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A TECHNIQUE FOR EXTRACTING THE POLES AND RESIDUES OF A SYSTEM DIRECTLY FROM ITS TRANSIENT RESPONSE

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INTRODUCTION

The advent of the singularity expansion method (SEM) [1] for describing the transient characteristics of antennas and scatterers and the recent development of methods for the direct production of both numerical [2] and experimental [3] transient electromagnetic response data have generated considerable interest in the possibility of direct extraction of the poles and residues from given time-domain system response. The conventional approach for determining the singularities of a system is based on an iterative search procedure that seeks the zeros of the system determinant in the complex frequency plane. The extraction of the poles from the given time-domain response is not possible using this method. This paper presents a novel approach for systematically deriving the complex poles and residues from a set of time-domain data.* Numerical studies have demonstrated that the method, which is based on Prony's algorithm [4], [5] , is both efficient and accurate and that the results compare well with those derived independently using the conventional method. Finally, it will be shown that there are several numerical advantages in applying the method for computing spectral characteristics from given time domain data.

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*This technique was originally presented by R. Mittra and M. Van Blaricum, "A Novel Technique for Extracting SEM Poles and Residues of a System Directly from its Transient Response," 1974 Annual USNC-URSI Meeting, University of Colorado, Boulder, Colorado, p. 160, October 14-17, 1974.

THE NUMERICAL METHOD

In circuit theory, the impulse response of a linear circuit may be determined from the knowledge of the location of the poles of the response function in the complex frequency plane and their corresponding residues. The impulse response of the circuit is then simply a summation of all the residues multiplied by exponentially damped sinusoids. It has been shown in recent works [1], [6], [7] that for electromagnetic antennas and scatterers of finite size the impulse response of the induced currents and the scattered fields for time $t > 0^+$ may also be described by the sum of exponentially damped sinusoids.* Since the focus of this paper is to extract the poles and residues of an electromagnetic system directly from its impulse response data, an obvious place to start would be to write the given impulse response, I(t), as the summation of exponentially damped sinusoids, i.e.,

$$I(t) = \sum_{m=1}^{N} A_m \exp(s_m t) \tag{1}$$

where the s_m are the poles or singularities in the complex frequency plane and the A_m are their corresponding residues. It should be noted here that the poles, s_m, must be in complex conjugate pairs in order to ensure that I(t) is real. It will be assumed, without loss of generality, that I(t) is causal. Since, in practice, one almost always deals with a discrete set of sampled transient data, Equation (1) can be rewritten as

$$I(t_n) = I_n = \sum_{m=1}^{N} A_m \exp(s_m n\Delta t) \qquad n = 0,1,\ldots,2N-1 \tag{2}$$

where Δt is the size of the time-stepping interval used in obtaining the sampled data and $t_n = n\Delta t$. This set of Equations (2) is seen to be 2N nonlinear

*At time t = 0, a delta function exists due to the impulse source.

equations in 2N unknowns. Prony's [4], [5] method of analysis may be applied to this set of equations to obtain an exact solution, or a least-square fit if more than 2N data samples are used.

The problem of interpolation of a function using sums of exponentials with unknown exponents was solved by Prony in 1795 [4], for the case of equally spaced data samples. The method is based on the fact that the I_n in equation (2) must satisfy a difference equation of order N which may be written as

$$\sum_{p=0}^{N} \alpha_p I_{p+k} = 0 \qquad p + k = n = 0,1, \ldots, 2N - 1 \tag{3}$$

where the roots of the algebraic equations

$$\sum_{p=0}^{N} \alpha_p Z^p = 0 \tag{4}$$

are $\exp(s_m \Delta t) = Z_m$, m=1, 2,...N. If in equation (3) α_N is defined equal to 1, then the α_p's may be obtained by solving the equation

$$\sum_{p=0}^{N-1} \alpha_p I_{p+k} = -I_{N+k} \tag{5}$$

where the I_{p+k} and I_{N+k} are simply the known sampled transient data values. If 2N data samples are used, then (5) can be solved exactly for the α_p's. If more than 2N samples are desired, then one can obtain a least-squares type fit to equation (5). Once the α_p have been found, then the roots, $Z_m = \exp(s_m \Delta t)$, of (4) can simply be found and the poles are obtained by

$$s_m = \frac{\ln Z_m}{\Delta t} \tag{6}$$

It is now a simple procedure to obtain the residues, A_m, by solving the matrix equation embodied in (2) since the elements of the matrix which involve the s_m's are now known. The matrix contained in (2) which must be inverted is in the form of a transposed Vandermonde matrix whose inverse can be computed in closed form. Thus, Prony's algorithm simply involves the solution of two matrix equations and a solution of the zeros of an N^{th} degree polynominal, N being the number of desired poles. The method requires that in order to find N poles and residues it is necessary to have at least 2N equally spaced transient data samples. The realness of the transient response, I(t), requires that the N poles and residues come in complex conjugate pairs.

Once the poles, s_m, and the residues, A_m, have been determined, it is then possible to express the impulse response of the system for $t > 0^+$ using Equation (1). If, in addition, the frequency domain transfer function is desired, then using the poles and residues it can be simply expressed as

$$H(j\omega) = \sum_{m=1}^{N} \frac{A_m}{-\sigma_m + j(\omega-\omega_m)} + C \tag{9}$$

where the poles s_m have been written in terms of their real and imaginary part as

$$s_m = \sigma_m + j\omega_m \quad . \tag{10}$$

and where the constant C gives rise to a delta function at $t = 0$. Thus, the frequency domain transfer function can be obtained directly from the time-domain impulse response without having to perform a Fourier transform.

At the beginning of the paper, the statement was made that the impulse response of a distributed system for $t > 0^+$ could be written as a sum of exponentials. Since one does not usually have as an exciting function an impulse function, it is of interest to determine the form of the transient response due to an arbitrary exciting waveform.

A general response function $R(s)$ is given in the Laplace transform domain as

$$R(s) = F(s)\,H(s) \tag{11}$$

where $H(s)$ and $F(s)$ are the Laplace transforms of the system's impulse response for $t > 0^+$ and the arbitrary driving function, respectively. By expressing the transfer function $H(s)$ for the system under consideration as an infinite set of pole singularities, then according to the theory of complex variables, $H(s)$ may be written in a residue series as

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s-s_i} \quad , \; \mathrm{Re}\; s_i<0 \tag{12}$$

where s_i is the i^{th} complex pole of $H(s)$ and A_i is its corresponding residue. Thus, the response $R(s)$ can be written as

$$R(s) = F(s)\sum_{i=1}^{N} \frac{A_i}{s-s_i} \quad . \tag{13}$$

If the inverse Laplace transform of (13) is taken, the transient response function $r(t)$ is written as

$$r(t) = \sum_{i=1}^{N} B_i \exp(s_i t) + g(t) \tag{14}$$

where the B_i contains the A_i multiplied by some influence of the driving function and the added term $g(t)$ is dependent on the driving function.

Equation (14) shows that the transient response due to an arbitrary exciting waveform can, in general, be written as a sum of complex exponentials plus some added term $g(t)$. If the exciting waveform itself has pole singularities, as in the case of a step function or a sinusoidal function, then the $g(t)$ term also may be expressed as a sum of exponentials. If the exciting waveform is of finite duration, that is it is turned off after some time t_0, then it may be shown that the term g(t) is identically zero for $t>t_0$. When the driving function is not finite in time and has no pole singularities, as in the case of the Gaussian pulse, then the g(t) term cannot be written as a sum of exponentials and Prony's method cannot be applied directly to the transient response. However, this difficulty may be circumvented by simply deconvolving the response function r(t) in a standard manner. Equation (11) gives for instance

$$H(s) = \frac{R(s)}{F(s)} .$$

Thus, H(s) may be computed by taking the ratio of the known frequency spectra of r(t) and f(t). The inevitable presence of experimental and computational noise limits the upper frequency for which the deconvolved spectrum H(s) is accurate and, in practice, the computed spectrum must be truncated beyond this frequency. Care also must be taken to exclude t = 0 from the deconvolution because of the presence of the delta function discussed previously.

THE NUMERICAL RESULTS

Results will be presented here for numerical studies in which the method was used both on transient data obtained from a numerical time-domain computer code[8] and from a transient electromagnetic measurement facility[9].

In the case using data from the time domain computer code, a 1.0 meter dipole antenna with a half length-to-radius ratio of 100 was modeled using 60 equal length segments and the exciting field was a Gaussian pulse which was applied across the center two segments of the antenna model. The Gaussian-pulse time variation was exp $-a^2(t-t_{max})^2$ with a, the Gaussian spread parameter, equal to $5 \times 10^9 \sec^{-1}$. It was assumed that since the Gaussian pulse was very narrow, it approximated an impulse and thus the transient response function was not deconvolved to obtain the true impulse response. A time step Δt of 5.556 $x\ 10^{-11}$ sec was used, and t_{max} was $10\Delta t$. The induced current at the center of each segment was calculated for 500 time steps. In order to calculate the poles, the current on the center or source segment was used. Of the 500 current values calculated, only 80 sampled values were actually used and these were taken from the first 160 current samples at every second time step. Figure 1a shows the source current for the first 160 time steps as originally generated and the 80 sampled values are indicated by dots. These 80 current samples were used with Prony's algorithm to produce 40 poles and residues. These 40 poles and residues were then used in Equation (2) to reproduce the transient response of the source current at 250 time steps where the time step used to reproduce the response was twice the original time step. Figure 1b shows a plot of this reproduced transient response. It is interesting to note here that although only the information in the first 160 original time steps was used the method reproduced the current for 500 time steps to within 2×10^{-4} milliamps of the original response. Figure 2 is a plot of the pole locations in the second quadrant of the complex frequency plane. Only the second quadrant is shown because as stated previously the poles must come in complex conjugate pairs or lie on the negative real axis. Of the poles generated, only the ten plotted as x's in Figure 2 correspond to the true poles of this system. These ten correspond to the first ten even poles of this dipole as calculated by Tesche [6]. The fact that only the first ten even poles were generated is not surprising if one looks at the original model which was used. The original model was a thin-wire approximation and was driven with even symmetry by a Gaussian pulse. Because of the thin wire model and the width of the Gaussian pulse, the expected spectral response has an upper frequency limit of about $\lambda = 10L$ [10], where L

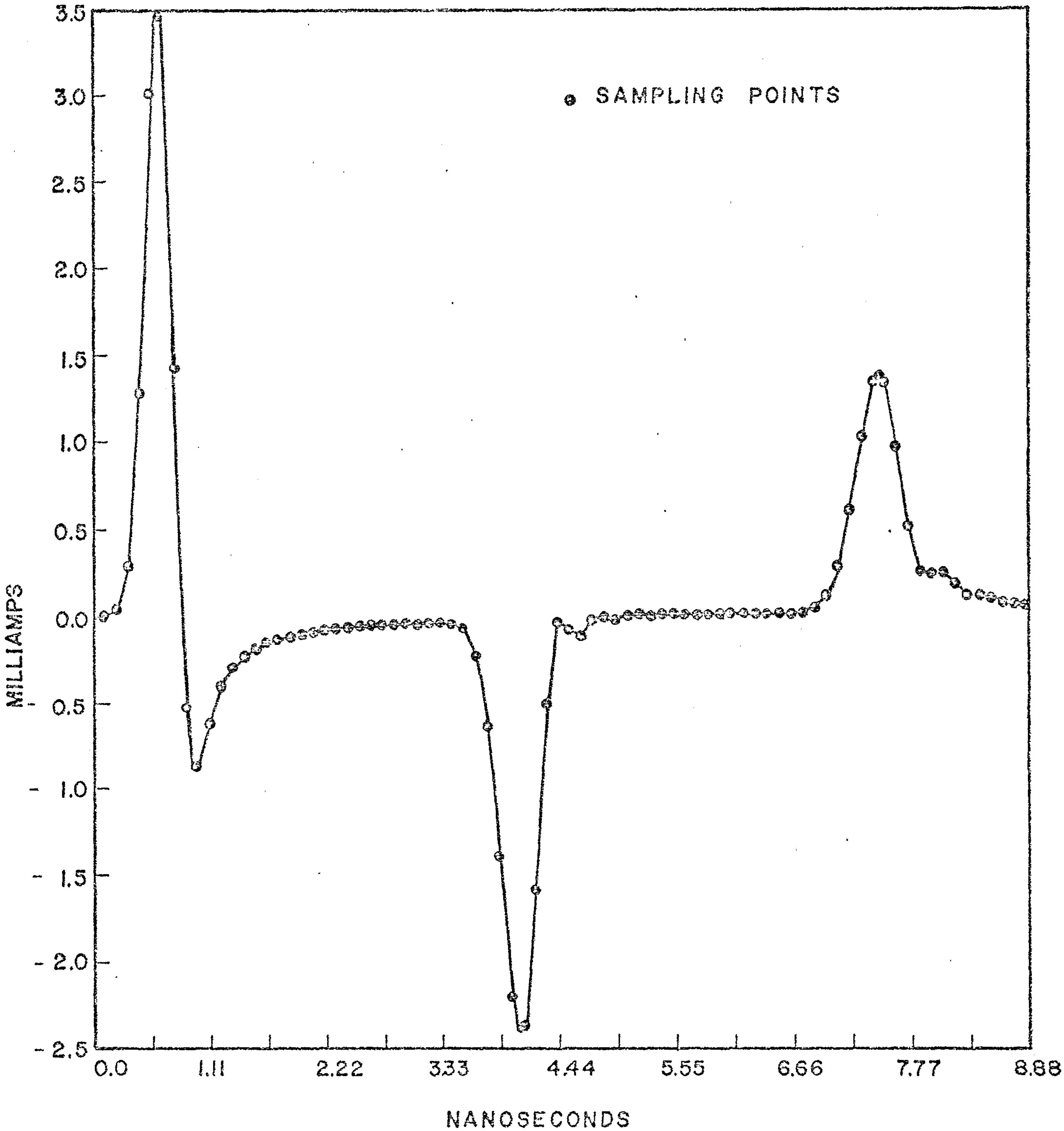

Fig. 1 (a)

Original Transient Data Showing the 80 Sampling Points

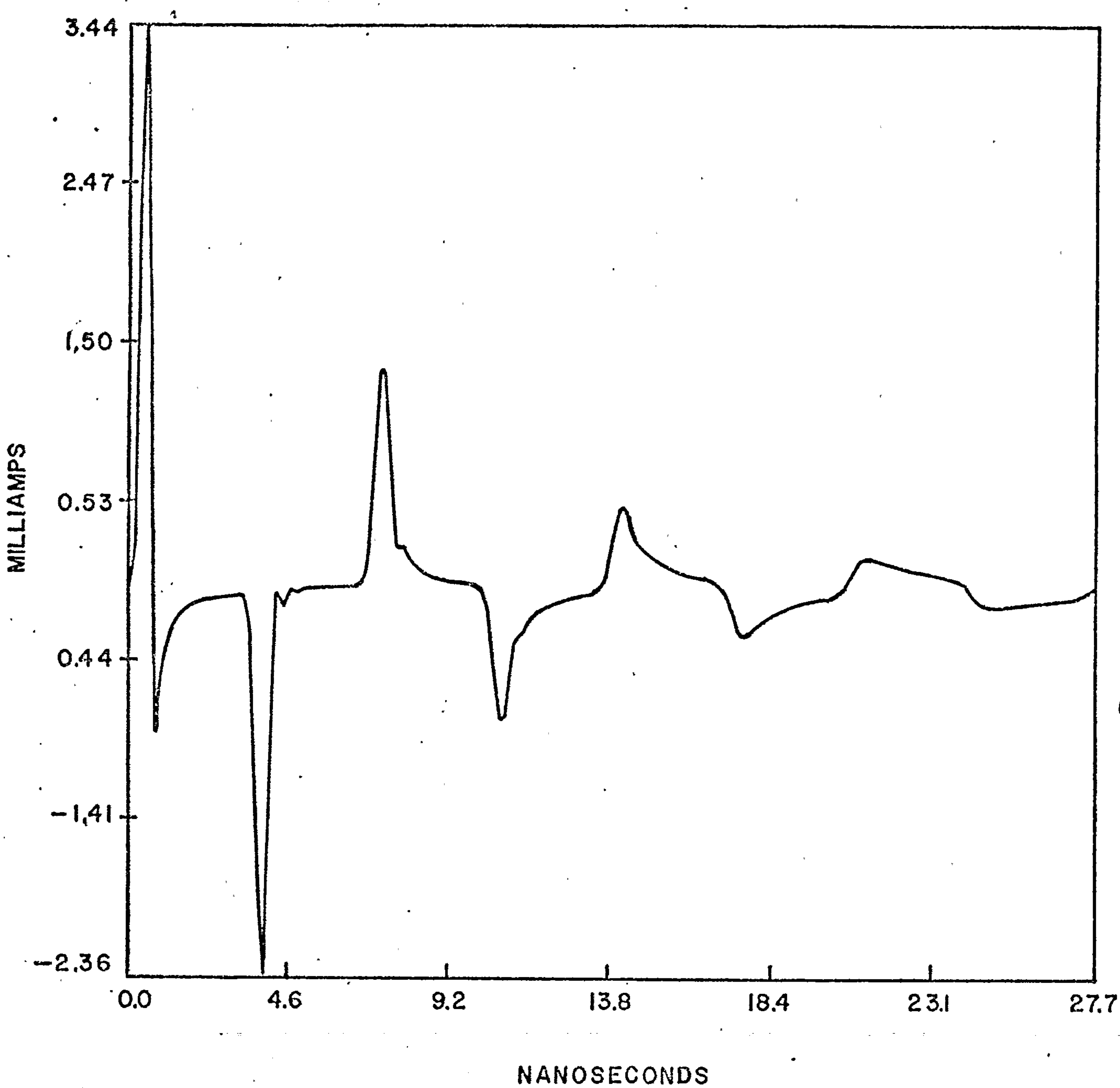

Fig. 1 (b)

The Reproduced Transient Response Function for 1.0 Meter Dipole

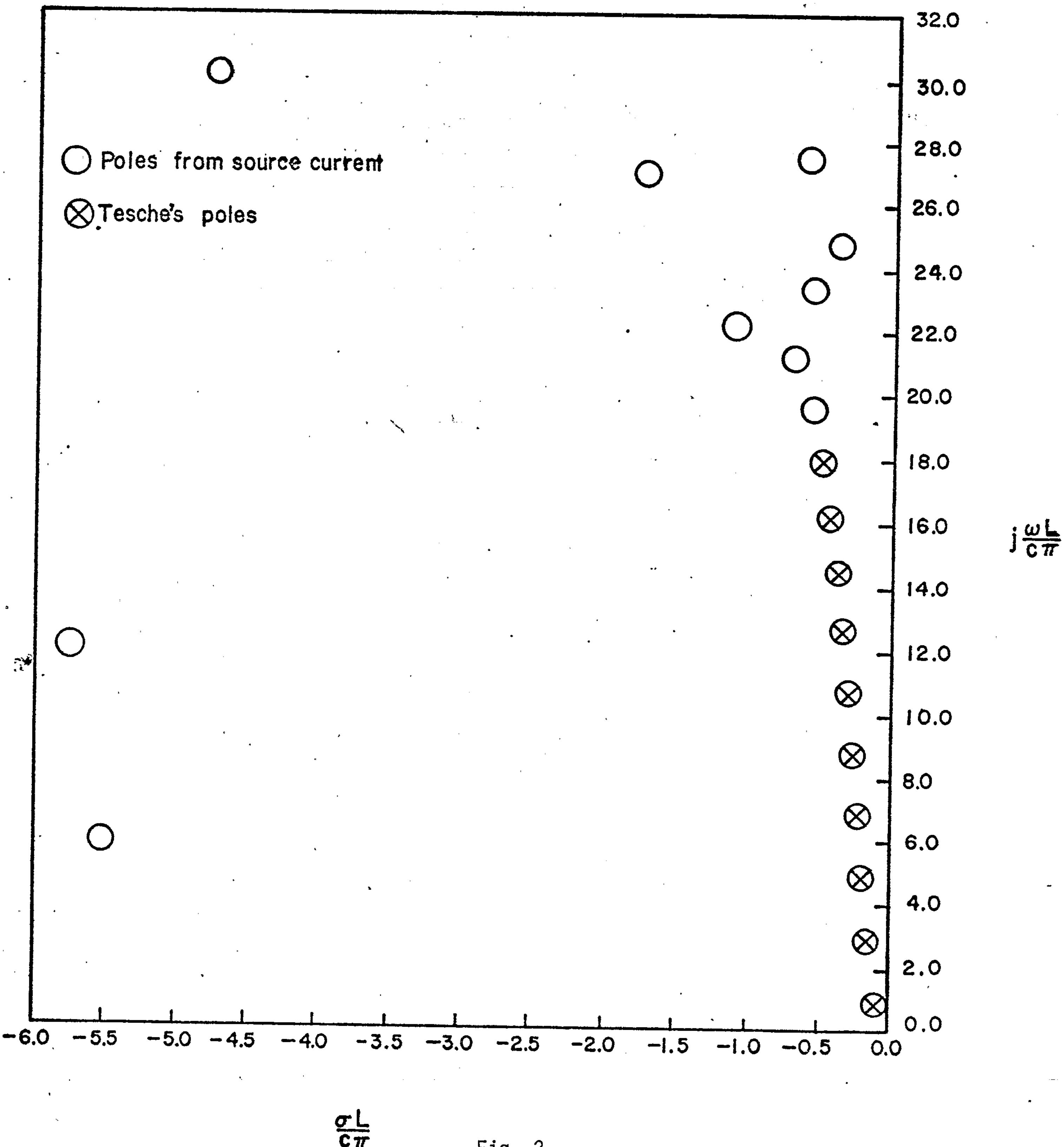

Fig. 2

Pole Locations in the Complex Frequency Plane for 1.0 Meter Dipole equals the antenna length. The resonances for a dipole occur at λ/L = 1/2,3/2,5/2..., thus, with the upper frequency limit given, only the first ten even resonances can occur. The extra ten pole pairs occurred because in the method used when 2N sampled values were taken the method returned N poles and residues. Thus, the extra ten pole pairs are partially a result of the numerical aspects of the method used and possibly due to the fact that the response is not the true impulse response of the structure but is an approximation to it since the driving function is a Gaussian pulse. It has been found that the transient response can still be reproduced using only the ten physical pole pairs.

The input admittance for this dipole was obtained by dividing Equation (9) by $V(j\omega)$, the frequency spectrum of the input Gaussian pulse. The input admittance, which is plotted in Figures 3a and 3b, compares closely for all but the higher frequencies with the input admittance obtained by taking the Fast Fourier Transform of the original data.

The experimental data used were generated on the transient electromagnetic measurement range at Lawrence Livermore Laboratory [9]. The response used here was that of a 1.0 meter monopole located on a ground plane and excited by an approximation to a Gaussian-pulse plane wave. The monopole was loaded at its base with a 50-ohm load, and the voltage across this load was measured with a sampling oscilloscope. A total of 512 samples were taken at a time interval of $\Delta t = 0.4 \times 10^{-10}$ sec. Of the 512 measured values, only 100 samples at every fifth-time step were used. Figure 4a shows the measured response in terms of the current through the load with the dots showing where the 100 samples were taken. These 100 current samples were used with Prony's method and 41 poles and residues were produced. The method has been adapted so that if any

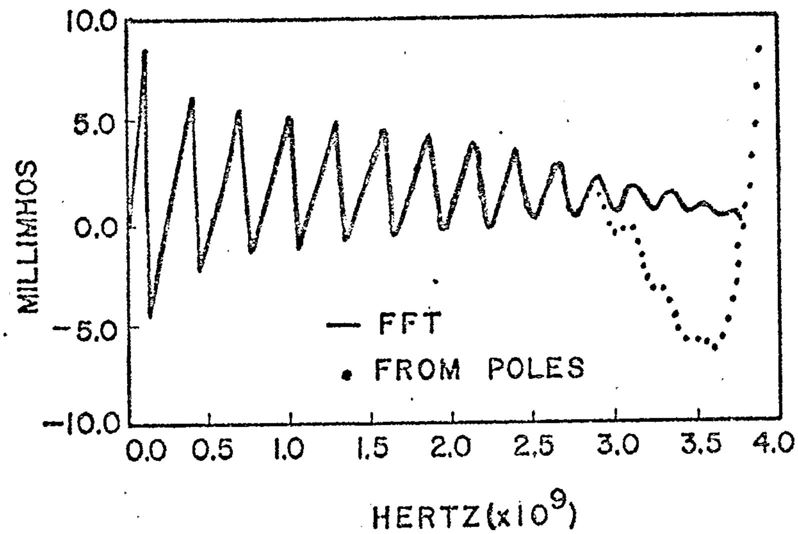

Fig. 3 (a)

The Imaginary Part of Input Admittance for 1.0 Meter Dipole

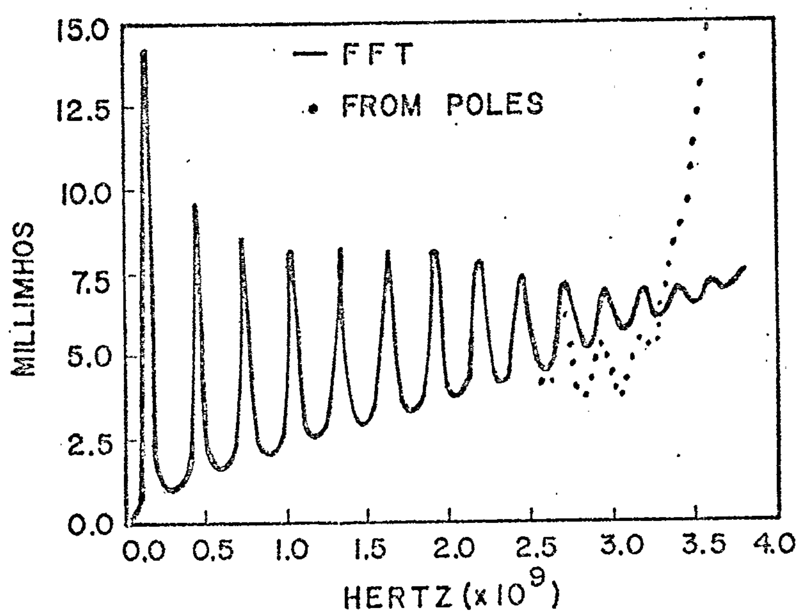

Fig. 3 (b)

The Real Part of Input Admittance for 1.0 Meter Dipole

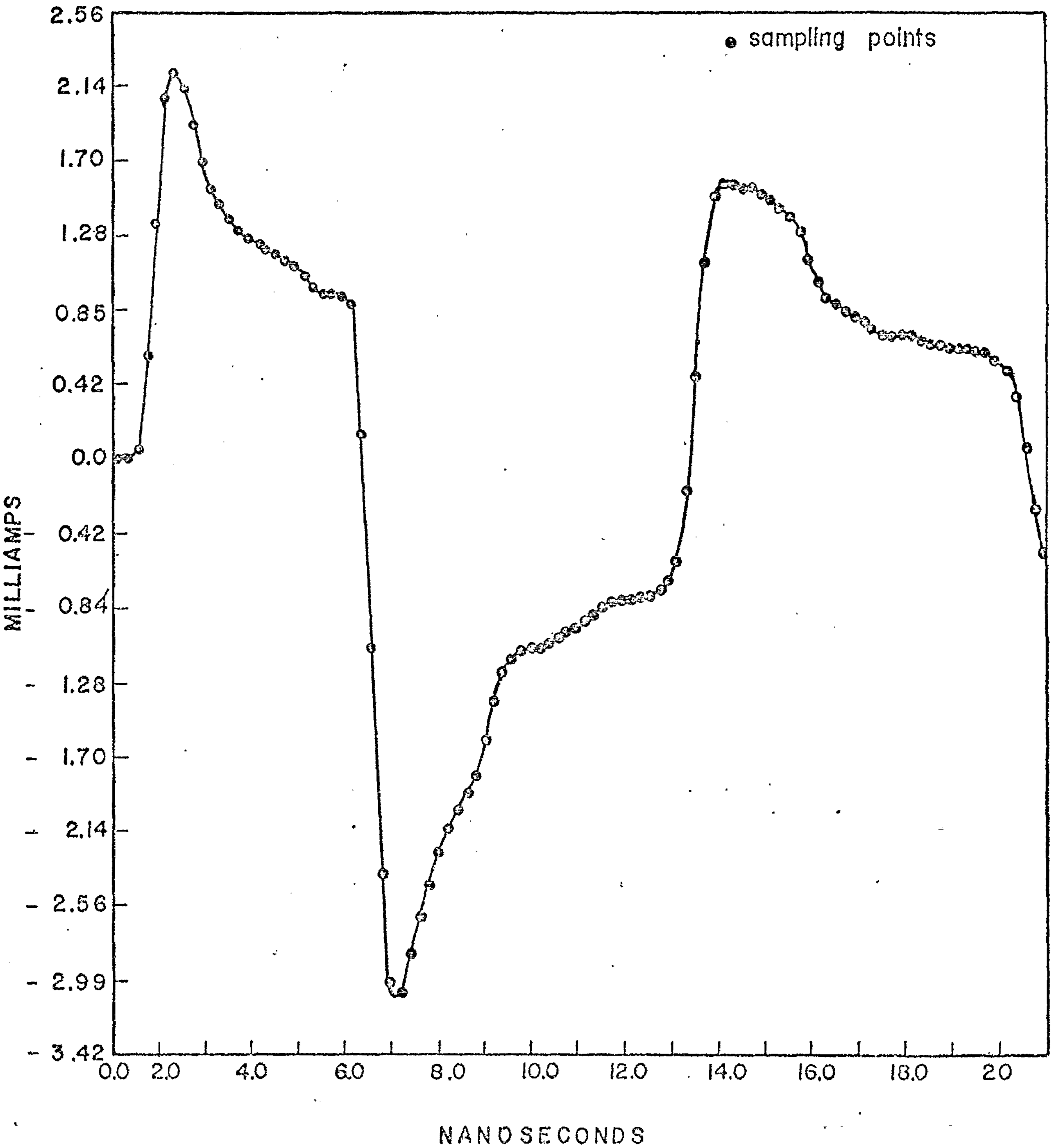

Fig. 4 (a)

Original Experimental Data Showing the 100 Sampling Points poles show up in the right half plane they are discarded before the residues are calculated. This explains why only 41 poles were obtained when 100 samples could give 50 poles. These 41 poles and residues were then used in Equation (2) to reproduce the transient response of the current for 500 time steps where the time step used was five times the original time step. It is seen in Figure 4b that this method has been used to extrapolate the measured response to very late time values. Thus, with the storage of only 82 complex numbers the original measured time response was reproduced and extrapolated to late values. Figure 5 is a plot of the generated poles found in the second quadrant of the complex plane. The first thing that is apparent about these poles is that they tend to fall along a curve running about parallel to the imaginary axis. This is typical of the pole locations for a dipole. The frequency of the first nine poles in this layer correspond to the first nine complex resonant frequencies of a 1 meter monopole. The fact that the remaining poles do not correspond to physical poles again relates to the fact that the pulse used did not contain frequencies higher than that of the ninth resonance. The real part of these poles seems to oscillate around the correct value, which is probably due to the sensitivity of the real part to the noise in the data. The measured data were indeed noisy and no attempt was made to smooth it. A pole sits on the real axis close to the origin. This is probably because there was a late time dc level present in the measurement system.

CONCLUSIONS

From a set of numerically generated transient current response data of a dipole, the method presented here was used to generate the physical poles of the system. If these physical poles are used, the transient response could be recovered

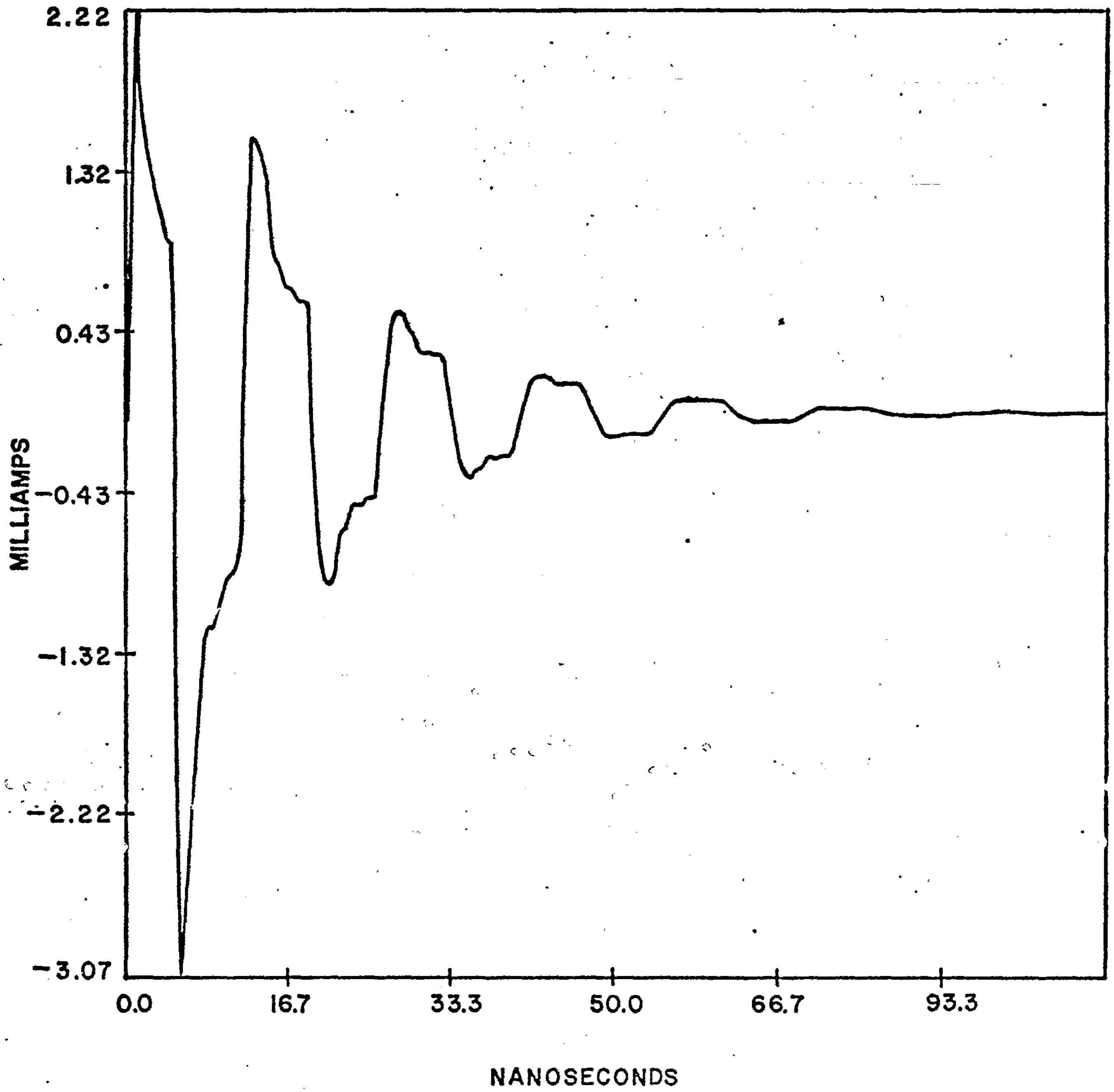

Fig. 4 (b)

The Extrapolated Values of the Transient Response for Experimental 1.0 Meter Whip

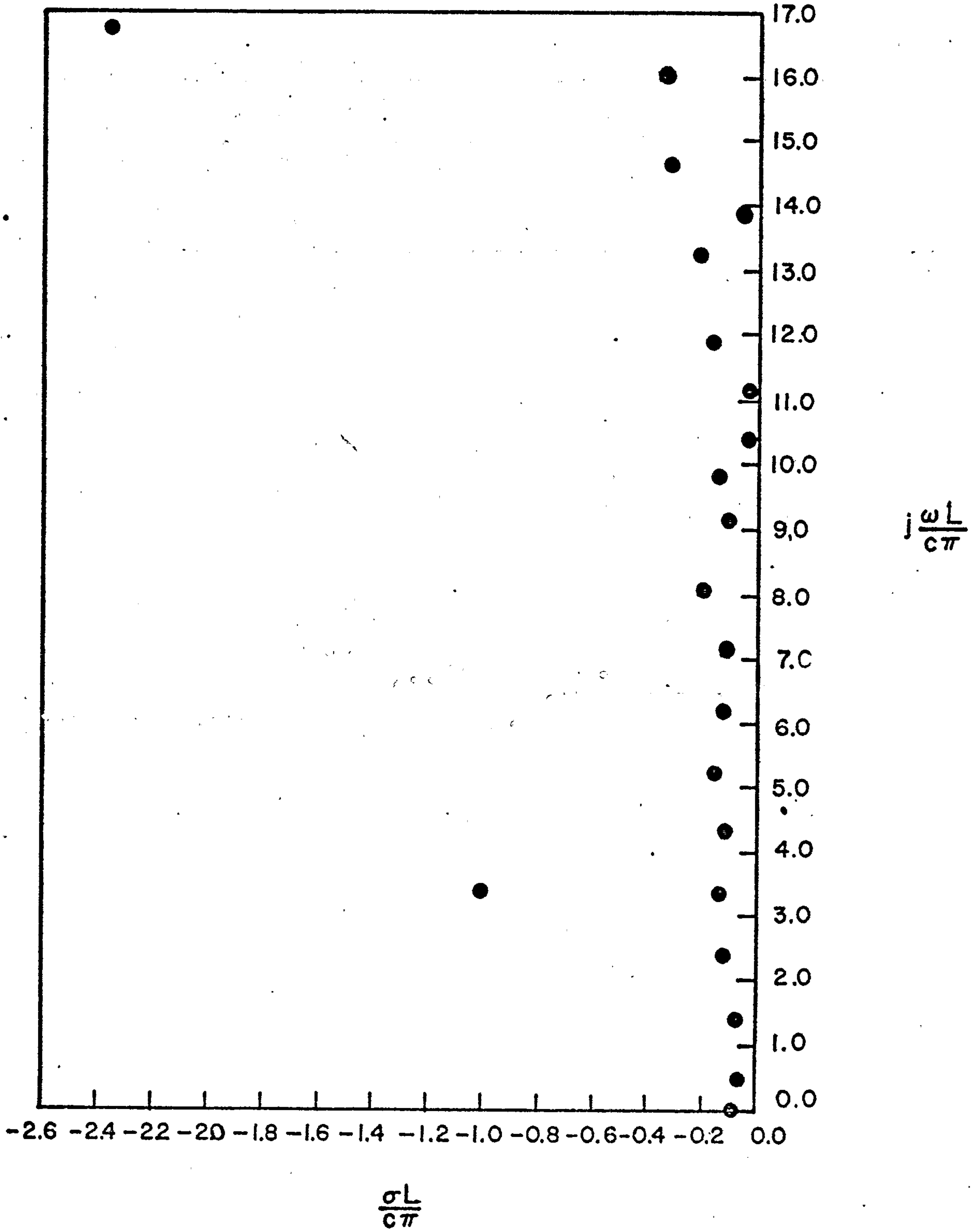

Fig. 5

Pole Locations in the Complex Frequency Plane for the 1.0 Meter Experimental Whip for all time and the spectral characteristics of the system could be generated. It was possible to reconstruct the measured response as well as extrapolate to the late time values, using a set of experimentally produced transient current data for a monopole. From these studies, a number of observations can be made about this method. These may be summarized as follows:

1) From relatively few transient data samples, it is possible to obtain the transient response of the system for both early and late times. This is particularly useful when only a few samples can be taken or only the early response is known. This also allows for compact storage of the transient response.

2) The spectral characteristics can be analytically obtained without the use of a Fast Fourier Transform (FFT). This allows one to obtain spectral information at any frequency desired rather than at the complete set of discrete frequencies give by the FFT.

3) The method allows for the direct extraction of the singularities of the system from its experimental transient response. Thus, the transfer function of an experimental system can be obtained allowing the system to be modeled from a lumped circuit viewpoint. As a result, this method can be used in radar image identification problems [11].

4) When 2N transient data samples are used, the method tries to return N poles even when fewer than N poles may be present in the data. This can conceivably be corrected by using something like a least-square fit to solve Equation (2).

5) The method, as presented here, is applicable to systems possessing only simple poles. However, the method can be extended to systems with multiple poles, as will be shown in a future publication.

Work is being done to further refine this method and to make it even more suitable for practical applications. Studies are also being undertaken to investigate the potential uses of this method, some of which were mentioned above.

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