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Electromagnetic Penetration Through Apertures of Arbitrary Shape: Formulation and Numerical Solution Procedure

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Abstract

A set of integral equations are derived for the equivalent magnetic current in an aperture of arbitrary shape lying in an infinite ground screen. The integral equations are cast into a matrix equation using the method of moments. An algorithm is described for the solution of the matrix equation which minimizes the machine storage required for the computation.

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Coaxial Cables

CONTENTS

ILLUSTRATIONS

SECTION I

INTRODUCTION

The problem of analyzing the electromagnetic diffraction of a finite two-dimensional planar surface of negligible thickness, either a perfectly conducting disk or the complementary problem of an aperture in an infinite, perfectly conducting plane, has been studied by many investigators. For the most part, however, the work has been restricted to the consideration of the diffraction of scalar fields or of predicting the far-fields of the system under consideration. Furthermore, a common approach is to assume a distribution of current or fields on the conductor surface or aperture, respectively, and to calculate the diffracted fields using the Huygen-Kirchoff integrals.

Another approach has been to use variational methods to obtain low order solutions of the aperture fields or conductor currents. Only in the case of circular apertures or discs is an exact solution available and it takes the form of an expansion in terms of oblate spheriodal wavefunctions. For low frequencies, a solution has been obtained by Bouwkamp [1] for the first two terms in the power series expansion in the wavenumber for the circular aperture field or circular disk current.

The reader is referred to an extensive review and bibliography of early work in this area found in the classic paper by Bouwkamp.

Not until the advent of the modern high speed digital computers, along with advances in numerical techniques for solving systems of simultaneous integral equations, has significant progress been made in calculating the distribution of fields induced on such planar surfaces. First attempts in this area modeled the diffracting surface by a grid of electrically thin wires (or its electrical dual for apertures), and made use of coupled integral equations and the method of moments to solve for the induced currents (or aperture fields) [2].

Mittra, et al. [3] begin with a set of uncoupled partial integro-differential equations derived by Bouwkamp [1, pp. 75-76], which they subsequently integrate to eliminate the derivatives and obtain a set of uncoupled Hallen-like integral equations. They then numerically solve them for the induced currents on a rectangular perfectly conducting plate.

We consider here the problem of an aperture in an infinite perfectly conducting plane using a technique similar to Mittra's, but which differs in several important respects. The present method is similar to that used in [4] for the quasi-static solution for apertures. We also consider the case of non-rectangular apertures.

In Section II, the development of the uncoupled integral equations is given. Section III is concerned with numerical considerations in solving the simultaneous integral equations on a digital computer where storage limitations make a direct solution prohibitive. In the Appendix, a solution is derived for the solution of the inhomogeneous two-dimensional Helmholtz equation.

SECTION II

FORMULATION OF INTEGRAL EQUATIONS

Physical system description

The configuration under consideration consists of a vanishingly thin sheet of perfect electric conductor, or ground screen, extending over an entire infinite plane in free space, with the exception of an arbitrarily shaped aperture. The aperture is bounded by a contour C, which encloses an area S. For convenience we choose a coordinate system such that the ground screen lies in the x-y plane, and the origin is enclosed by C. This system is depicted in Figure 2.1.

Electromagnetic fields

All independent sources of electromagnetic fields are assumed to exist in the half-space $z>0$, the region $z<0$ being source free. The time-harmonic electric and magnetic field intensities are $\bar{E}(\bar{r},\omega)$ and $\bar{H}(\bar{r},\omega)$, respectively, with the time variation $\exp(j\omega t)$ suppressed. We may relate $\bar{E}(\bar{r},\omega)$ to the corresponding time domain quantity $\bar{\mathcal{E}}(r,t)$ by the Fourier transform pair

$$\bar{E}(\bar{r},\omega) = \int_{-\infty}^{\infty} \bar{\mathcal{E}}(\bar{r},t)e^{-j\omega t}dt \quad ,$$

$$\bar{\mathcal{E}}(\bar{r},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{E}(\bar{r},\omega)e^{j\omega t}d\omega$$

with a similar relationship for $\bar{H}(\bar{r},\omega)$ and $\bar{\mathcal{H}}(\bar{r},t)$ and for other field quantities. We shall solve Maxwell's equations for this problem using the transformed quantities $\bar{E}(\bar{r},\omega)$ and $\bar{H}(\bar{r},\omega)$ only.

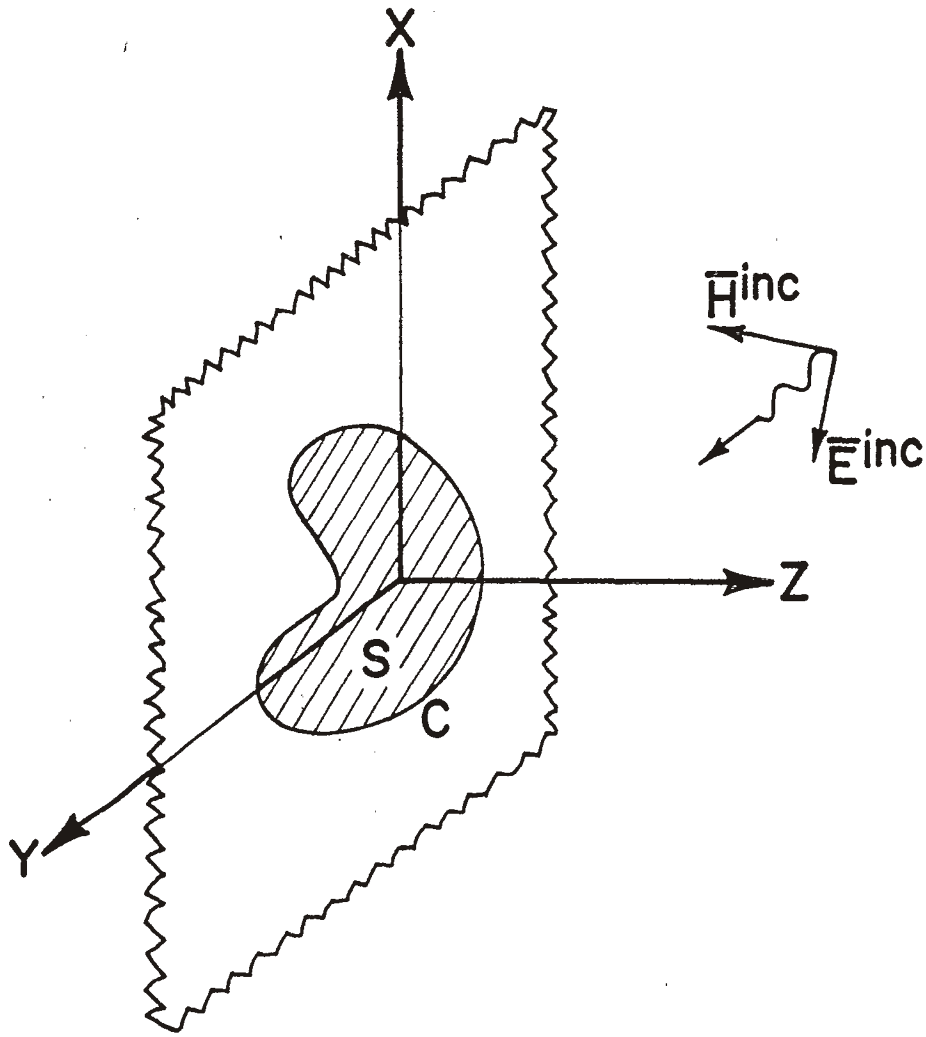


Figure 2.1. Infinite ground screen with aperture.

Boundary conditions

The configuration of Figure 2.1 imposes several boundary conditions on the electromagnetic fields. These conditions are [1]

$$\hat{u}_z \times \bar{E} = 0 \text{ on the surface of the ground screen} \quad (2.1)$$

$$\hat{u}_z \times \bar{E} \text{ is continuous through the aperture} \quad (2.2)$$

$$\hat{u}_z \times \bar{H} \text{ is continuous through the aperture .} \quad (2.3)$$

Development of an equivalent model of the system

Considering, for the moment, only the fields in the region z>0, we invoke the surface equivalence principle [5, pp. 106-110] to replace the entire ground screen, aperture and half-space z<0, as far as their effect in the half-space z>0 is concerned, by an imaginary surface in the x-y plane on which flow equivalent surface current densities $\bar{J}_s^+ = \hat{u}_z \times \bar{H}$ and $\bar{M}_s^+ = \bar{E} \times \hat{u}_z$. The "+" superscript denotes the z>0 region. Note that boundary condition (2.1) implies that $\bar{M}_s^+$ is non-zero only over that portion of the x-y plane originally occupied by the aperture, that is, over the surface S.

The application of the surface equivalence principle in the z>0 region causes zero fields, due to the equivalent sources and the incident fields, to exist in the region z<0. Therefore, as far as the fields in the z>0 region are concerned, it does not matter what material exists in the z<0 region. Consequently, we may place an infinite, vanishingly thin sheet of perfect electric conductor immediately behind the surface currents in the x-y plane. This eliminates the contribution of the electric surface current density $\bar{J}_s^+$ to the fields in the z>0 region, as may be easily seen from considering the image of $\bar{J}_s^+$ due to the infinite ground screen.

The development of this equivalent system from the z>0 region is represented pictorially in Figure 2.2. The use of entirely analogous arguments leads to an equivalent system for the z<0 region, as depicted in Figure 2.3. Note that the unit normal to S for the z<0 region is $-\hat{u}_z$.

Application of boundary condition (2.2), along with the fact that the unit surface normal vectors are oppositely directed for the two half-spaces, leads to the conclusion that $\bar{M}_s^- = -\bar{M}_s^+$. This gives a final composite equivalent system for the ground screen with aperture, as depicted in Figure 2.4.

The quantities $\bar{J}^{inc}$ and $\bar{M}^{inc}$ in Figures 2.2, 2.3, and 2.4 represent the independent sources of the electromagnetic fields.

The total fields $\bar{E}$ and $\bar{H}$ in the z>0 region are composed of incident fields $\bar{E}^{inc}$ and $\bar{H}^{inc}$, due to the independent sources, which would exist in free space, i.e. without the presence of the ground screen and aperture, plus fields $\bar{E}^r$ and $\bar{H}^r$ reflected from the infinite non-perforated ground screen of the equivalent model, and scattered fields $\bar{E}^{s+}$ and $\bar{H}^{s+}$ radiated by the equivalent surface current $\bar{M}_s^+$ in the presence of the ground screen. Note that in some of the literature the sum of the incident field and the reflected field is called the short-circuit field, written as

$$\bar{E}^{sc} = \bar{E}^{inc} + \bar{E}^r \tag{2.4}$$

and

$$\bar{H}^{sc} = \bar{H}^{inc} + \bar{H}^r \ . \tag{2.5}$$

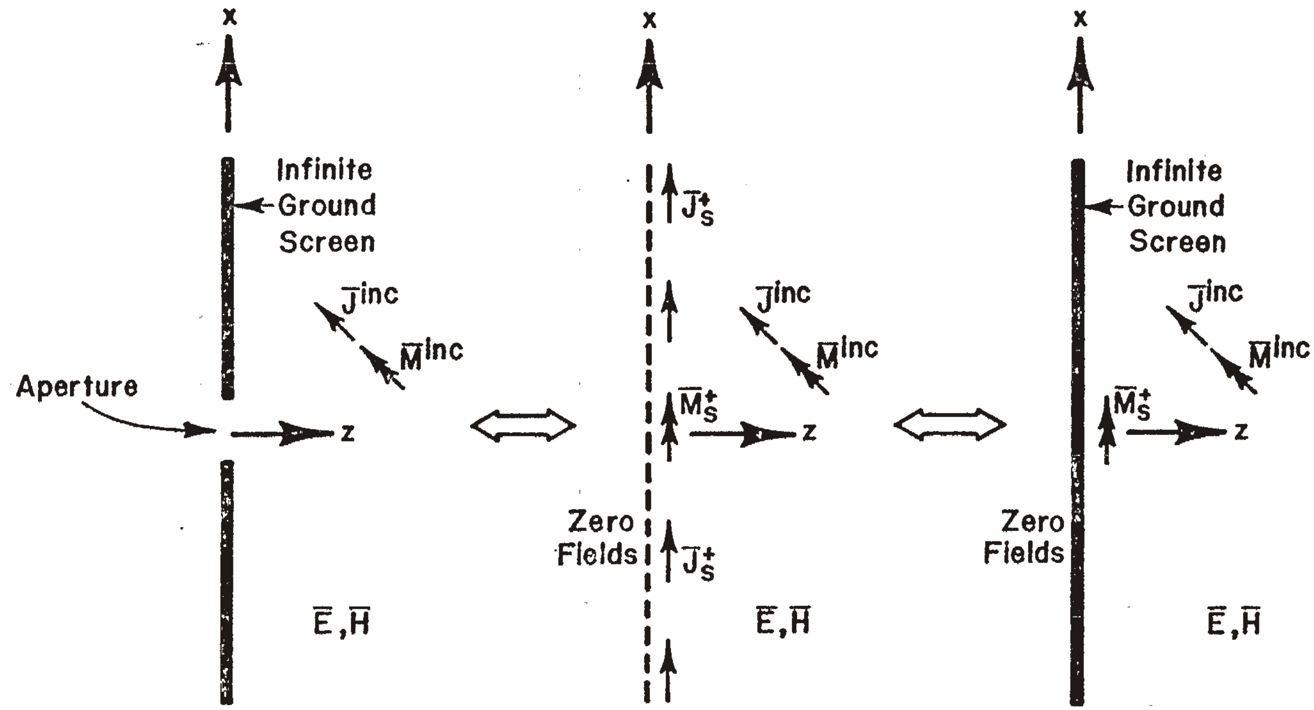


Figure 2.2. Equivalent system for z>0 region.

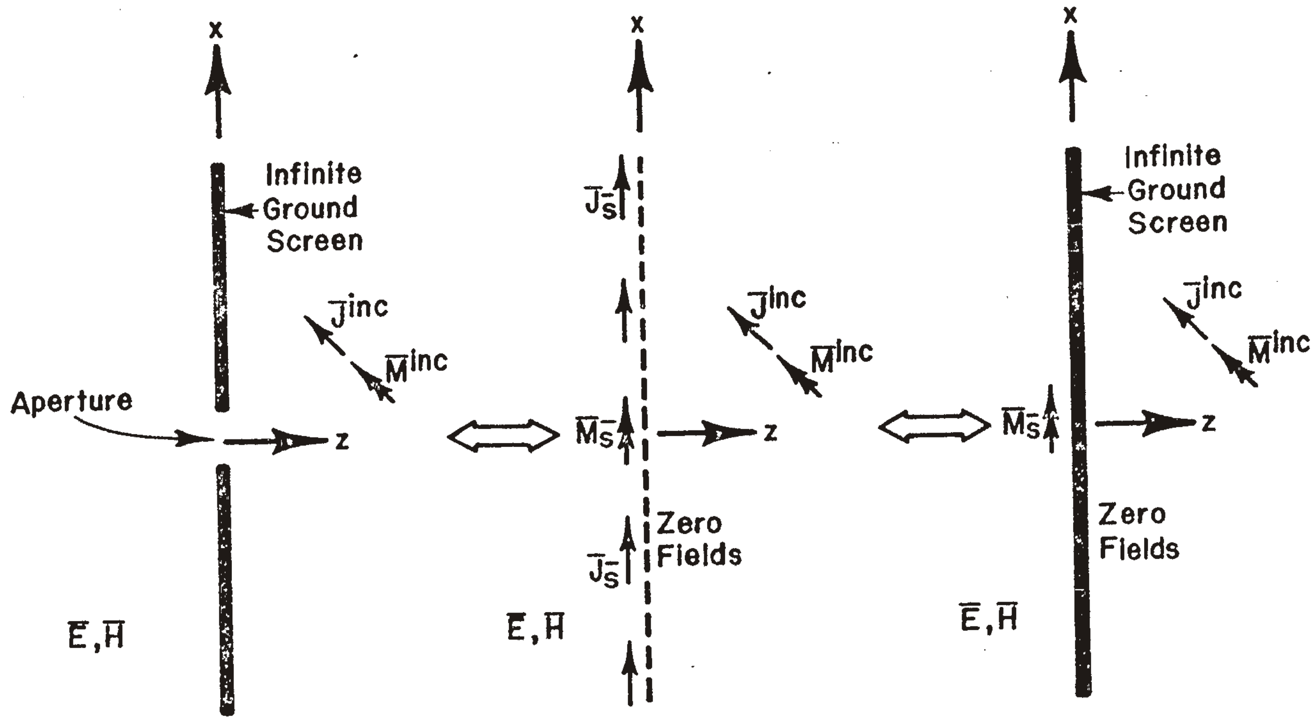


Figure 2.3. Equivalent system for $z < 0$ region.

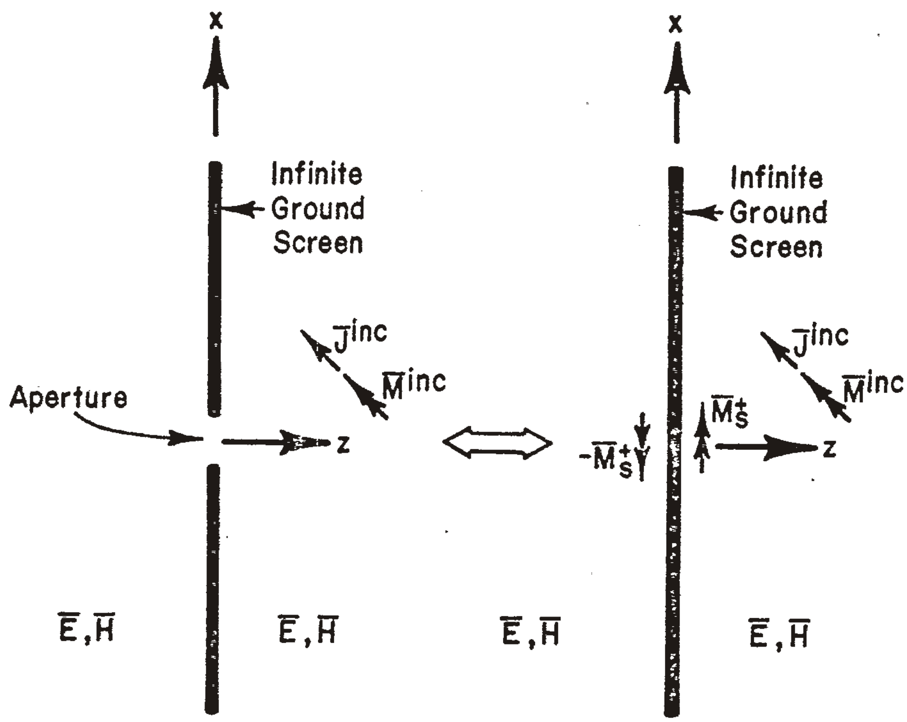


Figure 2.4. Equivalent system valid in both half-spaces.

In the z<0 region the total fields are simply the scattered fields $\bar{E}^{s-}$ and $\bar{H}^{s-}$ radiated by the equivalent magnetic surface current density M_s^- in the presence of the ground screen.

The fact that $\bar{M}_s^- = -\bar{M}_s^+$ implies that we need only solve for $\bar{M}_s^+$. We note at this point that with this restriction the equivalent system of Figure 2.4 satisfies boundary conditions (2.1) and (2.2). We have only to satisfy the condition on continuity of the tangential $\bar{H}$ field in the aperture, condition (2.3). This condition will finally determine the actual distribution of the equivalent current $\bar{M}_s^\pm$ in the aperture. In order to enforce this condition in terms of the currents $\bar{M}_s^\pm$, we must find the fields $\bar{H}^{s\pm}$ that they radiate.

Calculation of $\bar{H}$ fields

In order to facilitate expressing the fields $\bar{H}^{s\pm}$ in terms of $\bar{M}_s^+$ in the region z>0, we employ image theory, removing the infinite ground screen of the equivalent system and replacing its effect in the z>0 region by an image magnetic surface current density identical to the original and located an infinitesimally small distance away. This leaves, in effect, a magnetic surface current distribution $2\bar{M}_s^+$, which is non-zero only over the surface S, which resides in free space, and which, together with the incident fields, yields the correct fields for z>0 and zero fields for z<0. This free space equivalent system for z>0 is depicted in Figure 2.5. A similar equivalent system for the z<0 region is easily derived, and differs from that of Figure 2.5 only by the fact that the current distribution is the negative of that for

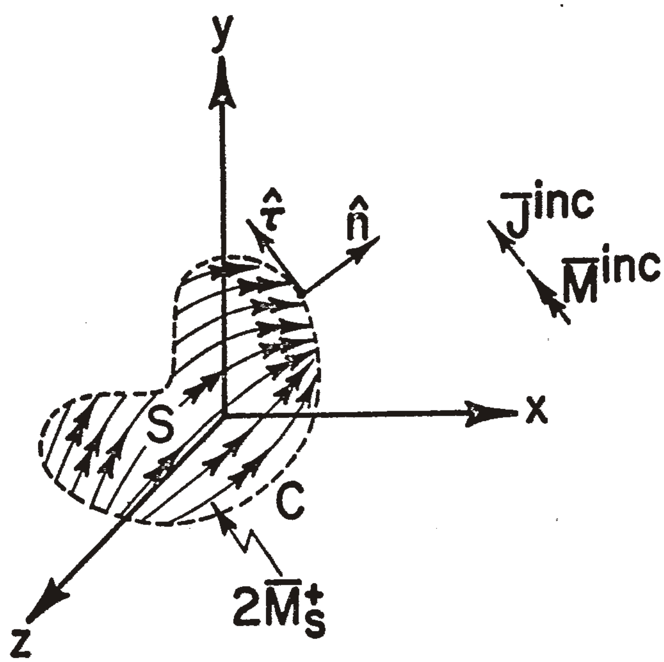


Figure 2.5. Equivalent system in free space for scattered fields in the region z>0.

z>0, and the sources of the incident field are not included.

An expression for $\bar{H}^{s\pm}$ is easily obtained in each region by using the free-space equivalent systems derived, along with the free-space electric vector potential integral and Maxwell's curl equation for $\bar{E}$, which results in [5, pp. 125-131]:

$$\left(\frac{1}{j\omega\mu_0}\right)(\nabla\nabla\cdot\bar{F}^{\pm} + k^2\bar{F}^{\pm}) = \bar{H}^{s\pm} \quad , \quad z \gtrless 0 \qquad (2.6)$$

where the electric vector potential in this case is:

$$\bar{F}^{\pm} = \frac{1}{4\pi}\iint_S \frac{(2\bar{M}_s^{\pm})\, e^{-jk|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} dS' \qquad (2.7)$$

and

$$\bar{r} = x\hat{u}_x + y\hat{u}_y + z\hat{u}_z$$

$$\bar{r}' = x'\hat{u}_x + y'\hat{u}_y + z'\hat{u}_z \quad .$$

Substitution of Equation (2.5) and (2.6) into the boundary condition (2.3) gives

$$\hat{u}_z \times \left[\bar{H}^i + \bar{H}^r + \left(\frac{1}{j\omega\mu_0}\right)(\nabla\nabla\cdot\bar{F}^+ + k^2\bar{F}^+)\right]_{z=0^+} = \hat{u}_z \times \left(\frac{1}{j\omega\mu_0}\right)(\nabla\nabla\cdot\bar{F}^- + k^2\bar{F}^-)\Big|_{z=0^-}$$

(2.8)

On the surface of the perfect electric conductor $u_z \times (\bar{H}^i - \bar{H}^r) = 0$. In addition let $\bar{M}_s^+ = -\bar{M}_s^- \equiv \bar{M}_s$, which implies that $\bar{F}^+\big|_{z=0^+} = -\bar{F}^-\big|_{z=0^-} \equiv \bar{F}\big|_{z=0}$. Substitution of these expressions into Equation (2.8) gives:

$$\hat{u}_z \times \left[2\bar{H}^i + \left(\frac{1}{j\omega\mu_0}\right)(\nabla\nabla\cdot\bar{F} + k^2\bar{F})\right]_{z=0} = -\hat{u}_z \times \left(\frac{1}{j\omega\mu_0}\right)(\nabla\nabla\cdot\bar{F} + k^2\bar{F})\Big|_{z=0}$$

which we can write since all derivatives on the surviving components are in the plane of the aperture, where $\bar{F}^{+} = -\bar{F}^{-}$. Rearranging, combining terms, and rewriting the above equation as two scalar equations, we have:

$$\left[\frac{\partial^2}{\partial x^2} + k^2\right] F_x + \frac{\partial^2}{\partial x \partial y} F_y = -j\omega\mu_0 H_x^i \quad \text{(on S)} \tag{2.9}$$

$$\left[\frac{\partial^2}{\partial y^2} + k^2\right] F_y + \frac{\partial^2}{\partial y \partial x} F_x = -j\omega\mu_0 H_y^i \quad \text{(on S)} \tag{2.10}$$

We note that Equations (2.9) and (2.10) with the substitution of (2.7) for $\bar{F}$, are the duals of the integral equations obtained for calculating scattering by a vanishingly thin sheet of perfect electric conductor of the same size and shape as the aperture. Babinet's principle is a consequence of this duality [5, pp. 365-367]. In fact, the two scalar equations may also be expressed as

$$\hat{u}_z \times (\bar{H}^s + \bar{H}^i) = 0 \quad \text{(on S)} \tag{2.11}$$

which is the boundary condition for scattering from a perfect magnetic conductor. While Equations (2.9) and (2.10) are a valid set of integral equations for the determination of the magnetic surface current density $\bar{M}_s$, it is possible to reduce them to an uncoupled form as was orginally done by Bouwkamp [1].

Uncoupling the integral equations

Application of Maxwell's curl equation on $\bar{H}$ (and an appropriate vector identity) in Equation (2.11) leads to

$$\hat{u}_z \cdot (\bar{E}^s + \bar{E}^i) = 0 \quad \text{(on S)}$$

which is dual to the requirement that the normal component of $\bar{H}$ must vanish on a perfect conductor. Writing $\hat{u}_z \cdot \bar{E}^s$ in terms of the electric vector potential yields

$$E_z^i + \frac{\partial}{\partial y}F_x - \frac{\partial}{\partial x}F_y = 0 \quad \text{(on S)}. \qquad (2.12)$$

Using Equation (2.12) we may eliminate F_y in(2.9) and F_x in (2.10) to obtain

$$\{\nabla_t^2 + k^2\}F_x + \frac{\partial}{\partial y}E_z^i = -j\omega\mu_0 H_x^i \quad \text{(on S)} \qquad (2.13)$$

$$\{\nabla_t^2 + k^2\}F_y - \frac{\partial}{\partial x}E_z^i = -j\omega\mu_0 H_y^i \quad \text{(on S)} \qquad (2.14)$$

where the transverse Laplacian is defined as

$$\nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad .$$

But by Maxwell's curl equation on $\bar{E}$, $-j\omega\mu_0 H_x^i = \frac{\partial}{\partial y}E_z^i - \frac{\partial}{\partial z}E_y^i$, and $-j\omega\mu_0 H_y^i = \frac{\partial}{\partial z}E_x^i - \frac{\partial}{\partial x}E_z^i$, which we substitute into (2.13) and (2.14) to obtain the vector equation

$$\{\nabla_t^2 + k^2\}\bar{F} = \hat{u}_z \times \frac{\partial}{\partial z}\bar{E}^i \quad \text{(on S)} . \qquad (2.15)$$

Note, however, that Equation (2.15) no longer satisfies the boundary condition of Equation (2.12), which must now be imposed as an auxiliary condition in order to uniquely determine $\bar{M}_s$ [3]. Equation (2.15) is the form obtained by Bouwkamp [1], and its dual is the form discussed subsequently by Mittra, et al. [3] in their treatment of scattering by a perfectly conducting plate. In their treatment, Mittra, et al. integrate Equation (2.15) and solve the resulting integral equations. We follow generally the same approach in the following development, but our procedure differs from theirs in a number of details.

Comparing Equation (2.15) and Equations (1) and (7) in the Appendix, we find that a solution to Equation (2.15) may be written as the vector equation

$$\bar{F}(\bar{\rho}) = -\hat{u}_z \times \left(\frac{1}{4j}\right) \iint_S \left(\frac{\partial}{\partial z}\bar{E}^i\right) H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|dS' \qquad (2.16)$$

$$+\hat{u}_z \times \left(\frac{1}{4j}\right) \oint_C \bar{S}(\bar{\rho}') H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' \quad (\bar{\rho}\varepsilon S)$$

$\bar{\rho} = x\hat{u}_x + y\hat{u}_y$ and $\bar{\rho}' = x'\hat{u}_x + y'\hat{u}_y$, and the unknown function $\bar{S}$ is defined by

$$\bar{S} = S_n\hat{n} + S_\tau\hat{\tau}$$

where $\hat{n}$ is a unit normal to C in the plane of S and pointing out of S, and $\hat{\tau}$ is the unit tangent to C and is related to $\hat{n}$ by (Figure 2.5)

$$\hat{n} \times \hat{\tau} = \hat{u}_z \quad .$$

These unit vectors may be written in terms of rectangular coordinate components as

$$\hat{n} = n_x\hat{u}_x + n_y\hat{u}_y \quad \text{and} \quad \hat{\tau} = -n_y\hat{u}_x + n_x\hat{u}_y \quad .$$

Furthermore $S_x = S_n n_x - S_\tau n_y$ and

$$S_y = S_n n_y + S_\tau n_x \quad .$$

The substitution of Equation (2.7) with $z = z' = 0$ into Equation (2.16) gives:

$$\frac{1}{2\pi}\iint_S \bar{M}_s \frac{e^{-jk|\bar{\rho}-\bar{\rho}'|}}{|\bar{\rho}-\bar{\rho}'|} dS' \; -\hat{u}_z \times (\frac{1}{4j}) \oint_C \bar{S}(\bar{\rho}') H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) d\ell'$$

$$= -\hat{u}_z \times (\frac{1}{4j}) \iint_S (\frac{\partial}{\partial z}\bar{E}^i) H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) dS' \quad (\bar{\rho}\varepsilon S) \qquad (2.17)$$

The various integrands appearing in Equation (2.17) all have integrable singularities providing the field quantities appearing therein are sufficiently well behaved. Equation (2.17) represents two uncoupled integral equations for the scalar components M_{sx} and M_{sy} of the equivalent magnetic surface current density $\bar{M}_s$. The coupling of the equations is now found in the relation between the components S_n and S_τ of the unknown function $\bar{S}$.

Relating the components of the function $\bar{S}$

The substitution of Equation (2.16) into the auxiliary condition, Equation (2.12), yields a relationship between the components of the function $\bar{S}$ and the incident field:

$$E_z^i + \frac{1}{4j}\frac{\partial}{\partial z}\iint_S \bar{E}_t^i(\bar{\rho}') \cdot \nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) dS' \qquad (2.18)$$

$$- \frac{1}{4j}\oint_C \bar{S}(\bar{\rho}') \cdot \nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) d\ell' = 0, \quad (\bar{\rho}\varepsilon S).$$

Equation (2.18) can be manipulated into a form involving only contour integrals over C. First, note that $\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) = -\nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)$. Using the scalar form of Green's first identity in two dimensions, the surface integral from Equation (2.18) is

$$- \frac{1}{4j}\frac{\partial}{\partial z}\iint_S \bar{E}_t^i(\bar{\rho}')\cdot\nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS' = - \frac{1}{4j}\frac{\partial}{\partial z}\oint_C \bar{E}_t^i(\bar{\rho}')\cdot\hat{n}' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell'$$

$$+ \frac{1}{4j}\frac{\partial}{\partial z}\iint_S \nabla_t'\cdot\bar{E}_t^i(\bar{\rho}') H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS'. \qquad (2.19)$$

We define the del operators

$$\nabla' = \hat{u}_x\frac{\partial}{\partial x'} + \hat{u}_y\frac{\partial}{\partial y'} + \hat{u}_z\frac{\partial}{\partial z}$$

$$\nabla = \hat{u}_x\frac{\partial}{\partial x} + \hat{u}_y\frac{\partial}{\partial y} + \hat{u}_z\frac{\partial}{\partial z}$$

the first of which operates on coordinates $(\bar{\rho}',z)$ and the second on coordinates $(\bar{\rho},z)$.

Because of the divergence condition on $\bar{E}(\bar{\rho}',z)$, we have $\nabla'\cdot\bar{E}^i = \frac{\partial}{\partial x'}E_x^i + \frac{\partial}{\partial y'}E_y^i + \frac{\partial}{\partial z}E_z^i = 0$, which gives $\nabla_t'\cdot\bar{E}_t^i(\bar{\rho}') = \frac{\partial}{\partial x'}E_x^i + \frac{\partial}{\partial y'}E_y^i = -\frac{\partial}{\partial z}E_z^i$ and which may be used to replace the transverse divergence of $\bar{E}^i$ in Equation (2.19). We further note that since $\{\nabla^2+k^2\}E_z^i = 0$, then

$$\{\nabla_t^2 + k^2\}E_z^i = -\frac{\partial^2}{\partial z^2}E_z^i \quad \text{(on S)} \qquad (2.20)$$

from which we can write, using Equation (6) of the Appendix,

$$E_z^i = \frac{1}{4j}\iint_S \left(\frac{\partial^2}{\partial z^2}E_z^i\right) H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\,dS' + \frac{1}{4j}\oint_C \{H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\nabla_t' E_z^i$$
$$- E_z^i \nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\}\cdot\hat{n}'\,d\ell' \qquad (2.21)$$

Substitution of Equation (2.21) into Equation (2.18) yields

$$\frac{1}{4j}\iint_S \left(\frac{\partial^2}{\partial z^2}E_z^i(\bar{\rho}')\right) H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\,dS' + \frac{1}{4j}\oint_C H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\nabla_t' E_z^i\cdot\hat{n}'\,d\ell'$$
$$- \frac{1}{4j}\oint_C E_z^i \nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\cdot\hat{n}'\,d\ell' - \frac{1}{4j}\frac{\partial}{\partial z}\oint_C \bar{E}_t^i(\bar{\rho}')\cdot n' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\,d\ell'$$
$$- \frac{1}{4j}\frac{\partial^2}{\partial z^2}\iint_S E_z^i(\bar{\rho}') H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\,dS' - \frac{1}{4j}\oint_C \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\,d\ell'$$
$$= 0 \qquad (\bar{\rho}\in S) \qquad (2.22)$$

The two surface integrals cancel, and we have finally

$$\oint_C \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\,d\ell' = -\oint_C \Big[E_z^i \nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)$$
$$+ \left[\frac{\partial}{\partial z}\bar{E}_t^i - \nabla_t' E_z^i\right] H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\Big]\cdot\hat{n}'\,d\ell' \qquad (\bar{\rho}\in S) \qquad (2.23)$$

This equation may be simplified slightly by noting that Maxwell's curl equation on $\bar{E}$ allows us to write $\frac{\partial}{\partial z}\bar{E}_t^i - \nabla_t' E_z^i = -(j\omega\mu)\hat{u}_z \times \bar{H}_t^i$ which, upon substitution into Equation (2.23), gives us

$$\oint_C \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' = -\oint_C \Big[E_z^i \nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)$$

$$+ \left[j\omega\mu_0 \hat{u}_z \times \bar{H}_t^i\right] H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\Big]\cdot\hat{n}'d\ell' \qquad (\bar{\rho}\varepsilon S) \qquad (2.24)$$

Equation (2.24) must be true for all $\bar{\rho}\varepsilon S$. However, this results in an overspecification of the problem. One may easily verify that (2.24) is a solution to the source-free wave equation for points $\bar{\rho}\varepsilon S$. As such, the analyticity of solutions to the wave equation implies that if the equation is satisfied for all $\bar{\rho}$ on some closed contour contained completely within S, then by an analytic continuation procedure [6], one can argue that it is satisfied throughout the interior of S. This argument is similar to that used by Waterman in his derivation of integral equations using the so-called extended boundary conditions [7]. In numerical procedures, it is more convenient, as well as more numerically stable, to choose the contour of integration to be the contour C approached from the interior. The singular behavior of certain terms in the integrands of Equation (2.24) requires that we exercise care in letting $\bar{\rho}$ approach C in the treatment of the integrals.

Handling singularities in the contour integrals

If we assume the contour C is smooth, the following method [8] can be applied.

The contour C is broken into two regions, one, C', involving that portion of C not in a neighborhood of the observation point $\bar{\rho}$, for $\bar{\rho}\varepsilon C$, and the other, C_0, being the remainder of C, which is chosen to be centered about the point $\bar{\rho}_0$, the limit of $\bar{\rho}$ as $\bar{\rho}$ approaches C.

The left side of Equation (2.24) becomes

$$\oint_C \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' = \int_{C'}\left[S_n\hat{n}+S_\tau\hat{\tau}'\right]\cdot\nabla H^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell'$$

$$+\int_{C_0}\left[S_n\hat{n}'+S_\tau\hat{\tau}\right]\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' \qquad (2.25)$$

As $\bar{\rho}\to C$ and $C_0\to 0$, the limiting case of the integral over C' is an improper, convergent integral given by

$$\lim_{\substack{C_0\to 0\\ \bar{\rho}\to C}}\int_{C'}\left[S_n\hat{n}'+S_\tau\hat{\tau}'\right]\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' = \oint\!\!\!\!-_C S(\bar{\rho})\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell'$$

where the symbol $-_C$ denotes a deleted integral, that is, the integral is performed over all points on C except that point $\bar{\rho}' = \bar{\rho}$.

The remaining integral is evaluated as C_0 tends to zero length by considering C_0 to be a straight line segment, and by employing the small argument approximation for the gradient of a Hankel function

$$\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) \underset{\rho\to\rho'}{\to} -\frac{j2}{\pi}\frac{(\bar{\rho}-\bar{\rho}')}{|\bar{\rho}-\bar{\rho}'|^2} \quad .$$

Thus $$\int_{C_0}\bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' \simeq -\frac{j2}{\pi}\int_{C_0}\left[S_n\hat{n}+S_t\hat{t}'\right]\cdot\frac{(\bar{\rho}-\bar{\rho}')}{|\bar{\rho}-\bar{\rho}'|}\frac{d\ell'}{|\rho-\bar{\rho}'|}$$

Refer to Figure 2.6 to observe that

$$\frac{(\bar{\rho}-\bar{\rho}')\cdot\hat{n}'}{|\bar{\rho}-\bar{\rho}'|} = -\cos\theta \quad \text{and} \quad \frac{(\bar{\rho}-\bar{\rho}')\cdot\hat{\tau}'}{|\bar{\rho}-\bar{\rho}'|} = \cos\phi$$

In addition, we have

$$d\alpha \simeq \sin d\alpha \simeq \frac{\cos\theta\, d\ell'}{|\bar{\rho}-\bar{\rho}'|}$$

since $d\ell'$ and $d\alpha$ are very small. Hence, the integral over C_0 becomes

$$\int_{C_0} \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' \simeq j\frac{2}{\pi}S_n(\bar{\rho}_0)\int_{C_0} d\alpha - j\frac{2}{\pi}S_\tau(\bar{\rho}_0)\int_{C_0}\frac{\cos\phi\, d\ell'}{|\bar{\rho}-\bar{\rho}'|} \qquad (2.26)$$

The value of $\bar{S}(\bar{\rho}')$ is assumed to vary slowly along C_0 and is approximated by its value at $\bar{\rho}_0$, where $\bar{\rho}_0$ denotes the point on C_0 which $\bar{\rho}$ approaches as $\bar{\rho}\to C$ (see Figure 2.6). Since C_0 is chosen so that $\bar{\rho}_0$ is in the center of segment C_0, then the second integral on the right side of the approximate Equation (2.26) is zero because the integrand is odd. The first integral approaches π in the limit as $\bar{\rho}$ approaches $\bar{\rho}_0$. The left side of Equation (2.24) is thus

$$\lim_{\rho\to C}\oint_C \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' = 2j\bar{S}(\bar{\rho})\cdot\hat{n}(\bar{\rho}) + ⨍_C \bar{S}(\bar{\rho}')\cdot\nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' \qquad (2.27)$$

Similarly, we also have that

$$\lim_{\rho\to C^-}\oint_C E_z^i\nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\cdot\hat{n}'d\ell' = 2jE_z^i(\bar{\rho}) - ⨍_C E_z^i\nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\cdot\hat{n}'d\ell' \qquad (2.28)$$

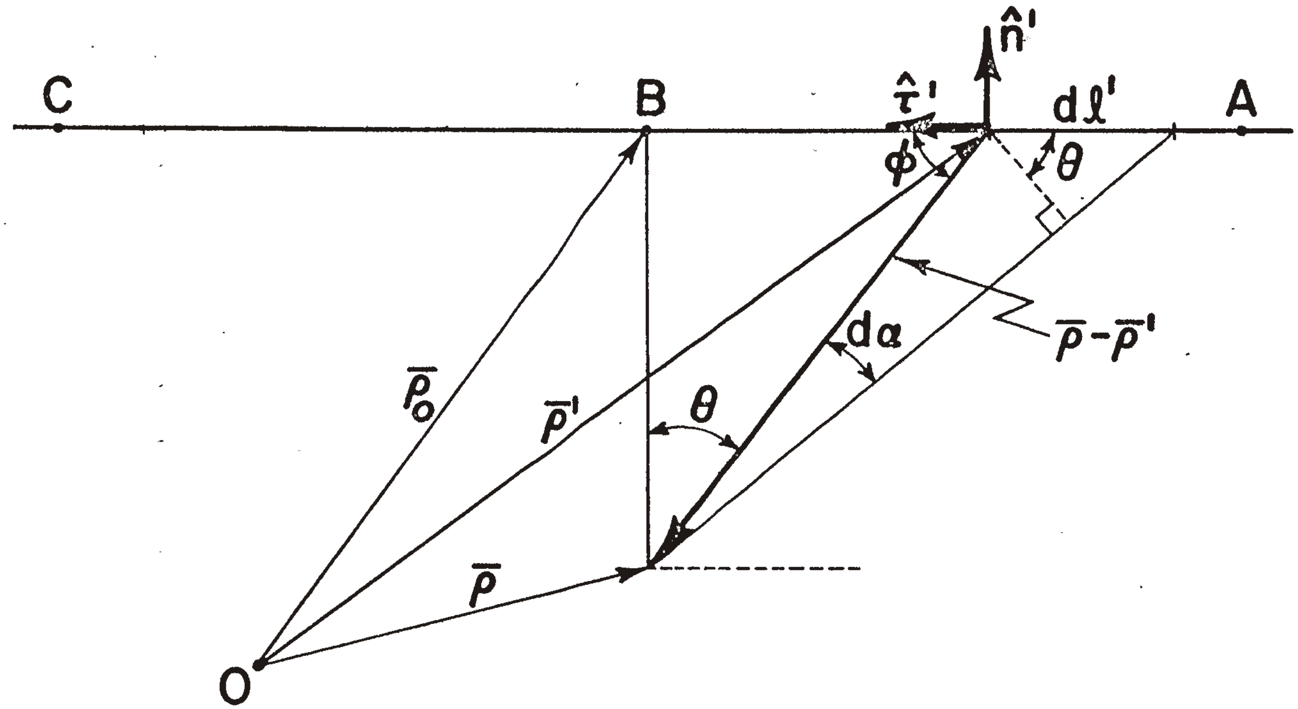


Figure 2.6. Geometry relevant to the evaluation of the principal part of the contour integral.

The integral involving $\bar{H}_t^i$ in (2.24), due to the integrable nature of the singularity of its integrand, is well-behaved as we take the limit $\bar{\rho} \to C$, and may be written as a deleted integral or not, as is convenient. This allows us finally to write Equation (2.24) as

$$2j\bar{S}(\bar{\rho}) \cdot \hat{n}(\bar{\rho}) + \fint_C \bar{S}(\bar{\rho}') \cdot \nabla_t H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\, d\ell'$$

$$= 2jE_z^i(\bar{\rho}) - \fint_C E_z^i \nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) \cdot \hat{n}' \, d\ell'$$

$$- j\omega\mu_0 \int (\hat{u}_z \times \bar{H}_t^i) H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) \cdot \hat{n}' \, d\ell' \ , \quad (\bar{\rho} \varepsilon S) \quad (2.29)$$

With the addition of the condition that the normal component of magnetic surface current density $\bar{M}_s$ vanish at the boundary contour

$$\bar{M}_s(\bar{\rho}) \cdot \hat{n}(\bar{\rho}) = 0 \ , \quad (\bar{\rho} \varepsilon C) \ , \quad (2.30)$$

Equations (2.17) and (2.29) form a complete system for the unique determination of the unknown $\bar{M}_s$ current distribution.

Specialization to plane wave incidence

While Equations (2.17), (2.29), and (2.30) are valid for any general incident field, for practical purposes we shall restrict our attention to the case of uniform plane wave incidence. Figure 2.7 gives the geometry of the field quantities. Zero phase reference is chosen at the origin of the coordinate system, and the vector $\bar{k}$ has a magnitude k (free space wavenumber) and

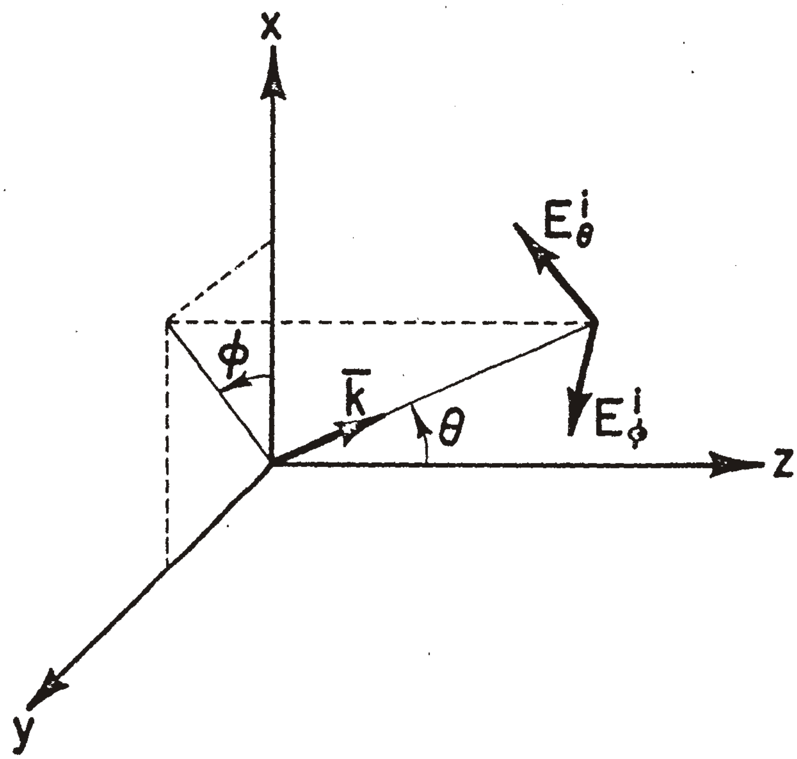


Figure 2.7. Incident field quantities.

points in the direction from which the wave propagates, i.e. in the direction of the source.

The incident electric field intensity may be expressed as

$$\bar{E}^i = \Big[(E_\theta^i \cos\theta\cos\phi - E_\phi^i \sin\phi)\hat{u}_x + (E_\theta^i \cos\theta\sin\phi + E_\phi^i \cos\phi)\hat{u}_y - E_\theta^i \sin\theta\,\hat{u}_z\Big] e^{jk\sin\theta\cos\phi x}\, e^{jk\sin\theta\sin\phi y}\, e^{jk\cos\theta z} \qquad (2.31)$$

The incident magnetic field intensity may be expressed in terms of the incident electric field intensity as

$$H_\phi^i = -\frac{E_\theta^i}{\eta_0} \quad \text{and} \quad H_\theta^i = \frac{E_\phi^i}{\eta_0} \qquad (2.32)$$

where η_0 is the intrinsic impedance of free space.

SECTION III

SOLUTION OF THE INTEGRAL EQUATIONS BY NUMERICAL METHODS

In order to numerically solve the system of integral equations developed in Section II we employ the well known method of moments [9]. We consider first a scheme for approximating an arbitrarily shaped aperture by a set of subdomains.

This problem requires two sets of subdomains, one associated with the surface integrals and one with the contour integrals. We choose the first set, which approximates the surface S of the aperture (refer to Figure 2.1), to be composed of rectangular subdomains, or patches, of equal size and shape, which are contained entirely within the bounding contour C of the aperture. We choose the other set, which approximates the smooth contour C, to be composed of straight-line segments, of as nearly equal length as possible, connecting points on the actual contour C. The normal and tangential unit vectors to C, $\hat{n}$ and $\hat{\tau}$ respectively, are approximated over the length of any arc of C by the normal and tangential unit vectors of the straight-line segments which approximate that arc. These subdomain regions and normal unit vectors are represented in Figure 3.1.

Derivation of the matrix equation

The two components of the vector integral equation (2.17) are

$$\frac{1}{2\pi}\iint_S M_{sx}\frac{e^{-jk|\bar{\rho}-\bar{\rho}'|}}{|\bar{\rho}-\bar{\rho}'|}dS' + \frac{1}{4j}\oint_C [S_n(\bar{\rho}')n_y(\bar{\rho}')+S_\tau(\bar{\rho}')n_x(\bar{\rho}')]H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell'$$

$$= \frac{k_z E_y^i}{4}\iint_S e^{jk_x x'} e^{jk_y y'} H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS' \qquad (3.1)$$

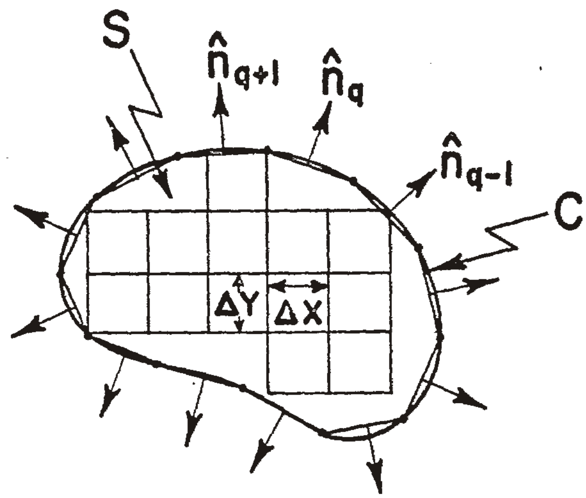


Figure 3.1. Approximation of an arbitrary-shaped aperture.

and

$$\frac{1}{2\pi}\iint_S M_{sy}\frac{e^{-jk|\bar{\rho}-\bar{\rho}'|}}{|\bar{\rho}-\bar{\rho}'|}dS' - \frac{1}{4j}\oint_C [S_n(\bar{\rho}')n_x(\bar{\rho}')-S_\tau(\bar{\rho}')n_y(\bar{\rho}')]H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell'$$

$$= -\frac{k_z E_x^i}{4}\iint_S e^{jk_x x'} e^{jk_y y'} H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS' \qquad (3.2)$$

where the incident field of Equation (2.31) has been used with the derivative operations performed.

We expand the unknown functions M_{sx} and M_{sy} in finite series of weighted two-dimensional unit pulse functions. If there are J total interior subdomains on S, then let

$$M_{s\binom{x}{y}}(\bar{\rho}) \simeq \sum_{j=1}^{J} M_{s\binom{x}{y}j} P_j(\bar{\rho}) \qquad (3.3)$$

where $M_{s\binom{x}{y}j}$ is the constant coefficient of the series associated with the x- or y-component of equivalent magnetic surface current in the j^{th} interior subdomain, and $P_j(\bar{\rho})$ is the two-dimensional pulse function

$$P_j(\bar{\rho}) = \begin{cases} 1 \ , & \text{for } \bar{\rho} \in \text{ in the } j^{th} \text{ interior subdomain} \\ 0 \ , & \text{for } \bar{\rho} \text{ elsewhere} \end{cases}$$

We also expand the unknown functions S_n and S_τ in finite series of weighted one-dimensional unit pulse functions. If there are P total straight-line subdomains approximating C, then let

$$S_{\binom{n}{\tau}}(\bar{\rho}) \simeq \sum_{p=1}^{P} S_{\binom{n}{\tau}p} Q_p(\bar{\rho}) \tag{3.4}$$

where $S_{\binom{n}{\tau}p}$ is the constant coefficient of the series associated with the normal or tangential component of the unknown vector function $\bar{S}(\bar{\rho})$ on the p^{th} boundary subdomain, and $Q_p(\bar{\rho})$ is the unit pulse function

$$Q_p(\bar{\rho}) = \begin{cases} 1 \ , & \text{for } \bar{\rho} \text{ on the } p^{th} \text{ boundary subdomain} \\ 0 \ , & \text{for } \bar{\rho} \text{ elsewhere} \end{cases}$$

Substitution of (3.3) and (3.4) into (3.1) and (3.2) gives us

$$\sum_{j=1}^{J} M_{sxj}\left(\frac{1}{2\pi}\right)\iint_{S_j}\frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}}{|\bar{\rho}_m-\bar{\rho}'|}dS' + \sum_{p=1}^{P}\left[S_{np}\left(\frac{1}{4j}\right)n_{yp}\int_{C_p} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)d\ell'\right.$$

$$\left. + S_{\tau p}\left(\frac{1}{4j}\right)n_{xp}\int_{C_p} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)d\ell'\right]$$

$$= \sum_{j=1}^{J}\frac{k_z E_y^i}{4}\iint_{S_j} e^{jk_x x'} e^{jk_y y'} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)dS' \tag{3.5}$$

and

$$\sum_{j=1}^{J} M_{syj}\left(\frac{1}{2\pi}\right)\iint_{S_j}\frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}}{|\bar{\rho}_m-\bar{\rho}'|}dS' + \sum_{p=1}^{P}\Bigg[S_{np}\left(\frac{1}{4j}\right)(-n_{xp})\int_{C_p} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)d\ell'$$

$$+ S_{\tau p}\left(\frac{1}{4j}\right)n_{yp}\int_{C_p} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)d\ell'\Bigg]$$

$$= \sum_{j=1}^{J}\frac{-k_z E_x^i}{4}\iint_{S_j} e^{jk_x x'} e^{jk_y y'} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)dS' \qquad (3.6)$$

where S_j denotes the surface of the j^{th} interior subdomain, C_p denotes the p^{th} straight-line segment in the approximation of the contour C, and n_{xp} and n_{yp} represent the x- and y-components, respectively, of the unit vector normal to segment C_p.

The subscripts m on the independent variable $\bar{\rho}$ in (3.5) and (3.6) refer to the fact that we evaluate both equations for observation points $\bar{\rho}_m$, m=1, ... , J, located at the center of each interior subdomain. This procedure is commonly called point-matching or collocation [9].

The four surface integrals appearing in (3.5) and (3.6) all have singular integrands which present numerical difficulties when j=m, that is, when integration is to be performed over the interior subdomain containing the match point of the equation.

The singularity in the integrand of the vector potential surface integral can be treated in the following fashion. First, we expand $\exp(-jk|\bar{\rho}_m-\bar{\rho}'|)/|\bar{\rho}_m-\bar{\rho}'|$ in a Taylor series about $\bar{\rho}_m$ and note that the first term alone contains the singular behavior:

$$\frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}}{|\bar{\rho}_m-\bar{\rho}'|} = \frac{1}{|\bar{\rho}_m-\bar{\rho}'|} + (-jk) + \frac{(-jk)^2}{2!}|\bar{\rho}_m-\bar{\rho}'| + \frac{(-jk)^3}{3!}|\bar{\rho}_m-\bar{\rho}'|^2 + \ldots \tag{3.7}$$

If we subtract and add the singular term to the integrand, we obtain a well-behaved (non-singular) function $(\exp(-jk|\bar{\rho}_m-\bar{\rho}'|)-1)/|\bar{\rho}_m-\bar{\rho}'|$ and a singular function $1/|\bar{\rho}_m-\bar{\rho}'|$. The first function may be integrated numerically with no difficulty (provided the function is defined to be equal to $-jk$ when $\bar{\rho}' = \bar{\rho}_m$), while the second function may be integrated analytically to obtain

$$\int_{x'=x_j-\frac{\Delta x}{2}}^{x_j+\frac{\Delta x}{2}} \int_{y'=y_j-\frac{\Delta y}{2}}^{y_j+\frac{\Delta y}{2}} \frac{1}{|\bar{\rho}_m-\bar{\rho}'|}\,dy'dx' = 2\left[\Delta x\,\ell n\left[\frac{\Delta y}{\Delta x} + \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\right] + \Delta y\,\ell n\left[\frac{\Delta x}{\Delta y} + \sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}\right]\right] \tag{3.8}$$

where Δx and Δy are the dimensions of an interior subdomain (Figure 3.1). Thus we have

$$\iint_{S_j} \frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}}{|\bar{\rho}_m-\bar{\rho}'|}dS' = \begin{cases} \displaystyle\iint_S \frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}}{|\bar{\rho}_m-\bar{\rho}'|}dS' & \text{, for } j \neq m \\ \displaystyle\iint_S \frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}-1}{|\bar{\rho}_m-\bar{\rho}'|}dS' + 2\left[\Delta x\,\ell n\left[\frac{\Delta y}{\Delta x} + \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\right] + \Delta y\,\ell n\left[\frac{\Delta x}{\Delta y} + \sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}\right]\right] & \text{, for } j=m \end{cases} \tag{3.9}$$

The singularity in the integrand of the surface integral involving the incident field and the Hankel function may be treated in a similar fashion. First, we approximate the phase variation of the incident field across one interior subdomain by its value at the center of the subdomain to obtain

$$\iint_{S_j} e^{jk_x x'} e^{jk_y y'} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)\,dS' \simeq e^{jk_x x_j} e^{jk_y y_j} \iint_{S_j} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)\,dS' \tag{3.10}$$

We now note the small argument approximation for the zeroth order Hankel function of the second kind

$$H_0^{(2)}(k\sqrt{x^2+y^2}) \simeq 1 - j\frac{2}{\pi}\ln\frac{k\gamma\sqrt{x^2+y^2}}{2} \quad , \quad \gamma = 1.7810\ldots$$

and again subtract and add the singular term to obtain

$$\iint_{S_j} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)\,dS' = \iint_{S_j}\left[H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|) + j\frac{2}{\pi}\ln\frac{k\gamma}{2}|\bar{\rho}_m-\bar{\rho}'|\right]dS'$$
$$- j\frac{2}{\pi}\Delta x\Delta y\ln(\frac{k\gamma}{2}) - j(\frac{1}{\pi})\iint_{S_j}\ln|\bar{\rho}_m-\bar{\rho}'|\,dS' \tag{3.11}$$

for the case j=m. The first integral on the right side of (3.11) is non-singular and may be integrated numerically. The last integral may be integrated analytically to obtain

$$\iint_{S_j}\ln|\bar{\rho}_m-\bar{\rho}'|\,dS' = \Delta x\Delta y\ln\left[\frac{\Delta x^2+\Delta y^2}{4}\right] - 3\Delta x\Delta y$$
$$+ \Delta x^2\tan^{-1}(\frac{\Delta y}{\Delta x}) + y^2\tan^{-1}(\frac{\Delta x}{\Delta y}) \quad . \tag{3.12}$$

Hence we write

$$\iint_{S_j} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)dS' = \begin{cases} \displaystyle\iint_{S_j} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)dS' \quad , \text{ for } j \neq m \\ \displaystyle\iint_{S_j} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|) + j\frac{2}{\pi}\ell n(\frac{k\gamma}{2}|\bar{\rho}_m-\bar{\rho}'|)\ dS' \\ \quad -j\frac{2}{\pi}\Delta x\Delta y\ell n(\frac{k\gamma}{2}) - \frac{j}{\pi}\Big[\Delta x\Delta y\ell n(\frac{\Delta x^2+\Delta y^2}{4}) - 3\Delta x\Delta y \\ \quad + \Delta x^2\tan^{-1}(\frac{\Delta y}{\Delta x}) + \Delta y^2\tan^{-1}(\frac{\Delta x}{\Delta y})\Big] \quad , \text{ for } j=m. \end{cases} \tag{3.13}$$

The surface integrals are performed numerically using a Gaussian quadrature scheme in two dimensions. All the contour integrals are performed numerically using a Gaussian quadrature scheme which integrates along the straight-line segments C_p in the x-y plane. The Hankel functions in the integrands are generated from the power series expansions of the Bessel and Neumann functions found in Abramowitz and Stegun [10].

The auxiliary condition, Equation (2.29), with the substitution of (3.4) for $\bar{S}$, becomes

$$\sum_{p=1}^{P}\Bigg[S_{np}\Bigg[2j\delta_{pq}-k(1-\delta_{pq})\int_{C_p}\left(\frac{n_{xp}(x_{cq}-x')+n_{yp}(y_{cq}-y')}{|\bar{\rho}_{cq}-\bar{\rho}'|}\right)H_1^{(2)}(k|\bar{\rho}_{cq}-\bar{\rho}'|)\,d\ell'\Bigg]$$

$$+\,S_{\tau p}\Bigg[-k(1-\gamma_{pq})\int_{C_p}\left(\frac{n_{xp}(y_{cq}-y')-n_{yp}(x_{cq}-x')}{|\bar{\rho}_{cq}-\bar{\rho}'|}\right)H_1^{(2)}(k|\bar{\rho}_{cq}-\bar{\rho}'|)\,d\ell'\Bigg]\Bigg]$$

$$=\sum_{p=1}^{P}\Bigg[E_z^i\Bigg[2j\delta_{pq}-k(1-\delta_{pq})\int_{C_p}e^{jk_xx'}e^{jk_yy'}\left(\frac{n_{xp}(x_{cq}-x')+n_{yp}(y_{cq}-y')}{|\bar{\rho}_{cq}-\bar{\rho}'|}\right)$$

$$\cdot H_1^{(2)}(k|\bar{\rho}_{cq}-\bar{\rho}'|)\,d\ell'\Bigg]$$

$$+\,j\omega\mu_0(1-\delta_{pq})\left(H_y^i n_{xp}-H_x^i n_{yp}\right)e^{jk_xx_p}e^{jk_yy_p}\int_{C_p}H_0^{(2)}(k|\bar{\rho}_{cq}-\bar{\rho}'|)\,d\ell'\Bigg] \qquad (3.14)$$

where δ_{pq} is the Kronecker delta $\delta_{pq}=\begin{cases}1 , & p=q\\ 0 , & p\neq q\end{cases}$ and the subscript q refers to the fact that we evaluate (3.14) for observation points $\bar{\rho}_{cq}$, q=1, ..., P, located at the center, or match point, of each boundary subdomain C_q.

Note that the deleted integrals of Equation (2.29) are numerically handled by not performing the integration over the subdomain in which the independent variable $\bar{\rho}_q$ and the integration variable $\bar{\rho}'$ coincide. The numerical contour integration is performed with the same Gaussian quadrature routine mentioned previously.

The magnetic current boundary condition, Equation (2.30), with the substitution of (3.3) for $\bar{M}_s$, becomes

$$\sum_{q=1}^{P}\left[M_{sxj(q)}n_{xq} + M_{syj(q)}n_{yq} \; P_{j(q)}\right] = 0 \quad , \qquad (3.15)$$

where j(q) is the index of the interior subdomain adjacent to the q^{th} boundary subdomain. Note that for apertures with right angle corners whose sides are parallel to the x and y axes, each corner subdomain has two boundary conditions imposed. That is, the current components normal to each face of such a corner subdomain are forced to zero.

Examining Equations (3.5), (3.6), (3.14), and (3.15), we find that we have a system of (2J + 2P) equations to solve for (2J + 2P) unknown constants. In matrix form this system is

$$\left[\begin{array}{c|c|c|c} A & 0 & B_y & B_x \\ \hline 0 & A & -B_x & B_y \\ \hline 0 & 0 & E & F \\ \hline G & H & 0 & 0 \end{array}\right] \left[\begin{array}{c} M_{sx} \\ \hline M_{sy} \\ \hline S_n \\ \hline S_\tau \end{array}\right] = \left[\begin{array}{c} Q \\ \hline R \\ \hline T \\ \hline 0 \end{array}\right] \qquad (3.16)$$

where $Q=CD_x$, $R=CD_y$, $B_x=BN_x$, $B_y=BN_y$, and $T=EE_z+LH_{tan}$.

Typical elements of the submatrices and vectors identified above are listed in the following:

$$A = \left[A_{mj}\right] \quad , \quad A_{mj} = \frac{1}{2\pi}\iint_{S_j} \frac{e^{-jk|\bar{\rho}_m-\bar{\rho}'|}}{|\bar{\rho}_m-\bar{\rho}'|} dS' \quad ;$$

$$B = \left[B_{mp}\right] \quad , \quad B_{mp} = \frac{1}{4j}\int_{C_p} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)\,d\ell' \quad ;$$

$$C = \left[C_{mj}\right] \quad , \quad C_{mj} = \iint_{S_j} H_0^{(2)}(k|\bar{\rho}_m-\bar{\rho}'|)\,dS' \quad ;$$

$$D_x = \left[D_{xj}\right] \quad , \quad D_{xj} = \frac{k_z E_y^i}{4}\, e^{jk_x x_j} e^{jk_y y_j} \quad ;$$

$$D_y = \left[D_{yj}\right] \quad , \quad D_{yj} = -\frac{k_z E_x^i}{4}\, e^{jk_x x_j} e^{jk_y y_j} \quad ;$$

$$E = \left[E_{qp}\right] \quad , \quad E_{qp} = 2j\delta_{pq} - k(1-\delta_{pq})\int_{C_p}\left(\frac{n_{xp}(x_{cq}-x') + n_{yp}(y_{cq}-y')}{|\bar{\rho}_{cq}-\bar{\rho}'|}\right)$$

$$\cdot H_1^{(2)}(k|\bar{\rho}_{cq}-\bar{\rho}'|)\,d\ell' \quad ;$$

$$E_z = \left[E_{zp}\right] \quad , \quad E_{zp} = E_z^i\, e^{jk_x x_{cp}} e^{jk_y y_{cp}} \quad ;$$

$$F = \left[F_{qp}\right] \quad , \quad F_{qp} = -k(1-\delta_{pq})\int_{C_p}\left(\frac{n_{xp}(y_{cq}-y') - n_{xp}(x_{cq}x')}{|\bar{\rho}_{cq}-\bar{\rho}'|}\right)$$

$$\cdot H_1^{(2)}(k|\bar{\rho}_{cq}-\bar{\rho}'|)\,d\ell \quad ;$$

$$G = \left[G_{qj}\right] \quad , \quad G_{qj} = n_{xq} P_j(q) \quad ;$$

$$H = \left[H_{qj}\right] \quad , \quad H_{qj} = n_{yq} P_j(q)$$

$$H_{tan} = \left[H_{tan\ p}\right], \quad H_{tan\ p} = (H_y^i n_{xp} - H_x^i n_{yp}) e^{jk_x x_p} e^{jk_y y_p} \quad ;$$

$$L = \left[L_{qp}\right] \quad , \quad L_{qp} = j\omega\mu_0 (1-\delta_{pq}) \int_{C_p} H_0^{(2)}(k|\bar{\rho}_{cq} - \bar{\rho}'|) d\ell' \quad ;$$

$$M_{sx} = \left[M_{sxj}\right] \quad , \quad M_{sxj} = M_{sxj} \quad ;$$

$$M_{sy} = \left[M_{syj}\right] \quad , \quad M_{syj} = M_{syj} \quad ;$$

$$S_n = \left[S_{np}\right] \quad , \quad S_{np} = S_{np} \quad ;$$

$$S_\tau = \left[S_{\tau p}\right] \quad , \quad S_{\tau p} = S_{\tau p} \quad .$$

The matrices N_x and N_y are defined as

$$N_x = \begin{bmatrix} n_{x1} & & & 0 \\ & n_{x2} & & \\ & & \ddots & \\ 0 & & & n_{xP} \end{bmatrix} \quad \text{and} \quad N_y = \begin{bmatrix} n_{y1} & & & 0 \\ & n_{y2} & & \\ & & \ddots & \\ 0 & & & n_{yP} \end{bmatrix}$$

<u>Algorithm for the solution of the matrix equation</u>

Due to limitations in the amount of storage available on the digital computer used in solving this problem, a direct solution by inversion of the large left-hand side matrix of Equation (3.16) is not considered feasible. Furthermore, the sparse nature of the matrix resulting from the decoupling of the unknowns M_{sx} and M_{sy} in the formulation makes such a procedure inefficient, as well. Consequently, we derive an algorithm which efficiently solves the partitioned system and which allows the use of the same storage areas for different submatrices at different points in the computations.

First, we express Equation (3.16) as four separate simultaneous matrix equations as follows:

$$AM_{sx} + B_y S_n + B_x S_\tau = CD_x = Q \tag{3.17}$$

$$AM_{sy} - B_x S_n + B_y S_\tau = CD_y = R \tag{3.18}$$

$$ES_n + FS_\tau = EE_z + LH_{tan} = T \tag{3.19}$$

$$GM_{sx} + HM_{sy} = 0 \tag{3.20}$$

Solving (3.19) for S_n, we have

$$S_n = E^{-1}T - E^{-1}FS_\tau = E_z + E^{-1}LH_{tan} - E^{-1}FS_\tau \qquad (3.21)$$

We note that E^{-1} exists since it is obtained from the same integral operation as that appearing in the H-field equation for the TE scattering of conducting cylinders [9]. F^{-1}, on the other hand, does not exist in general. Solving (3.17) and (3.18) for M_{sx} and M_{sy} yields

$$M_{sx} = A^{-1}Q - A^{-1}B_y S_n - A^{-1}B_x S_\tau \qquad (3.22)$$

$$M_{sy} = A^{-1}R + A^{-1}B_x S_n - A^{-1}B_y S_\tau \qquad (3.23)$$

where we substitute, along with (3.21) into (3.20) to obtain

$$GA^{-1}Q - GA^{-1}B_y E^{-1}T + GA^{-1}B_y E^{-1}FS_\tau - GA^{-1}B_x S_\tau \qquad (3.24)$$

$$+ HA^{-1}R + HA^{-1}B_x E^{-1}T - HA^{-1}B_x E^{-1}FS_\tau - HA^{-1}B_y S_\tau = 0$$

Solving (3.24) for S_τ gives

$$S_\tau = \left[GA^{-1}B_y E^{-1}F - GA^{-1}B_x - HA^{-1}B_x E^{-1}F - HA^{-1}B_y\right]^{-1}$$

$$\cdot \left[-GA^{-1}Q + GA^{-1}B_y E^{-1}T - HA^{-1}R - HA^{-1}B_x E^{-1}T\right]$$

which may be substituted into (3.21) to obtain S_n, at which point everything necessary to solve for M_{sx} and M_{sy} by (3.22) and (3.23) is available. A flow chart of the algorithm used to solve this system of equations is given in Figure 3.2.

Figure 3.2

Flow Chart of the Algorithm for Solution of the Matrix Equation

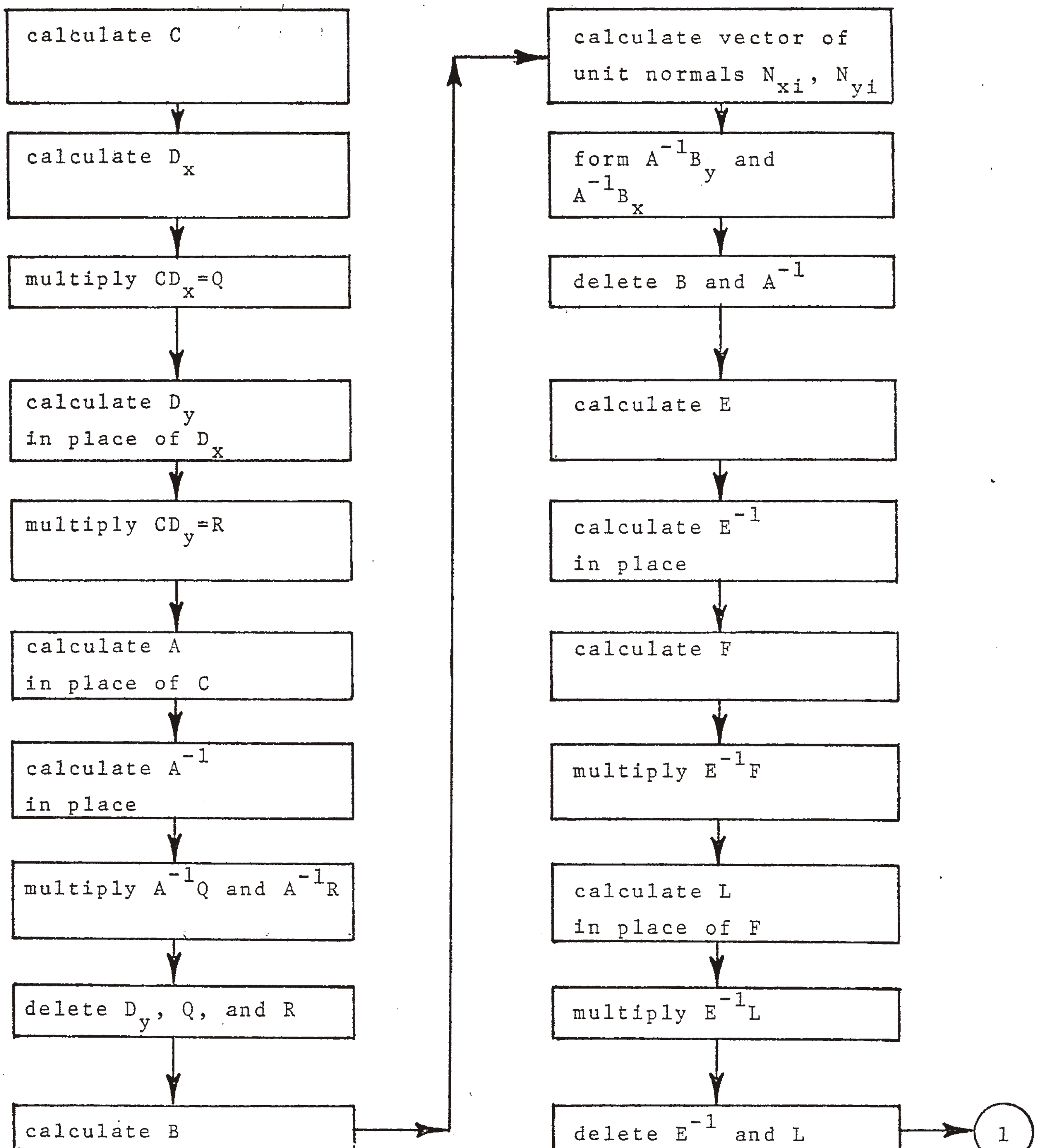


Figure 3.2 continued

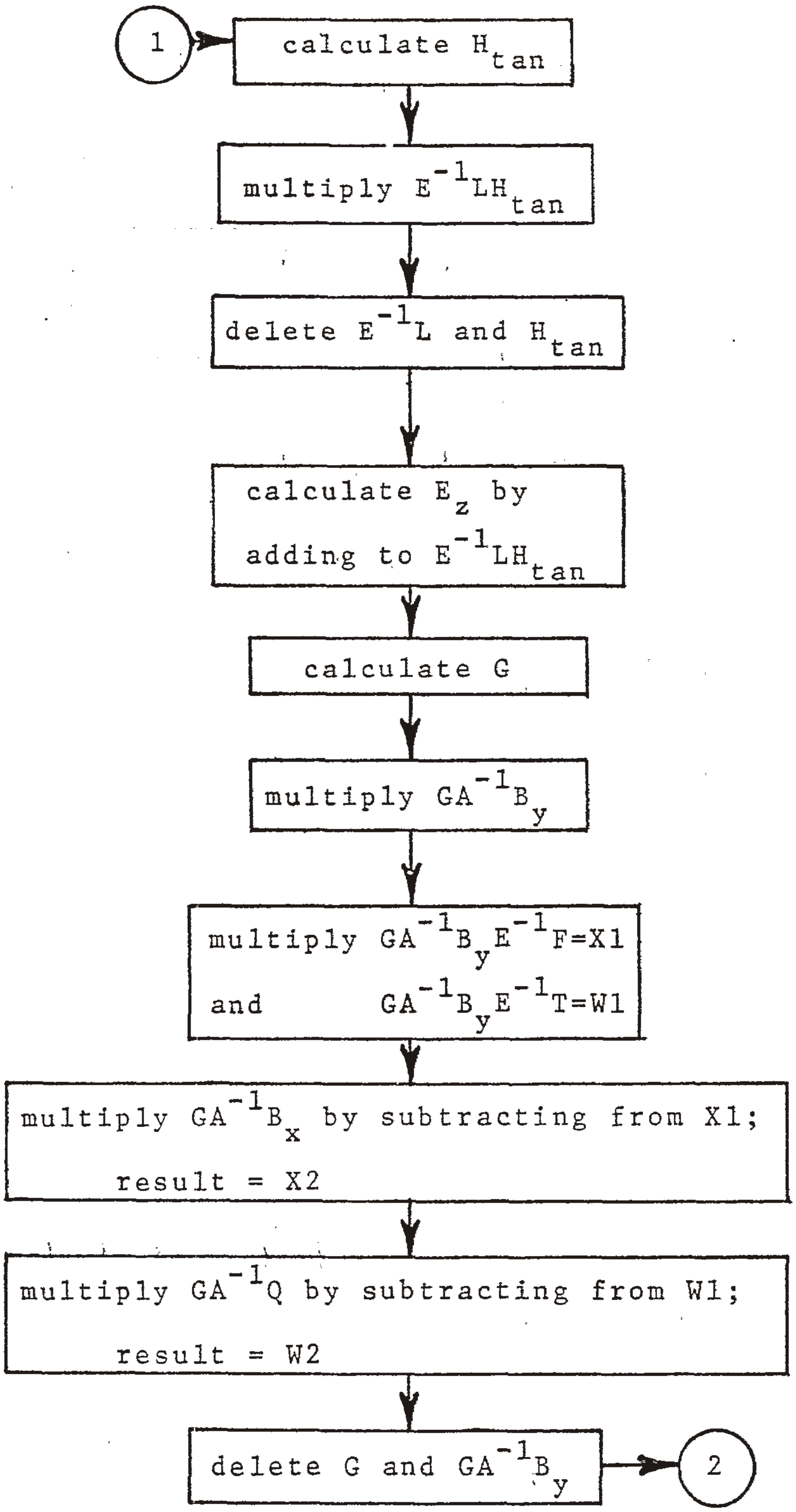


Figure 3.2 continued

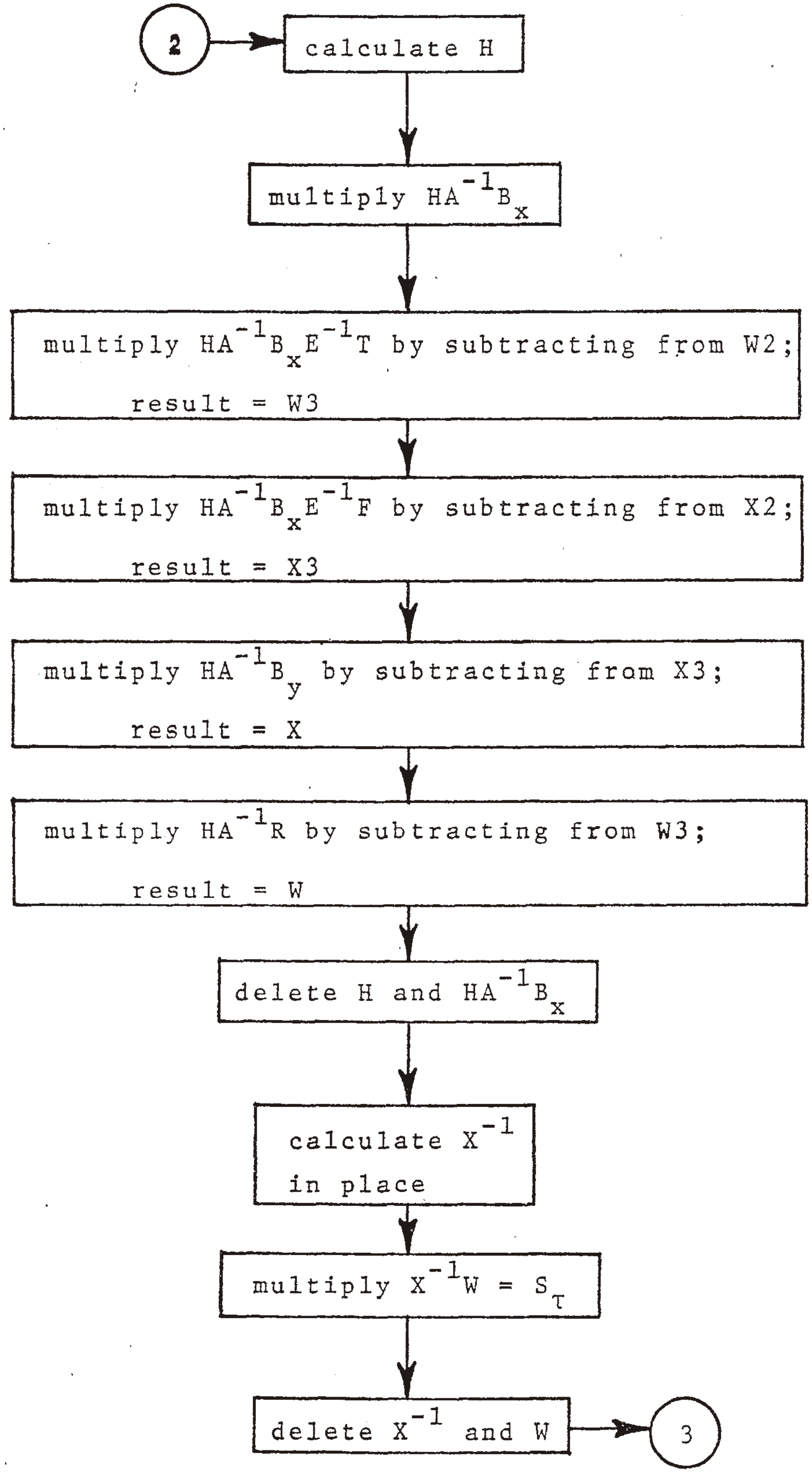
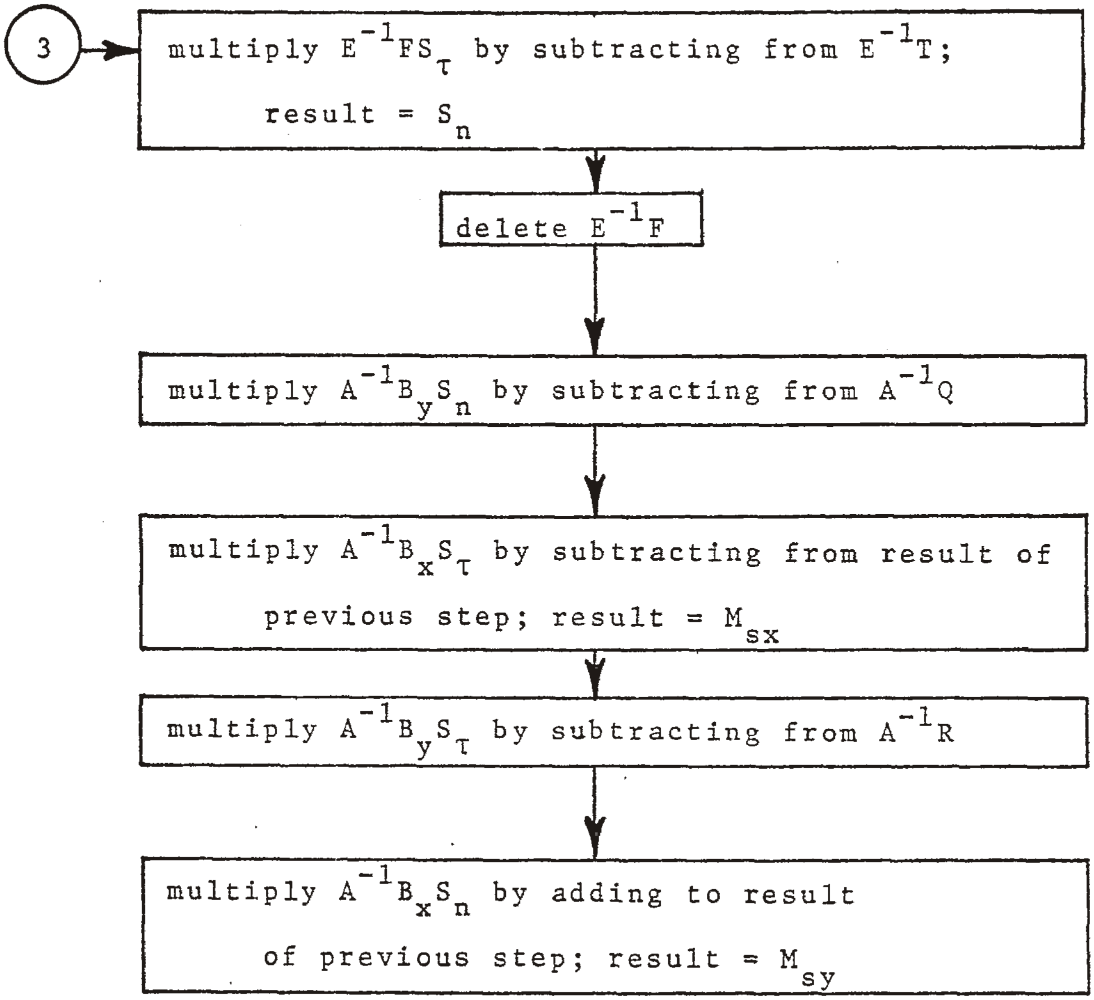


Figure 3.2 continued



The flow chart refers to matrix multiplication by adding to (subtracting from) a given quantity. This simply means that elements of a product of two matrices are added to (subtracted from) elements of a third matrix as they are computed. This eliminates the need for a separate storage area for a matrix product which is to be added to (subtracted from) another matrix. Calculations done "in place" refer to operations on a matrix which store the resultant matrix in the same storage area as the original matrix or overwrites the storage area of a previously used matrix.

All matrix inversion is performed numerically using a Gauss-Jordan elimination algorithm which uses the largest element of the remaining unreduced array as the pivot element. Numerical data are given in a separate report.

APPENDIX

SOLUTION OF THE TWO-DIMENSIONAL INHOMOGENEOUS WAVE EQUATION

We wish to consider a representation for a general solution to the two-dimensional inhomogeneous wave equation

$$\left[\nabla_t^2 + k^2\right] U = f \quad , \quad (\bar{\rho}\varepsilon S) \tag{1}$$

where S is a region in space bounded by a contour C whose outward normal is $\hat{n}$. Each of the vector components of Equation (2.15), for example, satisfy such an equation. We construct the desired representation by beginning with a fundamental solution to the problem

$$\left[\nabla_t^2 + k^2\right] G = -\delta(\bar{\rho}-\bar{\rho}') \quad , \tag{2}$$

for which one solution is

$$G = \frac{1}{4j} H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|) \quad . \tag{3}$$

Making use of Green's second identity in two dimensions,

$$\iint_S \left[A\nabla_t^2 B - B\nabla_t^2 A\right] dS' = \oint_C \left[A\nabla_t' B - B\nabla_t' A\right] \cdot \hat{n}' d\ell'$$

setting A=G and B=U, together with (1) and (2), we immediately obtain

$$\frac{1}{4j}\iint_S f H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS' - \frac{1}{4j}\oint_C \Big[H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\nabla_t' U$$
$$- U\nabla_t' H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\Big] \cdot \hat{n}' d\ell' = \begin{cases} -U(\bar{\rho}) & , \quad (\rho\varepsilon S) \\ 0 & , \quad \rho\notin S) \end{cases} \tag{4}$$

At this point we define the surface S^c, the complement of the surface S. That is, S^c is the entire plane except for the surface S. We further define a general function V which satisfies $\left[\nabla_t^2+k^2\right]V = 0$, $(\bar{\rho}\varepsilon S^c)$, and specify that V satisfy the radiation condition

$$\lim_{\rho\to\infty} \sqrt{\rho}\left[jkV + \frac{\partial V}{\partial \rho}\right] = 0$$

We again apply the scalar form of Green's second identity in two dimensions, this time over S^c, taking into account the direction of $\hat{n}$ into S^c, to obtain

$$-\frac{1}{4j}\oint_C\left[H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\nabla_t'V - V\nabla_t'H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\right]\cdot\hat{n}'d\ell' = \begin{cases} V(\bar{\rho}) & , \bar{\rho}\varepsilon S^c \\ 0 & , \bar{\rho}\varepsilon S \end{cases} . \tag{5}$$

We add Equations (4) and (5) to obtain

$$\frac{1}{4j}\iint_S fH_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS' - \frac{1}{4j}\oint_C H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\left[\nabla_t'U+\nabla_t'V\right]\cdot\hat{n}'d\ell' \tag{6}$$

$$+\frac{1}{4j}\oint_C (U+V)\nabla_t'H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)\cdot\hat{n}'d\ell' = \begin{cases} -U(\bar{\rho}) & , (\bar{\rho}\varepsilon S) \\ V(\bar{\rho}) & , (\bar{\rho}\varepsilon S^c) \end{cases} .$$

Since V is arbitrary, we may simplify (6) by specifying $-U=V$ for $\bar{\rho}$ on C, which gives finally

$$\frac{1}{4j}\iint_S f(\bar{\rho}')H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)dS' - \frac{1}{4j}\int_C T(\bar{\rho}')H_0^{(2)}(k|\bar{\rho}-\bar{\rho}'|)d\ell' = \begin{cases} -U(\bar{\rho}) & , (\bar{\rho}\varepsilon S) \\ V(\bar{\rho}) & , (\bar{\rho}\varepsilon S^c) \end{cases} . \tag{7}$$

where

$$T(\bar{\rho}') = \left[\nabla_t' U + \nabla_t' V\right] \cdot \hat{n}(\bar{\rho}')$$

This is the desired representation for $U(\bar{\rho})$ when $\bar{\rho}$ is on S. We note that Equation (7) is an equivalence theorem which states that the contribution to the field in a two-dimensional region S is given by an integral over the sources inside S (a particular solution to (1) plus an integration over a set of like sources distributed on the boundary of S, which set accounts for the sources outside S (a homogeneous solution of (1)).

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