Interaction Notes

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Analysis of Antennas and Scatterers with Non-Linear Loads

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Abstract

Two methods for analyzing antennas or scatterers having non-linear resistive loads are discussed. The first is a direct time domain integral equation approach, whereas the second involves the use of frequency domain data to compute the time dependent currents and voltages across the non-linear load. Both transient and time-harmonic excitations are considered in the sample problems illustrated here which involve a center-loaded linear antenna. Although it is difficult to do so with the direct time domain method, the second method of analysis may be readily applied to an arbitrary antenna whose frequency response is either measurable or calculable.

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I. Introduction

Many physical phenomena are inherently non-linear in nature. In the area of electromagnetics, non-linearities may be completely unforeseen and undesirable in the design of a particular system, whereas others may be essential in the functioning of the system. Non-linear effects are important for antenna systems containing semiconductors, integrated circuits and voltage limiters when they may be illuminated by an extremely strong signal, such as that produced by a lightning strike or a nuclear EMP.

Currently, for a general system, most cases involving non-linearities have been considered using direct time-domain techniques such as the state-space method. In addition, specialized problems involving weak non-linearities have been solved in the frequency domain [6] to obtain spectral components of the solution at harmonic, sub-harmonic and combination frequencies. Such frequency domain solutions almost invariably employ the techniques of perturbation or iteration. Some of these mathematical methods have been applied to antenna problems involving non-linear loads.

The determination of the behavior of an antenna or scatterer with a general non-linear load attached would usually be achieved by solving either a set of coupled partial differential equations or an integral equation directly in the time domain. Indeed, Schuman [8] has recently treated a non-linear antenna problem using a space-time domain integro-differential equation. Sarkar and Weiner [7] have solved the same problem using the Volterra series method to directly obtain harmonic responses of the time-harmonic excited system. Due to analytical complexity, the method is useful only if the non-linearity is not too strong. The utilization of a frequency domain method for finding the response of the antenna or scatterer with a general non-linear load is not usually employed, but would be very useful in view of the large amount of frequency domain data presently available for antennas and scatterers.

In this paper, two methods for analyzing antennas having non-linear resistive loads are described. The first, to be discussed in Section II, is a direct time domain approach which involves solving a space-time domain integral equation. The discussion of this method will be restricted to the case of a linear wire antenna having a non-linear load. The integral equation and the method of treating the non-linear loads are different to those of Schuman.

The second method, to be described in Section III, is a technique for obtaining the response of the antenna by making use of frequency domain data such as the short-circuit current and the driving-point admittance, both being solutions to the antenna problem in the absence of non-linearities. This method yields a non-linear integral equation containing a convolution integral. The numerical procedures of evaluating convolution integrals have been discussed by Baum* in the study of buried transmission line simulators.

Throughout this paper, the latter method will be illustrated using the same wire antenna as in the direct time domain solution. However, it is to be emphasized that this method can be readily applied to any antenna system for which the frequency domain behavior is known, either by calculation or by measurement. The advantages of employing this method are that it can make use of the large amount of existing frequency domain data and computer programs for antennas, and it also permits the consideration of certain problems which are too complicated to be treated by the direct time domain approach.

One such problem is a non-linearly loaded antenna over a lossy ground. Another example is that of a non-linear microwave diode mounted under a conducting post inside a waveguide. Such a configuration is often encountered in microwave circuits, such as Gunn oscillators, mixers, and the like. The harmonics generated by the non-linear device significantly affect the performance of the circuit [1], making the solution of this non-linear boundary-value problem essential. Although this problem is relatively difficult to solve in the time domain, frequency domain data for the linear portions of the problem exist [3]. Thus, the second method described in this paper is useful in solving this important microwave problem.

It should be noted in passing that another method is currently being used for certain non-linear antenna problems [5]. This involves solving the time dependent differential forms of Maxwell's equations, as opposed to the

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*Baum, C. E., "The buried-transmission-line simulator driven by multiple capacitive sources," *Sensor and Simulation Note* 49, August 1967.

integral equations discussed here. Such a method provides a solution for the unknown fields in a volume surrounding the antenna, instead of just on the surface. This necessitates longer computation time and much larger storage for a computer solution, but is useful when the volume surrounding the antenna has a non-linear character. An example is the source region EMP excitation problem which involves field-dependent air conductivity near the antenna. For the present discussion of an antenna with a non-linear load, which is localized at a single point on the antenna, the integral equation approaches are more efficient for obtaining a solution.

Throughout this paper, we shall consider the antenna configuration depicted in Fig. 1. Most of the numerical results will be for the scattering problem, but the case of the antenna driven through the non-linear load is also discussed. The wire antenna has a length L and radius a and is excited by an electric field E^{inc} tangential to the wire. The non-linear resistive element contained in the load, which is located at $z = z_o$, has a $v - i$ characteristic defined by

$$v_L(t) = F[i_L(t)] \tag{1}$$

where F is a known function and v_L and i_L are the instantaneous voltage across and current through the load element. Such a device, having an explicit $v - i$ characteristic given by Eq. (1), can be considered as a current-controlled device.

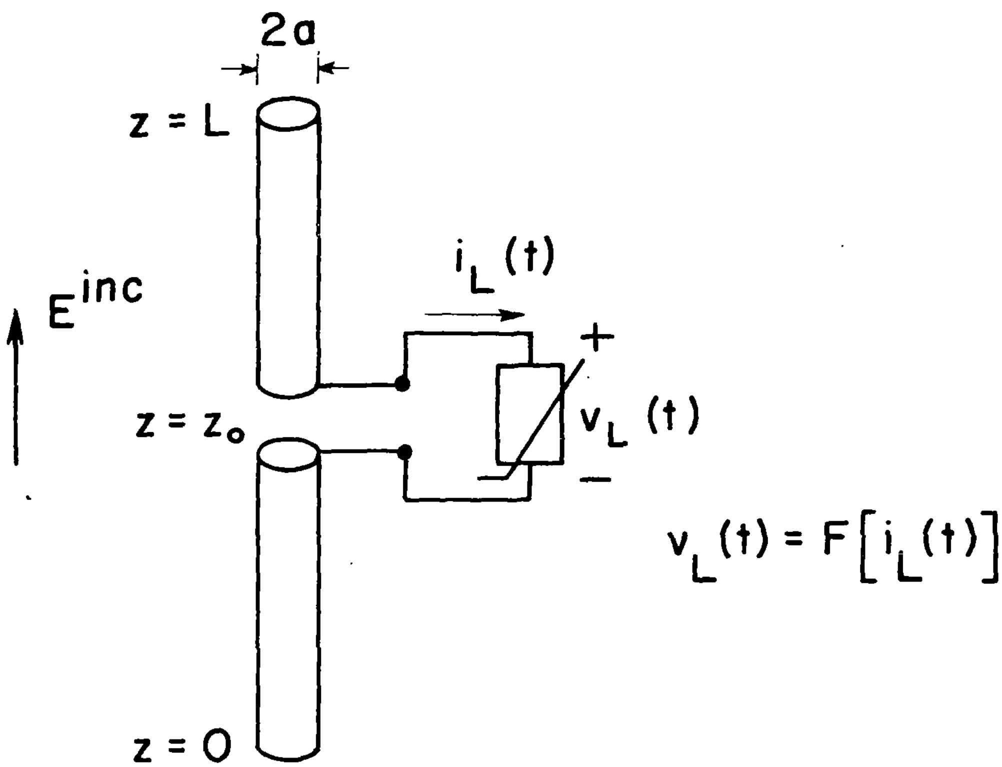

Fig. 1. A linear antenna with a non-linear load.

II. Non-Linear Analysis Using Direct Time Domain Approach

The advantages of using the space-time domain integral equation of Hallén's type for solving linear antenna problems have been discussed in a previous paper [4]. We shall use this formulation for the solution of the induced current on the antenna with a non-linear load. For the configuration illustrated in Fig. 1, the integral equation for the current $i(z,t)$ is given by

$$\int_0^L i(z',t-|z-z'|/c)K(z,z')dz' = \frac{1}{2Z_o}\left[\int_0^L E^{inc}(z',t-|z-z'|/c)dz' - v_L(t-|z-z_o|/c)\right]$$

$$+ f_1(ct-z) + f_2(ct+z) \tag{2}$$

where the kernel is

$$K(z,z') = \frac{1}{2\pi a}\int_0^{2\pi} \frac{a d\theta}{4\pi[(z-z')^2 + \{2a\sin(\theta/2)\}^2]^{\frac{1}{2}}} . \tag{3}$$

The functions $f_1(ct-z)$ and $f_2(ct+z)$ are invariant along their respective characteristic curves, $dz/dt = c$ and $dz/dt = -c$. These two functions are determined by the end conditions $i(0,t) = i(L,t) = 0$. In Eq. (2), $Z_o \simeq 377\,\Omega$ is the intrinsic impedance of free-space.

The numerical procedure of solving Eq. (2) is the same as that outlined in reference [4], except at $z = z_o$. Basically it is a step-by-step time-marching method, utilizing the property of Eq. (2) that the current, f_1, f_2 and v_L along the characteristic curves through the point (z,t) are known prior to time t. Following the usual numerical procedures of approximating various quantities in Eq. (2), we obtain the following equation in the unknown current $i(z,t)$:

$$\alpha i(z,t) = S(z,t) - (2Z_o)^{-1}\, v_L(t-|z-z_o|/c) \tag{4}$$

where α is the value resulting from numerical approximation of the integral of the kernel $K(z,z')$ over the zone surrounding the point $z' = z$, and

$S(z,t)$ is the numerical approximation of all the other terms except the term $(2Z_o)^{-1}\ v_L(t-|z-z_o|/c)$. The exact expressions of α and $S(z,t)$ depend on the numerical scheme (collocation method, Galerkin's method, etc.) adopted and will not be detailed here. However, it is important to mention that α is a positive quantity. A careful examination of Eq. (2) and Eq. (4) reveals that $S(z,t)$ is a known quantity.

For $z \neq z_o$, due to the time retardation effect, the quantity $v_L(t-|z-z_o|/c)$ is known. Hence $i(z,t)$ is simply given by Eq. (4).

For $z = z_o$, the second term on the right-hand-side of Eq. (4) is now $v_L(t) = F[i_L(t)]$ which is unknown. In fact, for the configuration under study, $i(z_o,t) = i_L(t)$ and Eq. (4) is a non-linear algebraic equation, the solution of which may be obtained by a numerical procedure such as the Newton-Raphson method. In the solution of this non-linear algebraic equation, one has to be careful about the uniqueness of the solution, as more than one value of $i_L(t)$ could exist. It is useful to consider the solution of the set of simultaneous equations, Eq. (1) and Eq. (4), i.e.,

$$\alpha i_L(t) = S(z_o,t) - (2Z_o)^{-1}\ v_L(t) \tag{5a}$$

$$v_L(t) = F[i_L(t)]. \tag{5b}$$

Eq. (5a) can be represented as a load line with a negative slope in the $v_L - i_L$ plane (see Fig. 2) whereas Eq. (5b) describes the device characteristic. The solution of Eqs. (5a) and (5b) is always unique if the two curves intersect at only one point for any fixed value of $S(z_o,t)$. This condition may be violated if the device exhibits negative dynamic resistance behavior, such as in the case of a tunnel diode. This situation of possible multiple solutions of i_L is illustrated in Fig. 2. As can also be observed in Fig. 2, if the device characteristic can be described by a monotonically-increasing function, i.e., the device has only positive dynamic resistance such as in the case of an ordinary rectifying diode, the solution would be unique.

For the special case that the non-linear $v - i$ relation can be described by two piecewise-linear curves through the origin such that

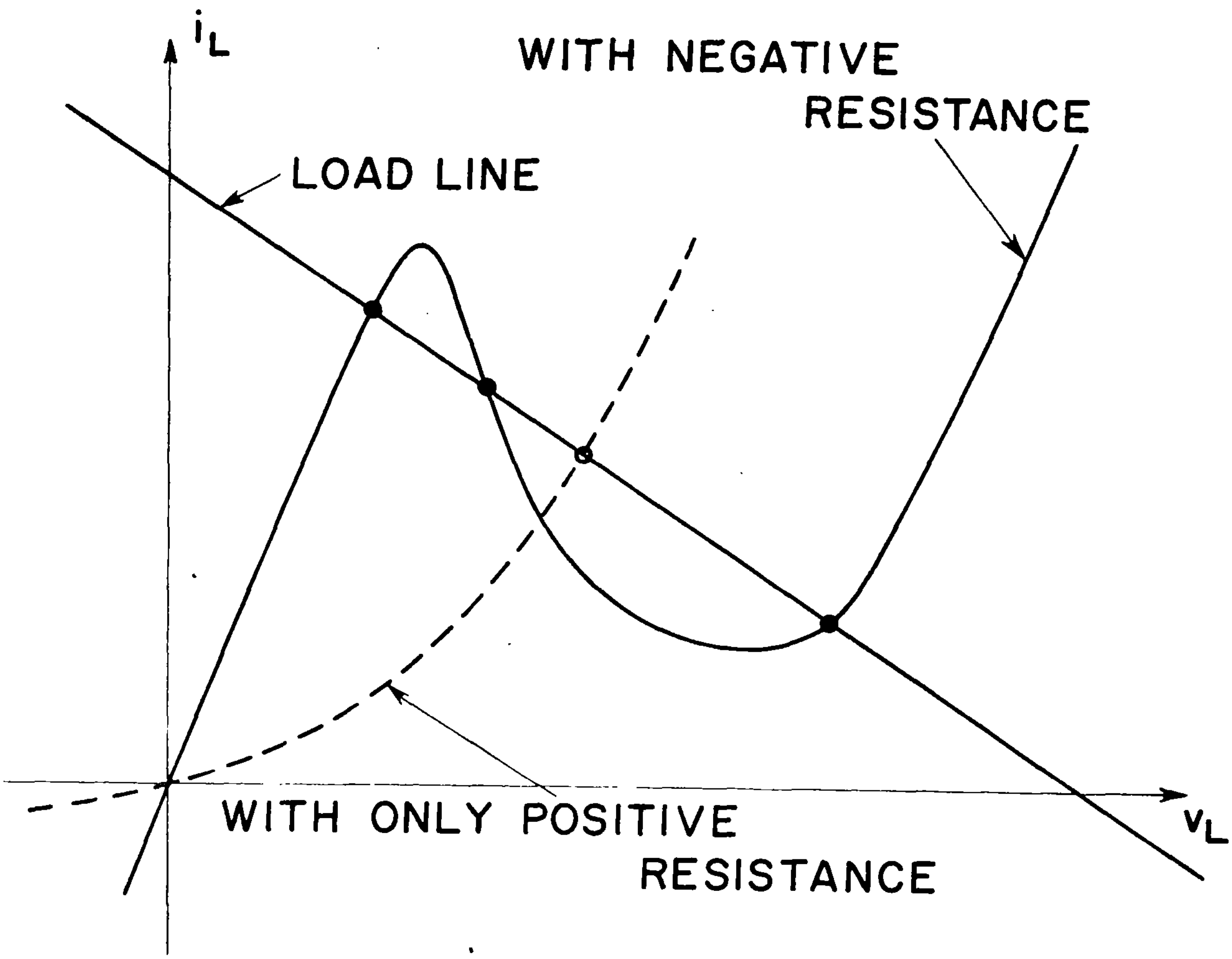

Fig. 2. Graphical illustration of the solution of the non-linearly loaded antenna problem at $z = z_0$ and time t. The load line represents the antenna characteristic without the non-linear load. The two non-linear device characteristics, namely, that with negative resistance and that with only positive resistance, help to illustrate the uniqueness of the solution.

$$v_L(t) = R_1\, i_L(t), \quad i_L(t) \geq 0$$

$$v_L(t) = R_2\, i_L(t), \quad i_L(t) < 0$$

it can be readily shown, by substituting these equations into Eq. (4), that the resulting equation gives explicit solution of the unknown current $i_L(t)$, i.e.

$$i_L(t) = S(z_o,t)[\alpha + R_1/2Z_o]^{-1}, \quad S(z_o,t) \geq 0$$

$$i_L(t) = S(z_o,t)[\alpha + R_2/2Z_o]^{-1}, \quad S(z_o,t) < 0\ .$$

It should be noted, as pointed out earlier, that α is a positive quantity. With appropriate values of R_1 and R_2, this special $v-i$ characteristic crudely approximates that of a rectifying diode. Such a loading will be studied in detail numerically in Section VI.

Special techniques exist so that one can avoid solving the non-linear algebraic equation (4). In the circuit modeling, a physically-real small inductance is added in series with a current-controlled non-linear device, such as that described by (1); or a small capacitance is added in shunt with a voltage-controlled device. The presence of such elements yields a differential equation containing a non-linear term in place of Eq. (4); the differential equation can then be solved uniquely using some numerical procedures. This technique is illustrated in Appendix A for the case involving a voltage-controlled device. It should be pointed out that in a physical device, small lead inductance and small package or junction capacitance are present so that the above technique is justified.

It should also be mentioned here that the method of solution for the case that an antenna is driven via a non-linear element is very similar to that described above and will not be detailed.

III. Non-linear Analysis Using Frequency Domain Data

Referring again to the non-linear scattering problem illustrated in Fig. 1, an alternate approach for determining the load current is to first solve the boundary-value problem in the frequency domain for the antenna without the non-linear load, and then use that solution to treat the entire non-linear problem in the time domain.

It is well known that the frequency domain behavior of the plane wave excited antenna can be represented by the Norton equivalent circuit shown in Fig. 3. The term $I_{sc}(s)$ is the short-circuit current at the antenna input and $Y_{in}(s)$ is the input admittance of the antenna, both being defined as a function of the complex frequency s. The calculation of these quantities for the linear antenna is outlined in Appendix B. Despite the non-linearity, it is still possible to formally define the Laplace transforms of $v_L(t)$ and $i_L(t)$ to be $V_L(s)$ and $I_L(s)$. From circuit analysis, the load current in the frequency domain can then be written as

$$I_L(s) = I_{sc}(s) - Y_{in}(s)\, V_L(s). \tag{6}$$

If the antenna load were linear, $V_L(s)$ could be related directly to $I_L(s)$ in terms of an impedance. This would then enable $I_L(s)$ to be obtained from Eq. (6). The transient response $i_L(t)$ could then be determined by an inverse Laplace transform. For the non-linear load, a different approach must be taken.

Noting that the product of two functions in the frequency domain implies convolution in the time domain, Eq. (6) may be Laplace transformed to yield

$$i_L(t) = i_{sc}(t) - \int_{-\infty}^{t} y_{in}(t-\tau)\, v_L(\tau)d\tau \tag{7}$$

where $y_{in}(t)$ is the inverse Laplace transform of the input admittance of the antenna. Physically, this corresponds to the current flowing into the antenna at the load point $z = z_o$ due to a delta-function voltage source at $t = 0$. The term $i_{sc}(t)$ is the time dependent short-circuit current at the load point $z = z_o$ induced by the time varying incident plane wave, striking the antenna at $t = 0$.

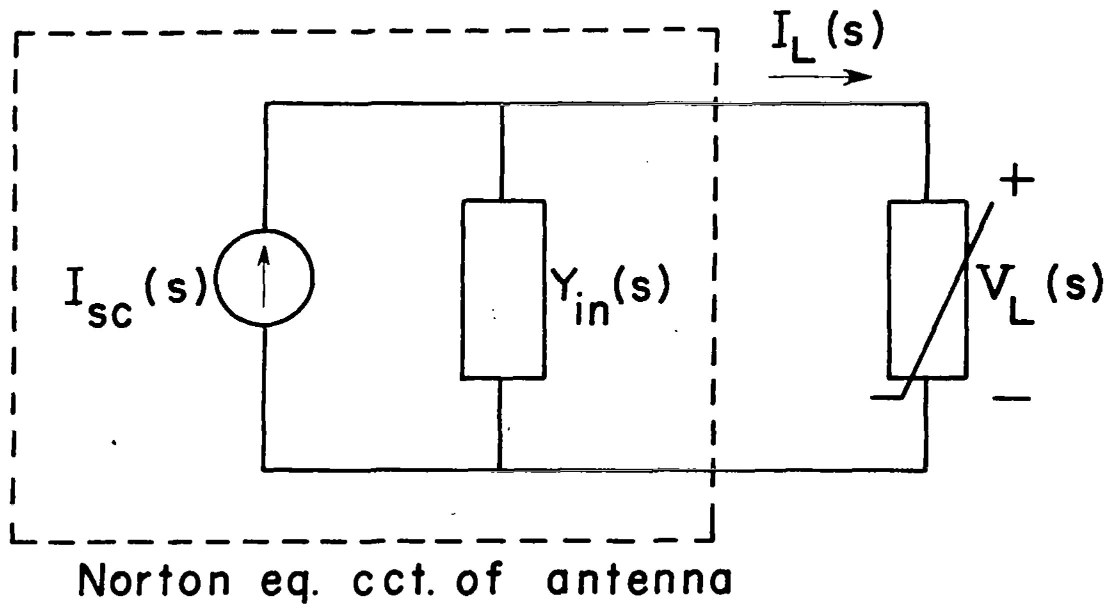

Fig. 3. Norton equivalent circuit representation of the antenna with a non-linear load.

By substituting in the $v-i$ relation (1) for the load of the antenna, the following non-linear integral equation of the Volterra type results for the unknown load current $i_L(t)$:

$$i_L(t) = i_{sc}(t) - \int_{-\infty}^{t} y_{in}(t-\tau)\ F[i_L(\tau)]d\tau. \tag{8}$$

One method of solving this equation iteratively is described by Pipes [6]*. By choosing an initial guess $i_L^{(0)}(t)$ for the load current, successive corrections can be made of the form:

$$\begin{aligned}
i_L^{(1)}(t) &= i_{sc}(t) - \int_{-\infty}^{t} y_{in}(t-\tau)\ F[i_L^{(0)}(\tau)]d\tau \\
i_L^{(2)}(t) &= i_{sc}(t) - \int_{-\infty}^{t} y_{in}(t-\tau)\ F[i_L^{(1)}(\tau)]d\tau \\
&\cdots\cdots\cdots\cdots \\
i_L^{(n)}(t) &= i_{sc}(t) - \int_{-\infty}^{t} y_{in}(t-\tau)\ F[i_L^{(n-1)}(\tau)]d\tau
\end{aligned} \tag{9}$$

such that the solution to Eq. (8) is given by

$$i_L(t) = \lim_{n\to\infty} i_L^{(n)}(t). \tag{10}$$

An alternate form of this equation is obtained by using the convolution theorem on the integral in Eq. (8). This yields

$$i_L(t) = i_{sc}(t) - \mathcal{L}^{-1}\left[Y_{in}(s)\ \mathcal{L}\{F[i_L(t)]\}\right] \tag{11}$$

which can also be solved iteratively for $i_L(t)$, but by using a numerical

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*Care should be exercised in interpreting Pipes' equations due to his unconventional definition of the Laplace transform as $H(s) = \mathcal{L}\,h(t) = s\int_0^\infty h(\tau)\ e^{-s\tau}d\tau$.

transform algorithm instead of a direct evaluation of the convolution integral. Here, the symbols $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and inverse Laplace transform operators, respectively.

The convergence properties of Eq. (9) have been discussed by Davis [2]. For cases of relatively small loading (i.e. $F[i_L(t)]$ being small for all t), Eq. (9) can be expected to converge quickly to the proper functional form. For very large loads (for example, a nearly open-circuit) the convergence of Eq. (9) or Eq. (11) is not guaranteed. As an example, using Eq. (9) it was found that the solution for a 100 Ω linear load required approximately 65 iterations before convergence was achieved. A 50 Ω /100 Ω piecewise-linear load required approximately 200 iterations and the 50 Ω /5 kΩ piecewise-linear load case did not converge, but seemed to oscillate about a solution. Thus, it is desirable to find another method for treating this class of non-linear antenna problems.

Another approach is to evaluate $i_L(t)$ by a time-marching procedure. At any time t, Eq. (8) can be written as

$$i_L(t) = i_{sc}(t) - \int_{-\infty}^{t-\delta} y_{in}(t-\tau)\, F[i_L(\tau)]d\tau - \int_{t-\delta}^{t} y_{in}(t-\tau)\, F[i_L(\tau)]d\tau \qquad (12)$$

where δ is small. Letting $\delta \to 0$, the last integral may be expressed as

$$F[i_L(t)]\, \bar{y}_{in}(0) \qquad (13)$$

where

$$\bar{y}_{in}(0) = \lim_{\delta\to 0} \int_{t-\delta}^{t} y_{in}(t-\tau)d\tau. \qquad (14)$$

Thus, Eq. (12) can be written as

$$i_L(t) + \bar{y}_{in}(0)\, F[i_L(t)] = i_{sc}(t) - \int_{-\infty}^{t-} y_{in}(t-\tau)\, F[i_L(\tau)]d\tau\ . \qquad (15)$$

Since the quantities $i_{sc}(t)$ and $y_{in}(t)$ are calculated from inverse Laplace transforms of frequency domain data and are known for all time, and since i_L is assumed known for all times prior to the current calculation time t, it is noted that the right-hand-side of Eq. (15) has a known value. This results, therefore, in a non-linear algebraic equation for i_L at the current calculation time t, and is easily treated using a standard root-solving algorithm. The solution is then stepped along in time using this technique.

The uniqueness of the solution of Eq. (15) has to be considered. Eq. (15) can be re-written as

$$i_L(t) = A - \overline{y}_{in}(0)\ v_L(t)$$

with

$$v_L(t) = F[i_L(t)].$$

This set of equations is identical in form to that of Eqs. (5). However, because of the different methods of evaluation, the parameters, namely, $s(z_o,t)/\alpha$, $(2\alpha\ Z_o)^{-1}$, A and $\overline{y}_{in}(0)$ may be somewhat different in value. Of course, as pointed out earlier, Eqs. (5) are strictly for a linear wire antenna, whereas the above equations are more general. The discussions on uniqueness in Section II applies equally well in this case. In addition, the technique described in Section II of adding a small inductance in series with, or a small capacitance is shunt with the non-linear device can also be useful for this method.

When this method is applied to the transient scattering problem, it is assumed that the antenna is initially relaxed and that the incident field strikes the scatterer at $t=0$. Thus, the lower limit $-\infty$ in Eq. (15) is replaced by 0 and the condition $i_L(0)=0$ is used to begin the time-stepping procedure.

Once the load current is determined, the load voltage is easily found from $v_L = F(i_L)$. With the knowledge of this voltage, it is possible to find the transient behavior of the current at an arbitrary point z_1, $z_1 \neq z_o$, on the antenna structure, provided the quantities $i_s(z_1,t)$ (the field-induced current flowing at z_1 with the input short-circuited) and $y_t(z_1,t)$ (the current at z_1 due to an impulsive voltage excitation at the antenna terminals) are known. This relationship takes a form similar to Eq. (7) as

$$i(z_1,t) = i_s(z_1,t) - \int_0^t y_t(z_1,t-\tau)\, v_L(\tau)d\tau \tag{16}$$

where as before it is assumed that $v_L(t) = 0$ for $t \le 0$. Due to the fact that both y_t and v_L are known, the evaluation of this integral is a straightforward task, compared with the solution of Eq. (8).

In certain circumstances, it may be desirable to compute the response of a driven antenna through a non-linear load. This case is handled trivially using the proceeding formalism, except that the short-circuit current $i_{sc}(t)$ is replaced by the current flowing into the antenna when excited by a local source $v_o(t)$ with no non-linear device present. This may be achieved by defining $i_{sc}(t)$ as

$$i_{sc}(t) = \mathcal{L}^{-1}\{Y_{in}(s)\mathcal{L}[v_o(t)]\} \tag{17}$$

and using Eq. (7) directly.

Using the frequency domain method, the response of the antenna to a steady-state excitation can also be computed. By starting a periodic excitation at $t = 0$ and solving Eq. (15) for times $t > 0$, it will be noted that the transient portions of the response soon die out, leaving the steady-state response. As an indication to how long one must carry out the computation to obtain the steady-state response, we can look at the rate of decay of the currents on the unloaded antenna. For the relatively high "Q" wire antenna, the steady-state response may be obtained for roughly $ct/L > 20$.

IV. Numerical Results

To investigate the effects of non-linear loading on antennas, the structure shown in Fig. 1 has been analyzed. The antenna is assumed to be excited by a step-function, broadside incident field, and it is center-loaded by a non-linear resistor. In this example, an antenna with $\Omega = 2\,\ell n(L/a) = 10$ is chosen. The $v-i$ relationship of the load is represented by a piecewise-linear function through the origin, having a $50\,\Omega$ resistance in one direction and a $5\,k\Omega$ resistance in the other direction. The device has only positive dynamic resistance.

The application of the frequency domain method described in the previous section requires a knowledge of the properties of the antenna with no load. By solving Eq. (A4) for the scattering case at a number of frequencies, the spectrum of the short-circuit current is obtained. Fig. 4 shows the magnitude of the quantity. Note that this is the spectrum for a delta function excitation. Hence, multiplying by $1/j\omega$ and taking an inverse Fourier transform*, the step excited short-circuit current, $i_{sc}(t)$, results and is shown in Fig. 5.

In a similar manner, the input admittance $Y_{in}(\omega)$ may be calculated again from Eq. (A4) for the driven case. Fig. 6 shows this quantity as a function of frequency, and its inverse transform, $y_{in}(t)$, is shown in Fig. 7.

Using $i_{sc}(t)$ and $y_{in}(t)$ in Eq. (15), along with the specified load characteristic, the load current can be calculated. Results are shown in Fig. 8 for the case that the load is so placed that the current initially flows through the $50\,\Omega$ section. Also shown in Fig. 8 are the results obtained using the direct time domain approach. The two sets of results agree favorably.

For comparison purposes, the load current for the case that the antenna is loaded by a $50\,\Omega$ linear resistor is also presented in Fig. 8. Careful inspection reveals that for $ct/L > 1$, the non-linear load current oscillates at about twice the rate of the linear load current. This phenomenon can be readily explained. During the first transit period $ct/L < 1$, the current flows through the $50\,\Omega$ portion of the load and behaves in much the same way as the short-circuit current (Fig. 5). As the current is reflected at the ends of the

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*It is more convenient numerically to use Fourier transform in place of Laplace transform.

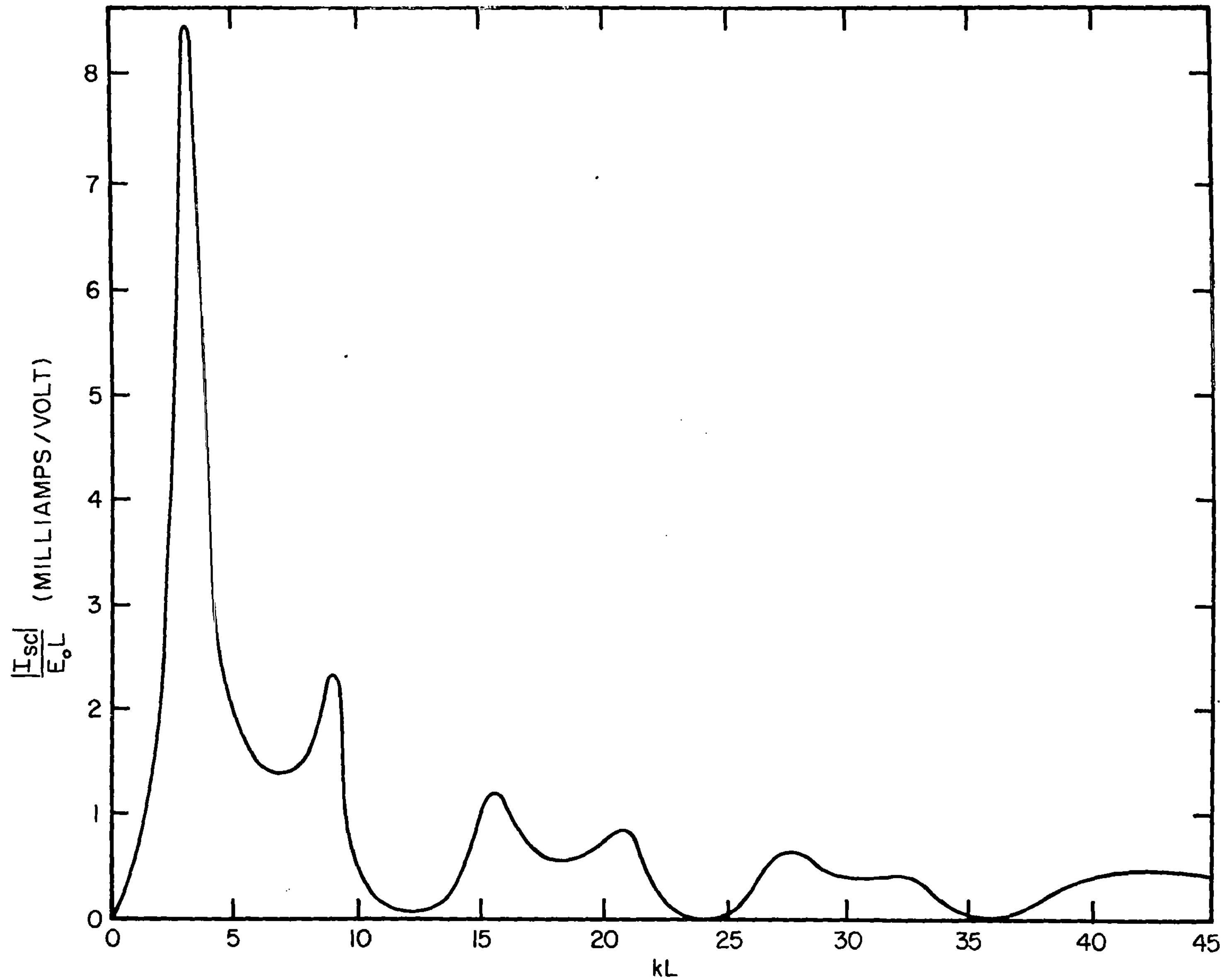

Fig. 4. Frequency spectrum of the magnitude of the short-circuit current at the mid-point of the antenna excited by a broadside incident impulsive field. $\Omega = 2\, \ell n(L/a) = 10$.

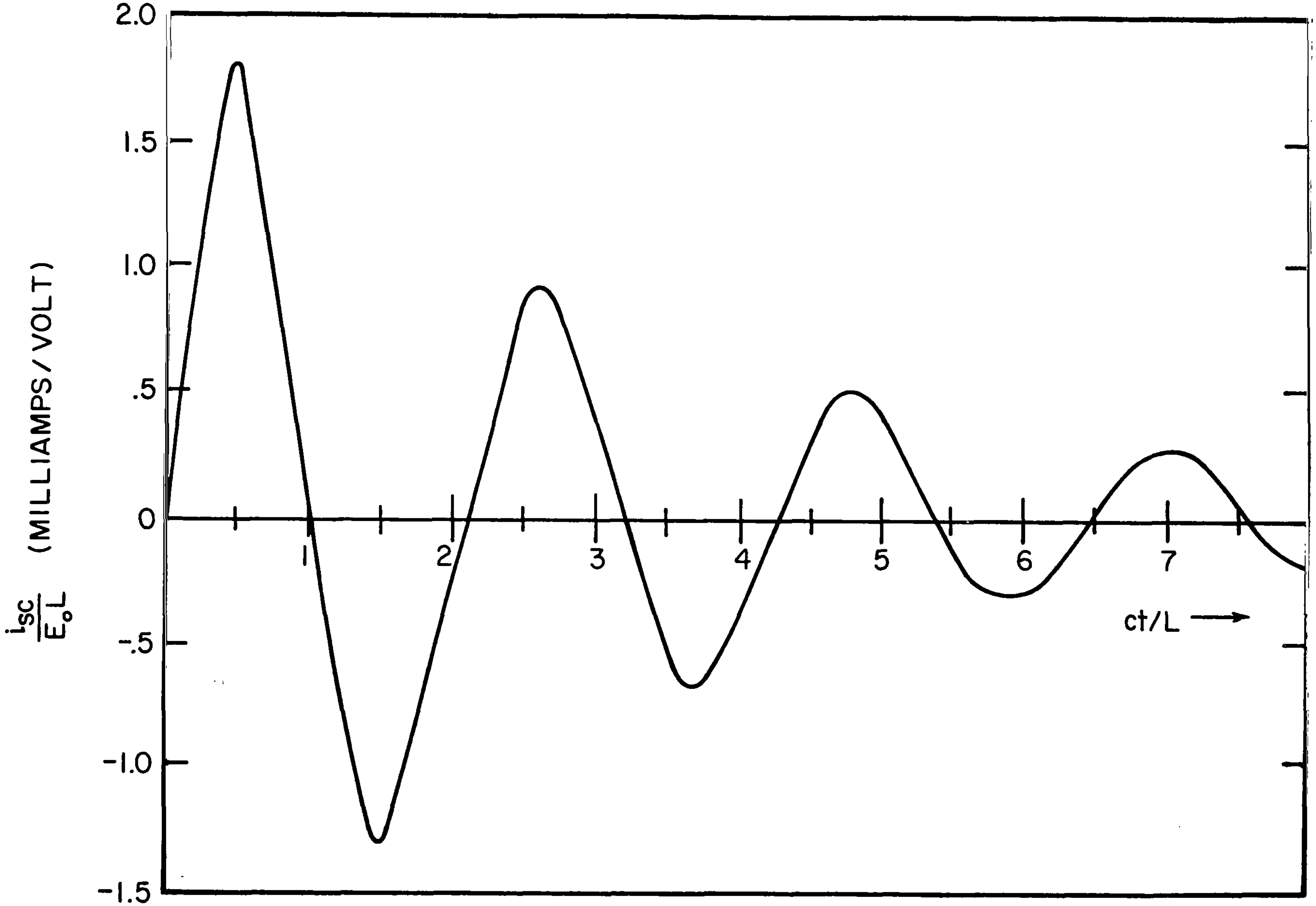

Fig. 5. Time response of the short-circuit current at the mid-point of the antenna excited by a step-function, broadside incident electric field. Ω = 10.

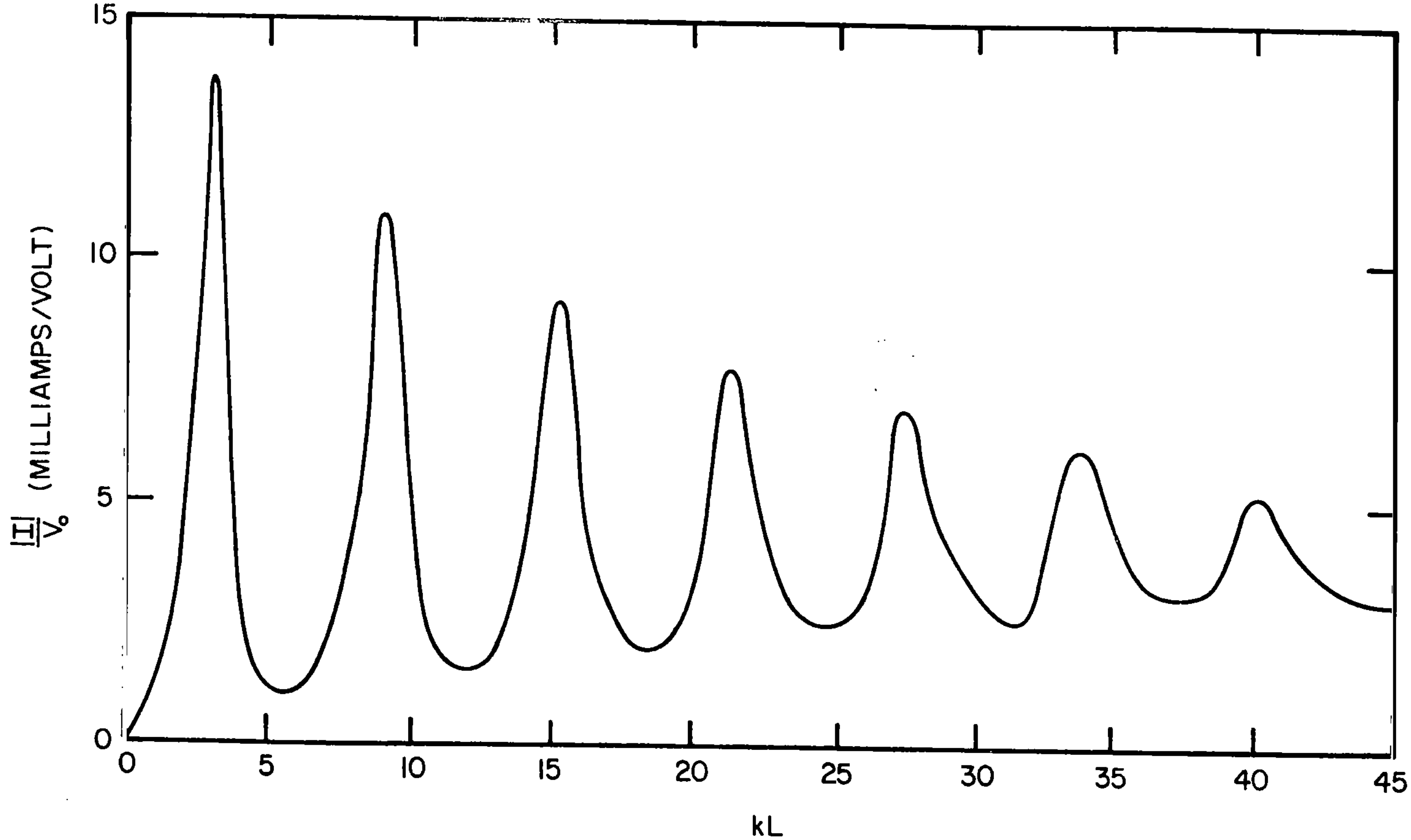

Fig. 6. Frequency spectrum of the input admittance of the center-driven antenna. Ω = 10.

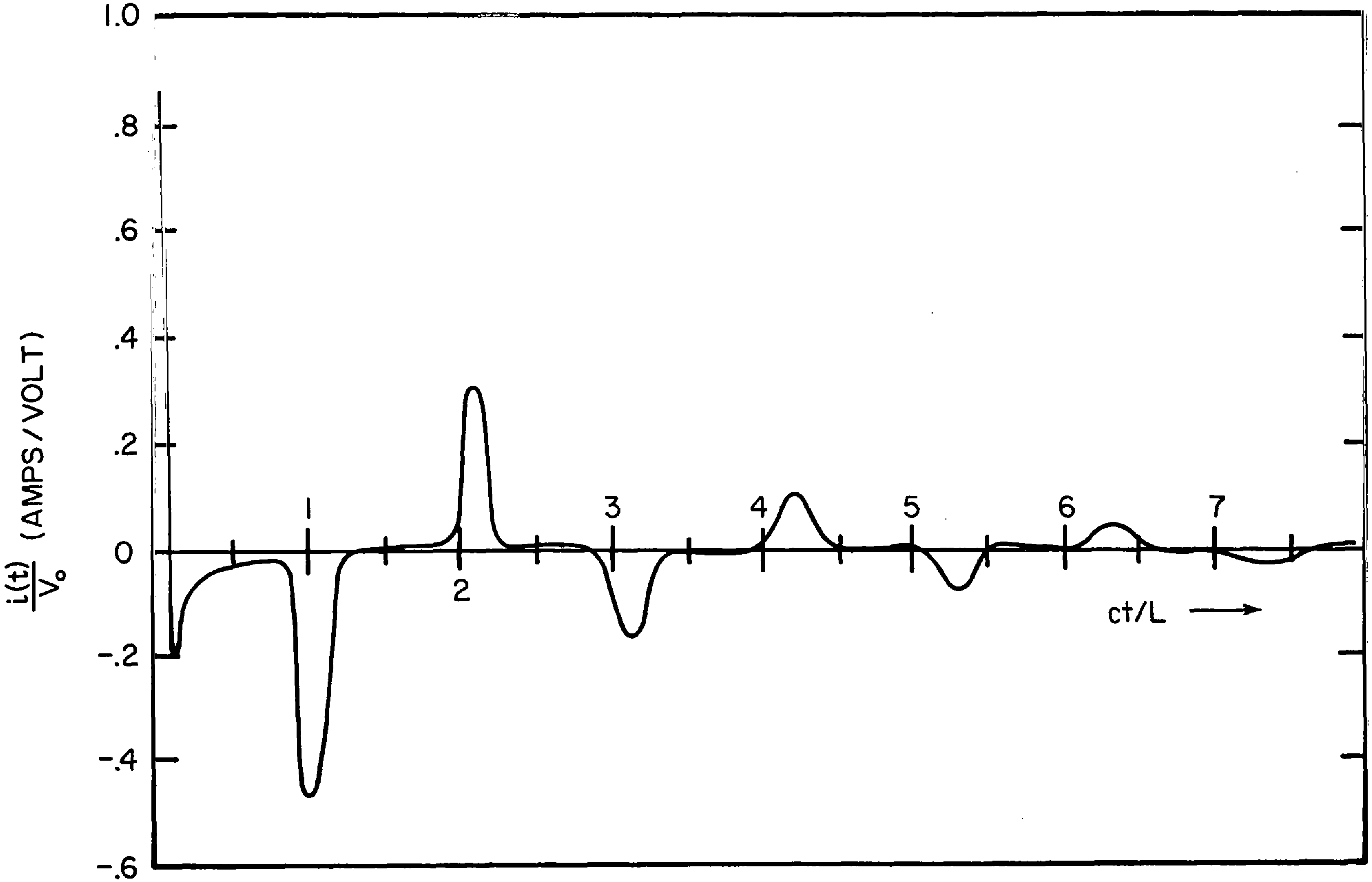

Fig. 7. Time response of the driving-point current of the antenna center-driven by an impulsive voltage source. Ω = 10.

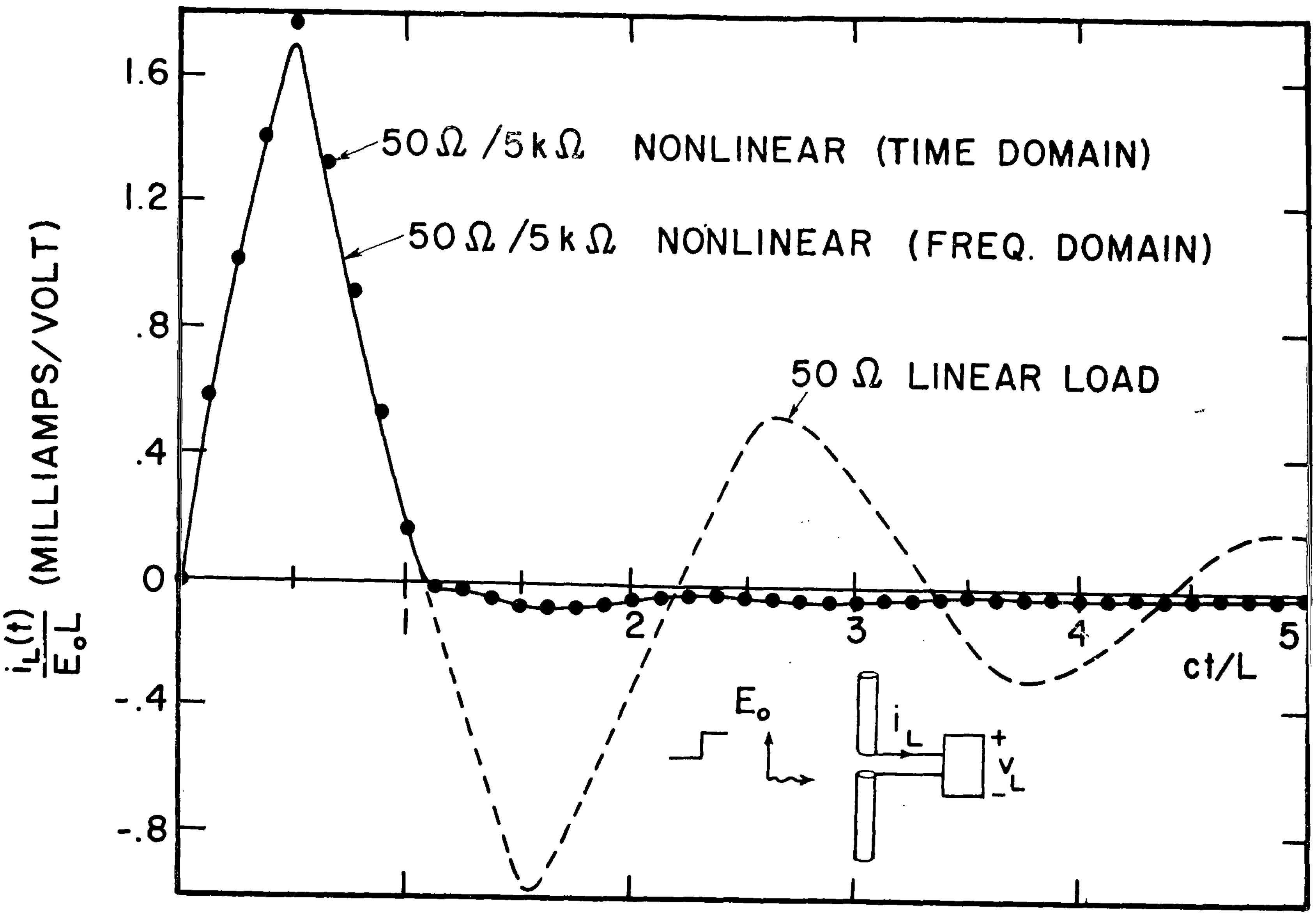

Fig. 8. Current through the 50 Ω /5 k Ω non-linear load. Both the time domain and the frequency domain methods are presented. Also included is the current through a 50 Ω linear load.

antenna with reversed polarity, it sees the $5\,k\Omega$ portion of the load which is large enough to reduce the amount of current flow. The $5\,k\Omega$ portion in effect behaves like an open-circuit and the current is reflected at $z = L/2$ with reversed polarity and with reduced magnitude. Subsequent reflections at the ends of the antenna again reverse the current polarity and the current again sees the $5\,k\Omega$ portion at $z = L/2$. This process continues and the antenna effectively behaves like two coupled collinear antennas each with length $L/2$. This behavior then accounts for the more rapid oscillation of the current.

The increase in the oscillation rate can be more easily observed in the plot of the time response of the load voltage, as shown in Fig. 9. The non-linear load voltage is obtainable from the $v - i$ relation of the load. Also included in Fig. 9 is the voltage across a linear $50\,\Omega$ resistor which loads the antenna at $z = L/2$. As may be noted, there is more than a 400% increase in the maximum load voltage when the non-linearity is encountered. This effect of increased voltage level should be taken into consideration in design to prevent any possible damage to the loading network.

The current at $z = L/4$ is shown in Fig. 10. Results from both the direct time domain method and the frequency domain method are shown, the latter being computed from Eq. (16). The results are in agreement, but do not agree quite as well as in the load current case (Fig. 8). The reason for this is that in the computation of the quantities $i_s(z_1,t)$ and $y_t(z_1,t)$, a spatial interpolation must be performed because the moment method solution of the integral equation (A4) does not give the current at precisely $z = L/4$. This interpolation procedure gives rise to errors in the antenna parameters used in Eq. (16). It should be mentioned that the necessity of interpolation could have been avoided had the number of zones been chosen properly. Superimposed in Fig. 10 is the current at $z = L/4$ when the antenna is center-loaded by a linear $50\,\Omega$ resistor. The more rapid oscillation behavior of the non-linear case is again evident.

As mentioned earlier, the problem that an antenna being driven via a non-linear load can be readily solved by methods identical to those used for the scattering case. The case of an antenna center-driven by a step-function voltage source with magnitude V_o via a $50\,\Omega / 5\,k\Omega$ piecewise-linear resistor

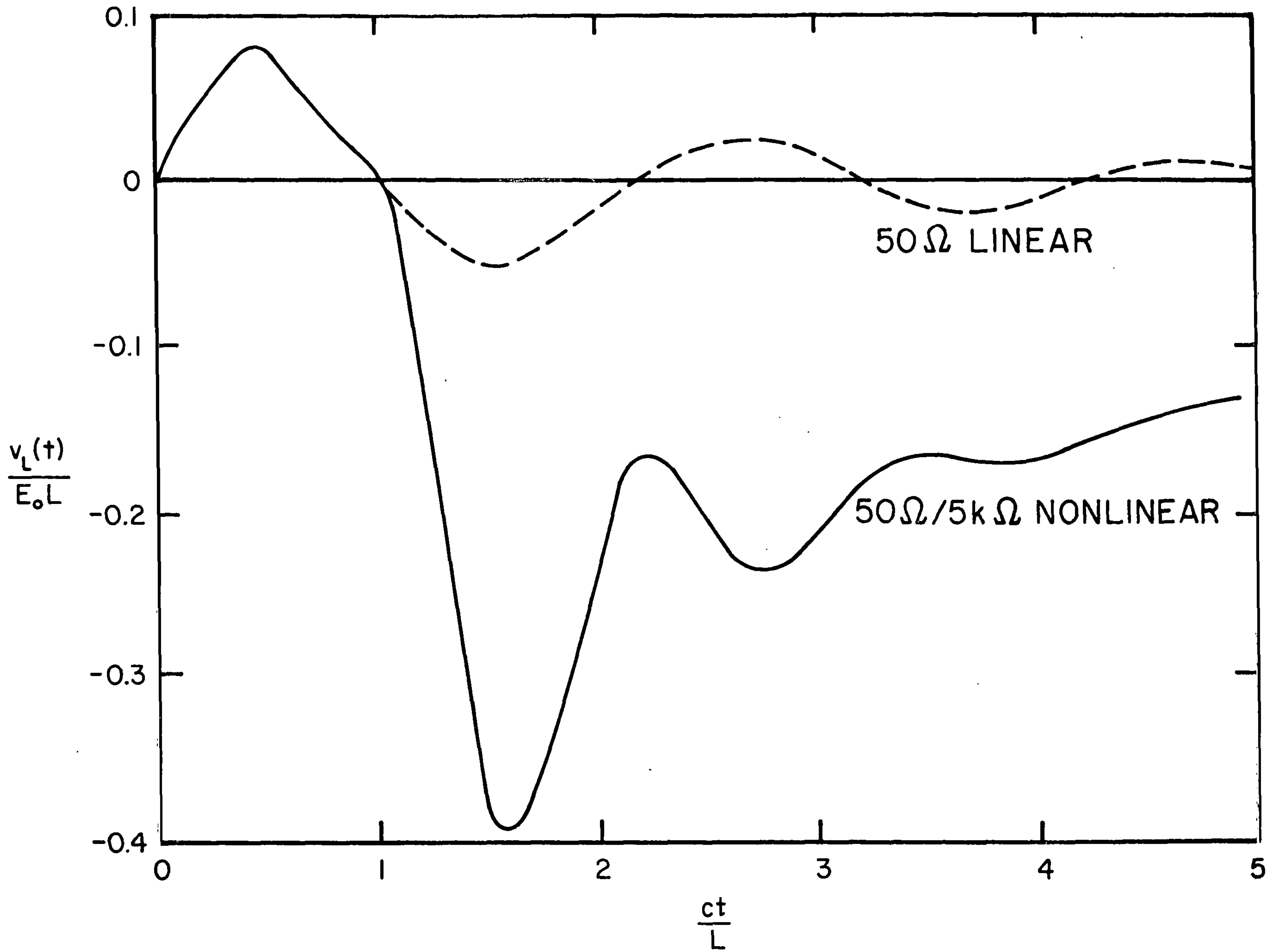

Fig. 9. Time response of the load voltage across the 50 Ω /5 k Ω non-linear load and the 50 Ω linear load.

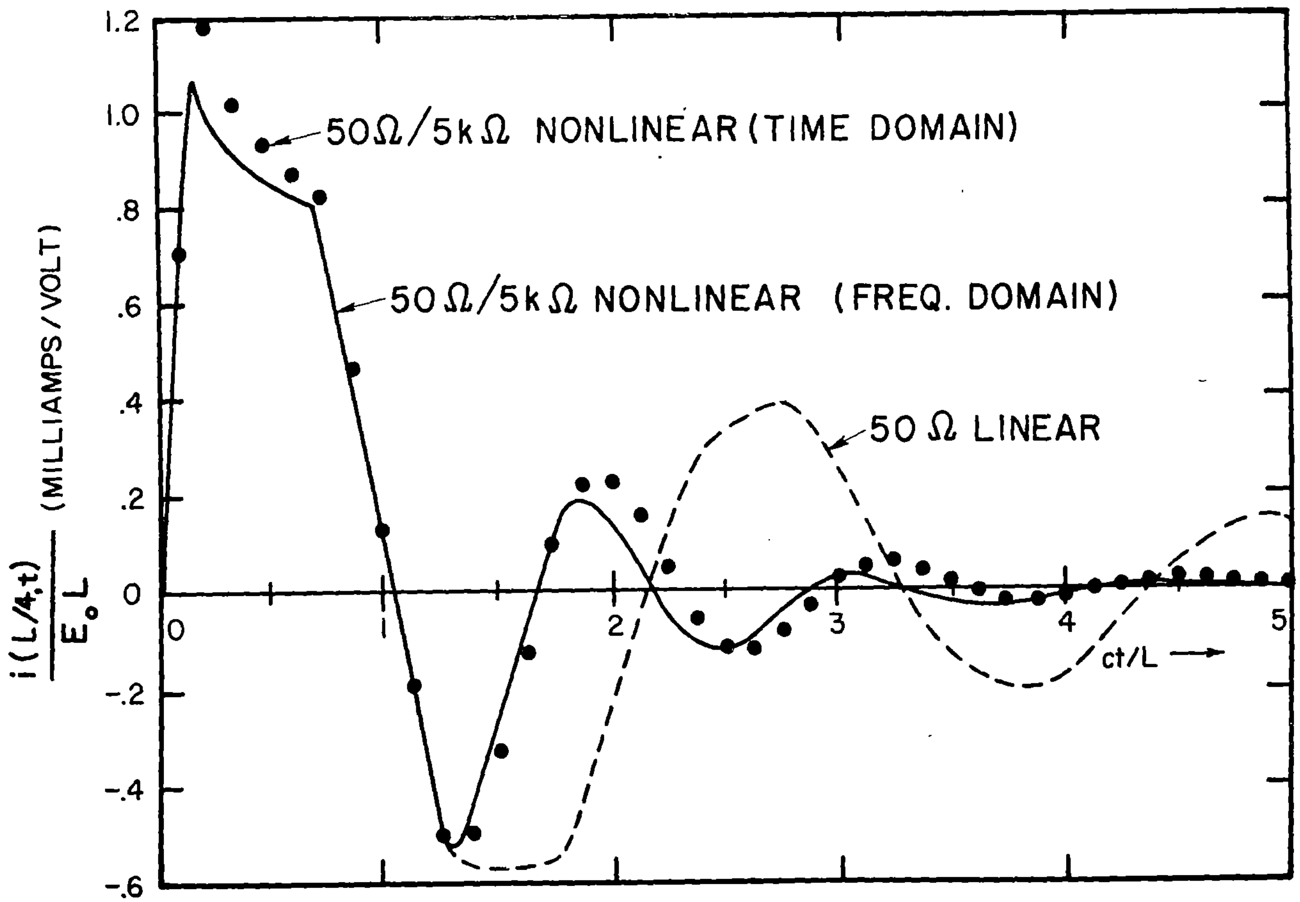

Fig. 10. Time response of the induced current at $z = L/4$.

has been analyzed. Again, the initial current is assumed to flow through the 50 Ω portion of the non-linear element. The driving-point current and the voltage across the non-linear element are presented in Fig. 11. Also included in Fig. 11 is the driving point current in the absence of the non-linear element. As expected from the reasons given above for the scattering problems, the non-linear case exhibits more rapid oscillations.

A more general class of non-linear loads is that whose response is dependent on the magnitude of the exciting field, E_o, or the exciting voltage V_o. A simple example for the scattering case is the same piecewise-linear load treated above, but with an additional break point at $v = -10$ volts. For $V_L < -10$ volts, the load resistance is assumed to be 10 Ω, instead of 5 kΩ. This roughly models the reverse breakdown of a zener diode. Fig. 12 shows the load current for three different values of E_oL. For $E_oL = 10$ volts, the result is the same as shown in Fig. 8 because the excitation is not strong enough to drive the diode to breakdown. For $E_oL = 50$ and 100 volts, the effects of the reverse breakdown are evident.

Fig. 13 illustrates the steady-state response over one period of the load current to a time-harmonic signal. The incident field is of the form $E^{inc}(\omega) = E_o \sin(k_oL \cdot ct/L)$ with $k_oL = 1.718$, where k_o is the free-space wave number. Thus, at $t = 0$, the incident field is just beginning to grow in time.

The same zener diode as employed in the previous figure is assumed to be connected to the antenna. For $E_oL = 10$ volts, the diode does not break down and the rectifying nature of the diode is obvious. As the excitation increases, the zener breakdown begins to substantially modify the load current behavior.

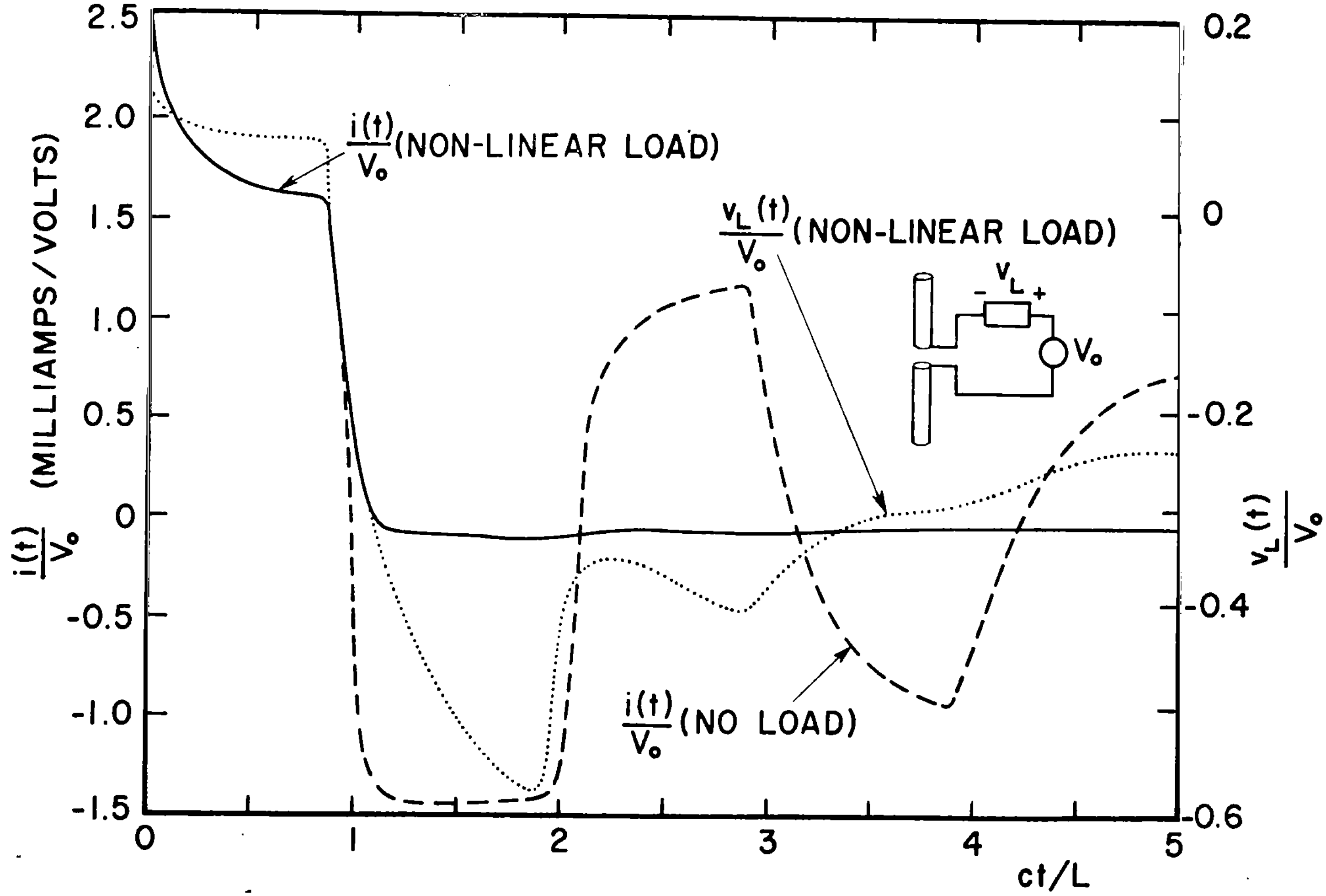

Fig. 11. Time response of the driving-point current and the load voltage when the antenna is center-driven by a step-function voltage source (with magnitude V_o) through a $50\,\Omega/5\,k\Omega$ non-linear load. Also included is the driving-point current in the absence of the non-linear element. $\Omega = 2\,\ell n(L/a) = 10$.

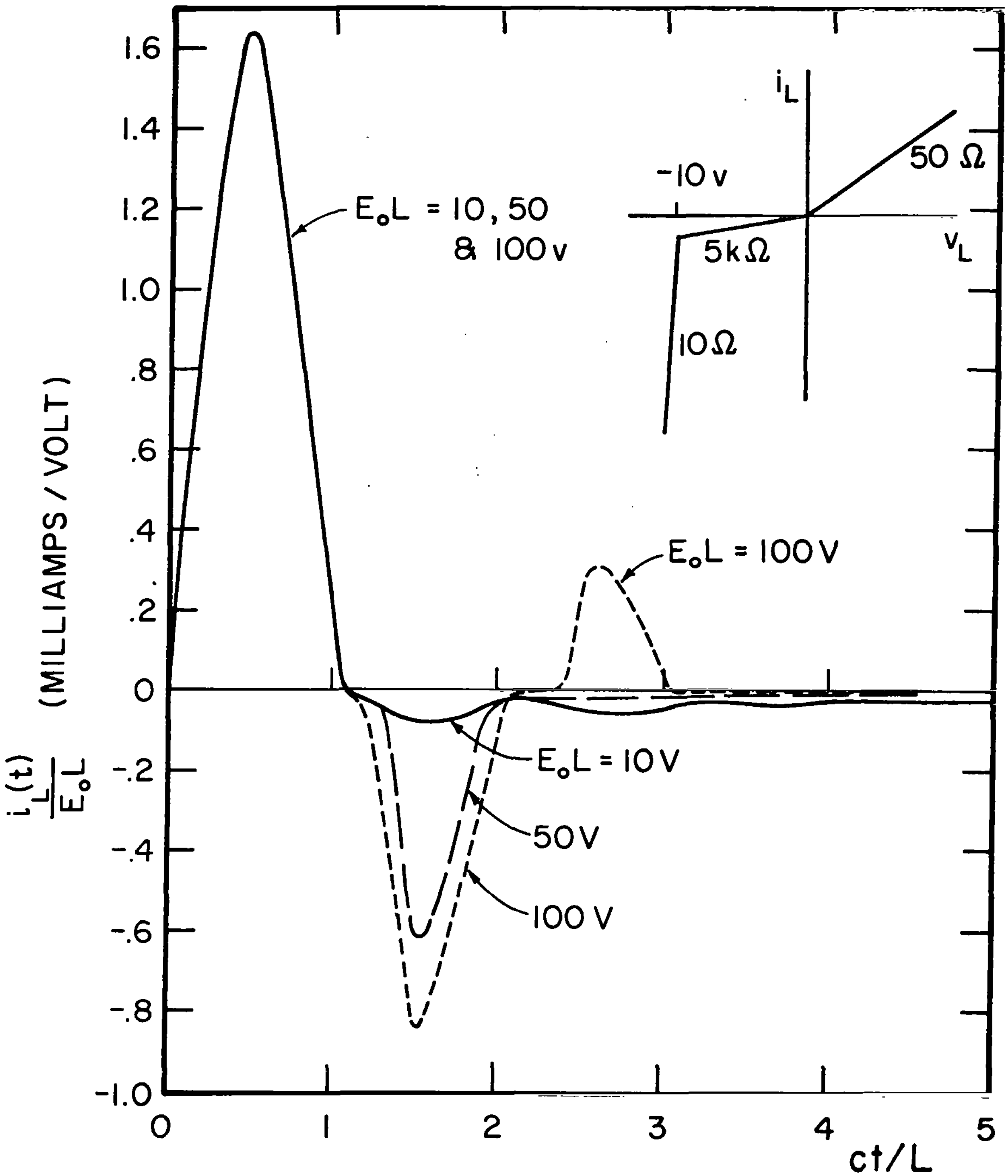

Fig. 12. Time response of the load current of an antenna center-loaded by a 50 Ω/5 kΩ /10 Ω non-linear load, excited by a step-function, broadside incident electric field with various field strength E_o. Ω = 10.

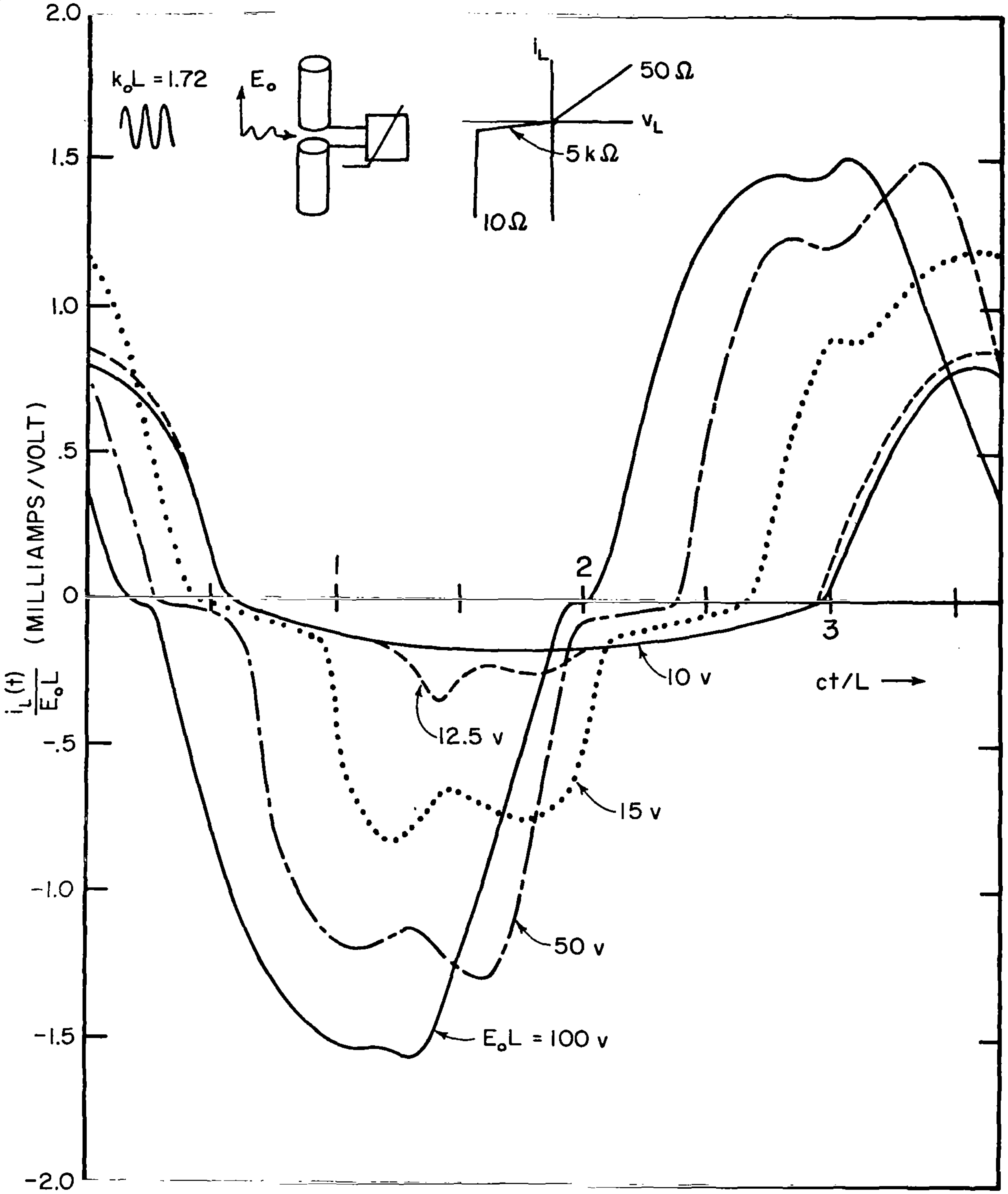

Fig. 13. Time response of the load current at steady-state of an antenna center-loaded by a 50 Ω/5 kΩ/10 Ω non-linear load, excited by a time-harmonic (with frequency k_oL = 1.72), broadside incident electric field with various field strength E_o. Ω = 10.

V. Conclusions

Two methods of analyzing antennas loaded by non-linear resistors have been described. The results obtained using these different methods agree very well. The second method, which makes use of frequency domain data determined for the unloaded antenna is extremely valuable in view of the fact that large amounts of frequency domain data currently are available for many different types of antennas. This method can be easily implemented for many practical antenna systems.

The idealized examples chosen fall into the category that the solution of the non-linear algebraic equation (15) is unique. For a practical problem, small package or junction capacitances and small lead inductances of the non-linear device are present. As have been pointed out, these quantities help to yield a non-linear differential equation which can be solved uniquely.

Both methods of solution can be easily extended to treat the case that the load is either a non-linear capacitance or a non-linear inductance. Such an extension will be useful in dealing with problems involving, for example, varactor diodes or ferromagnetic devices.

VI. Appendices

Appendix A. Voltage-Controlled Device in Shunt with a Capacitance

In this appendix we illustrate how to describe the problem of an antenna with a non-linear voltage-controlled device by a non-linear differential equation, of which a unique solution exists.

As mentioned in the text, a small but physically-real capacitance C_s is assumed to be in shunt with the non-linear load whose $v-i$ relation is given by

$$i_L(t) = F'[v_L(t)] \tag{A1}$$

where F' is a known function. This configuration is depicted in Fig. A1. For $z = z_o$, the antenna current $i(z_o,t)$ is found to be

$$i(z_o,t) = i_L(t) + C_s \frac{d}{dt} v_L(t). \tag{A2}$$

Substituting Eqs. (A1) and (A2) into Eq. (4) yields

$$\alpha C_s \frac{d}{dt} v_L(t) = S(z,t) - (2Z_o)^{-1} v_L(t) - \alpha F'[v_L(t)]. \tag{A3}$$

Eq. (A3) is a non-linear differential equation in the unknown $v_L(t)$ and can be readily solved at each time step, using a suitable algorithm for solving ordinary differential equations. For $z \neq z_o$, the antenna current is given by Eq. (4) where the right-hand-side is known.

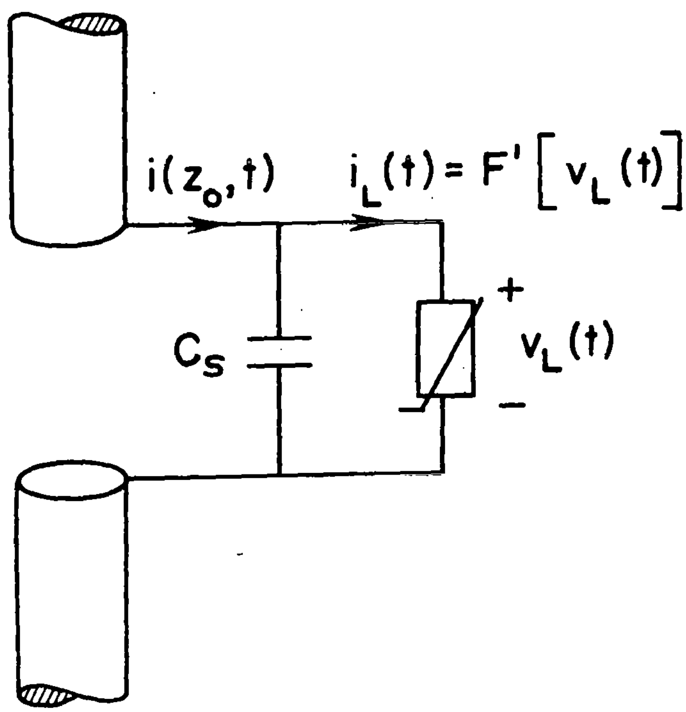

Fig. A1. Antenna with a non-linear load in shunt with a capacitance C_s.

Appendix B. The Frequency Domain Integral Equation

To obtain a solution to Eq. (8) it is necessary to have the time domain quantities $i_{sc}(t)$ and $y_{in}(t)$. These can be obtained by the Fourier transform of the corresponding frequency domain quantities $I_{sc}(\omega)$ and $Y_{in}(\omega)$.

These quantities can be computed for the linear antenna of length L and radius a by using the Pocklington integro-differential equation

$$-j\omega\varepsilon E^{inc}(z,\omega) = \left(\frac{d^2}{dz^2} + k^2\right) \int_0^L I(z',\omega)\, G(z,z';\omega)dz' \tag{A4}$$

where the Green's function G has the form

$$G(z,z';\omega) = \frac{1}{2\pi a} \int_0^{2\pi} \frac{e^{-jk[(z-z')^2 + \{2a \sin(\theta/2)\}^2]^{\frac{1}{2}}}\, ad\theta}{4\pi[(z-z')^2 + \{2a \sin(\theta/2)\}^2]^{\frac{1}{2}}} \,. \tag{A5}$$

In this expression, $k = \omega/c$, $I(z',\omega)$ is the total axial current flowing in the antenna and $E^{inc}(z,\omega)$ is the frequency domain component of the exciting electric field tangential to the antenna. For an accurate high frequency response, the exact kernel is employed in Eq. (A5) instead of the easier to evaluate thin-wire kernel [9].

The solution of Eq. (A4) is well documented in the literature [10], [11] and involves the use of the moment method to form a system of linear, algebraic equations with unknowns being related to the current distribution along the antenna. Once the matrix of this system of equations has been inverted to find the unknown current, it is possible to change the excitation of the antenna and obtain the resulting current with very little additional effort.

Thus, at a particular frequency, Eq. (A4) is solved first for the scattering case which involves an incident electric field defined over the entire structure to yield the short-circuit current at the antenna terminals, $I_{sc}(\omega)$. Then the driven case is considered which involves a constant electric field existing only over the antenna source region; the source region is assumed to be finite in size to insure a finite capacitance [9]. The resulting current distribution is again evaluated and the input admittance of the antenna calculated from the current at the terminals by

$$Y_{in}(\omega) = I_{in}(\omega)/V \tag{A6}$$

where V is the exciting voltage defined as

$$V = \int_{GAP} E^{inc}(z)dz. \tag{A7}$$

The previous calculations are repeated for a wide range of frequencies (typically from $kL = 0$ to $kL = 45$ for the problems under consideration) so that reasonably accurate time responses can be found using the Fourier transform method.

In an analogous manner, the current induced at any point on the short-circuited scatterer and the transfer admittance of the current at the same point due to an applied voltage across the antenna gap can be computed. Again, time domain responses can be found using the Fourier transform method.

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