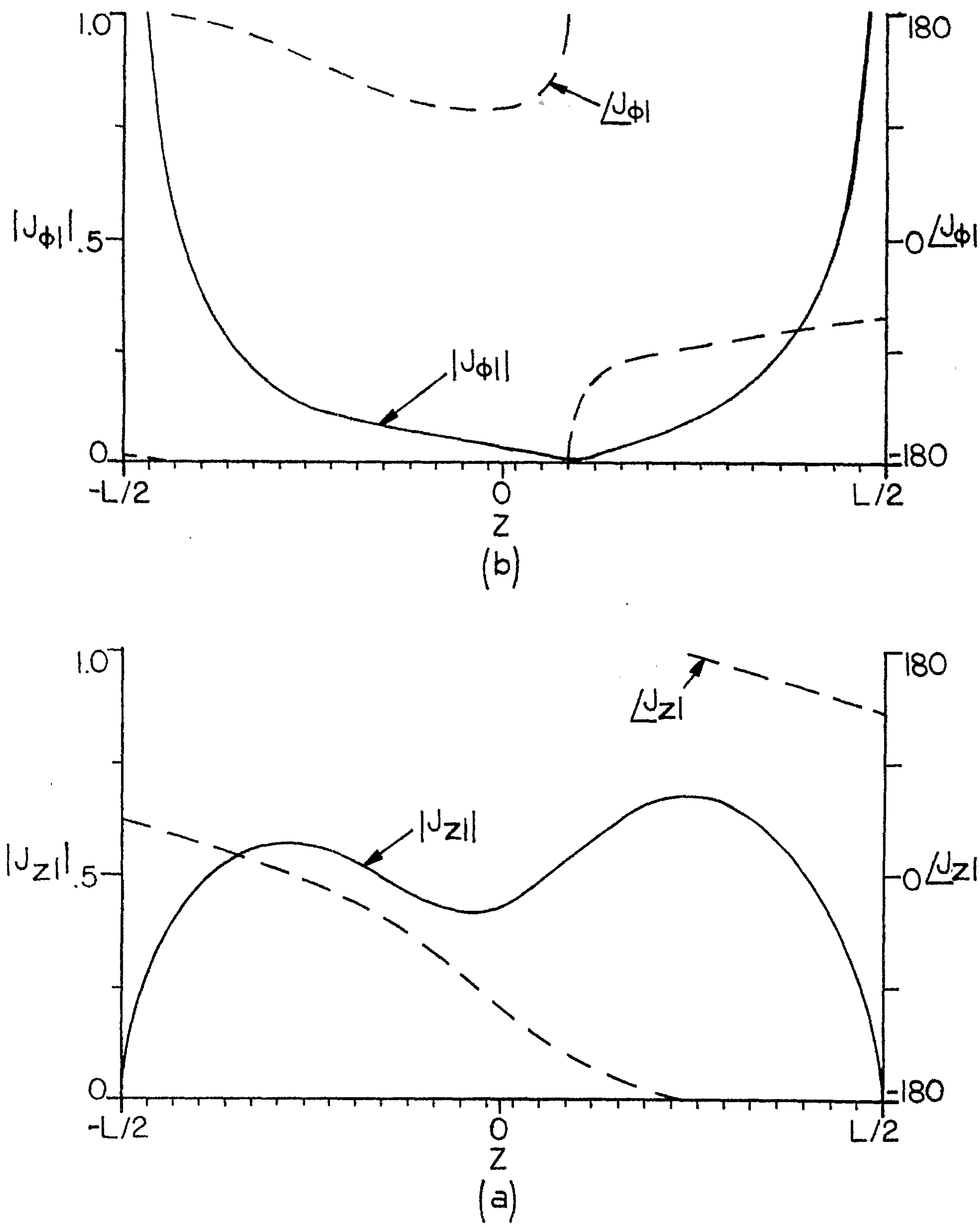

Figure 4. First-harmonic currents on a one-wavelength cylinder using the hybrid equations for ka=1. The incident field is axial incident with $E_{\rho 1} = \eta/4$ at z=0.

with a solution of the form

$$H_{\rho n} = A \cos \sqrt{k^2 - \frac{n^2}{a^2}}\, z + B \sin \sqrt{k^2 - \frac{n^2}{a^2}}\, z. \tag{13}$$

Hence one only needs to set $H_{\rho n} = 0$ at two values of z not separated by an integral multiple of $\pi/\sqrt{k^2 - n^2/a^2}$. To stress the need for this constraint, consider E - polarized normal incidence with n = ka, ka not a Bessel function zero. In this case both the forcing functions of (10) become zero. Without setting $H_{\rho n} = 0$, one might incorrectly assume that the current solution is zero. Such an assumption is avoided by application of the constraints.

In addition to the hollow cylinder problem, the hybrid equations have been used for the thin-plate scattering problem. In the latter, only the first term of (9) remains and the equations may be written as

$$-\frac{\partial H_x}{\partial z} = (k^2 + \nabla_t^2) \int_{x,y} (J_y \Phi)\, ds' \tag{14a}$$

and

$$\frac{\partial H_y}{\partial z} = (k^2 + \nabla_t^2) \int_{x,y} (J_x \Phi)\, ds'. \tag{14b}$$

These equations were solved using a Hallen's type of approach with the 2-dimensional Helmholtz operator, using the constraints on $\bar{J}$ and H_z to relate the resulting coefficients of integration. Further information on the thin plate problem is available in a previous article by Rahmat-Samii and Mittra [3].

CONCLUSION

It has been shown that the hybrid equations offer a significant advantage over other equations in decoupling the unknowns for integral equation solution. The only constraints on the problem are applied to the normal magnetic field and the current perpendicular to the edge. The hybrid equation approach is potentially capable of stablizing the solutions for many of the thin surface problems currently being considered in the industry. The hollow cylinder and thin plate scattering problems have been treated here to explicitly demonstrate the advantages of decoupling.

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