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ANALYSIS OF CROSSED WIRES IN A PLANE-WAVE FIELD

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ABSTRACT

The currents and charges induced in a pair of electrically thin crossed wires by a normally incident plane electromagnetic wave are derived by analytical methods. The boundary conditions at the junction are explained and compared with the somewhat different ones used in the past. The solution of a new integro-differential equation for the currents is obtained in terms of trigonometric and integral-trigonometric functions. Depending on the electrical lengths of the crossed elements and the location of their junction a variety of quite different distributions of current and charge obtain. These determine the scattered near and far fields.

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1. Introduction

A knowledge of the distributions of current and charge induced on crossed conductors illuminated by an incident plane electromagnetic wave is prerequisite to the determination of the scattered field. At distant points this is of interest in radar; near the surfaces of the conductors it can be used to estimate the field in the interior of the conductors when these have holes or slots. An important structure consisting of crossed conductors is an aircraft with its' wings and fuselage. Since the cross sections of these members are generally not electrically small, the determination of the distributions of surface current and charge per unit area is a formidable problem. Attempts have been made to simulate an aircraft by crossed circular cylinders and to determine the distributions of current and charge on these by thin-wire antenna theory. Since transverse currents are ignored in the thin-wire approximation and these are significant on electrically thick conductors, the currents and charges calculated for crossed thin conductors are not at all representative of those that obtain on thick cylinders. Nevertheless, a complete and accurate determination of their properties could serve as a valuable first step in learning to understand the behavior of currents and charges on crossed electrically thick conductors. This is true in particular of the junction region.

Crossed electrically thin wires excited by an incident plane wave have received the attention of numerous investigators but so far only through the application of well-known numerical methods [1]-[4]. These have been applied to solve coupled integral equations subject to a set of boundary and junction conditions which do not adequately characterize the junction. Graphs of numerically computed distributions of the induced currents have been displayed [2]-[4] for a small number of crosses constructed of relatively short wires that never exceed 0.3 wavelength, measured from the junction to the end of the longest arm. However, even if the correct conditions at the junction had been used, these would be quite inadequate to provide a general insight into the behavior of currents and charges on crossed wires, especially near the junction under conditions of resonance and antiresonance with their quite different standing-wave patterns. It is, for example, possible to locate the junction at points of minimum current and maximum charge per unit length, maximum current and minimum charge per unit length, or minimum current and minimum charge per unit length with corresponding very significant differences in the coupling among the arms of the cross.

In order to obtain a generally useful understanding of the distributions of current and charge per unit length on parasitic crossed wires, an analytical solution is desirable. This should provide physically meaningful formulas which reveal the dependence of the distributions on the lengths of the arms and the location of the junction.

2. Formulation of the Problem: Boundary and Junction Conditions

In order not to complicate the problem unnecessarily it will be assumed that the plane of the crossed wires coincides with a wave front of a normally incident plane electromagnetic wave with its electric vector parallel to one of the wires. For mutually perpendicular wires, the solution with the electric vector parallel to the second wire is obtained by a simple change in notation. A superposition of the solutions for the two polarizations provides the solution for an arbitrarily polarized normally incident wave. A relatively simple extension of the theory can be made to deal with other than mutually perpendicular wires.

The crossed wires in their relations to the incident electromagnetic wave are shown in Fig. 1. The vertical wire extends from $z = -h_1$ to $z = h_2$, the horizontal wire from $x = -\ell_1$ to $x = \ell_2$; the center of the junction is at $x = 0$, $y = 0$, $z = 0$. The wires all have the same radius a and this is sufficiently small so that the following inequalities are satisfied:

$$ka = 2\pi a/\lambda << 1 \quad , \quad h_i/a >> 1 \quad , \quad \ell_i/a >> 1 \tag{1}$$

where $i = 1$ or 2 and $k = \omega/c = 2\pi/\lambda$ is the wave number. The incident field is $E_z^{inc}(y) = E_z^{inc} e^{-jky}$ where E_z^{inc} is the value at $y = 0$, i.e., along the axis of the vertical conductor.

Under the action of the incident field a standing-wave distribution of current and charge is induced in the vertical conductor and these, in turn, induce a distribution in the horizontal conductor. The currents and charges in all conductors are so distributed that the total tangential electric field vanishes on the surfaces of the perfect conductors. In general, the distributions of current and charge per unit length are different in the four arms of the cross. Since the conductors are assumed to be perfect, the currents and charges are confined to thin layers on the surfaces. Subject to the condition $ka << 1$ the transverse currents induced on the horizontal conductor and near and at the ends and junction of the vertical conductor are negligible and may be ignored. Since the excitation by the incident field and by mutual interaction is not rotationally symmetrical, the induced axial currents and associated charges per unit length also depart from rotational symmetry. However, when $ka << 1$, this departure is very small and can be disregarded. This means that the components $K_x(x)$ and $K_z(z)$ of the surface density of current and the associated surface densities of charge $\eta(x)$ and $\eta(z)$ are functions of the axial coordinates only as indicated. The total axial currents and charges per unit length and the equations of continuity they

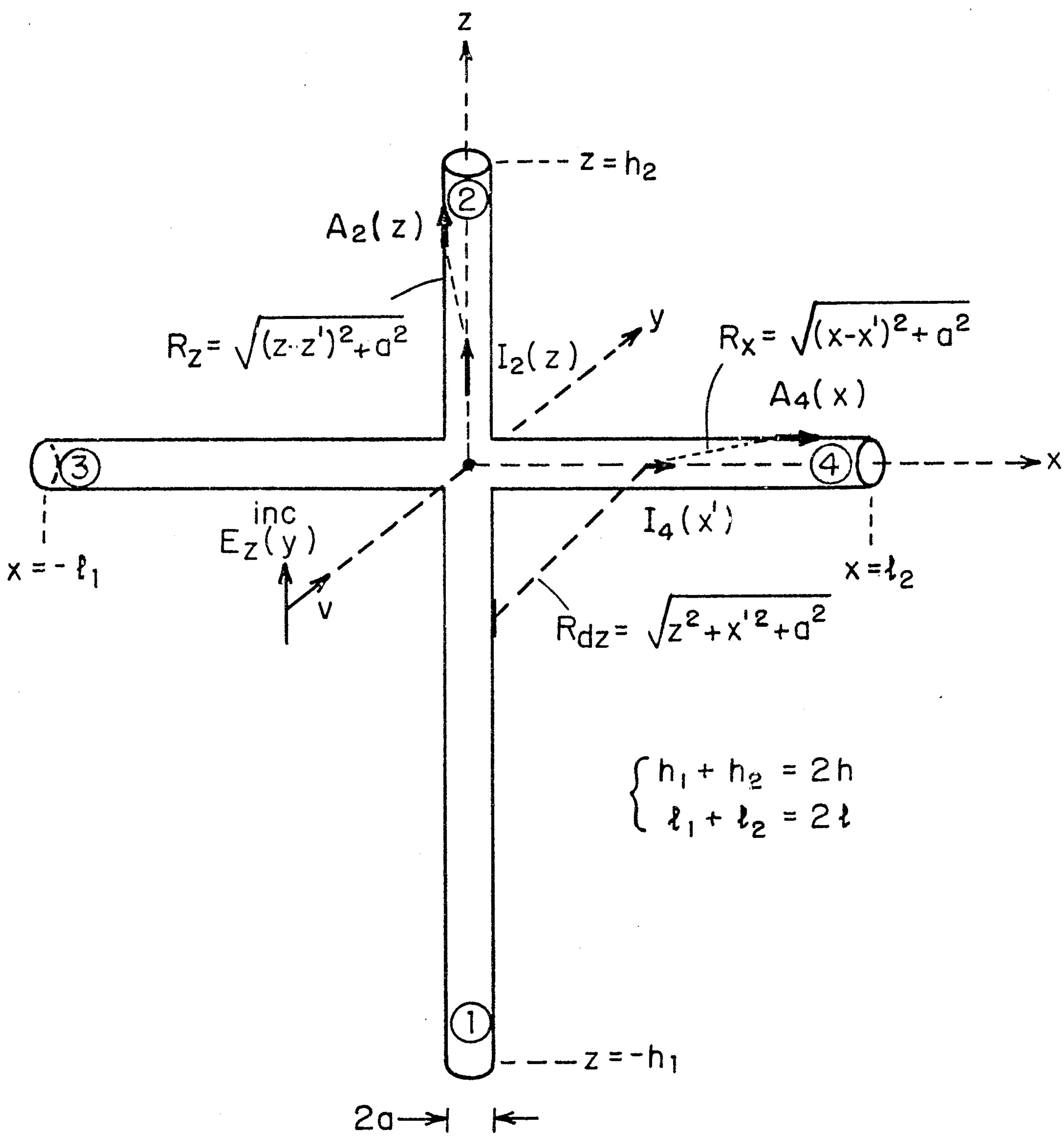

FIG. 1 CROSSED WIRES IN AN INCIDENT PLANE-WAVE FIELD

satisfy are:

$$I_z(z) = 2\pi a K_z(z) \quad ; \quad q(z) = 2\pi a \eta(z) \quad ; \quad \frac{\partial I_z(z)}{\partial z} + j\omega q(z) = 0 \tag{2a}$$

$$I_x(x) = 2\pi a K_x(x) \quad ; \quad q(x) = 2\pi a \eta(x) \quad ; \quad \frac{\partial I_x(x)}{\partial x} + j\omega q(x) = 0 \tag{2b}$$

The four sets of current and charge are $I_{1z}(z)$, $q_1(z)$ in the range $-h_1 \le z \le 0$; $I_{2z}(z)$, $q_2(z)$ in the range $0 \le z \le h_2$; $I_{3x}(x)$, $q_3(x)$ in the range $-\ell_1 \le x \le 0$; and $I_{4x}(x)$, $q_4(x)$ in the range $0 \le x \le \ell_2$. In order to determine these currents appropriate boundary conditions must be formulated to specify their behavior at the respective ends of the arms. At the open ends of tubular conductors, the total current must vanish so that

$$I_{1z}(-h_1) = I_{2z}(h_2) = I_{3x}(-\ell_1) = I_{4x}(\ell_2) = 0 \tag{3}$$

The specification of the behavior of the currents and charges at the junction is made difficult by the complicated geometry that makes the boundaries between chargeable surfaces belonging to the four arms and the surface belonging to the junction ambiguous. However, since these latter have an area of the order a^2 which is electrically negligible under the condition $ka << 1$ of thin-wire theory, each of the conductors can be assumed to end at $x = z = 0$ and the small overlapping areas can be ignored. Alternatively and equivalently, the currents and charges may be treated as if concentrated at average locations on the axes of the conductors instead of on their surfaces. For them the junction is a single point at $x = z = 0$ with no chargeable surface.

To complement the four conditions (3) on the currents at the outer ends of the conductors, four additional conditions must be established at the inner ends, i.e., at the junction at $x = z = 0$. In view of the requirement ka << 1, all interactions associated with charges and currents near the junction and characteristic of its properties are in the very near zone, i.e., they are quasi-stationary. As a consequence, the conditions familiar from low-frequency electric circuits obtain. With the continuity properties of the electric field it follows that

$$I_{1z}(0) - I_{2z}(0) + I_{3x}(0) - I_{4x}(0) = 0 \tag{4}$$

$$q_1(0) = q_2(0) = q_3(0) = q_4(0) \tag{5}$$

With the equations of continuity in (2a,b), an alternative form of (5) is

$$\left[\frac{\partial I_{1z}(z)}{\partial z}\right]_{z=0} = \left[\frac{\partial I_{2z}(z)}{\partial z}\right]_{z=0} = \left[\frac{\partial I_{3x}(x)}{\partial x}\right]_{x=0} = \left[\frac{\partial I_{4x}(x)}{\partial x}\right]_{x=0} \tag{6}$$

The first condition is a necessary consequence of the conservation of electric charge and the absence of significant chargeable surfaces on the junction as distinct from the ends of the four conductors. The second condition is usually not expressed in low-frequency circuit theory since there are no charges on the surfaces of the conductors (except on the inner surfaces of condensers). In effect, in low-frequency circuits all conductors and their junctions are at a maximum of current and a zero of charge per unit length in a standing-wave pattern. In circuits like transmission lines and antennas that are not electrically short, a junction may be located at an arbitrary point in a standing-wave pattern, so that significant concentrations of charge per unit length may exist on the conductors at and near the junction. The condition (5) assures that discontinuities in charge per unit length are ruled out in moving from one conductor to another across the junction. Such discontinuities could not exist in the absence of delta-function generators.

The condition (5) in a superficially different form is regularly applied at the junction of four mutually perpendicular transmission lines. Note that with the equations of continuity in (2a,b) and one of the familiar first-order transmission-line equations, the following relations obtain:

$$j\omega q_1(0) = -\left[\frac{\partial I_1(z)}{\partial z}\right]_{z=0} = yV_1(0) \tag{7}$$

where $V_1(0)$ is the voltage across the end of transmission line no. 1 at the junction, $I_1(z)$ is the current, $q_1(z)$ the charge per unit length on the reference conductor (the inner conductor of a coaxial line), y is the admittance per unit length. Evidently, the equivalent of (5) for transmission lines is

$$V_1(0) = V_2(0) = V_3(0) = V_4(0) \tag{8}$$

This equivalence exists because in balanced two-wire and coaxial transmission lines

$$q(z) = -(jy/\omega)V(z) \tag{9}$$

Note that in a balanced two-wire line $V(z) = 2\phi_1(z)$, in a coaxial line $V(z) = \phi_1(z)$ where $\phi_1(z)$ is the scalar potential on the surface of the reference conductor at the point z.

On an electrically thin antenna the charge per unit length at a point z is approximately proportional to the scalar potential on the surface of the conductor at z when this point is (a) not near a minimum of charge in the standing-wave pattern, and (b) not near an open end or a junction. For the crossed antennas under study, the condition

$$\phi_1(0) = \phi_2(0) = \phi_3(0) = \phi_4(0) \tag{10}$$

is not equivalent to (5). That the difference is significant is made evident by a study of the distributions of current in Figs. 1 and 3 of Reference 4 which were computed using (10) and not (5). At the junction the slopes of the large imaginary parts of the several currents differ significantly. This means that the charges per unit length on these conductors are discontinuous across the junction which is obviously incorrect.

One author [3] has used not only the conditions (10) in place of (5) but has supplemented these with additional conditions of continuity on the components of the vector potential along each wire at the junction, i.e., $A_x(x)$ and $A_z(z)$ are required to be continuous at $x = z = 0$. Since $A_x(x)$ and $A_z(z)$ [like $\phi(z)$] are in any case continuous at all points along both conductors for all distributions of current including those that are discontinuous, these conditions do not characterize the particular properties of the junction. If (10) is used as an approximation of (5), the Lorentz conditions, $\partial A_z(z)/\partial z + j(k^2/\omega)\phi(z) = 0$, $\partial A_x(x)/\partial x + j(k^2/\omega)\phi(x) = 0$, show that it is equivalent to:

$$\left[\frac{\partial A_{1z}(z)}{\partial z}\right]_{z=0} = \left[\frac{\partial A_{2z}(z)}{\partial z}\right]_{z=0} = \left[\frac{\partial A_{3x}(x)}{\partial x}\right]_{x=0} = \left[\frac{\partial A_{4x}(x)}{\partial x}\right]_{x=0} \qquad (11)$$

No other condition on $A_x(x)$ and $A_z(z)$ at the junction is required. This is also evident in [3] where ten conditions are imposed to determine eight constants of integration. The redundant two are those requiring continuity of $A_x(x)$ and $A_z(z)$ at the junction. They are no more necessary at the junction than at the ends of the conductors or any point along them.

In concluding this introductory discussion of conditions at the junction of electrically thin wires, it is important to emphasize that the eight requirements in (3), (4) and (5) are the necessary and sufficient conditions for determining the currents in the four arms of the crossed antennas. The substitution of (10) for (5) or (11) for (6) is an approximation that involves significant errors in the calculated charges on the conductors near and at the junction and, therefore, in the near electric field. The magnitude of these errors depends somewhat on the location of the junction in the standing-wave patterns that obtain on the two crossed wires.

3. Analytical Formulation

Since the correct boundary conditions at both the ends of the crossed antennas and their junctions involve not only the currents but the derivatives of these, an integral-equation formulation in which the constants of integration appear in the expressions for the potentials is not convenient. New and somewhat different integral equations for the currents are, therefore, sought. Their derivation necessarily begins with the following boundary conditions on the surfaces of the two perfect conductors:

$$E_z(z) = E_z^{inc} - \frac{\partial\phi(z)}{\partial z} - j\omega A_z(z) = 0 \quad ; \quad -h_1 \le z \le h_2 \tag{12a}$$

$$E_x(x) = -\frac{\partial\phi(x)}{\partial x} - j\omega A_x(x) = 0 \quad ; \quad -\ell_1 \le x \le \ell_2 \tag{12b}$$

where the vector and scalar potentials on the surfaces of the conductors are:

$$A_z(z) = \frac{\mu_0}{4\pi}\int_{-h_1}^{h_2} I_z(z')K(z,z')\,dz' \tag{13a}$$

$$A_x(x) = \frac{\mu_0}{4\pi}\int_{-\ell_1}^{\ell_2} I_x(x')K(x,x')\,dx' \tag{13b}$$

$$\phi(z) = \frac{1}{4\pi\epsilon_0}\left[\int_{-h_1}^{h_2} q(z')K(z,z')\,dz' + \int_{-\ell_1}^{\ell_2} q(x')K(z,x')\,dx'\right] \tag{14a}$$

$$\phi(x) = \frac{1}{4\pi\epsilon_0}\left[\int_{-\ell_1}^{\ell_2} q(x')K(x,x')\,dx' + \int_{-h_1}^{h_2} q(z')K(x,z')\,dz'\right] \tag{14b}$$

The average kernels are:

$$K(z,z') = \frac{e^{-jkR_z}}{R_z} \quad , \quad R_z = \sqrt{(z - z')^2 + a^2} \tag{15a}$$

$$K(z,x') = \frac{e^{-jkR_{cz}}}{R_{cz}} \quad , \quad R_{cz} = \sqrt{z^2 + x'^2 + a^2} \tag{15b}$$

Note that

$$K(z,z') = K_R(z,z') + jK_I(z,z') \tag{16a}$$

where

$$K_R(z,z') = \frac{\cos kR_z}{R_z} \quad , \quad K_I(z,z') = -\frac{\sin kR_z}{R_z} \tag{16b}$$

and

$$k = \omega\sqrt{\mu_0\varepsilon_0} \tag{17}$$

The following notation is used: $A_z(z) = A_1(z)$, $\phi(z) = \phi_1(z)$, $I_z(z) = I_1(z)$, $q(z) = q_1(z)$ when $-h_1 \leq z \leq 0$; $A_z(z) = A_2(z)$, $\phi(z) = \phi_2(z)$, $I_z(z) = I_2(z)$, $q(z) = q_2(z)$ when $0 \leq z \leq h_2$; $A_x(x) = A_3(x)$, $\phi(x) = \phi_3(x)$, $I_x(x) = I_3(x)$, $q(x) = q_3(x)$ when $-\ell_1 \leq x \leq 0$; $A_x(x) = A_4(x)$, $\phi(x) = \phi_4(x)$, $I_x(x) = I_4(x)$, $q(x) = q_4(x)$ when $0 \leq x \leq \ell_2$.

In the analysis of single and parallel conductors it is customary to introduce the Lorentz condition, $\nabla \cdot \vec{A} + j(k^2/\omega)\phi = 0$, explicitly in (12a) to eliminate the scalar potential. This procedure was also followed by Butler [3] in his formulation of the integral equations for crossed wires. When this is done, the scalar potential $\phi(z)$ on the surface of the vertical wire is, in effect, separated into the two parts expressed by the two integrals in (14a). These are treated separately and differently. The first is combined directly with $A_z(z)$ to form a simple differential equation, the latter remains as the inhomogeneous term. In the solution, the first integral in (14a) is then included in the complementary function, the second in a particular integral. This procedure is not followed here. Instead the integrals in (13a,b) and (14a,b) are inserted directly into (12a,b) to obtain the following basic equations:

$$\int_{-h_1}^{h_2} I(z')K(z,z')\,dz' - \frac{j\omega}{k^2}\frac{\partial}{\partial z}\left[\int_{-h_1}^{h_2} q(z')K(z,z')\,dz' + \int_{-\ell_1}^{\ell_2} q(x')K(z,x')\,dx'\right]$$

$$= -\frac{j4\pi}{\omega\mu}E_z^{inc} \tag{18a}$$

$$\int_{-\ell_1}^{\ell_2} I(x')K(x,x')\,dx' - \frac{j\omega}{k^2}\frac{\partial}{\partial x}\left[\int_{-\ell_1}^{\ell_2} q(x')K(x,x')\,dx' + \int_{-h_1}^{h_2} q(z')K(x,z')\,dz'\right]$$

$$= 0 \tag{18b}$$

These are to be solved for the currents and charges per unit length in the two conductors subject to the boundary conditions (3) at the ends of the wires and the junction conditions (4) and (5) where they intersect. The continuity equations in (2a,b) relate the currents and charges per unit length.

4. Formal Solution of the Integral Equations

Before obtaining a solution of (18a,b) it is convenient to make use of the equations of continuity (2a,b) in the middle integrals in (18a) and (18b). Note also that $\partial K(z,z')/\partial z = -\partial K(z,z')/\partial z'$. The desired relations are:

$$J(z) \equiv j\omega\frac{\partial}{\partial z}\int_{-h_1}^{h_2} q(z')K(z,z')\,dz' = -\frac{\partial}{\partial z}\int_{-h_1}^{h_2}\frac{\partial I(z')}{\partial z'}K(z,z')\,dz'$$

$$= \int_{-h_1}^{h_2}\frac{\partial I(z')}{\partial z'}\frac{\partial}{\partial z'}K(z,z')\,dz' \tag{19a}$$

Integration by parts now yields:

$$J(z) = -j\omega[q(h_2)K(z,h_2) - q(-h_1)K(z,-h_1)] - \int_{-h_1}^{h_2} \frac{\partial^2 I(z')}{\partial z'^2} K(z,z')\, dz' \tag{19b}$$

Similarly,

$$J(x) = -j\omega[q(\ell_2)K(x,\ell_2) - q(-\ell_1)K(x,-\ell_1)] - \int_{-\ell_1}^{\ell_2} \frac{\partial^2 I(x')}{\partial x'^2} K(x,x')\, dx' \tag{19c}$$

With these expressions (18a,b) become:

$$\int_{-h_1}^{h_2} \left[\frac{\partial^2 I(z')}{\partial z'^2} + k^2 I(z') \right] K(z,z')\, dz' - F_2(z) - F_3(z) = \frac{-j4\pi k^2 E_z^{inc}}{\omega\mu} \tag{20a}$$

$$\int_{-\ell_1}^{\ell_2} \left[\frac{\partial^2 I(x')}{\partial x'^2} + k^2 I(x') \right] K(x,x')\, dx' - F_2(x) - F_3(x) = 0 \tag{20b}$$

where

$$F_2(z) = j\omega \frac{\partial}{\partial z} \int_{-\ell_1}^{\ell_2} q(x')K(z,x')\, dx' \tag{21a}$$

$$F_3(z) = -j\omega[q(h_2)K(z,h_2) - q(-h_1)K(z,-h_1)] \tag{21b}$$

The functions $F_2(x)$ and $F_3(x)$ are obtained from (21a,b) with the substitution of x for z and ℓ for h.

The equations (20a,b) can be simplified greatly if use is made of the peaking property of the real parts of the kernels in the integrals. These occur at $z' = z$ and $x' = x$ where $K_R(z,z)$ and $K_R(x,x)$ become very large when $ka << 1$. As a consequence, the relation

$$\int_{-h_1}^{h_2} f(z')K_R(z,z')\, dz' = \int_{-h_1}^{h_2} f(z') \frac{\cos k\sqrt{(z - z')^2 + a^2}}{\sqrt{(z - z')^2 + a^2}}\, dz' \doteq \Psi f(z) \tag{22a}$$

where Ψ is a constant is an excellent approximation for any function $f(z)$ such as $(\partial^2/\partial z^2 + k^2)I(z)$. [Note that (22a) is not valid when ka is not small.] The constant parameter Ψ is defined by

$$\Psi = f^{-1}(z_m) \int_{-h_1}^{h_2} f(z')K_R(z_m,z')\, dz' \tag{22b}$$

where z_m is a point near the maximum of $f(z)$. This integral is evaluated in Appendix A for the case at hand. It is there shown that Ψ depends only on ka and not on h_1 or h_2. Accordingly, the same parameter Ψ applies to the transverse conductor if it has the same radius as the vertical one.

With (19a,b,c) the coupled integral equations (20a,b) become:

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) I(z) = Ak^2 + \Psi^{-1}[F_1(z) + F_2(z) + F_3(z)] \tag{23a}$$

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) I(x) = \Psi^{-1}[F_1(x) + F_2(x) + F_3(x)] \tag{23b}$$

where

$$A = -\frac{j4\pi E_z^{inc}}{\omega\mu\Psi} = \frac{-j}{60\pi\Psi}(E_z^{inc}\lambda) \tag{24a}$$

and

$$F_1(z) = -j \int_{-h_1}^{h_2} \left[\frac{\partial^2 I(z')}{\partial z'^2} + k^2 I(z')\right] K_I(z,z')\, dz' \tag{24b}$$

The function $F_1(x)$ is obtained from (24b) with the substitution of x for z and ℓ for h.

The solution of the inhomogeneous one-dimensional wave equations (23a,b) consist of the simple solutions of the homogeneous equations plus sums of particular integrals due to the inhomogeneous terms. The formal solutions for the currents on the four parts of the crossed wires are:

$$I_1(z) = A[C_1' \cos kz + C_1'' \sin kz + 1] + H_h(z)/\Psi \quad ; \quad -h_1 \le z \le 0 \tag{25a}$$

$$I_2(z) = A[C_2' \cos kz + C_2'' \sin kz + 1] + H_h(z)/\Psi \quad ; \quad 0 \le z \le h_2 \tag{25b}$$

$$I_3(x) = A[C_3' \cos kx + C_3'' \sin kx] + H_\ell(x)/\Psi \quad ; \quad -\ell_1 \le x \le 0 \tag{25c}$$

$$I_4(x) = A[C_4' \cos kx + C_4'' \sin kx] + H_\ell(x)/\Psi \quad ; \quad 0 \le x \le \ell_2 \tag{25d}$$

where the C's are arbitrary constants of integration and

$$H_h(z) = T_1(z) + T_2(z) + T_3(z) \quad , \quad H_\ell(x) = T_1(x) + T_2(x) + T_3(x) \tag{26a}$$

with

$$T_i(z) = \frac{1}{k}\int_0^z F_i(s) \sin k(z - s)\, ds \quad , \quad i = 1, 2, 3 \tag{26b}$$

The functions F_i are defined in (21a,b) and (24b). The particular integral due to the first term on the right in (23a) is obtained from (26b) with k^2A substituted for $F_i(s)$. It contributes the term 1 in (25a,b) and to the arbitrary constants C_1' and C_2'. The other particular integrals are the functions $T_i(z)$ and $T_i(x)$ with $i = 1$, 2, and 3.

The distributions of charge per unit length are obtained from the currents in (25a-d) with the equations of continuity in (2a,b). With $\partial H(z)/\partial z$ denoted by $H'(z)$, the formulas are:

$$q_1(z) = \frac{jk}{\omega} A[-C_1' \sin kz + C_1'' \cos kz] + \frac{j}{\omega\Psi} H_h'(z) \tag{27a}$$

$$q_2(z) = \frac{jk}{\omega} A[-C_2' \sin kz + C_2'' \cos kz] + \frac{j}{\omega\Psi} H_h'(z) \tag{27b}$$

$$q_3(x) = \frac{jk}{\omega} A[-C_3' \sin kx + C_3'' \cos kx] + \frac{j}{\omega\Psi} H_\ell'(x) \tag{27c}$$

$$q_4(x) = \frac{jk}{\omega} A[-C_4' \sin kx + C_4'' \cos kx] + \frac{j}{\omega\Psi} H_\ell'(x) \tag{27d}$$

Since the currents and charges appear in the integrands of the particular integrals, (25a-d) and (27a-d) are not solutions but rearranged coupled integral equations. Approximate solutions can be obtained by iteration. Suitable zero-order solutions are given by the square brackets in (25a-d) and (27a-d). First-order solutions are then obtained by the substitution of the zero-order solutions in the integrands in the terms $H_h(z)$ and $H_\ell(x)$ and their derivatives. Second-order solutions can be generated by the substitution of first-order values in $H_h(z)$ and $H_\ell(x)$. For present purposes first-order solutions are adequate.

5. First-Order Solutions

First-order currents and charges are obtained from (25a-d) and (27a-d) when the explicit zero-order values are substituted in (23a-c) and the explicit values of the F's so obtained are used to evaluate $H_h(z)$ and $H_\ell(x)$ as defined by (26a,b). Thus,

$$F_1(z) = -jk^2 A \int_{-h_1}^{h_2} K_I(z,z')\, dz' \quad ; \quad F_1(x) = 0 \tag{28}$$

$$F_2(z) = -kA \frac{\partial}{\partial z} \Big[\int_{-\ell_1}^{0} (-C_3' \sin kx' + C_3'' \cos kx')K(z,x')\, dx' + \int_{0}^{\ell_2} (-C_4' \sin kx' + C_4'' \cos kx')K(z,x')\, dx' \Big] \tag{29a}$$

$$F_2(x) = -kA \frac{\partial}{\partial x} \Big[\int_{-h_1}^{0} (-C_1' \sin kz' + C_1'' \cos kz')K(x,z')\, dz' + \int_{0}^{h_2} (-C_2' \sin kz' + C_2'' \cos kz')K(x,z')\, dz' \Big] \tag{29b}$$

$$F_3(z) = kA[(-C_2' \sin kh_2 + C_2'' \cos kh_2)K(z,h_2) - (C_1' \sin kh_1 + C_1'' \cos kh_1)K(z,-h_1)] \tag{30a}$$

$$F_3(x) = kA[(-C_4' \sin k\ell_2 + C_4'' \cos k\ell_2)K(x,\ell_2) - (C_3' \sin k\ell_1 + C_3'' \cos k\ell_1)K(x,-\ell_1)] \tag{30b}$$

When these quantities are used in (26a,b), the functions $H_h(z)$ and $H_\ell(x)$ can be evaluated in terms of sines, cosines and integral sines and cosines. The computations are carried out in Appendices B, C and D where explicit formulas are obtained for $T_i(z)$ and $T_i(x)$, $i = 1, 2, 3$. They are given in the forms

$$T_1(z) = -At_1(z) \quad ; \quad T_1(x) = 0 \tag{31}$$

$$T_2(z) = -A[C_3'G_s(z,\ell_1) - C_4'G_s(z,\ell_2) + C_3''G_c(z,\ell_1) + C_4''G_c(z,\ell_2)] \tag{32a}$$

$$T_2(x) = -A[C_1'G_s(x,h_1) - C_2'G_s(x,h_2) + C_1''G_c(x,h_1) + C_2''G_c(x,h_2)] \tag{32b}$$

$$T_3(z) = A[\vartheta(z,h_2)(-C_2' \sin kh_2 + C_2'' \cos kh_2) - \vartheta(z,-h_1)(C_1' \sin kh_1 + C_1'' \cos kh_1)] \tag{33a}$$

$$T_3(x) = A[\vartheta(x,\ell_2)(-C_4' \sin k\ell_2 + C_4'' \cos k\ell_2) - \vartheta(x,-\ell_1)(C_3' \sin k\ell_1 + C_3'' \cos k\ell_1)] \tag{33b}$$

Formulas for the functions $G_s(z,\ell_1)$, $G_c(z,\ell_1)$, $\vartheta(z,h_2)$, $\vartheta(z,-h_1)$, etc. are given in Appendices B, C and D.

When (31)-(33) are substituted in (26a), (25a-d) and (27a-d), explicit first-order solutions for the currents and charges per unit length are obtained. It remains to evaluate the constants C_i' and C_i'' from the boundary and junction conditions (3)-(5).

6. Evaluation of Constants of Integration

Since the charges appear only in the first-order terms of the expressions (25a-d) for the currents, zero-order values of the charges per unit length are adequate to obtain first-order currents. This suggests that the application of (5) is advantageously carried out first. With $z = 0$ and $x = 0$ in the zero-order parts of (27a-d), it follows directly that (5) gives:

$$C_1'' = C_2'' = C_3'' = C_4'' \equiv C'' \tag{34}$$

Since $H_h(0) = H_\ell(0) = 0$, the condition (4) acts only on the zero-order terms for the currents. The resulting equation is

$$C_1' - C_2' + C_3' - C_4' = 0 \tag{35}$$

The conditions (3) which require the currents to vanish at the four ends of the antenna are more complicated since they involve all of the first-order terms. The following four simultaneous equations are obtained for the C_j', $j = 1, 2, 3, 4$:

$$\sum_{j=1}^{4} C_j' a_{ij} = R_i \quad ; \quad i = 1, 2, 3, 4 \tag{36a}$$

where

$$R_1 = -1 + \theta(-h_1) + C''(\sin kh_1 - b_1) \quad ; \quad R_3 = C''(\sin k\ell_1 - b_3) \tag{36b}$$

$$R_2 = -1 + \theta(h_2) - C''(\sin kh_2 + b_2) \quad ; \quad R_4 = -C''(\sin k\ell_2 + b_4) \tag{36c}$$

The following quantities are involved:

$$a_{11} = \cos kh_1 - \Psi^{-1}\vartheta(-h_1,-h_1)\sin kh_1 \; ; \; a_{12} = -\Psi^{-1}\vartheta(-h_1,h_2)\sin kh_2 ;$$

$$a_{13} = -\Psi^{-1}G_s(-h_1,\ell_1) \; ; \; a_{14} = \Psi^{-1}G_s(-h_1,\ell_2) \; ; \tag{37a}$$

$$a_{21} = -\Psi^{-1}\vartheta(h_2,-h_1)\sin kh_1 \; ; \; a_{22} = \cos kh_2 - \Psi^{-1}\vartheta(h_2,h_2)\sin kh_2 \; ;$$

$$a_{23} = -\Psi^{-1}G_s(h_2,\ell_1) \; ; \; a_{24} = \Psi^{-1}G_s(h_2,\ell_2) \; ; \tag{37b}$$

$$a_{31} = -\Psi^{-1}G_s(-\ell_1,h_1) \; ; \; a_{32} = \Psi^{-1}G_s(-\ell_1,h_2); \; a_{33} = \cos k\ell_1 - \Psi^{-1}\vartheta(-\ell_1,-\ell_1)$$

$$\times \sin k\ell_1 \; ; \; a_{34} = -\Psi^{-1}\vartheta(-\ell_1,\ell_2)\sin k\ell_2 \; ; \tag{37c}$$

$$a_{41} = -\Psi^{-1}G_s(\ell_2,h_1) \; ; \; a_{42} = \Psi^{-1}G_s(\ell_2,h_2) \; ; \; a_{43} = -\Psi^{-1}\vartheta(\ell_2,-\ell_1)\sin k\ell_1 \; ;$$

$$a_{44} = \cos k\ell_2 - \Psi^{-1}\vartheta(\ell_2,\ell_2)\sin k\ell_2 \tag{37d}$$

$$b_1 = -\Psi^{-1}[\vartheta(-h_1,-h_1)\cos kh_1 - \vartheta(-h_1,h_2)\cos kh_2] - \Psi^{-1}[G_c(-h_1,\ell_1) + G_c(-h_1,\ell_2)] \tag{38a}$$

$$b_2 = -\Psi^{-1}[-\vartheta(h_2,h_2)\cos kh_2 + \vartheta(h_2,-h_1)\cos kh_1] - \Psi^{-1}[G_c(h_2,\ell_1) + G_c(h_2,\ell_2)] \tag{38b}$$

$$b_3 = -\Psi^{-1}[\vartheta(-\ell_1,-\ell_1)\cos k\ell_1 - \vartheta(-\ell_1,\ell_2)\cos k\ell_2] - \Psi^{-1}[G_c(-\ell_1,h_1) + G_c(-\ell_1,h_2)] \tag{38c}$$

$$b_4 = -\Psi^{-1}[-\vartheta(\ell_2,\ell_2)\cos k\ell_2 + \vartheta(\ell_2,-\ell_1)\cos k\ell_1] - \Psi^{-1}[G_c(\ell_2,h_1) + G_c(\ell_2,h_2)] \tag{38d}$$

$$\theta(z) = t_1(z)/\Psi \tag{39}$$

Note that the quantities a_{ij}, b_i and θ are of order $1/\Psi$, the quantities a_{ii} are of order 1.

The solutions of the simultaneous equations in (36a) can be carried out numerically for any special case. General analytical formulas can be obtained quite simply when the basic condition underlying the present solution is satisfied, viz., $\Psi >> 1$, which is a necessary consequence of $ka << 1$. If the solutions are expressed in the form $C_j' = A_j/D$, $j = 1, 2, 3, 4$, and terms of order Ψ^{-2} are neglected, the determinant of the coefficients reduces to the diagonal terms. Thus,

$$D \doteq a_{11}a_{22}a_{33}a_{44} \tag{40}$$

This contains terms of the order Ψ^{-1}. These are important since the leading terms in a_{ii} can vanish when the cosine is zero.

Without further approximation

$$A_1 \doteq R_1a_{22}a_{33}a_{44} - R_2a_{12}a_{33}a_{44} - R_3a_{13}a_{22}a_{44} - R_4a_{14}a_{22}a_{33} \tag{41a}$$

$$A_2 \doteq R_2a_{11}a_{33}a_{44} - R_1a_{21}a_{33}a_{44} - R_3a_{23}a_{11}a_{44} - R_4a_{24}a_{11}a_{33} \tag{41b}$$

$$A_3 \doteq R_3a_{11}a_{22}a_{44} - R_1a_{31}a_{22}a_{44} - R_2a_{32}a_{11}a_{44} - R_4a_{34}a_{11}a_{22} \tag{41c}$$

$$A_4 \doteq R_4a_{11}a_{22}a_{33} - R_1a_{41}a_{22}a_{33} - R_2a_{42}a_{11}a_{33} - R_3a_{43}a_{11}a_{22} \tag{41d}$$

It follows that:

$$C_1' = \frac{A_1}{D} = \frac{1}{a_{11}}\left[R_1 - R_2\frac{a_{12}}{a_{22}} - R_3\frac{a_{13}}{a_{33}} - R_4\frac{a_{14}}{a_{44}}\right] \tag{42a}$$

$$C_2' = \frac{A_2}{D} = \frac{1}{a_{22}}\left[R_2 - R_1\frac{a_{21}}{a_{11}} - R_3\frac{a_{23}}{a_{33}} - R_4\frac{a_{24}}{a_{44}}\right] \tag{42b}$$

$$C_3' = \frac{A_3}{D} = \frac{1}{a_{33}}\left[R_3 - R_1\frac{a_{31}}{a_{11}} - R_2\frac{a_{32}}{a_{22}} - R_4\frac{a_{34}}{a_{44}}\right] \tag{42c}$$

$$C_4' = \frac{A_4}{D} = \frac{1}{a_{44}}\left[R_4 - R_1\frac{a_{41}}{a_{11}} - R_2\frac{a_{42}}{a_{22}} - R_3\frac{a_{43}}{a_{33}}\right] \tag{42d}$$

These expressions involve the constant C" which occurs in the R's. It can be determined with (35) and the simplifying notation:

$$n_1 = \frac{a_{21}}{a_{22}} - \frac{a_{31}}{a_{33}} + \frac{a_{41}}{a_{44}} \quad , \quad n_2 = \frac{a_{32}}{a_{33}} - \frac{a_{42}}{a_{44}} + \frac{a_{12}}{a_{11}} \tag{43}$$

$$n_3 = \frac{a_{43}}{a_{44}} - \frac{a_{13}}{a_{11}} + \frac{a_{23}}{a_{22}} \quad , \quad n_4 = \frac{a_{14}}{a_{11}} - \frac{a_{24}}{a_{22}} + \frac{a_{34}}{a_{33}} \tag{44}$$

With (43) and (44), (35) becomes:

$$\frac{R_1}{a_{11}} (1 + n_1) - \frac{R_2}{a_{22}} (1 + n_2) + \frac{R_3}{a_{33}} (1 + n_3) + \frac{R_4}{a_{44}} (1 + n_4) = 0 \tag{45}$$

The substitution for the R's from (36b,c) yields an explicit formula for C",
viz.,

$$C'' = (T + M)^{-1}(a_{11}^{-1} - a_{22}^{-1} + N) \tag{46}$$

where

$$N = a_{11}^{-1}[n_1 - \theta(-h_1)] - a_{22}^{-1}[n_2 - \theta(h_2)] \tag{47}$$

$$M = a_{11}^{-1}[n_1 \sin kh_1 - b_1(1 + n_1)] + a_{22}^{-1}[n_2 \sin kh_2 + b_2(1 + n_2)]$$
$$+ a_{33}^{-1}[n_3 \sin k\ell_1 - b_3(1 + n_3)] + a_{44}^{-1}[n_4 \sin k\ell_2 + b_4(1 + n_4)] \tag{48}$$

$$T = a_{11}^{-1} \sin kh_1 + a_{22}^{-1} \sin kh_2 + F(\ell_1, \ell_2) \tag{49}$$

and

$$F(\ell_1, \ell_2) = a_{33}^{-1} \sin k\ell_1 + a_{44}^{-1} \sin k\ell_2 \tag{50}$$

If (46) and (36a-c) are used in (42a), these yield explicit formulas for the four C's. It is convenient to separate the leading and higher-order terms as follows:

$$C_1' = a_{11}^{-1}[-1 + c_1' + C''(\sin kh_1 + c_1'')] \tag{51a}$$

$$C_2' = a_{22}^{-1}[-1 + c_2' - C''(\sin kh_2 + c_2'')] \tag{51b}$$

$$C_3' = a_{33}^{-1}[c_3' + C''(\sin k\ell_1 + c_3'')] \tag{51c}$$

$$C_4' = a_{44}^{-1}[c_4' - C''(\sin k\ell_2 + c_4'')] \quad (51d)$$

where

$$c_1' = \theta(-h_1) + a_{12}a_{22}^{-1}[1 - \theta(h_2)] \quad ; \quad c_2' = \theta(h_2) + a_{21}a_{11}^{-1}[1 - \theta(-h_1)] \quad (52a)$$

$$c_3' = a_{31}a_{11}^{-1}[1 - \theta(-h_1)] + a_{32}a_{22}^{-1}[1 - \theta(h_2)] \quad ;$$

$$c_4' = a_{41}a_{11}^{-1}[1 - \theta(-h_1)] + a_{42}a_{22}^{-1}[1 - \theta(h_2)] \quad ; \quad (52b)$$

$$c_1'' = -b_1 + a_{12}a_{22}^{-1}(\sin kh_2 + b_2) - a_{13}a_{33}^{-1}(\sin k\ell_1 - b_3) + a_{14}a_{44}^{-1}(\sin k\ell_2 + b_4) \quad (52c)$$

$$c_2'' = b_2 + a_{23}a_{33}^{-1}(\sin k\ell_1 - b_3) - a_{24}a_{44}^{-1}(\sin k\ell_2 + b_4) + a_{21}a_{11}^{-1}(\sin kh_1 - b_1) \quad (52d)$$

$$c_3'' = -b_3 + a_{34}a_{44}^{-1}(\sin k\ell_2 + b_4) - a_{31}a_{11}^{-1}(\sin kh_1 - b_1) + a_{32}a_{22}^{-1}(\sin kh_2 + b_2) \quad (52e)$$

$$c_4'' = b_4 + a_{41}a_{11}^{-1}(\sin kh_1 - b_1) - a_{42}a_{22}^{-1}(\sin kh_2 + b_2) + a_{43}a_{33}^{-1}(\sin k\ell_1 - b_3) \quad (52f)$$

Since C'' is given explicitly in (46), the four constants C_1', C_2', C_3' and C_4' have been determined.

7. The Distributions of Current

The substitution of the expressions (51a-d) and (46) for the C's into (25a-d) gives the first-order currents. They are:

$$I_1(z) = A[T + M]^{-1}\left\{\frac{\sin kz + \sin kh_1}{a_{11}} - \frac{\sin kz - \sin kh_2}{a_{22}} - \frac{\sin kh_1 + \sin kh_2}{a_{11}a_{22}}\right.$$

$$\times \cos kz + [F(\ell_1,\ell_2) + M]\left(1 - \frac{\cos kz}{a_{11}}\right) + N\left[\sin kz + \frac{\sin kh_1}{a_{11}}\cos kz\right]$$

$$\left. + \left[c_1'(T + M) + c_1''\left(\frac{1}{a_{11}} - \frac{1}{a_{22}} + N\right)\right]\frac{\cos kz}{a_{11}}\right\} + \frac{H_h(z)}{\Psi} \quad (53a)$$

$$I_2(z) = A[T+M]^{-1}\left\{\frac{\sin kz + \sin kh_1}{a_{11}} - \frac{\sin kz - \sin kh_2}{a_{22}} - \frac{\sin kh_1 + \sin kh_2}{a_{11}a_{22}}\right.$$

$$\times \cos kz + [F(\ell_1,\ell_2) + M]\left(1 - \frac{\cos kz}{a_{22}}\right) + N\left[\sin kz - \frac{\sin kh_2}{a_{22}}\cos kz\right]$$

$$\left. + \left[c_2'(T+M) - c_2''\left(\frac{1}{a_{11}} - \frac{1}{a_{22}} + N\right)\right]\frac{\cos kz}{a_{22}}\right\} + \frac{H_h(z)}{\Psi} \tag{53b}$$

$$I_3(x) = A[T+M]^{-1}\left\{\left(\frac{1}{a_{11}} - \frac{1}{a_{22}} + N\right)\left(\sin kx + \frac{\sin k\ell_1}{a_{33}}\cos kx\right)\right.$$

$$\left. + \left[c_3'(T+M) + c_3''\left(\frac{1}{a_{11}} - \frac{1}{a_{22}} + N\right)\right]\frac{\cos kx}{a_{33}}\right\} + \frac{H_\ell(x)}{\Psi} \tag{53c}$$

$$I_4(x) = A[T+M]^{-1}\left\{\left(\frac{1}{a_{11}} - \frac{1}{a_{22}} + N\right)\left(\sin kx - \frac{\sin k\ell_2}{a_{44}}\cos kx\right)\right.$$

$$\left. + \left[c_4'(T+M) - c_4''\left(\frac{1}{a_{11}} - \frac{1}{a_{22}} + N\right)\right]\frac{\cos kx}{a_{44}}\right\} + \frac{H_\ell(x)}{\Psi} \tag{53d}$$

With (26a), (31), (32a,b) and (33a,b), it follows that

$$H_h(z) = -A\{t_1(z) + C_1'\vartheta(z,-h_1)\sin kh_1 + C_2'\vartheta(z,h_2)\sin kh_2 + C_3'G_s(z,\ell_1)$$

$$- C_4'G_s(z,\ell_2) + C''[\vartheta(z,-h_1)\cos kh_1 - \vartheta(z,h_2)\cos kh_2 + G_c(z,\ell_1)$$

$$+ G_c(z,\ell_2)]\} \tag{54a}$$

where the C' are given by (51a-d) and C" by (46). The functions t, G and ϑ are defined in the Appendices B, C and D. The corresponding formula for $H_\ell(x)$ is:

$$H_\ell(x) = -A\{C_3'\vartheta(x,-\ell_1)\sin k\ell_1 + C_4'\vartheta(x,\ell_2)\sin k\ell_2 + C_1'G_s(x,h_1) - C_2'G_s(x,h_2)$$

$$+ C''[\vartheta(x,-\ell_1)\cos k\ell_1 - \vartheta(x,\ell_2)\cos k\ell_2 + G_c(x,h_1) + G_c(x,h_2)]\} \tag{54b}$$

Note that $H_h(0) = H_\ell(0) = 0$.

When the electrical lengths of the four arms of the cross differ from integral multiples of a quarter wavelength, simple zero-order forms may be adequate. These are obtained by neglecting all terms with Ψ^{-1} as a factor. They are:

$$[I_1(z)]_0 = -A[\sin k(h_1 + z) + \sin k(h_2 - z) - \sin k(h_1 + h_2) + F(\ell_1,\ell_2) \times \cos kh_2(\cos kz - \cos kh_1)][\sin k(h_1 + h_2) + F(\ell_1,\ell_2) \times \cos kh_1 \cos kh_2]^{-1} \quad ; \quad -h_1 \leq z \leq 0 \tag{55a}$$

$$[I_2(z)]_0 = -A[\sin k(h_1 + z) + \sin k(h_2 - z) - \sin k(h_1 + h_2) + F(\ell_1,\ell_2) \times \cos kh_1(\cos kz - \cos kh_2)][\sin k(h_1 + h_2) + F(\ell_1,\ell_2) \times \cos kh_1 \cos kh_2]^{-1} \quad ; \quad 0 \leq z \leq h_2 \tag{55b}$$

$$[I_3(x)]_0 = -A\left\{\frac{\cos kh_2 - \cos kh_1}{\sin k(h_1 + h_2) + F(\ell_1,\ell_2)\cos kh_1 \cos kh_2}\right\} \frac{\sin k(\ell_1 + x)}{\cos k\ell_1} \quad ; \quad -\ell_1 \leq x \leq 0 \tag{55c}$$

$$[I_4(x)]_0 = A\left\{\frac{\cos kh_2 - \cos kh_1}{\sin k(h_1 + h_2) + F(\ell_1,\ell_2)\cos kh_1 \cos kh_2}\right\} \frac{\sin k(\ell_2 - x)}{\cos k\ell_2} \quad ; \quad 0 \leq x \leq \ell_2 \tag{55d}$$

where A is defined in (24a) and $F(\ell_1,\ell_2) = \tan k\ell_1 + \tan k\ell_2 = \sin k(\ell_1 + \ell_2)/\cos k\ell_1 \cos k\ell_2$.

8. The Distributions of Charge Per Unit Length

The first-order distributions of charge per unit length are obtained directly from the currents with the help of the equation of continuity, $q(z) = (j/\omega)[\partial I(z)/\partial z]$. Thus,

$$q_1(z) = \frac{jkA}{\omega}[T+M]^{-1}\left\{\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}\right)\cos kz - \frac{\sin kh_1 + \sin kh_2}{a_{11}a_{22}}\sin kz + [F(\ell_1,\ell_2)+M]\frac{\sin kz}{a_{11}} + N\left[\cos kz - \frac{\sin kh_1}{a_{11}}\sin kz\right] - \left[c_1'(T+M) + c_1''\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}\right)\right]\frac{\sin kz}{a_{11}}\right\} + \frac{jH_h'(z)}{\omega\Psi} \tag{56a}$$

$$q_2(z) = \frac{jkA}{\omega}[T+M]^{-1}\left\{\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}\right)\cos kz - \frac{\sin kh_1 + \sin kh_2}{a_{11}a_{22}}\sin kz + [F(\ell_1,\ell_2)+M]\frac{\sin kz}{a_{22}} + N\left[\cos kz + \frac{\sin kh_2}{a_{22}}\sin kz\right] - \left[c_2'(T+M) - c_2''\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}\right)\right]\frac{\sin kz}{a_{22}}\right\} + \frac{jH_h'(z)}{\omega\Psi} \tag{56b}$$

$$q_3(x) = \frac{jkA}{\omega}(T+M)^{-1}\left\{\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}+N\right)\left(\cos kx - \frac{\sin k\ell_1}{a_{33}}\sin kx\right) - \left[c_3'(T+M) + c_3''\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}\right)\right]\frac{\sin kx}{a_{33}}\right\} + \frac{jH_\ell'(x)}{\omega\Psi} \tag{56c}$$

$$q_4(x) = \frac{jkA}{\omega}(T+M)^{-1}\left\{\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}+N\right)\left(\cos kx + \frac{\sin k\ell_2}{a_{44}}\sin kx\right) - \left[c_4'(T+M) - c_4''\left(\frac{1}{a_{11}}-\frac{1}{a_{22}}\right)\right]\frac{\sin kx}{a_{44}}\right\} + \frac{jH_\ell'(x)}{\omega\Psi} \tag{56d}$$

$H'_h(z)$ and $H'_\ell(x)$ are the derivatives with respect to the indicated arguments of the functions $H_h(z)$ in (54a) and $H_\ell(x)$ in (54b). The differentiation is discussed in Appendix E. The corresponding zero-order charges per unit length are:

$$[q_1(z)]_0 = \frac{-jkA}{\omega}\left\{\frac{\cos k(h_1 + z) - \cos k(h_2 - z) - F(\ell_1,\ell_2)\cos kh_2 \sin kz}{\sin k(h_1 + h_2) + F(\ell_1,\ell_2)\cos kh_1 \cos kh_2}\right\} \quad (57a)$$

$$[q_2(z)]_0 = \frac{-jkA}{\omega}\left\{\frac{\cos k(h_1 + z) - \cos k(h_2 - z) - F(\ell_1,\ell_2)\cos kh_1 \sin kz}{\sin k(h_1 + h_2) + F(\ell_1,\ell_2)\cos kh_1 \cos kh_2}\right\} \quad (57b)$$

$$[q_3(x)]_0 = \frac{-jkA}{\omega}\left\{\frac{\cos kh_2 - \cos kh_1}{\sin k(h_1 + h_2) + F(\ell_1,\ell_2)\cos kh_1 \cos kh_2}\right\} \frac{\cos k(\ell_1 + x)}{\cos k\ell_1} \quad (57c)$$

$$[q_4(x)]_0 = \frac{-jkA}{\omega}\left\{\frac{\cos kh_2 - \cos kh_1}{\sin k(h_1 + h_2) + F(\ell_1,\ell_2)\cos kh_1 \cos kh_2}\right\} \frac{\cos k(\ell_2 - x)}{\cos k\ell_2} \quad (57d)$$

9. Special Cases

In order to gain insight into the numerous possible distributions of current and charge on the crossed dipoles, it is advantageous to treat certain special cases associated with conditions of resonance and antiresonance in the six possible circuits each consisting of two arms. These will be outlined only briefly here and discussed in greater detail in conjunction with measurements in a companion paper [5]. Before these are examined, the symmetrical case and the vertical section alone without side arms are of interest.

1) Junction at the center of the vertical element, $h_2 = h_1 = h$.

When the junction of the crossed antennas is at the center of the vertical element, all currents and charges on the horizontal element are zero since it is in the neutral plane. The vertical section behaves as if isolated. Specifically, in this case

$$I(z) = A\{1 + C_1' \cos kz - \theta(z) - C_1'\Psi^{-1}[\vartheta(z,-h) + \vartheta(z,h)]\sin kh\} \quad ; \quad -h \leq z \leq 0 \tag{58}$$

$$q(z) = \frac{-jkA}{\omega}\{C_1' \sin kz + \frac{\theta'(z)}{k} + [C_1'/k\Psi][\vartheta'(z,-h) + \vartheta'(z,h)]\sin kh\} \quad ; \quad -h \leq z \leq 0 \tag{59}$$

where the prime on θ and ϑ denotes differentiation with respect to z and where

$$C_1' = \frac{\theta(-h) - 1}{a_{11} + a_{12}} \quad , \quad \theta(z) = t_1(z)/\Psi \tag{60}$$

In this simple case the complete determinant, viz., $D = a_{11}a_{22} - a_{12}a_{21}$, is used instead of the approximate form given in (40). Note that

$$I(-z) = I(z) \quad ; \quad q(-z) = -q(z) \tag{61}$$

$$I(x) = 0 \quad ; \quad q(x) = 0 \tag{62}$$

Since the horizontal member at $z = 0$ has no effect, the above formulas also apply to the vertical section when the horizontal member is absent.

When $kh = \pi/2$, the antenna is near resonance. In this case, $a_{11} = -\vartheta(h,h)/\Psi$, $a_{12} = -\vartheta(-h,h)/\Psi$, $C_1' = -\Psi[\theta(-h) - 1]/[\vartheta(h,h) + \vartheta(-h,h)]$. The zero-order terms are:

$$[I(z)]_0 = \frac{A\Psi \cos kz}{\vartheta(h,h) + \vartheta(-h,h)} \tag{63}$$

$$[q(z)]_0 = \frac{-(jkA\Psi \sin kz)\omega}{\vartheta(h,h) + \vartheta(-h,h)} \tag{64}$$

When $kh = \pi$, $a_{11} = a_{22} = -1$, $C_1' = 1 - t_1(-h)/\Psi$, so that

$$I_1(z) = A\{1 + \cos kz - \Psi^{-1}[t_1(z) + t_1(-h)\cos kz]\} \; ; -h \leq z \leq 0 \tag{65}$$

where

$$t_1(z) = -j\{Si(\pi + kz) + Si(\pi - kz) + (1/2)[Cin\, 2(\pi + kz) - Cin\, 2(\pi - kz)]$$

$$\times \sin kz + (1/2)[Si\, 2(\pi + kz) + Si\, 2(\pi - kz) - 4\, Si\, \pi - 2\, Si\, 2\pi]$$

$$\times \cos kz\}$$

and

$$t_1(-h) = (j/2)[4\, Si\, \pi - Si\, 4\pi] = j2.95$$

Also,

$$q_1(z) = \frac{-jkA}{\omega}\{\sin kz - \Psi^{-1}[t_1'(z)/k - t_1(-h)\sin kz]\} \; ; -h \leq z \leq 0 \tag{66}$$

Note that $I_2(z) = I_1(-z)$, $q_2(z) = -q_1(-z)$ for $0 \leq z \leq h$; $A = -j4\pi E_z^{inc}/\omega\mu\Psi$.

2) All elements resonant; junction at minima of charges per unit length and maxima of currents along horizontal and vertical elements: $kh_1 = 5\pi/2$, $kh_2 = k\ell_1 = k\ell_2 = \pi/2$.

For the specified lengths, the currents and charges involve the following parameters:

$$a_{11} = -\vartheta(5\lambda/4, 5\lambda/4)/\Psi \quad ; \quad a_{22} = a_{33} = a_{44} = -\vartheta(\lambda/4, \lambda/4)/\Psi \tag{67}$$

With (46) and (51a-d), it is readily verified that C" is of order 1, whereas the C' are of order $\Psi >> 1$. It follows that the leading terms in all of the currents and charges are simply:

$$I_1(z) \doteq AC_1' \cos kz \quad ; \quad q_1(z) \doteq \frac{-jkA}{\omega} C_1' \sin kz \quad ; \quad -h_1 \leq z \leq 0 \tag{68}$$

$$I_2(z) \doteq AC_2' \cos kz \quad ; \quad q_2(z) = \frac{-jkA}{\omega} C_2' \sin kz \quad ; \quad 0 \leq z \leq h_2 \tag{69}$$

$$I_3(x) \doteq AC_3' \cos kx \quad ; \quad q_3(x) = \frac{-jkA}{\omega} C_3' \sin kx \quad ; \quad -\ell_1 \leq x \leq 0 \tag{70}$$

$$I_4(x) \doteq AC_4' \cos kx \quad ; \quad q_4(x) = \frac{-jkA}{\omega} C_4' \sin kx \quad ; \quad 0 \leq x \leq \ell_2 \tag{71}$$

where the C' are given by (51a-d) with each of the four sines equal to unity. The parameters c_i', c_i'' and C'' are all of order unity. Note that all of the boundary conditions (3), (4) and (5) are satisfied by these simple expressions. In particular, $q_1(0) = q_2(0) = q_3(0) = q_4(0) = 0$.

3) Junction at minimum of charge per unit length and maximum of current along vertical element, maximum of charge and minimum of current along horizontal element: $kh_1 = 3\pi$, $kh_2 = k\ell_1 = k\ell_2 = \pi$.

The following parameters are involved: $a_{11} = a_{22} = a_{33} = a_{44} = -1$, $T = 0$ and $F(\ell_1,\ell_2) = 0$. The leading terms in the currents are simply

$$I_1(z) \doteq A[1 + \cos kz + (N/M)\sin kz] \quad ; \quad I_2(z) \doteq A[1 + \cos kz + (N/M)\sin kz] \tag{72}$$

$$I_3(x) \doteq A(N/M)\sin kx \quad ; \quad I_4(x) \doteq A(N/M)\sin kx \tag{73}$$

The associated charges per unit length are

$$q_1(z) = \frac{-jkA}{\omega}[\sin kz - (N/M)\cos kz] \quad ; \quad q_2(z) = \frac{-jkA}{\omega}[\sin kz - (N/M)\cos kz] \tag{74}$$

$$q_3(x) = \frac{jkA}{\omega}(N/M)\cos kx \quad ; \quad q_4(x) = \frac{jkA}{\omega}(N/M)\cos kx \tag{75}$$

Note that at the junction $I_1(0)$ and $I_2(0)$ have maxima, $q_1(0)$ and $q_2(0)$ minima. On the other hand, $I_3(0)$ and $I_4(0)$ have minima, $q_3(0)$ and $q_4(0)$ maxima.

4) Junction at minima of charge and current along vertical element, maximum of charge and minimum of current along horizontal elements: $kh_1 = 4\pi$, $kh_2 = 2\pi$, $k\ell_1 = k\ell_2 = \pi$.

For the specified lengths, the following parameters are involved: $a_{11} = a_{22} = 1$, $a_{33} = a_{44} = -1$; $T = 0$; $F(\ell_1,\ell_2) = 0$. The leading terms in the currents are:

$$I_1(z) \doteq A[1 - \cos kz + (N/M)\sin kz] \; ; \; I_2(z) \doteq A[1 - \cos kz + (N/M)\sin kz] \tag{76}$$

$$I_3(x) \doteq A(N/M)\sin kx \quad ; \quad I_4(x) \doteq A(N/M)\sin kx \tag{77}$$

The associated charges per unit length are:

$$q_1(z) \doteq \frac{jkA}{\omega}[\sin kz + (N/M)\cos kz] \; ; \; q_2(z) \doteq \frac{jkA}{\omega}[\sin kz + (N/M)\cos kz] \tag{78}$$

$$q_3(x) \doteq \frac{jkA}{\omega}(N/M)\cos kx \quad ; \quad q_4(x) = \frac{jkA}{\omega}(N/M)\cos kx \tag{79}$$

where $M = -b_1 + b_2 + b_3 - b_4$, $N = a_{31} - a_{41} + a_{32} - a_{42}$. The b's are given by (38a-d), the a's by (37). Note that at the junction $I_1(0)$, $I_2(0)$, $q_1(0)$ and $q_2(0)$ all have minima, whereas $I_3(0)$ and $I_4(0)$ have minima, $q_3(0)$ and $q_4(0)$ maxima. Nevertheless, $q_1(0) = q_2(0) = q_3(0) = q_4(0) = jkAN/\omega M$.

5) Currents axially discontinuous at the junction: $kh_1 = 4\pi$, $kh_2 = k\ell_1 = k\ell_2 = \pi$.

The following parameters occur when the above lengths are used: $a_{11} = 1$, $a_{22} = a_{33} = a_{44} = -1$; $T = 0$; $F(\ell_1,\ell_2) = 0$; $M = -b_1 - b_2 + b_3 - b_4$; $N = n_1 + n_2 - \theta(-h_1) - \theta(h_2)$. The b's are given by:

$$b_1 = -\Psi^{-1}[\vartheta(2\lambda,2\lambda) + \vartheta(-2\lambda,\lambda/2) - 2G_c(2\lambda,\lambda/2)] \quad (80a)$$

$$b_2 = -\Psi^{-1}[\vartheta(\lambda/2,\lambda/2) + \vartheta(\lambda/2,-2\lambda) + 2G_c(\lambda/2,\lambda/2)] \quad (80b)$$

$$b_3 = -\Psi^{-1}[-\vartheta(\lambda/2,\lambda/2) + \vartheta(-\lambda/2,\lambda/2) - G_c(\lambda/2,2\lambda) - G_c(\lambda/2,\lambda/2)] \quad (80c)$$

$$b_4 = -\Psi^{-1}[\vartheta(\lambda/2,\lambda/2) - \vartheta(\lambda/2,-\lambda/2) + G_c(\lambda/2,2\lambda) + G_c(\lambda/2,\lambda/2)] \quad (80d)$$

The leading terms in the currents are:

$$I_1(z) \doteq A\{1 - \cos kz - [(2 + N)/M](b_1 \cos kz - \sin kz)\} + H_h(z)/\Psi \quad (81a)$$

$$I_2(z) \doteq A\{1 + \cos kz + [(2 + N)/M](b_2 \cos kz + \sin kz)\} + H_h(z)/\Psi \quad (81b)$$

$$I_3(x) \doteq [A(2 + N)/M](b_3 \cos kx + \sin kx) + H_\ell(x)/\Psi \quad (82a)$$

$$I_4(x) \doteq [A(2 + N)/M](b_4 \cos kx + \sin kx) + H_\ell(x)/\Psi \quad (82b)$$

The associated charges per unit length are:

$$q_1(z) = \frac{jkA}{\omega}\{(1 + 2b_1/M)\sin kz + [(2 + N)/M]\cos kz + H_h'(z)/k\Psi\} \quad (83a)$$

$$q_2(z) = \frac{jkA}{\omega}\{-(1 + 2b_2/M)\sin kz + [(2 + N)/M]\cos kz + H_h'(z)/k\Psi\} \quad (83b)$$

$$q_3(x) = \frac{jkA}{\omega M}\{-2b_3 \sin kx + (2 + N)\cos kx + H_\ell'(x)/k\Psi\} \quad (84a)$$

$$q_4(x) = \frac{jkA}{\omega M}\{-2b_4 \sin kx + (2 + N)\cos kx + H_\ell'(x)/k\Psi\} \quad (84b)$$

At the junction ($x = 0$, $z = 0$), $I_1(0) \doteq -2Ab_1/M$; $I_2(0) \doteq A(2 + 2b_2/M)$; $I_3(0) \doteq 2Ab_3/M$; $I_4(0) \doteq 2Ab_4/M$. Also, $q_1(0) = q_2(0) = q_3(0) = q_4(0) \doteq (jkA/\omega M)(2 + N)$. Note that $I_1(0) - I_2(0) + I_3(0) - I_4(0) = A[(2/M)(-b_1 - b_2 + b_3 - b_4) - 2] = 0$. At the junction the largest current is in the upper arm 2 of the vertical member. The vanishing of the currents at the ends $z = -h_1$, h_2 and $x = -\ell_1$, ℓ_2 is accomplished only with the inclusion of the higher-order terms $H_h(z)$ and $H_\ell(x)$.

6) Horizontal element asymmetrical: $kh_1 = 5\pi/2$, $kh_2 = \pi$, $k\ell_1 = \pi/2$, $k\ell_2 = \pi$.

Owing to the large currents and charges per unit length in two of the arms and the small ones in the other two, a simple approximate representation is not satisfactory in this case. Actually, all terms must be retained with the currents given by (53a-d) and the charges per unit length by (56a-d). The special formulas for the several parameters are:

$$a_{11} = -\psi^{-1}\mathcal{J}(-h_1,-h_1) \ , \ a_{22} = -1 \ , \ a_{33} = -\psi^{-1}\mathcal{J}(-\ell_1,-\ell_1) \ , \ a_{44} = -1 \qquad (85a)$$

where $|a_{11}^2| << 1$, $|a_{33}^2| << 1$. Also,

$$T = a_{11}^{-1} + a_{33}^{-1} \quad ; \quad F(\ell_1,\ell_2) = a_{33}^{-1} \qquad (85b)$$

$$M = -\left[\frac{a_{13} + a_{31}}{a_{11}a_{33}} + \frac{a_{21} + a_{41}}{a_{11}} + \frac{a_{23} + a_{43}}{a_{33}}\right] \qquad (85c)$$

$$N = -\left(\frac{a_{21} + a_{41}}{a_{11}}\right) + \frac{a_{32}}{a_{33}} - \frac{a_{31}}{a_{11}a_{33}} + a_{42} \qquad (85d)$$

$$c_1' = 0 \ , \ c_2' = \frac{a_{21}}{a_{11}} \ , \ c_3' = \frac{a_{31}}{a_{33}} - a_{32} \ , \ c_4' = \frac{a_{41}}{a_{11}} - a_{42} \qquad (86a)$$

and

$$c_1'' = -\frac{a_{13}}{a_{33}}, \quad c_2'' = \frac{a_{23}}{a_{33}} + \frac{a_{21}}{a_{11}}, \quad c_3'' = -\frac{a_{31}}{a_{11}}, \quad c_4'' = \frac{a_{41}}{a_{11}} + \frac{a_{43}}{a_{33}} \tag{86b}$$

The a_{ij}'s are given by (37a-d) with $\sin kh_1 = \sin k\ell_1 = 1$.

The terms $H_h(z)$ and $H_\ell(x)$ also contribute significantly, especially near the ends of the arms. Note that $H_h(0) = H_\ell(0) = 0$. The terms $H_h(z)/\Psi$ and $H_\ell(x)/\Psi$ can be expressed in the following forms for the case at hand:

$$H_h(z) = -A\{t_1(z) + C_1'\vartheta(z,-h_1) + C_3'G_s(z,\ell_1) - C_4'G_s(z,\ell_2)$$

$$+ C''[\vartheta(z,h_2) + G_c(z,\ell_1) + G_c(z,\ell_2)]\} \tag{87a}$$

$$H_\ell(x) = -A\{C_3'\vartheta(x,-\ell_1) + C_1'G_s(x,h_1) - C_2'G_s(x,h_2)$$

$$+ C''[\vartheta(x,\ell_2) + G_c(x,h_1) + G_c(x,h_2)]\} \tag{87b}$$

where

$$C_1' = a_{11}^{-1}[-1 + c_1' + C''(1 + c_1'')] \tag{88a}$$

$$C_2' = 1 - c_2' + C''c_2'' \tag{88b}$$

$$C_3' = a_{33}^{-1}[c_3' + C''(1 + c_3'')] \tag{88c}$$

$$C_4' = -c_4' + C''c_4'' \tag{88d}$$

where

$$C'' = (a_{11}^{-1} + N)/(T + M) \tag{89}$$

The above solutions satisfy the eight conditions (3), (4) and (5). For the currents: $$I_1(-h_1) = I_2(h_2) = I_3(-\ell_1) = I_4(\ell_2)$$

$$I_1(0) - I_2(0) + I_3(0) - I_4(0) = 0$$

For the charges:

$$q_1(0) = q_2(0) = q_3(0) = q_4(0) = \frac{jkA}{\omega(T + M)} \left(1 + N + \frac{1}{a_{11}}\right)$$

An examination of the currents in the four arms shows that very large resonant currents with leading terms $(\cos kz)/a_{11}a_{33}$ and $(\cos kx)/a_{11}a_{33}$ are induced in arms 1 and 3 which have the lengths $kh_1 = 5\pi/2$ and $k\ell_1 = \pi/2$ so that $k(h_1 + \ell_1) = 3\pi$ or $h_1 + \ell_1 = 3\lambda/2$. These two arms form a single resonant circuit with a standing wave that has a maximum current and a minimum of charge at the junction. As a consequence, this mode of oscillation is largely independent and only loosely coupled to other possible circuits. These are arms 1 and 2 with $k(h_1 + h_2) = 7\pi/2$, arms 1 and 4 with $k(h_1 + \ell_2) = 7\pi/2$, arms 2 and 3 with $k(h_2 + \ell_1) = 3\pi/2$ and arms 3 and 4 with $k(\ell_1 + \ell_2) = 3\pi/2$. They all have lengths that are an odd instead of even number of quarter wavelengths long so that they are antiresonant and induced currents are small. The circuit consisting of the arms 2 and 4 has the length $k(h_2 + \ell_2) = 2\pi$, so that it is potentially resonant. However, its standing-wave pattern locates a maximum of charge per unit length at and near the junction. This must distribute itself equally among the four conductors at the junction and so act to excite the other two arms that are parts of antiresonant circuits for this mode. The four currents have the form $(1+N+a_{11}^{-1})\sin kz$, $(1+N+a_{11}^{-1})\sin kx$. The fact that the resonant circuit consisting of arms 2 and 4 is closely coupled at the junction to the other two arms with which it forms antiresonant circuits, means that the resonant amplitude in circuit 2 and 4 is severely damped. Other components of current with suitable distributions are required to satisfy the boundary conditions $E_z(z) = 0$ on the surfaces of the conductors and the conditions on the current and its derivative.

10. Discussion and Conclusion

A complete analytical solution has been obtained for the currents and charges per unit length on the perfectly conducting, mutually perpendicular and electrically thin arms of a crossed dipole antenna when excited by a normally incident plane electromagnetic wave. The solution is obtained specifically when the incident electric vector is parallel to one of the mutually perpendicular conductors. However, since there are no restrictions on the lengths of the conductors or the location of the point of intersection, the corresponding solution for the currents induced when the electric field is parallel to the second conductor can be written down directly. The superposition of the solutions for the two mutually perpendicular polarizations yields the currents and charges in the conductors in an arbitrarily polarized normally incident plane wave. In the analysis the correct boundary conditions on the currents and their derivatives at the junction have been used. The commonly used condition for the continuity of scalar and vector potentials at the junction are not sufficient conditions. They are satisfied automatically as a consequence of the definitions of the potential functions. The substitution of continuity of scalar potential for continuity of charge per unit length at the junction is a rough approximation the accuracy of which depends on the degree in which the scalar potential is proportional to the charge per unit length - this improves near maxima of charge in a standing-wave pattern. In general, the condition of continuity of scalar potential at the junction permits large, physically unacceptable discontinuities in the charge per unit length at the junction and results in an incorrect electric field.

The currents and charges obtained in the new solution determine the complete scattered field from a pair of mutually perpendicular crossed wires in a normally incident field. The method of solution can be generalized to treat wires that cross at other than a 90° angle and to an incident plane wave with an arbitrary angle of incidence.

The current and charge distributions in their leading terms can be visualized in terms of two important principal modes: (1) those of resonant circuits with maxima of current and minima of charges at the junction and (2) those of resonant circuits with maxima of charge per unit length at the junction. The former may include any combination in which the length from the end of one conductor in the cross to the end of another is an integral number of half wavelengths. Oscillations in any two (three or four) arms that together form a resonant circuit are largely independent of currents in the others since coupling to them is minimal for any mode for which the charge per unit length at the junction is zero. Thus, large amplitudes can obtain in any two (three or four) arms that form a resonant circuit. The second principal type of oscillation occurs when the length of any one arm is such that a maximum of charge per unit length is located at the junction. As a consequence of the continuity of charge per unit length at the junction, all four arms are excited and oscillate closely coupled. If all are resonant, the amplitude in all is large; if one or more are not resonant, the amplitude in all is correspondingly reduced. In general, superpositions of the two types of oscillation obtain.

Appendix A: The Expansion Parameters

The parameter Ψ defined in (22b) can be evaluated when the zero-order form of the function f(z) is substituted in it. The function in the case at hand is $(\partial^2/\partial z^2 + k^2)I(z)$. Its zero-order value is the constant $k^2A = -j4\pi k^2 E_z^{inc}/\omega\mu\Psi$. Since this is then also the value at any point z_m, the formula (22b) becomes:

$$\Psi = \int_{-h_1}^{h_2} K_R(0,z')\,dz' = \int_{-kh_1}^{kh_2} \frac{\cos W}{W}\,dZ' = \left(\int_0^{kh_1} + \int_0^{kh_2}\right) \frac{\cos W}{W}\,dZ' \qquad \text{(A-1)}$$

where $z_m = 0$ has been chosen for simplicity and $W = k[z'^2 + a^2]^{1/2}$, $Z = kz$. This is a generalized cosine integral given by

$$\Psi = 2 \ln \frac{2\sqrt{h_1h_2}}{a} - \text{Cin } kh_1 - \text{Cin } kh_2 \qquad \text{(A-2)}$$

where the condition $k^2a^2 << 1$ has been invoked and $\text{Cin } x = \int_0^x [(1 - \cos u)/u]du = C + \ln x - \text{Ci } x$. When kh_1 and kh_2 exceed about $\pi/2$, the approximation $\text{Cin } x \doteq C + \ln x$ is acceptable. With it

$$\Psi = \ln \frac{4h_1h_2}{a^2} - 2C - \ln k^2h_1h_2 = 2\left[\ln \frac{2}{ka} - C\right] \qquad \text{(A-3)}$$

where $C = 0.5772$. It is clear that Ψ has the same value for the horizontal as for the vertical elements if their radii are the same.

Appendix B: Evaluation of $T_1(z)$

The function $T_1(z)$ is defined by (26) with (24b). Specifically,

$$T_1(z) = \frac{1}{k}\int_0^z F_1(s)\sin k(z-s)\,ds \qquad \text{(B-1)}$$

where

$$F_1(z) = -j\int_{-h_1}^{h_2}\left(\frac{\partial^2}{\partial z'^2} + k^2\right) I(z')K_I(z,z')\,dz' \qquad \text{(B-2)}$$

In the evaluation of the first-order functions, zero-order values of currents and charges may be used. This means that

$$\left(\frac{\partial^2}{\partial z'^2} + k^2\right) I(z') = \frac{-j4\pi k^2}{\omega\mu\Psi} E_z^{inc} = k^2 A \qquad \text{(B-3)}$$

Also, $K_I(z,z') = -(\sin k\sqrt{(z-z')^2 + a^2}/\sqrt{(z-z')^2 + a^2}$. With these values

$$F_1(z) = jk^2A\int_{-h_1}^{h_2}\frac{\sin k\sqrt{(z'-z)^2 + a^2}}{k\sqrt{(z'-z)^2 + a^2}}\,k\,dz' = -jk^2A\int_{-k(h_1+z)}^{k(h_2-z)}\frac{\sin W}{W}\,dU \qquad \text{(B-4)}$$

where $W = \sqrt{U^2 + k^2a^2}$. This is a generalized sine integral.

$$F_1(z) = jkA\{S[ka,k(h_2-z)] + S[ka,k(h_1+z)]\} \qquad \text{(B-5)}$$

Since $(ka)^2 \ll 1$,

$$F_1(z) \doteq jk^2A[\text{Si}\,k(h_1+z) + \text{Si}\,k(h_2-z)] \qquad \text{(B-6)}$$

This expression can now be substituted in (B-1) and integrated as follows:

$$T_1(z) = jkA\int_0^z[\text{Si}\,k(h_1+s) + \text{Si}\,k(h_2-s)]\sin k(z-s)\,ds \qquad \text{(B-7)}$$

The result is

$$T_1(z) = -At_1(z)$$

$$t_1(z) = -j\{\mathrm{Si}\ k(h_1 + z) - \frac{1}{2}\sin k(h_1 + z)[\mathrm{Cin}\ 2k(h_1 + z) + 2\ \mathrm{Si}\ kh_1 \sin kh_1 - \mathrm{Cin}\ 2kh_1] - \frac{1}{2}\cos k(h_1 + z)[\mathrm{Si}\ 2k(h_1 + z) + 2\ \mathrm{Si}\ kh_1 \cos kh_1 - \mathrm{Si}\ 2kh_1] + \mathrm{Si}\ k(h_2 - z) - \frac{1}{2}\sin k(h_2 - z)[\mathrm{Cin}\ 2k(h_2 - z) + 2\ \mathrm{Si}\ kh_2 \sin kh_2 - \mathrm{Cin}\ 2kh_2] - \frac{1}{2}\cos k(h_2 - z)[\mathrm{Si}\ 2k(h_2 - z) + 2\ \mathrm{Si}\ kh_2 \cos kh_2 - \mathrm{Si}\ 2kh_2]\} \qquad \text{(B-8)}$$

Note that

$$t_1(0) = 0 \qquad \text{(B-9)}$$

Approximate forms useful near the junction when the arms of the cross are sufficiently long are obtained with $\mathrm{Si}\ x \doteq \pi/2$, $\mathrm{Cin}\ x \doteq C + \ln x$ when $x \geq \pi/2$ where $C = 0.577$ is Euler's constant. For $k(h_1 + z) \geq \pi/2$, $k(h_2 - z) \geq \pi/2$,

$$t_1(z) \doteq -j\{\pi(1 - \cos kz) - (1/2)[\sin k(h_1 + z)\ln(1 + z/h_1) + \sin k(h_2 - z)\ln(1 - z/h_2)]\} \qquad \text{(B-10)}$$

To supplement this formula, values at $z = -h_1$ and $z = h_2$ when $kh_1 \geq \pi/2$ and $kh_2 \geq \pi/2$ are useful. They are

$$t_1(-h_1) = -j[\pi(3/4 - \cos kh_1) - (1/2)\ln(1 + h_1/h_2)\sin k(h_1 + h_2)] \qquad \text{(B-11)}$$

$$t_1(h_2) = -j[\pi(3/4 - \cos kh_2) - (1/2)\ln(1 + h_2/h_1)\sin k(h_1 + h_2)] \qquad \text{(B-12)}$$

Appendix C: Evaluation of $T_2(z)$

The function $T_2(z)$ is defined by (26) with (21a). It is:

$$T_2(z) = \frac{1}{k} \int_0^z F_2(s) \sin k(z - s) \, ds \tag{C-1}$$

with

$$F_2(z) = j\omega \frac{\partial}{\partial z} \int_{-\ell_1}^{\ell_2} [q(x')]_0 K(z,x') \, dx'$$

$$= -kA \frac{\partial}{\partial z} \left\{ \int_{-\ell_1}^{0} [-C_3' \sin kx' + C_3'' \cos kx'] K(z,x') \, dx' + \int_0^{\ell_2} [-C_4' \sin kx' + C_4'' \cos kx'] K(z,x') \, dx' \right\} \tag{C-2}$$

The first two integrals can be expressed as follows:

$$\int_{-\ell_1}^{0} \sin kx' \, K(z,x') \, dx' = -\int_0^{\ell_1} \sin kx' \, K(z,x') \, dx' \tag{C-3a}$$

$$\int_{-\ell_1}^{0} \cos kx' \, K(z,x') \, dx' = \int_0^{\ell_1} \cos kx' \, K(z,x') \, dx' \tag{C-3b}$$

It follows that with $Z = kz$, $X = kx$, $L = k\ell$, $A = ka$,

$$F_2(z) = -kA[C_3' F_s(Z,L_1) - C_4' F_s(Z,L_2) + C_3'' F_c(Z,L_1) + C_4'' F_c(Z,L_2)] \tag{C-4}$$

where

$$F_s(Z,L) = \frac{\partial}{\partial Z} \int_0^L \sin X' \frac{e^{-j\sqrt{X'^2 + Z^2 + A^2}}}{\sqrt{X'^2 + Z^2 + A^2}} \, dX' \tag{C-5}$$

$$F_c(Z,L) = \frac{\partial}{\partial Z} \int_0^L \cos X' \frac{e^{-j\sqrt{X'^2 + Z^2 + A^2}}}{\sqrt{X'^2 + Z^2 + A^2}} \, dX' \tag{C-6}$$

With (C-4), (C-1) becomes

$$T_2(z) = -A[C_3'G_s(Z,L_1) - C_4'G_s(Z,L_2) + C_3''G_c(Z,L_1) + C_4''G_c(Z,L_2)] \qquad \text{(C-7)}$$

where

$$G_s(Z,L) = \int_0^L \sin X \, dX \int_0^Z \sin(Z - S) \frac{\partial}{\partial S} \frac{e^{-j\sqrt{S^2 + X^2 + A^2}}}{\sqrt{S^2 + X^2 + A^2}} \, dS \qquad \text{(C-8)}$$

$$G_c(Z,L) = \int_0^L \cos X \, dX \int_0^Z \sin(Z - S) \frac{\partial}{\partial S} \frac{e^{-j\sqrt{S^2 + X^2 + A^2}}}{\sqrt{S^2 + X^2 + A^2}} \, dS \qquad \text{(C-9)}$$

The second integrals in (C-8) and (C-9) can be integrated by parts to give:

$$G_s(Z,L) = - \sin Z \int_0^L \sin X \frac{e^{-j\sqrt{X^2 + A^2}}}{\sqrt{X^2 + A^2}} \, dX + I_s(Z,L) \qquad \text{(C-10)}$$

$$G_c(Z,L) = - \sin Z \int_0^L \cos X \frac{e^{-j\sqrt{X^2 + A^2}}}{\sqrt{X^2 + A^2}} \, dX + I_c(Z,L) \qquad \text{(C-11)}$$

where

$$I_s(Z,L) = \int_0^L \sin X \, dX \int_0^Z \cos(Z - S) \frac{e^{-j\sqrt{S^2 + X^2}}}{\sqrt{S^2 + X^2}} \, dS \qquad \text{(C-12)}$$

$$I_c(Z,L) = \int_0^L \cos X \, dX \int_0^Z \cos(Z - S) \frac{e^{-j\sqrt{S^2 + X^2}}}{\sqrt{S^2 + X^2}} \, dS \qquad \text{(C-13)}$$

In these last two integrals A^2 has been neglected in the radical since it contributes negligibly.

The integrals in (C-10) and (C-11) are generalized sine and cosine integrals. Since $k^2a^2 \equiv A^2$ is small, they reduce to the following:

$$G_s(Z,L) = -(1/2)\sin Z\ [\mathrm{Si}\ 2L - j\ \mathrm{Cin}\ 2L] + I_s(Z,L) \qquad \text{(C-14)}$$

$$G_c(Z,L) = -(1/2)\sin Z\ [2\ \sinh^{-1}(L/A) - \mathrm{Cin}\ 2L - j\ \mathrm{Si}\ 2L] + I_c(Z,L) \qquad \text{(C-15)}$$

The integrals $I_s(Z,L)$ and $I_c(Z,L)$ can be expressed as follows:

$$I_s(Z,L) = -(j/4)[e^{jZ}(J_1 - J_3) + e^{-jZ}(J_2 - J_4)] \qquad \text{(C-16a)}$$

$$I_c(Z,L) = (1/4)[e^{jZ}(J_1 + J_3) + e^{-jZ}(J_2 + J_4)] \qquad \text{(C-16b)}$$

where

$$J_1 = \int_0^L dx \int_0^Z dy\ e^{jx}e^{-jy}\ \frac{e^{-j\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\ ;\ J_2 = \int_0^L dx \int_0^Z dy\ e^{jx}e^{jy}\ \frac{e^{-j\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \qquad \text{(C-17a)}$$

$$J_3 = \int_0^L dx \int_0^Z dy\ e^{-jx}e^{-jy}\ \frac{e^{-j\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\ ;\ J_4 = \int_0^L dx \int_0^Z dy\ e^{-jx}e^{jy}\ \frac{e^{-j\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

(C-17b)

With the substitution $j = -i$, these four integrals are special cases of the general integral,

$$J(a,b) = \int_0^a dx \int_0^b dy\ e^{ix}e^{iy}\ \frac{e^{i\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \qquad \text{(C-18)}$$

in the forms:

$$J_1 = -J(-L,Z)\ ,\ J_2 = J(-L,-Z)\ ,\ J_3 = J(L,Z)\ ,\ J_4 = -J(L,-Z) \qquad \text{(C-19)}$$

The general double integral can be reduced to a single integral with the substitution $x = y \sinh \theta$ and the appropriate change in the limits of integration. The single integral in θ can be evaluated explicitly. It is

$$J(a,b) = i\{(1 - e^{ia})\sinh^{-1}(b/a) + (1 - e^{ib})\sinh^{-1}(a/b) - 2[\text{Cin}(a + b + u)$$

$$- i\ \text{Si}(a + b + u)] + \text{Cin}\ 2b - i\ \text{Si}\ 2b + \text{Cin}\ 2a - \text{Si}\ 2a$$

$$+ e^{ib}[\text{Cin}(a + u) - i\ \text{Si}(a + u) - \text{Cin}\ b + i\ \text{Si}\ b] + e^{ia}[\text{Cin}(b + u)$$

$$- i\ \text{Si}(b + u) - \text{Cin}\ a + i\ \text{Si}\ a]\} \qquad \text{(C-20)}$$

where $u = (a^2 + b^2)^{1/2}$. With this formula and (C-19), $I_s(Z,L)$ and $I_c(Z,L)$ as defined in (C-16a,b) are readily evaluated and substituted in (C-14) and (C-15) to obtain the following formulas in which i has been replaced by -j:

$$G_s(Z,L) = (j/2)\sin Z\ [\text{Cin}\ 2L + j\ \text{Si}\ 2L] + j\cos Z\ [\sin L \sinh^{-1}(Z/L) + \text{Si}\ 2Z]$$

$$+ j\sin Z\ [\text{Cin}\ 2Z + \text{Cin}\ 2L - \cos L\ \text{Cin}\ L - \sin L\ \text{Si}\ L] - j\ \text{Si}\ Z$$

$$- (1/2)e^{jZ}\{\text{Cin}(-L + Z + U) + j\ \text{Si}(-L + Z + U) + \text{Cin}(L + Z + U)$$

$$+ j\ \text{Si}(L + Z + U) - [\text{Cin}(Z + U) + j\ \text{Si}(Z + U)]\cos L\}$$

$$+ (1/2)e^{-jZ}\{\text{Cin}(-L - Z + U) + j\ \text{Si}(-L - Z + U) + \text{Cin}(L - Z + U)$$

$$+ j\ \text{Si}(L - Z + U) - [\text{Cin}(-Z + U) + j\ \text{Si}(-Z + U)]\cos L\} \qquad \text{(C-21)}$$

$$G_c(Z,L) = -\sin Z \sinh^{-1}(L/A) + (1/2)\sin Z\ [\text{Cin}\ 2L + j\ \text{Si}\ 2L] - j\cos Z$$

$$\times (1 - \cos L)\sinh^{-1}(Z/L) + j(1 - \cos Z)\sinh^{-1}(L/Z) + j\sin Z$$

$$\times [\sin L\ \text{Cin}\ L - \cos L\ \text{Si}\ L + \text{Si}\ 2L] - (j/2)e^{jZ}\{\text{Cin}(-L + Z + U)$$

$$+ j\ \text{Si}(-L + Z + U) - \text{Cin}(L + Z + U) - j\ \text{Si}(L + Z + U)$$

$$- j[\text{Cin}(Z + U) + j\ \text{Si}(Z + U)]\sin L\} + (j/2)e^{-jZ}\{\text{Cin}(-L - Z + U)$$

(cont.)

$$+ j\,\mathrm{Si}(-L - Z + U) - \mathrm{Cin}(L - Z + U) - j\,\mathrm{Si}(L - Z + U)$$

$$- j[\mathrm{Cin}(-Z + U) + j\,\mathrm{Si}(-Z + U)]\sin L\} \quad \text{(C-22)}$$

In these formulas $U = (Z^2 + L^2)^{1/2}$ and $Z = kz$, $L = k\ell$, $A = ka$; $\mathrm{Si}\, v = \int_0^v [(\sin v)/v]dv$, $\mathrm{Cin}\, v = \int_0^v [(1 - \cos v)/v]dv$. Note that $\mathrm{Si}(-v) = -\mathrm{Si}\, v$, $\mathrm{Cin}(-v) = \mathrm{Cin}\, v$. Also,

$$G_s(-Z,L) = -G_s(Z,L) \quad , \quad G_c(-Z,L) = -G_c(Z,L) \quad \text{(C-23)}$$

$$G_s(0,L) = 0 \quad , \quad G_c(0,L) = 0 \quad \text{(C-24)}$$

Simplified approximate formulas for the functions $G_s(Z,L)$ and $G_c(Z,L)$ can be obtained for the region of particular interest near the junction when the arms are sufficiently long so that $L^2 \gg Z^2$ and $H^2 \gg X^2$. Subject to these conditions $\mathrm{Cin}(-L \pm Z + U) \doteq \mathrm{Cin}\, Z$, $\mathrm{Cin}(L \pm Z + U) \doteq \mathrm{Cin}\, 2L \pm (Z/2L) \times (1 - \cos 2L)$; $\mathrm{Si}(-L \pm Z + U) \doteq \pm \mathrm{Si}\, Z$, $\mathrm{Si}(L \pm Z + U) \doteq \mathrm{Si}\, 2L \pm (Z/2L)\sin 2L$; $\mathrm{Cin}(\pm Z + U) \doteq \mathrm{Cin}\, L \pm (Z/L)(1 - \cos L)$, $\mathrm{Si}(\pm Z + U) \doteq \mathrm{Si}\, L \pm (Z/L)\sin L$; $\sinh^{-1}(Z/L) \doteq Z/L$, $\sinh^{-1}(L/Z) \doteq \ln(2L/Z)$. With these approximations

$$G_s(Z,L) \doteq j \sin Z\, [(1/2)(\mathrm{Cin}\, 2L - j\,\mathrm{Si}\, 2L) + j\,\mathrm{Si}\, L\, e^{jL} + \ln 2]$$

$$G_c(Z,L) \doteq -\sin Z\, [\ln(2L/A) + (1/2)(\mathrm{Cin}\, 2L - j\,\mathrm{Si}\, 2L) + j\,\mathrm{Si}\, L\, e^{jL}]$$

$$+ j(1 - \cos Z)\ln(2L/Z)$$

Note that $(1 - \cos Z)\ln Z$ vanishes at $Z = 0$. These can be further simplified with the approximations valid for $L \geq \pi/2$, $\mathrm{Cin}\, L \doteq C + \ln L$ where $C = 0.5772$, $\mathrm{Si}\, L \doteq \pi/2$.

Appendix D: Evaluation of $T_3(z)$

The function $T_3(z)$ is defined by (26) with (21b). It is

$$T_3(z) = \frac{1}{k}\int_0^z F_3(s)\sin k(z-s)\,ds \tag{D-1}$$

with

$$F_3(z) = -j\omega[q(h_2)K(z,h_2) - q(-h_1)K(z,-h_1)]_0 \tag{D-2}$$

where the zero-order values of the charges per unit length are to be used. It follows that

$$T_3(z) = -(j\omega/k)\{[q(h_2)]_0\vartheta(z,h_2) - [q(-h_1)]_0\vartheta(z,-h_1)\} \tag{D-3}$$

where

$$\vartheta(z,h_2) = \int_0^z \frac{e^{-jk\sqrt{(s-h_2)^2+a^2}}}{\sqrt{(s-h_2)^2+a^2}}\sin k(z-s)\,ds \tag{D-4}$$

$$\vartheta(z,-h_1) = \int_0^z \frac{e^{-jk\sqrt{(s+h_1)^2+a^2}}}{\sqrt{(s+h_1)^2+a^2}}\sin k(z-s)\,ds \tag{D-5}$$

These integrals can be reduced to generalized sine and cosine integrals. Thus,

$$\vartheta(z,h_2) = -\sin k(h_2-z)\int_{k(h_2-z)}^{kh_2} W^{-1}(\cos W - j\sin W)\cos U\,dU$$
$$+\cos k(h_2-z)\int_{k(h_2-z)}^{kh_2} W^{-1}(\cos W - j\sin W)\sin U\,dU \tag{D-6}$$

where $W = (U^2 + A^2)^{1/2}$. With $A^2 \ll 1$ the following result is obtained:

$$\vartheta(z,h_2) = \{\sinh^{-1}[(h_2 - z)/a] - \sinh^{-1}(h_2/a)\}\sin k(h_2 - z)$$
$$- (1/2)[\mathrm{Si}\ 2k(h_2 - z) - \mathrm{Si}\ 2kh_2 - j\ \mathrm{Cin}\ 2k(h_2 - z) + j\ \mathrm{Cin}\ 2kh_2]$$
$$\times \exp[jk(h_2 - z)] \qquad \text{(D-7)}$$

Similarly,

$$\vartheta(z,-h_1) = \{\sinh^{-1}[(h_1 + z)/a] - \sinh^{-1}(h_1/a)\}\ \sin k(h_1 + z)$$
$$- (1/2)[\mathrm{Si}\ 2k(h_1 + z) - \mathrm{Si}\ 2kh_1 - j\ \mathrm{Cin}\ 2k(h_1 + z) + j\ \mathrm{Cin}\ 2kh_1]$$
$$\times \exp[jk(h_1 + z)] \qquad \text{(D-8)}$$

Note that

$$\vartheta(-z,h) = \vartheta(z,-h) \quad ; \quad \vartheta(-z,-h) = \vartheta(z,h) \qquad \text{(D-9)}$$

$$\vartheta(0,\pm h) = 0 \quad ; \quad \vartheta(-h,-h) = \vartheta(h,h) \quad ; \quad \vartheta(-h,h) = \vartheta(h,-h) \qquad \text{(D-10)}$$

$$\vartheta(h_2,h_2) = (1/2)[\mathrm{Si}\ 2kh_2 - j\ \mathrm{Cin}\ 2kh_2] \quad ;$$
$$\vartheta(-h_1,-h_1) = (1/2)[\mathrm{Si}\ 2kh_1 - j\ \mathrm{Cin}\ 2kh_1] \qquad \text{(D-11)}$$

$$\vartheta(-h_1,h_2) = \ln(1 + h_2/h_1)\sin k(h_1 + h_2) - (1/2)[\mathrm{Si}\ 2k(h_1 + h_2) - \mathrm{Si}\ 2kh_1$$
$$- j\ \mathrm{Cin}\ 2k(h_1 + h_2) + j\ \mathrm{Cin}\ 2kh_1]\exp[jk(h_1 + h_2)] \qquad \text{(D-12)}$$

$$\vartheta(h_2,-h_1) = \ln(1 + h_1/h_2)\sin k(h_1 + h_2) - (1/2)[\mathrm{Si}\ 2k(h_1 + h_2) - \mathrm{Si}\ 2kh_2$$
$$- j\ \mathrm{Cin}\ 2k(h_1 + h_2) + j\ \mathrm{Cin}\ 2kh_2]\exp[jk(h_1 + h_2)] \qquad \text{(D-13)}$$

When z is not too near the ends and the lengths h_1 and h_2 of the two sections of the vertical element are sufficient so that $k(h_2 - z) \geq \pi/2$, $k(h_1 + z) \geq \pi/2$, the approximations $\mathrm{Si} \doteq \pi/2$, $\mathrm{Cin}\ x \doteq C + \ln x$ where C is Euler's constant are acceptable with x standing for $2k(h_2 - z)$, $2k(h_1 + z)$, $2kh_2$ and $2kh_1$. With these simplifications (D-7) and (D-8) reduce to:

$$\vartheta(z,h_2) \doteq (j/2)\ln(1 - z/h_2)\exp[-jk(h_2 - z)] \quad ;$$

$$\vartheta(z,-h_1) \doteq (j/2)\ln(1 + z/h_1)\exp[-jk(h_1 + z)] \qquad \text{(D-14)}$$

Also,

$$\vartheta(-h_1,h_2) \doteq (j/2)\ln(1 + h_1/h_2)\exp[-jk(h_1 + h_2)] \quad ;$$

$$\vartheta(h_2,-h_1) \doteq (j/2)\ln(1 + h_2/h_1)\exp[-jk(h_1 + h_2)] \qquad \text{(D-15)}$$

From (D-11),

$$\vartheta(h_2,h_2) \doteq (1/2)[\pi/2 - j(C + \ln 2kh_2)] \quad ;$$

$$\vartheta(-h_1,-h_1) \doteq (1/2)[\pi/2 - j(C + \ln 2kh_1)] \qquad \text{(D-16)}$$

Appendix E: Derivatives of Integral Trigonometric Functions

The functions $H_h'(z) = \partial H_h(z)/\partial z$ and $H_\ell'(x) = \partial H_\ell(x)/\partial x$ involve the derivatives of the integral sine and cosine functions Si y and Cin y. These are readily obtained directly from the integral definitions. Thus,

$$\frac{\partial}{\partial y} \text{Si } y = \frac{\partial}{\partial y} \int_0^y \frac{\sin u}{u} du = \frac{\sin y}{y} \tag{E-1}$$

$$\frac{\partial}{\partial y} \text{Cin } y = \frac{\partial}{\partial y} \int_0^y \frac{1 - \cos u}{u} du = \frac{1 - \cos y}{y} \tag{E-2}$$

With these formulas the expressions for $H_h'(z)$ and $H_\ell'(x)$ involve only trigonometric and integral trigonometric functions.

References

[1] C. D. Taylor, "Electromagnetic scattering from arbitrary configurations of wires," IEEE Trans. Antennas Propagat., Vol. AP-17, pp. 662-663, September 1969.

[2] C. D. Taylor, S.-M. Lin and H. V. McAdams, "Scattering from crossed wires," IEEE Trans. Antennas Propagat., Vol. AP-18, pp. 133-136, January 1970.

[3] C. M. Butler, "Currents induced on a pair of skew crossed wires," IEEE Trans. Antennas Propagat., Vol. AP-20, pp. 731-736, November 1972.

[4] H. H. Chao and B. J. Strait, "Radiation and scattering by a configuration of bent wires with junctions," IEEE Trans. Antennas Propagat., Vol. AP-19, pp. 701-702, September 1971.

[5] R. W. Burton and R. W. P. King, "The crossed-dipole: Theory and experiment," (to be published).